FINITE DIFFERENCE METHOD AT ARBITRARY IRREGULAR MESHES IN NON-LINEAR PROBLEMS OF APPLIED MECHANICS

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SUMMARY

A new formulation of the finite difference method (FDM), based on an arbitrary irregular mesh of nodal points, is presented. Preserving the basic advantages of the classical FDM the avoidance of its main difficulties is allowed. These difficulties are namely: satisfaction of boundary conditions in the case of a domain of an irregular shape, and a local condensation of a mesh, e.g. in a zone of expected condensations of stresses.

These disadvantages are the main reason why the classical FDM is not as universal as the FEM. The use of an arbitrary irregular mesh, on the other hand, causes arising of some new problems, like:

— Mesh generation as in the FEM. In the present paper a semiautomatic method of generation is suggested. It combines: the specific of a considered problem, engineering intuition, the need of an automatic data preparation, and the economy of calculation according to which the mesh should be as regular as possible.

— Automatic search for the neighbourhood ω_k of the considered nodal point P_k. A review of various suggestions has been done. A two step procedure has appeared to be the best one. The first step yields a certain neighbourhood ω_k, due to a simple distance criterion only.

— An optimal choice of a star $P_{i(k)}$ out of the neighbourhood ω_k and an avoidance of singular as well as ill-conditioned schemas. A certain minimal number of nodal points is required to generate the FD scheme corresponding to considered differential operator. This minimum may not be sufficient if the nodal points $P_{i(k)}$ are disadvantageously situated. Then the FD formulas would be ill-conditioned or perhaps would not be derived at all. This must be taken into account when a criterion of a choice of the star $P_{i(k)}$ is considered. Various criteria were examined and a certain optimal procedure was worked out.

— Classification of stars $P_{i(k)}$. To save the computer time classes of typical stars were introduced.

— Automatic FD formulas generation for subsequent nodal points P_k. Various methods of the choice of the star $P_{i(k)}$ yield to the solution of a system of linear equations $[q]\{Du\}=\{f\}$. Solving these equations we get the derivatives $\{Du\}=[u_x, u_y, ..., u_{xx}, ...]^T$ of unknown function $u(P)$ at the nodal point P_k. Vector $\{f\}$ consists of the differences $u_{i(k)}-u_k$. The system of equations is obtained by expanding the function $u(P)$ in the Taylor series at the nodal points $P_{i(k)}$, with the respect to the point P_k.

Some ways of FD formulas generation were examined. The best appeared to be an authors' method, according to which the number of nodal points $P_{i(k)}$ in the star is greater then the number of the required derivatives $\{Du\}$. An over-determined system of equations, involving unknown values of the derivatives $u_x, u_y, ..., u_{xx}, ...$ at the nodal point P_k, is then obtained. Its solution is obtained by minimisation of a norm being the sum of weighted squares of the deviations arising due to the FD discretisation of the true $u(P)$ function.

— Reasonable "assignment of an area" to individual nodal points P_k. This is important in nonhomogeneous problems as well as in the case of the variational formulation of the FDM.

Due to the succesful solution of the above mentioned problems, both the local and variational versions of the FDM may be applied. Thus formulated FDM is universal enough to be competitive for the FEM, especially in non-linear cases, in optimisation as well as in time- and temperature- dependent problems.

The paper presents a theoretical frame of the method and numerical solutions of some real, physically and geometrically nonlinear problems of applied mechanics. These solutions were obtained upon a set of specially prepared computer programs. They are fully automatic like programs based on the FDM and are specially designed for the solution of large systems.

1. Introduction

It has been observed by Zienkiewicz [1] that one of the obvious critical differences between the finite element method (FEM) and finite difference technique (FDM) is the ability of the later to treat irregular domains. By using an arbitrary irregular mesh of nodal points one can preserve the basic advantages of the classical FDM and avoid its main difficulties. An arbitrary grid permits the satisfaction of boundary conditions in the case of a domain of an irregular shape and enables a local condensation of a mesh. On the other hand, arising from this are several new problems mainly associated with the automatic generation of FD formulas.

