CURVED FINITE ELEMENTS IN THE ANALYSIS OF SHELL STRUCTURES BY THE FINITE ELEMENT METHOD

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ABSTRACT

The application of the finite element method to shell analysis has received considerable attention is recent years. This paper reviews the current stage of development for curved elements whose properties are based on the assumption of the Kirchoff hypothesis and on the application of the minimum potential energy principle. Particular attention is paid to the requirements of assumed displacement fields for the curved elements. Some comparative numerical results of the finite-element studies are presented where appropriate.

1. INTRODUCTION

The thin shell is an important structural component in many branches of engineering including that of reactor technology. With the advent of the digital computer and the development of numerical procedures such as the finite element method, the solution of general shell problems has become a practical possibility.

In recent years a great deal of effort has been concentrated on the development of suitable finite element models for thin shell analysis and many papers have been published on the topic. A large majority of the finite-element analyses to date have been based on the minimum potential energy principle and on the Kirchoff hypothesis of maintenance of the shell normal. This paper reviews the considerable literature relating only to this type of analysis and involving elements of curved geometry. Because of space limitations no discussion is made of the use of flat plate elements in shell analysis and it is not possible to consider other types of finite-element approach involving the relaxation of the Kirchoff hypothesis, the use of three-dimensional elements or the use of variational principles other than that of minimum potential energy.

The analysis of shell structures by the finite element displacement method is complicated by the fact that there is no one system of equations in shell theory which is generally accepted. Typical shell equations are presented in Section 2 where brief discussion is made of the nature of the equations. The form of the shell element stiffness matrix is described in Section 3 and this is followed in the next section by a consideration of the requirements of element displacement fields. Since the great majority of available curved elements are of a specific geometry various element categories are considered separately in Sections 5 to 8. The categori s considered are shell-of-revolution elements, cylindrical-shell elements, elements based on shallow-shell theory and elements of a general geometry. Short conclusions in Section 9 complete the paper.

2. SHELL EQUATIONS

Love's first approximation theory of shells [1], [2] (and the version thereof due to Reissner [3]) has occupied a central position in shell theory since its inception despite some inconsistency with regard to small terms. Numerous modifications have been made to the theory by adding or omitting small terms and so various versions of the shell formulae exist (see the books of Washizu [4] and Kraus [5] for references on this topic) which differ only slightly one from another and which give very similar results in applications to many problems. No particular set of equations has met with universal approval.

In selecting shell theories for use with the finite element displacement method a particular consideration is the question as to whether the strain-displacement equations allow the proper representation of rigid-body motions. If a shell theory is consistent in this regard then the true rigid-body motions will satisfy the homogeneous strain-displacement equations. Love's theory does not meet this requirement for the general shell since the rigid-body rotation about a shell normal gives rise to non-zero additional twist.

The geometry of an arbitrary shell is characterised by a system of curvilinear coordinates α and β in the shell middle-surface and a normal coordinate ζ (see Fig. 1). It is assumed that the curvilinear coordinates form an orthogonal net on the shell surface and that the usual assumptions regarding isotropy, homogeneity, thinness and smallness of displacements apply. Koiter [6] has pointed out that the Kirchoff hypothesis in its usual form is contradictory; a consistent equivalent assumption is that the state of stress in the shell is approximately plane or, in other words, that the effect of transverse shear stresses and of transverse normal stress acting on surfaces parallel to the moddle surface may be neglected in the strain energy expression. The strain energy density, W, may then be expressed as the sum of the extensional and flexural energies as

$$W = \frac{Eh}{2(1-\nu^2)}\left[(\varepsilon_1+\varepsilon_2)^2 - 2(1-\nu)(\varepsilon_1\varepsilon_2 - \frac{\psi^2}{4}) + \frac{h^2}{12}\{(\chi_1+\chi_2)^2 - 2(1-\nu)(\chi_1\chi_2 - \tau^2)\}\right] \quad (1)$$

Here ε_1 and ε_2 are the extensions along the orthogonal parametric curves α, β and ψ is the shear strain between these curves; χ_1, χ_2 and τ are the physical components of the changes of curvature and twist referred to these curves. Most shell theories use an orthogonal curvilinear coordinate system which coincides with the lines of principal curvature of the surface. Koiter's theory, however, removes this restriction and yields the following strain-displacement equations:-

$$\begin{aligned}
\varepsilon_1 &= \frac{1}{A}\frac{\partial u}{\partial\alpha} + \frac{v}{AB}\frac{\partial A}{\partial\beta} - \frac{w}{R_1}, \\
\varepsilon_2 &= \frac{1}{B}\frac{\partial v}{\partial\beta} + \frac{u}{AB}\frac{\partial B}{\partial\alpha} - \frac{w}{R_2}, \\
\psi &= \frac{1}{A}\frac{\partial v}{\partial\alpha} + \frac{1}{B}\frac{\partial u}{\partial\beta} - \frac{u}{AB}\frac{\partial A}{\partial\beta} - \frac{v}{AB}\frac{\partial B}{\partial\alpha} - \frac{2w}{T}, \\
\chi_1 &= \frac{1}{A}\frac{\partial\phi_1}{\partial\alpha} + \frac{\phi_2}{AB}\frac{\partial A}{\partial\beta} + \frac{\Omega}{T}, \\
\chi_2 &= \frac{1}{B}\frac{\partial\phi_2}{\partial\beta} + \frac{\phi_1}{AB}\frac{\partial B}{\partial\alpha} - \frac{\Omega}{T}, \\
2\tau &= \frac{1}{A}\frac{\partial\phi_2}{\partial\alpha} + \frac{1}{B}\frac{\partial\phi_1}{\partial\beta} - \frac{\phi_1}{AB}\frac{\partial A}{\partial\beta} - \frac{\phi_2}{AB}\frac{\partial B}{\partial\alpha} - (\frac{1}{R_1} - \frac{1}{R_2})\,\Omega .
\end{aligned}$$

In these equations u and v are the tangential displacements along the α and β curves and w is the normal displacement; R_1, R_2 and T are the radii of curvature and torsion of the initial middle surface; A and B are the Lamé parameters such that $Ad\alpha$ and $Bd\beta$ are line elements along the α and β curves. The functions Φ_1 and Φ_2 represent the rotation of the tangents to the middle-surface oriented along α_1 and α_2 respectively whilst Ω is the rotation in the middle surface around the normal. These quantities are defined in terms of the displacements as

$$\Phi_1 = \frac{1}{A}\frac{\partial w}{\partial \alpha} + \frac{u}{R_1} + \frac{v}{T},$$

$$\Phi_2 = \frac{1}{B}\frac{\partial w}{\partial \beta} + \frac{v}{R_2} + \frac{u}{T}, \qquad (3a\text{-}c)$$

and

$$2\Omega = \frac{1}{A}\frac{\partial v}{\partial \alpha} - \frac{1}{B}\frac{\partial u}{\partial \beta} - \frac{u}{AB}\frac{\partial A}{\partial \beta} + \frac{v}{AB}\frac{\partial B}{\partial \alpha}.$$

The stress resultants and couples may be introduced as partial derivatives of the strain energy density with respect to the strains and curvature changes. Thus the symmetric physical stress resultants and couples are given as

$$N_1 = \frac{Eh}{(1-\nu^2)}(\varepsilon_1 + \nu\varepsilon_2), \quad N_2 = \frac{Eh}{(1-\nu^2)}(\varepsilon_2 + \nu\varepsilon_1), \quad S = \frac{Eh}{2(1+\nu)}\psi$$

$$M_1 = \frac{Eh^3}{12(1-\nu^2)}(\chi_1 + \nu\chi_2), \quad M_2 = \frac{Eh^3}{12(1-\nu^2)}(\chi_2 + \nu\chi_1), \quad W = \frac{Eh^3}{12(1+\nu)}\tau \qquad (4)$$

where $N_1 = \int_\zeta \sigma_1(1 + \frac{\zeta}{R_2})\, d\zeta, \quad M_1 = \int_\zeta \sigma_1(1 + \frac{\zeta}{R_2})\, \zeta\, d\zeta$ etc. (5)

The equilibrium conditions for the shell, which require the definition of modified asymmetric stress resultants, need not be presented here since they are not directly required in the variational formulation.

The above expressions for the strain energy and the first five of the strain-displacement equations are commonly accepted in shell theory (T being infinite, though, in those theories where the curvilinear coordinates are chosen to coincide with the lines of principal curvature). The expression for the additional twist, however, differs somewhat in the various theories by small terms. Koiter has pointed out that it is permissable to add terms of the type ε/R to expressions for the physical components of the changes in curvature and that curvature expressions which differ only by terms of this type are equivalent in Love's first approximation. However, such terms are significant in properly representing rigid-body motions. Koiter's theory leads to vanishing strains and curvatures in a general rigid-body motion whilst in Reissner's version [3] of Love's theory the last term on the right-hand side of the expression for change of twist is not present and this results in a non-zero twist for a general rigid-body rotation. The shell theory of Novozhilov [7] has been widely used in finite element analyses and, although differing in the last term of the twist expression, eq. (3f), (and in that T is infinite in the Novozhilov equations) the general rigid-body motion is correctly represented in this theory too.

3. ELEMENT STIFFNESS MATRIX

In an approximate analysis by the finite element displacement method a continuum structure is envisaged as an assemblage of individual regions or elements and displacement patterns are assumed to represent the behaviour within the element. The displacement patterns may be expressed in terms of the generalised displacements of the element either directly or indirectly via a number of coefficients which can be evaluated in terms of the generalised displacements using the element boundary (nodal) conditions. In either case the displacement field can ultimately be written in matrix form as

$$\{u\} = [A]\{q_e\} \tag{6}$$

where $\{u\} = \{u, v, w\}$, [A] is a matrix whose terms are functions of the coordinates and $\{q_e\}$ is a vector of element generalised displacements.

The strain-displacement relationships, eq. (2) and (3) for instance, can be expressed as

$$\{\varepsilon\} = [B]\{u\} \tag{7}$$

where $\{\varepsilon\}$ is the vector of membrane strains and curvatures and [B] is, for the general case, a 6x6 matrix whose terms contain first and second order differential operators. The strain energy density, as given by eq. (1) for instance for the general shell, can be written in the matrix form

$$W = \tfrac{1}{2}[\varepsilon][C]\{\varepsilon\} \tag{8}$$

where the 6x6 matrix [C] relates to the constitutive equations (eq. (4)) and contain terms which are functions of the elastic constants and the shell thickness.

Using equations (6), (7) and (8) the strain energy density can further be written in terms of the generalised displacements and from this the element stiffness matrix is obtained in the form

$$[k] = \int\int_{Ae} [A]^T [B]^T [C] [B] [A]\, AB d\alpha d\beta \tag{9}$$

where Ae denotes the surface area of the element.

It is seen that the stiffness matrix of the shell element is characterised by three matrices formed from the constitutive, the strain-displacement and the displacement-generalised displacement relationships. Of these the selection of the latter two sets of relationships is open to some choice. The range of strain-displacement equations which is available in the literature for the general shell has been briefly discussed in the previous section where attention has been directed to the desirability of selecting equations which are consistent with rigid-body motions. The basis for the selection of displacement patterns for the shell element will be discussed in what follows.

4. BASIC REQUIREMENTS FOR ELEMENT DISPLACEMENT FIELDS

The energy functional in a displacement analysis of shells under the Kirchoff hypothesis contains the first derivatives of the tangential displacement components and the second derivatives of the normal displacement. The continuity requirements for a flat or curved shell element are thus that the membrane displacements together with the normal displacement and its first derivatives must conform at the inter-element boundaries. The matching of the slopes across these boundaries can present considerable difficulty but if slope and displacement conformity is achieved the total functional is genuinely the sum of the contributions from all the regions. This condition must be met if bound criteria are to apply but it is, according to the heuristic proof of Zienkiewicz [8], neither a sufficient nor a necessary condition for convergence to the true minimum of the functional. A single condition which it is suggested is both necessary and sufficient to ensure such convergence is that the field variables and their derivatives which occur in the definition of the functional can assume any uniform values within an element. Included in this condition (which is the well-known uniform strain condition) is the case where the uniform value is zero and this corresponds to the representation of a general rigid body motion.