In [2 - 3], a 6 point control scheme with the use of two-dimensional Taylors series expansion was adopted for obtaining FD formulas with derivatives up to the order of two. Trybiło [4] extends this approach for higher order derivative terms for solution of a plate equation. Perrone and Kao [5] have suggested a 9 point control scheme with an averaging process to improve the accuracy of obtained FD formulas.

In this paper, we continue our investigations of irregular finite difference techniques [6 - 8]. A new "optimal" way of FD formulas generation has been introduced to obtain arbitrary high order derivatives. For two dimensional, second order problems,the 9 point control scheme has been used, but the procedure remains applicable for wider problems.

Some other problems caused by an irregular grid, like generating a mesh, assigning of an area for each point and satisfying boundary conditions expressed by derivatives have been presented in forthcoming sections of this paper.

The set of computer programs in Algol 1204 was tested by calculating several, linear and non-linear problems of mechanics. The programs are fully automatic like those based on the FEM and are specially designed for the solution of large systems by using a small computer "Odra 1204".

2. The mesh generation

An automatic mesh generation technique, analogous to that used in FEM can be used also in FDM to save manpower and schedule time. In the presented program a semiautomatic method is suggested. It allowes the automatic data preparation for regular (preferably square or rectangular) mesh as well as convenient generation of singular nodes.

It performs also identification of boundary lines in case of symmetry or periodicity of solution. On the contrary to FEM it is necessary here to identify the whole zones adjacent to these lines. For unique treatment during successive computation, it was made by the generation of additional nodes in the zones, which are going to be identyfied. Each additional node is referred to the same unknown with the assigned basic node, thus, the control scheme generated near the segment CD (fig.1.) containes numbers of unknowns from the neighbourhood of the AB, as though they were coexisting.

3. Automatic search for the neighbourhood

The most important problems caused by irregular mesh appear during the generation of FD formulas. They are mainly connected with automatic selection of the points for the control scheme. All the points included to the control scheme we call "a star" of nodes. The number and the positions of the nodes in each star are the decisive factors affecting FD formulas approximations.

It is intuitively clear that the nodes selected for the star should "surround" the central point in the star (ie., the point where FD formulas are determined). Several criteria for selection of stars were pointed out in [2 - 4] . The better one was suggested by Perrone and Kao [5]. "The domain in the vicinity of a given central point is broken into eight 45 degree pie shaped segments and the closest FD point is noted" (fig.2.). This criterion results in well surrounded central points, but it appears to be too rigorous, overly complicated and too time consuming for the computer.

The analogous one, but much simpler and quicker was applied with good results. The domain around the central point is divided into four quadrants, and two nearest points are noted in each one of them (fig.3.). The only calculations for checking the node are: distance computing; comparing the signs of the local coordinates of the node.

For saving the computer time, the selection of the stars is performed in two steps: firstly, 20 nearest points are selected; next, precisely selection using the mentioned criterion, among these 20 nodes is utilized. Strictly ordering the nodes, using their coordinates $(x_n < x_{n+1} \vee (x_n = x_{n+1} \wedge y_n < y_{n+1}))$ one can check only a given strip, not the entire domain (fig.4.). For very large domains (1000 and more nodes), it is effective to divide the entire domain into a few regions and perform the selection in each region separately.

4. Difference coefficients for irregular meshes

For any sufficiently differentiable function $f(x,y)$ in a given domain, its Taylor series expansion about a point, say (x_o,y_o), can be written as

$$f = f_o + h\frac{\partial f_o}{\partial x} + k\frac{\partial f_o}{\partial y} + \frac{h^2}{2}\frac{\partial^2 f_o}{\partial x^2} + \frac{k^2}{2}\frac{\partial^2 f_o}{\partial y^2} + h\,k\,\frac{\partial^2 f_o}{\partial x \partial y} + 0(\Delta^3) \qquad (1)$$

where

$$f=f(x,y),\quad f_o=f(x_o,y_o),\quad h=x-x_o,\quad k=y-y_o,\quad \Delta=\sqrt{(h^2+k^2)}.$$

The last term in this equation gives the error order of magnitude in f caused by premature termination of the series.