For flat elements (beams, plates, plane and three-dimensional bodies) the requirements of the uniform strain criterion are clear. For the flat plate in bending, for example, it requires that states of uniform curvature be representable when nodal conditions are compatible with such a state. There is considerable numerical evidence to verify the criterion for this type of element (see, for instance, Dawe [9]) and several useful plate bending elements exist in the literature which are non-conforming but which satisfy the criterion. For the flat plate under in-plane loading the uniform-strain condition requires the possibility of representing states of uniform displacement gradient throughout an element.

The criterion is more questionable when applied to curved elements such as arches or shells. It is clearly still most desirable that the special case of the criterion calling for the representation of strain-free motions by the element displacement patterns be met. Where a consistent shell theory is used the true rigid-body motion states coincide with states of zero strain but if an inconsistent theory is used then the true rigid-body motions will not precisely satisfy the homogeneous strain-displacement equations. In the latter case it is better to include strain-free states in the displacement field rather than the true rigid-body states. The inclusion of the strain-free states will mean that the macroscopic equations of equilibrium of the element will be satisfied. The need to represent finite states of uniform strain in a curved element is not so evident. Indeed it does not appear possible to develop uniform strain (including curvature) states in a region of a general shell although such states may perhaps be envisaged for shells of a more limited geometry. It certainly appears that no attempt has been made to invoke the uniform strain condition in the consideration of displacement patterns for shell elements. (It may be noted here that, in a non-linear study of shallow curved arches, Dawe [10] has developed a curved-beam element which can accommodate uniform states of both membrane strain and curvature and which exhibits excellent convergence characteristics. In this particular case the differential equations governing the problem indicate that the membrane strain should be uniform along the arch under any vertical loading.)

With these brief remarks in mind the consideration of displacement patterns will be continued in the remainder of the paper when shells of a specific geometry are considered.

5. ELEMENTS FOR SHELLS OF REVOLUTION

The solution of shell-of-revolution problems is of considerable practical importance and consequently numerous investigators have considered the application of the finite element method to the analysis of such problems. If the loading as well as the shell geometry is axisymmetric then the problem is essentially one-dimensional. Unsymmetrical loadings can be accommodated where such loads may be expressed in a practical number of terms of harmonic functions around the shell circumference.

The first published analyses of shells of revolution using doubly-curved elements appear to be due to Jones and Strome [11] and to Stricklin et al. [12] and a comparison of these two comparatively early approaches is of interest. The geometry of the curved shell element is illustrated in Fig. 2. In both papers similar physical approximations are made to the initial geometry of the shell. Second-degree polynomials in the coordinate ξ are assumed for the meridional slope [12] or for the sine of the slope [11]. In each case the discretised shell forms a smooth surface with a continuous slope, ϕ, and radii r and R_2 such that the nodal values of slope and radius match those of the actual shell but may depart slightly from the true structure between nodes. The meridional radius of curvature, R_1, is piecewise continuous with discontinuities at the nodes and this may lead to considerable physical idealisation errors in representing shells of rapidly-varying curvature with few elements. However, the number of curved elements needed for a solution will in general be less than if conical elements are used to approximate the shell geometry and the problem of residual meridional bending moment which is associated with the use of the latter type of element is eliminated.

Jones and Strome suggest a displacement field for the curved element which is an extended form of that used in previous analyses employing conical elements and they restrict their approach to axisymmetric loading. The complete displacement field is an interesting one.

$$u = A_1 + A_2\xi + A_7 \cos\phi + A_8 \sin\phi + A_9[(x - \bar{x})\sin\phi - (r - \bar{r})\cos\phi]$$
$$w = A_3 + A_4\xi + A_5\xi^2 + A_6\xi^3 - A_7 \sin\phi + A_8 \cos\phi + A_9[(x - \bar{x})\cos\phi + (r - \bar{r})\sin\phi] \qquad (10)$$

Here the A_7 terms are included to properly represent the only possible rigid-body motion of the element (a translation in the axial direction), the A_8 terms to represent a uniform radial growth of the element and the A_9 terms to represent a rotation about a circle of radius $\bar{r}$ corresponding to a point $\bar{\xi}$. The purpose of including the A_7 term is clear enough but the authors do not justify the inclusion of the A_8 and A_9 terms except by the statement that they significantly improve the accuracy of the method. Their inclusion seems logical, however. The terms are equivalent to a rigid-body translation in the radial direction and a rigid-body rotation of a corresponding beam element. Of the strain and curvature components corresponding to the motions represented by the A_8 and A_9 terms only the circumferential strain is non-zero; this is a sensible representation of these basic motions. Although nine coefficients are involved in the displacement field three constraints can be imposed to reduce the element degrees of freedom to six.

In the work of Stricklin et al. [12] the assumed displacement field corresponding to the n^{th} harmonic of an asymmetric loading is

$$u = (A_1 + A_2\xi)\cos n\theta ,$$
$$v = (A_3 + A_4\xi)\sin n\theta , \qquad (11)$$
and
$$w = (A_5 + A_6\xi + A_7\xi^2 + A_8\xi^3)\cos n\theta .$$

This field does not contain the rigid-body axial motion of the curved element but it is suggested that this is not necessary since this motion is represented in the limit as element size approaches zero. Haisler and Stricklin [13] have further considered this matter and purport to show through numerical examples that a slope change of two degrees or less allows rigid-body motion of the element and that in practical problems the slope change may be somewhat greater. Further usages of the curved element based on the simple displacement pattern of eq. (11) and including dynamic, nonlinear and stability applications are described in references [14], [15], [16] and [17]; extension to shells of revolution with variable properties in the circumferential direction is documented in references [18] and [19]. Although the rigid-body motion of the curved element is adequately represented when using a large number of elements it would clearly be advantageous, from the point of view of rate of convergence, if the displacement field included either an explicit representation of the rigid-body motion or a reasonable approximation thereto; the latter could be obtained by using a higher-order polynomial for the representation of the membrane components of displacement.

Khojasteh-Bahkt [20], [21], in an elasto-plastic study of axi-symmetrically loaded shells of revolution has suggested a curved element whose properties are based on local dimensionless rectilinear coordinates rather than on the surface coordinates which are normally used. To approximate the geometry of the meridional curve a comparatively high-order polynomial expansion is assumed for the height of the curve above the base line joining the nodes which lie on the true meridional curve. The approximation gives both slope and curvature continuity at the nodes. A major reason behind the choice of the local rectilinear coordinates is that the simplest displacement field (u and w as linear and cubic functions, respectively, of the rectilinear coordinate) can be chosen which includes the correct representation of the axial rigid-body motion. Associated with the rectilinear coordinate system is a set of strain-displacement equations which are derived from the usual equations by transforming the required variables. An elastic application to a pressurised torus problem demonstrates good accuracy in displacement and stress. Marcal [22] has also employed this element in an analysis of shells of revolution involving both geometric and physical non-linearity.