After writing equation (1) for each of m nodes in the star, we arrive at the set of linear equations

$$[A]\{Df\} - \{f\} = \{0\} \qquad (2)$$

where

$$[A] = \begin{bmatrix} h_1 & k_1 & h_1^2/2 & k_1^2/2 & h_1k_1 \\ h_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & & \\ h_m & k_m & \cdot & \cdot & \cdot \end{bmatrix}$$

$$\{f\}^T = \{f_1-f_o, f_2-f_o, \ldots \quad , f_m-f_o\}$$

with the set of five unknown derivatives at point (x_o,y_o)

$$\{Df\}^T = \left\{\frac{\partial f_o}{\partial x}, \frac{\partial f_o}{\partial y}, \frac{\partial^2 f_o}{\partial x^2}, \frac{\partial^2 f_o}{\partial y^2}, \frac{\partial^2 f_o}{\partial x \partial y}\right\}.$$

The main obstacles to the success of previous attempts are the difficulties in avoiding a singularity or an ill condition in matrix $\left[A \right]$ of eq.(2) and in obtaining acceptable derivatives. After assuming m=5, it is likely to obtain one or more dependent equations in eq.(2). By selecting more nodes in a star (m 5), it is more likely to have 5 independent equations and obtain good approximation of derivatives. Then, the eq.(2) becomes an overdetermined set of linear equations. Its solution is obtained by minimization of a norm

$$B = \sum_{i=1}^{m} \left[(f_0 - f_i + \frac{\partial f_0}{\partial x} h_i + \frac{\partial f_0}{\partial y} k_i + \ldots \qquad) \frac{1}{\Delta_i^3} \right]^2 = \min \qquad (3)$$

After writing

$$\frac{\partial B}{\partial \{Df\}} = 0 \qquad (4)$$

we arrive at a set of five equations with five unknowns.

The weight coefficients ($1/\Delta_i^3$) are inversely related to the error term in eq.(1) considering the nearest points have more influence on the results. For the regular square mesh and 9 point stars, this method produces values of the derivatives one order more accurate when compaired to classical FD formulas (fig.5.). In the case of irregular mesh, it results also in better solution(fig.9).

The value of m depends on the mesh, especially in the regions where it is regular. We assume square or rectangular mesh, if possible (see section 2.) so 9 point stars are suggested.

5. Assignement of an area

In nonhomogeneous problems it is neccessary to define the domain of an area assigned to each node. This problem becomes especially important for analysis based on a variational approach (energy minimization).

The simplest proposals assign each node the circle with radius dependent on the star size or the area of polygon circumscribing the star. Then, the reasonable coefficients are applied, to ensure the proper value of the sum of the areas, which should be equal to the area of the whole domain. Two interesting solutions are based on the triangulation of the entire domain (fig.6.). Subsequently, the nodes are located in the centers of gravity of the triangles or in the apexes of triangles (then, 1/3 of an area of surrounding triangles is assigned to the node). The independence of the area assignement and the star shape is the main imperfection of these two approaches.

6. Application to non-elliptic equations

For time-dependent problems, we usually arrive to parabolic or hyperbolic differential equations and at this point it becomes necessary to ensure the stability of the generated FD formulas. So far, there are no theoretical investigations in this problem, and, what's more, various FD formulas generated by using Taylors expansion produce often non-stable approximations.

Assuming the mesh regular in time, it is the possible remedy for these shortcomings. Then one can introduce into the program the stable formulas for the time derivatives and the expressions for the space derivatives (in x,y coordinates) may be still generated automatically.

Two good approximations of the time dependent problem of temperature distribution

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = a^2 \frac{\partial f}{\partial t} \qquad (5)$$

have been tested (fig.7). The first "open" formula becomes stably only for reasonably small value Δt (mesh size in time), which should be pick out according to the smallest value of mesh size in space. The second formula (fig.7b) causes more calculations (solving the set of linear equations for each time step) in return assuming larger time mesh size Δt.