The finite element applications discussed thus far have used the minimum number of degrees of freedom consistent with the requirements of kinematic admissibility. Webster [23], [24] and Adelman et al. [25], [26] have independently considered the effect of various extended displacement patterns on the accuracy of solution of vibration problems involving shells of revolution. In both approaches the elements described can represent the shell geometry exactly; no approximation to the slope of the shell is necessary. The form of the assumed displacements for the n^{th} harmonic is

$$u = \sum_{i=0}^{i=I} A_i \xi^i \cos n\theta \ , \quad v = \sum_{j=0}^{j=J} A_j \xi^j \sin n\theta \ , \quad \text{and } w = \sum_{k=0}^{k=K} A_k \xi^k \cos n\theta \ . \qquad (12)$$

This displacement field does not explicitly include the rigid-body translation in the axial direction but clearly can approximate it closely if sufficient terms are used.

Both authors have made detailed studies of the solution for the natural frequency, amplitudes of displacement and the stresses of the mode of vibration corresponding to the lowest natural frequency of a cylindrical shell. Webster [24] retains equal numbers of terms (up to $I = J = K = 7$) in each of the three displacement components except for the simplest displacement field where $I = J = 1$ and $K = 3$ (equivalent to eq. (11)). Adelman et al. [26] consider three displacement patterns with $(I,J,K) = (1,1,3)$, $(3,3,3)$ and $(3,3,5)$. Broadly similar conclusions are made by both authors; the results for frequency, and particularly, stress and moment distribution show that a small number of elements with high-order polynomial displacements generally gives, for the same overall number of degrees of freedom, more accurate results than does a larger number of elements using simpler polynomials. As a particular example for a simply-supported cylindrical shell Webster has shown that very accurate values of fundamental frequency, displacement amplitude and stress distribution are obtained using only a single sophisticated element ($I = J = K = 7$) whereas considerable in-accuracies are apparent when using up to 20 elements based on the simplest displacement pattern. In particular the stress distributions for the simple-element assemblage are highly discontinuous and fluctuate rapidly; similar behaviour is also noted by Adelman et al. [26].

6. CYLINDRICAL SHELL ELEMENTS

Circular cylindrical shells have already been considered in part in the last section as a special case of the shell of revolution loaded axisymmetrically or harmonically around the circumference. The development of cylindrical elements which can accommodate an arbitrary loading is described here.

Following the derivation of a conforming, 16-degree-of-freedom rectangular element for plate bending [27] the generation of a cylindrical shell element (shown in Fig. 3) was pursued by Bogner, Fox and Schmit [28], [29]. These authors mention a 24-degree-of-freedom element based on the following displacement field:

$$\begin{aligned} u &= A_1 + A_2\xi + A_3\eta + A_4\xi\eta\,, \\ v &= A_5 + A_6\xi + A_7\eta + A_8\xi\eta\,, \\ w &= A_9 + A_{10}\xi + A_{11}\eta + A_{12}\xi\eta + A_{13}\xi^2 + A_{14}\eta^2 + A_{15}\xi^3 + A_{16}\xi^2\eta + A_{17}\xi\eta^2 + A_{18}\eta^3 + A_{19}\xi^3\eta + \\ &\quad + A_{20}\xi^2\eta^2 + A_{21}\xi\eta^3 + A_{22}\xi^3\eta^2 + A_{23}\xi^2\eta^3 + A_{24}\xi^3\eta^3\,. \end{aligned} \tag{13}$$

The components of this field can be expressed in somewhat different form as products of Hermite interpolation polynomials [28]; the membrane displacements are equivalent to a bi-linear interpolation and the bending displacement to a bicubic interpolation. The resulting element is unsatisfactory, however, since it does not admit all six independent displacement states (rigid-body states) with associated strain energies which are effectively negligible compared with other states. To remedy this deficiency a bicubic interpolation is adopted for each displacement component to give a compatible 48-degree-of-freedom element which has, as its nodal displacements, each of the displacements together with their first derivatives and mixed second derivatives. The bicubic displacements do not explicitly represent the rigid-body motion states of the cylindrical element since such states are of a trigonometric nature. However, if the angle θ is small enough such that $\sin\theta$ and $\cos\theta$ can each be represented by the first two terms of a Taylor series ($\sin\theta \approx \theta - \theta^3/6$, $\cos\theta \approx 1 - \theta^2/2$) then the rigid-body motion may be expressed in a polynomial form which is a specialisation of the bicubic interpolation pattern. An eigenvalue analysis of the stiffness matrix shows that for practical subtended angles there exist six linearly independent displacement states which produce 'very little' strain energy compared with other states.

A different approach has been adopted by Cantin and Clough [30] who have given priority to the exact representation of the rigid-body motions. The true rigid-body motions are incorporated into a displacement field which is basically that of eq. (13). The six generalised displacements at each node are u, v, w, $\partial w/\partial\xi$, $(\partial w/\partial\eta - v/r)$ and $\partial^2 w/\partial\xi\partial\eta$. Unfortunately (but typically) the price paid for an exact representation of the rigid-body motions is a small violation of displacement conformity at inter-element boundaries. Nevertheless, numerical applications demonstrate the improvement in accuracy obtained by including the true representation of rigid-body motions in the element displacement field.

Olson and Lindberg [31] have developed a non-conforming element having 28 degrees of freedom which does not explicity allow for rigid-body motions. Since the element sides coinciding with a meridian are straight it is deemed sufficient to assume a linear dependence on the meridional coordinate for the in-plane displacement components whereas their dependence on the circumferential coordinate is taken to be of a higher order because of the shell curvature in that direction. The transverse displacement is represented by the wellknown non-conforming twelve term pattern and the degrees of freedom are w, $\partial w/\partial \xi$, $\partial w/\partial \eta$, u, $\partial u/\partial \eta$, v and $\partial v/\partial \eta$ at each node. An eigenvalue analysis suggests that the displacement field adequately represents the rigid-body modes. The vibration of a curved fan blade is investigated using the element.

In a later paper Cantin [32] has developed a method to include the rigid-body motions in a stiffness analysis which is based on a displacement field that is otherwise acceptable. The method is a general one but is specifically applied to the cylindrical element using as a basis the element whose displacement field is given by eq. (13). The number of degrees of freedom of the original matrix is expanded by a congruent transformation to include the components of the general rigid-body motion. Using the condition that such a motion should not lead to nodal forces a relationship is obtained between the components of this motion and the modified nodal coordinates which can be solved for the rigid-body components. The amended stiffness matrix (still 24x24) can then be expressed purely in terms of the nodal coordinates; the element is completely strain free under any rigid-body motion. The complete displacements are, however, not fully compatible at element boundaries.

Cantin [33] has also considered some of the various strain-displacement relationships for cylindrical shells with regard to their consistency in representing rigid-body motions and has demonstrated that some of those available in the literature do not lead to strain-free states for a general rigid-body motion. However, the kinematic equations of Novozhilov for cylindrical shells, which are used in the derivation of the elements described here, are consistent with rigid-body motions.

A numerical problem to which some of the above elements have been applied is illustrated in Fig. 4. The finite-element results shown have been obtained using

(a) The 24 degree-of-freedom conforming element without rigid-body motion [30]

(b) The 24 degree-of-freedom element with rigid-body motions [30]

(c) The 48 degree-of-freedom element [28]

(d) The 24 degree-of-freedom element adjusted for rigid-body motions [32].

In Figure 4 the results are further divided into different families which are marked 1 or 2 corresponding to the number of elements in the half-length of the cylinder. For comparison an inextensional solution due to Timoshenko (w = 0.1084 in) is available but this is known to be too stiff. Cantin's results [32] apparently converge to the solution w = 0.1139 in. and this is taken as the 'exact' solution here.

Of the solutions for the pinched-cylinder problem that obtained using the latest element of Cantin [32] shows good convergence qualities and is clearly superior in this application. There is little difference between elements (b) and (c) and both appear convergent. The results for element (a) are very poor indeed, with the discretised structure being remarkably stiff. The reason for this would appear to be the inability of this type of element to closely approximate the considerable rigid-body motion which occurs for elements around the cylinder circumference, (despite the fact that there are 49 of these elements around a quarter of the circumference in the finest mesh).

7. ELEMENTS BASED ON SHALLOW-SHELL THEORY

Numerous investigators have developed shallow curved elements for the displacement approach, usually on the basis of the shallow-shell theory of Vlasov [34]. A number of elements of both rectangular and triangular planform have been suggested and most of these have been documented by Brebbia and Deb Nath [35]. A typical triangular element is shown in Fig. 5.

In the development of an element of rectangular planform Connor and Brebbia [36] adopt a displacement field in which the two membrane components are bilinear whilst the transverse component is represented by the 12-term nonconforming polynomial; there are thus five degrees of freedom (u, v, w, $\partial w/\partial x$ and $\partial w/\partial y$) per node. For the curved element the rigid-body motions are not properly represented and, although the authors suggest that this deficiency can be allowed, it is clearly problem dependent and might result in considerable error in some applications. The authors have extended their work to include geometric non linearities [37]. Deb Nath and Petyt [38] use the conforming displacement field of eq. (13) and thus require an extra degree of freedom - the twist $\partial^2 w/\partial x\partial y$ - at each node. Their shallow element is used in a study of the vibration of rectangular curved plates and the rigid-body motions of the element are investigated by an eigenvalue analysis of free plates. It is shown that significant straining of curved plates occurs under rigid-body rotations about the coordinate axes.

Pecknold and Schnobrich [39] make an interesting choice of displacement field for a shallow curved element of parallelogrammic planform. Basically the u and v components are again bilinear whilst the w component is the conforming pattern taken from the work of Birkhoff and Garabedian [40]. To represent basic strain-free motions the patterns for u and v are extended by terms which couple these components with the w component. The strain-free motion are

$$\begin{aligned} u &= A_1 + A_2(rx + sy) + \frac{A_3}{2a}(rx^2 - ty^2) + \frac{A_4}{b}(rxy + sy^2) , \\ v &= A_5 + A_2(sx + ty) + \frac{A_3}{a}(sx^2 + txy) + \frac{A_4}{2b}(ty^2 - rx^2) , \\ w &= A_2 + A_3 \frac{x}{a} + A_4 \frac{y}{b} . \end{aligned} \qquad (14)$$

where r, s and t are $\partial^2 z/\partial x^2$, $\partial^2 z/\partial x\partial y$ and $\partial^2 z/\partial y^2$ respectively; 2a and 2b are the element dimensions in the x and y directions. These additional terms give rise to small discontinuities in the membrane displacement components at element boundaries but a numerical example concerning a hyperbolic parabolal shows good agreement with a series solution. It should be noted that the above motions are only free of strain for a constant-curvature element and they do not coincide with the representation of a general rigid-body motion.

Megard [41] suggests a shallow cylindrical element of rectangular planform derived on the basis of the Vlasov-type equations modified to cylindrical coordinates. The rigid-body motions of the curved element are explicitly included in the assumed displacement field as trigonometric functions but, once more, this leads to a lack of compatibility at element boundaries. Lastly, so far as elements of rectangular planform are concerned, a stiffness matrix is presented by Sabir and Ashwell [42] using the analogy between doubly-curved shells and plates on elastic foundations. The displacement field is identical with that used by Connor and Brebbia.

Strickland and Loden [43] describe a simple shallow triangular element with 15 degrees of freedom in which the membrane displacement components are assumed to vary linearly whilst the (non-conforming) normal displacement is assumed to vary cubically in the manner of Bazeley et al. [44]. Bonnes et al. [45] consider two versions of an element having higher order functions for the membrane components. In their 36 degree-of-freedom element all displacement components are represented by cubic functions; this element has nine generalised displacements (u, v, w, $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$, $\partial w/\partial x$, $\partial w/\partial y$) at each corner point and three (the normal derivatives of u, v and w) at each of the mid-side points. The presence of these mid-side nodes is something of a computational disadvantage and a 27 degree-of-freedom element is also presented which eliminates the associated freedoms but which restricts the normal slope variation along an edge to be linear.