7. Boundary conditions

For each node on the boundary of a domain, the boundary condition should be satisfyied. For the 1st type of boundary condition

$$f = u(x,y) \qquad (6)$$

this can be done without additional calculations. For other boundary conditions we receive for each boundary node the FD formulas by the same way as for inner nodes.

The star for boundary node would not be well selected because it is not possible to surround this node. It conduces worse approximation of derivatives on the boundary and limites the accuracy of results. By assuming fictitious nodes outside the domain, one can improve the accuracy of FD formulas. On the other hand, this approach increases the size of the problem, and, moreover, it can lead to erroneous solutions in the case of eigenvalue problem, as it was shown by Bushnell et al. [9].

8. Non-linear problems

The FDM is universally applicable to both linear and nonlinear problems. For nonlinear problems various iterative procedures can be applied. As it was tested, e.g. by Stricklin and Haisler [10], the Newton-Raphson and selfcorrecting procedures converge faster then other approaches. They have been adopted for the FDM at arbitrary irregular grids. Designation of the "slope matrix" was made by means of previously calculated coefficients of the FD formulas. The iterative method of solution of the set of linear equations was performed so the classical Newton-Raphson method with the slope matrix varying after each "nonlinear iteration" was assumed.

Using FD formulation for the nonlinear problems, like elastic - plastic torsion of a bar, leads to increased errors in the vicinity of an elastic - plastic boundary (fig.13). They are caused by discontinuous slope of proper solution at this boundary. The arbitrary mesh enables to improve those results by assuming additional nodes along the elastic - plastic boundary and introducing an additional boundary problem. The first assumption of an elastic boundary would not be correct, so the mesh should be adopted in spite of calculations to obtain more precisely results.

9. Numerical results

3 analogous computer programs have been tested to compare the accuracy and computer time consumption of various methods of the generation of FD formulas. The Poisson's equation

$$\nabla^2 f = 1 \qquad (7)$$

for the domain shown on fig.8. has been solved. Fig.9. illustrates the convergence of the solution with the number of nodes for these programs:

a) selecting the stars with distance criterion only, generating FD formulas by the averaging process suggested by Perrone and Kao [5];
b) distance criterion and minimization procedure for FD formulas;
c) "four quadrants" selected stars and minimization procedure.

The CPU time consumption for the best program (c version) for selection of stars and generation of FD formulas for 329 nodes was 177 seconds (= 0.8 CPU seconds of CYBER 70 computer).

Temperature distribution problem for the square bar with uniformly heated boundary along $y=\pm 1$ line, and constant zero temperature along $x=\pm 1$ line (fig. 10.) has been examined. The influence of additional nodes outside the boundary on the obtained results has been noted (fig. 11).

A physically nonlinear problem of a torsion of an elastic - plastic bars has been solved. Fig. 12 & 13 illustrate the elastic - plastic boundaries for several steps of loading of the bars. Geometrical nonlinearity has been presented in the case of an ideal membrane (fig. 14 & 15).

10. Conclusions

Future development of the presented set of programs will enable considerable increase of application of the method. Further research is also required for analysis based on a variational approach, mainly for precisely defining of the assigned areas for integration. The basic advantage of this approach is the fact that in the energy expression the highest order of the derivatives is only one - half of what is encountered in the differential equations.

As has been stated above, the application of the variational formulation will enable further expansion and simplification of the presented method, however the present day method permits using it for solving various real problems of applied mechanics. Due to the succesful solution of the difficulties, discussed in this paper, FDM at irregular meshes is universal enough to be competitive for the FEM, especially in non-linear cases, in optimization as well as in time- and temperature dependend problems.

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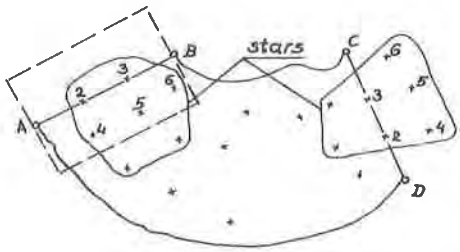


Fig.1.