The successful 18 degree-of-freedom triangular plate element of recent origin (see [46] for example) has been extended to the realm of shallow shells by Ford [47] and by Cowper et al. [48]. The generalised displacements corresponding to the conforming transverse component are w itself and its first and second derivatives at each of the three corner points; the associated displacement field contains all the terms of a complete quartic polynomial together with some higher-order terms. In the work of Ford the two membrane displacement components are represented only by linear patterns, which seems inadequate. Cowper et al. use cubic polynomials for the membrane components and the corresponding generalised displacements are, after some condensation, u and w and their first derivatives at the corner points. This 36 degree-of-freedom element is completely conforming for those shells which can be represented as smooth element assemblages. High accuracy of displacement, stress and bending moment prediction are demonstrated in numerical applications using only a few elements.

In the above analyses the individual elements are taken to have constant initial curvatures; these curvatures are either specified directly or calculated from as assumed quadratic function for the height above a base plane. The membrane displacement components corresponding to the strain-free motions of a constant-curvature element are representable by quadratic expressions. Therefore the assumption of quadratic or higher-order expressions for u and v in a displacement field should provide a good description of the strain-free modes. Clearly the geometry of the general shallow shell will only be approximated with the use of constant-curvature elements and discontinuities of initial slope will occur at inter-element boundaries. Thus the surface of the discretised shell will, in general, not be smooth and strict displacement continuity will be violated even for those patterns which are conforming for smooth shells unless the functions assumed for the displacement components u, v and w are of equal degree.

The numerical applications of the shallow shell elements have been somewhat limited in number and scope but, fortunately, one or two common applications have been considered using different elements and these enable comparison to be made between some of the various formulations. A cylindrical shell roof structure is illustrated in Fig. 6 and the value of a typical deflection is used as a measure of the accuracy of several types of shallow element. It can be seen that the element of Cowper et al. [48] gives the highest accuracy for this particular application and this is not surprising in view of the chosen displacement field. This element would seem to be the most suitable of the shallow elements presently available.

8. GENERAL SHELL ELEMENTS

Oden [49] has outlined in general terms a procedure for the development of stiffness matrices for curved shell elements. The shell is formed from a network of curvilinear quadrilateral elements whose boundaries follow lines of constant surface coordinate. The suggested example of a displacement field for this type of element has bilinear membrane components and a bicubic normal component (as eq. (13)). There are no numerical applications in the paper.

Gallagher and Yang [50] also consider a doubly-curved element whose edges lie along or parallel to the orthogonal curvilinear coordinate directions. Again the displacement field is of the form of eq. (13) so that the element has 24 degrees of freedom and is assumed to have constant initial curvatures. On the basis of a few illustrative examples the authors point out that the lack of explicit representation of all rigid-body motions and of complete compatibility appears to have little effect on the solutions obtained. However, with the membrane components represented by only bilinear patterns it would be dangerous to accept this as a general conclusion.

A further element of the same type is described by Greene et al. [51] and differs from that of the preceding paragraph in the physical approximation of the true shell and in the assumed displacements. Here the Lamé parameters A_1 and A_2 and the radii of curvature R_1 and R_2 are approximated with polynomials in the shell middle surface coordinate system to give a close approximation to the true shell in a numerical sense. The assumed displacement field uses bicubic polynomials for all three displacement polynomials, thus resulting in a 48 degree of freedom element which is displacement compatible if the radii of curvature are continuous across element boundaries. The authors contend that rigid-body nodes need not be explicitly included in the displacement field so long as the component patterns are of sufficiently high order and their numerical results tend to support this.

A family of very sophisticated curved triangular elements has been developed by Argyris and Scharpf [52] using the natural strain concept. This is the SHEBA family, the members of which have various numbers of node points; attention has been concentrated on the element type SHEBA 6 having six nodes. The elements have curvilinear boundaries and varying curvatures. In developing a consistent shell theory the geometry and displacements are expressed in an identical manner in contrast to the usual approach in which displacement components are in the directions of the principal curvatures and the normal to the middle surface. The theory leads to zero strains under any rigid-body motion and the element is fully compatible. In SHEBA 6 the three displacement components are each assumed to be complete polynomial functions of order five. The element has the displacements and their first and second derivatives as unknowns at the triangle vertices together with the first derivatives normal to the edges at an intermediate point along each edge. Some applications of this general shell element have been described by Argyris [53] but unfortunately these published results are concerned only with shells of a specific geometry (axisymmetric shells). High accuracy is demonstrated by the available results with the use of few elements. The SHEBA 6 element has considerable advantages in that its geometry is very general and in that compatibility and rigid-body motion considerations are met. Its possible practical disadvantages lie in the large number of degrees of freedom it incorporates and in the fact that some of these freedoms are associated with mid-side nodes.

Another set of curved triangular finite elements for shell analysis (this time with straight edges) is proposed in a recent interesting paper by Dupuis and Goël [54]. The development is based on the shell equations of Koiter expressed in terms of a Cartesian coordinate system rather than the usual curvilinear system and the element stiffness is derived in a general way that is valid for all mathematical models based on Kirchoff's normalcy assumption. It is shown that if the height of the original middle surface and the Cartesian displacement components are represented in a like manner the strain energy theoretically vanishes for, and only for, a rigid-body displacement of the shell middle surface. An element model having nine degrees of freedom (each displacement component and its first derivatives) at each corner node and two element models with eighteen degrees of freedom (the second derivatives of displacement in addition) per node are presented. Only one numerical study is included in the paper and although no results of comparative studies are quoted the finite element results appear to converge satisfactorily with mesh refinement. For the chosen Koiter shell model it is concluded that, of the three elements, that having eighteen degrees of freedom per node and corresponding to a fifth-order polynomial assumption for each displacement component is probably the most suitable.

9. CONCLUDING REMARKS

The curved shell elements currently available for use in the finite-element displacement method have been described and documented. Of these elements almost all are restricted to some specific geometry and many are not completely satisfactory in regard to the assumed displacements.

It would appear that the uniform-strain condition has no useful meaning when applied to two-dimensional shells except in that uniform states of zero strain must be accommodated. In selecting an element displacement field attention may then be concentrated on adequately meeting the requirements of continuity and of representation of the rigid-body motions. An accurate representation of these latter motions is required not so much to guarantee convergence to correct energy levels (this appears to occur ultimately anyway) but to attain a reasonable rate of convergence. In the early development of curved elements a popular assumption was of linear membrane components and a cubic bending component of displacement. Such an assumption generally provides a poor approximation to the rigid-body or strain-free modes and for problems in which elements experience a considerable amount of rigid-body motion this can lead to very substantial inaccuracies.