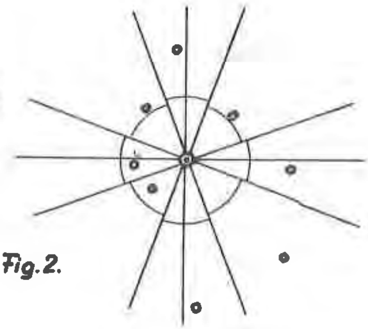


Fig.2.

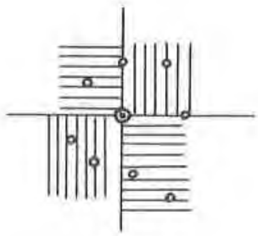
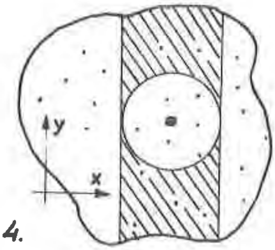


Fig.3.



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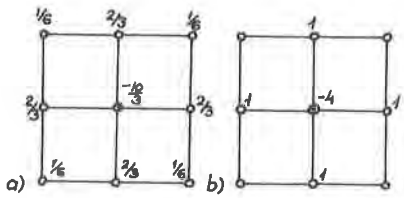


FD formulas for ∇^2 $(*h^2)$
a) obtained, b) classical.

Fig.5.

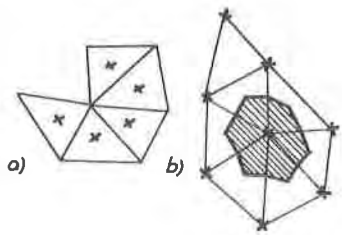


Fig.6.

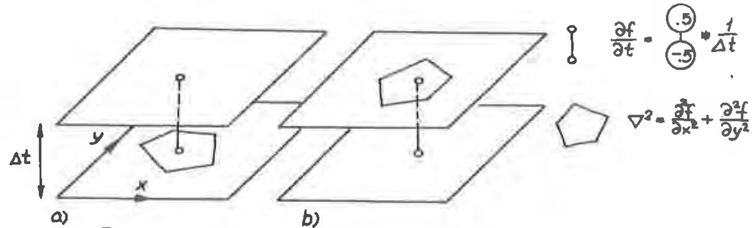


Fig.7.

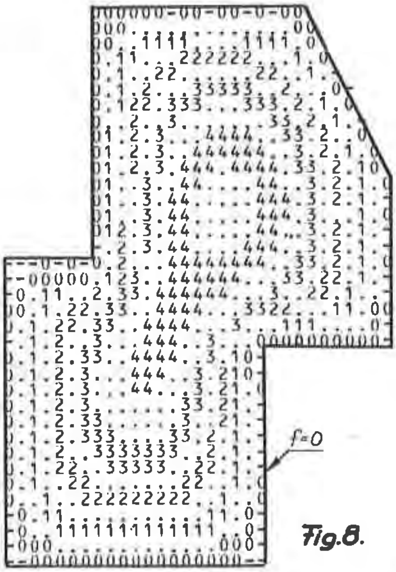


Fig. 8.

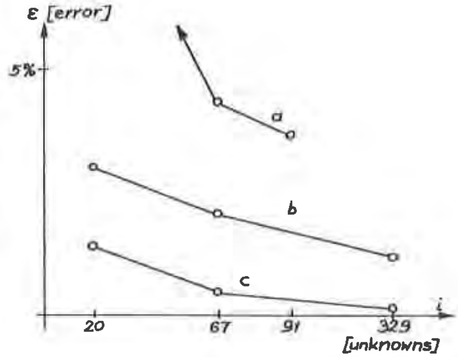


Fig. 9.