In most formulations the precise satisfaction of both continuity and rigid-body motion requirements is unfortunately not possible due to the nature of the rigid-body motions. In this situation the usual approach is to assume displacementswhich are compatible and which give a good approximation to the rigid-body motions. This generally requires that the membrane components of displacement as well as the normal component are assumed to be of a cubic or higher order. Where this is done the elements appear to have reasonably good convergence properties in practical applications despite the fact that the lack of exact zero-strain modes means that the equilibrium of the element is not precisely satisfied. In one or two instances the alternative approach, in which the exact rigid-body motions are included at the expense of precise satisfaction of continuity requirements, has been adopted with apparently good result. The sophisticated formulations of references [52] and [54] in which element geometry and displacements are expressed in the same way do apparently precisely satisfy both continuity and rigid-body motion requirements as well as accommodating a general geometry.

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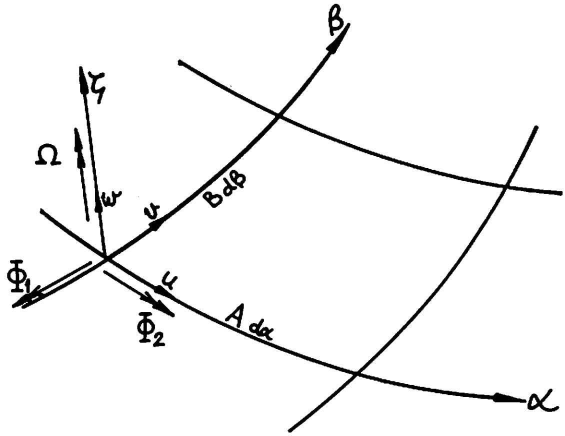