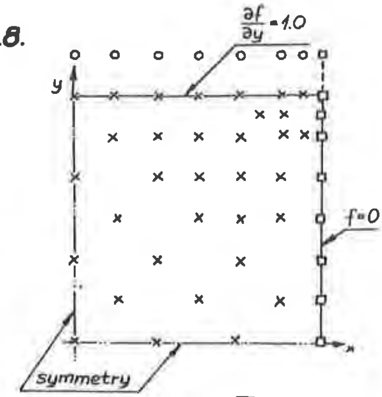


Fig. 10.

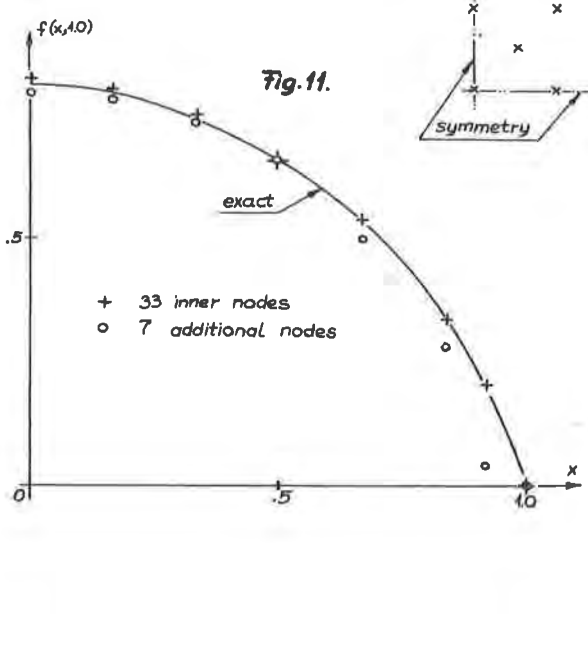
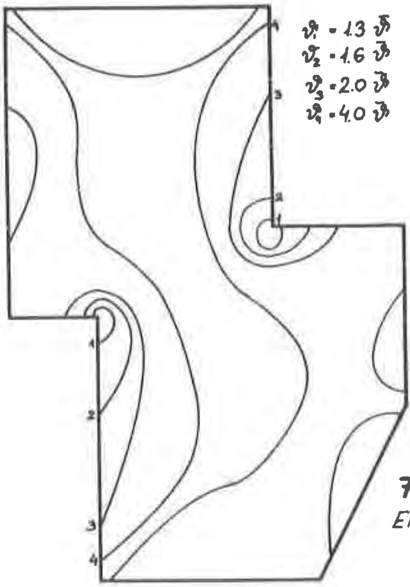


Fig. 11.



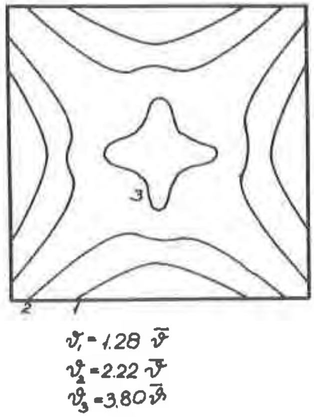


Fig. 12. *Fig. 13.*

Elastic-plastic boundaries of torsioned bars.
$\bar{\vartheta}$ - torsion at elastic capacity.

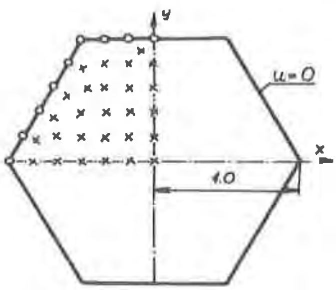


Fig. 14.

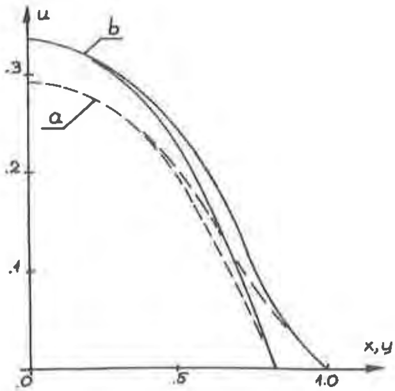


Fig. 15.

Deflection of an ideal membrane.
a) linear approximation,
b) non-linear case.