Figure 1. Geometry of arbitrary shell

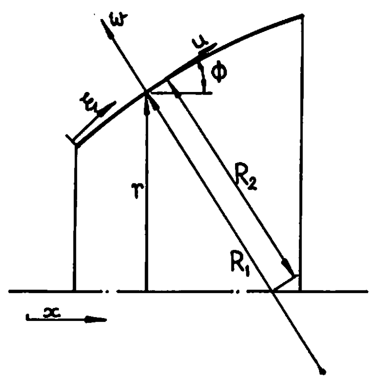

Figure 2. Shell-of-revolution element

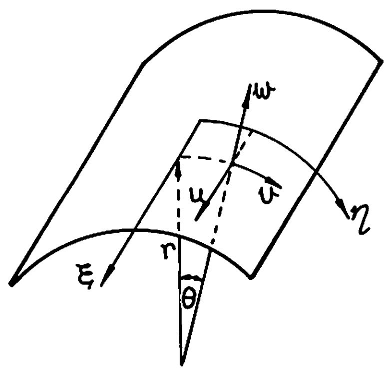

Figure 3. Cylindrical shell element

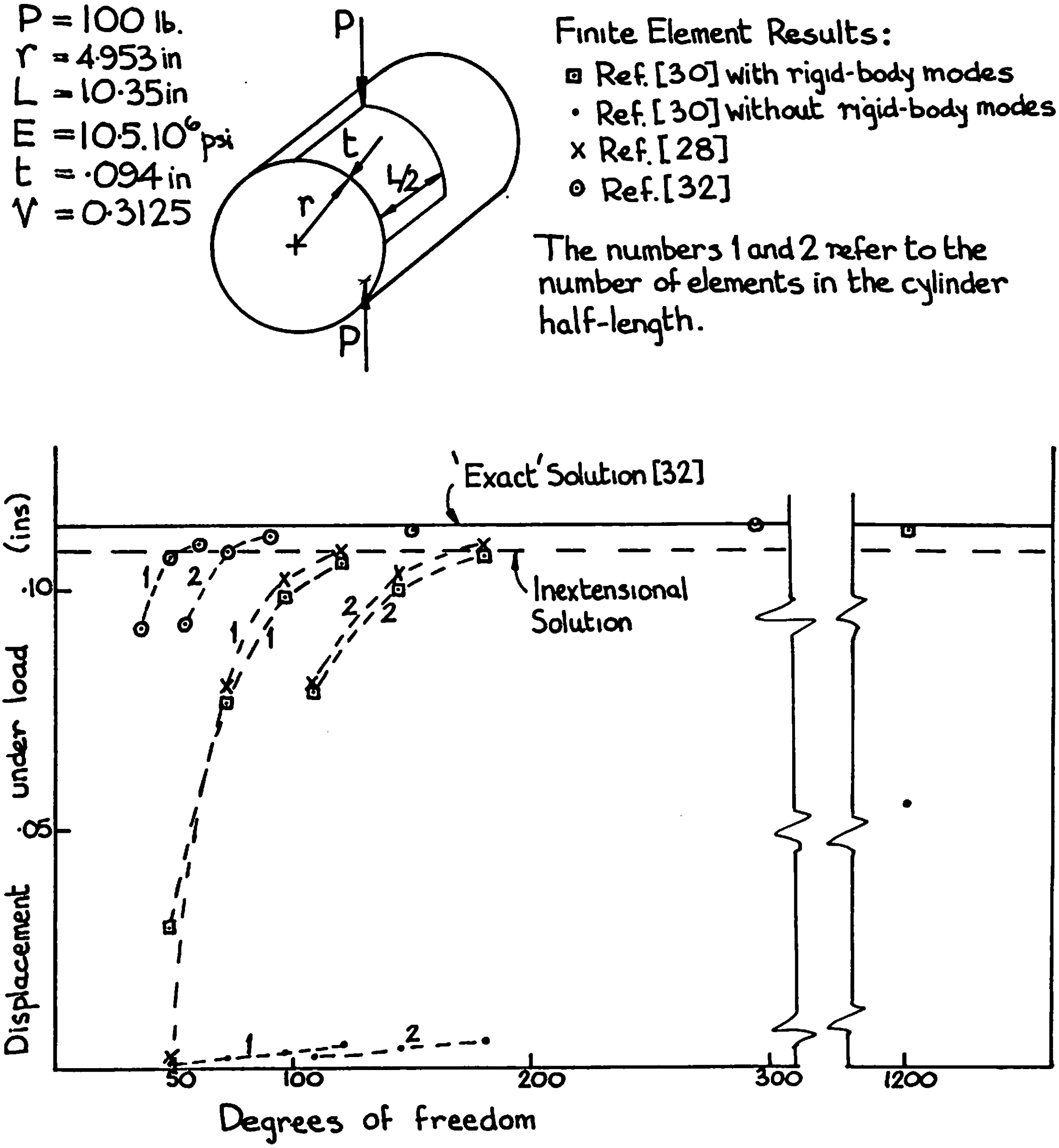

Figure 4. The pinched-cylinder problem

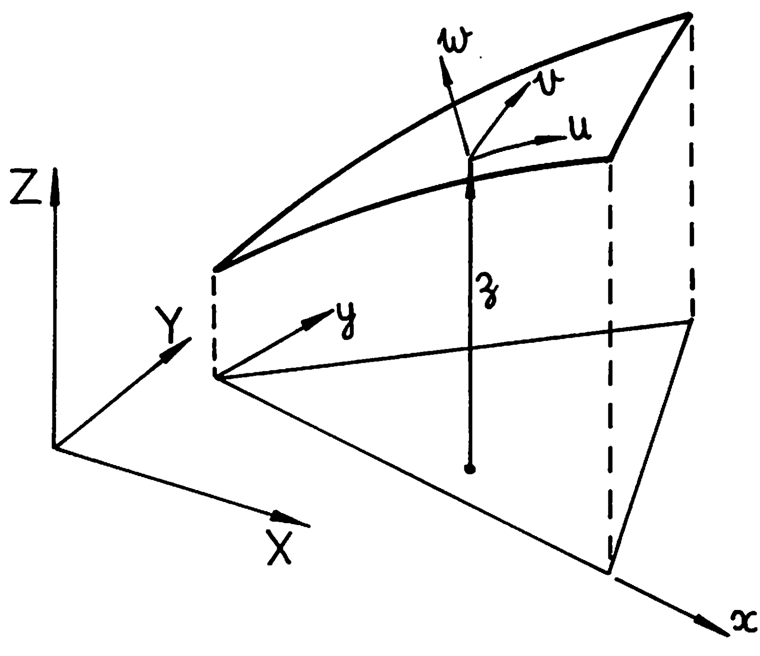

Figure 5. Shallow shell element

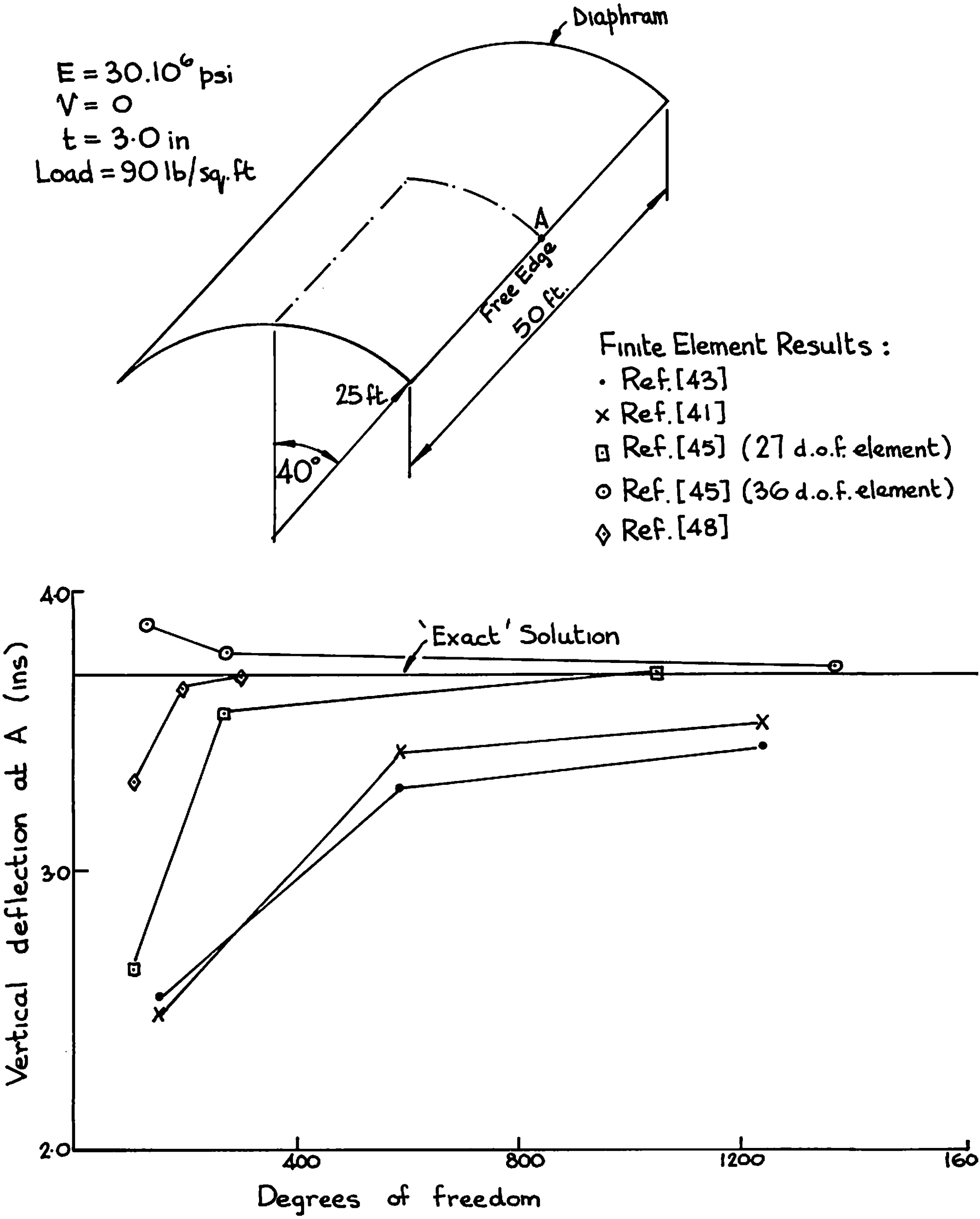

Figure 6. Cylindrical shell roof problem.