Abstract

Qian Zheng. Saturation Control of LTI Systems. (Under the direction of Dr. Fen Wu).

In this thesis, we consider the stabilization and disturbance attenuation problems for linear systems subject to actuator saturations using gain-scheduling output feedback control. By utilizing convex hull expression of saturating linear feedback law, we parameterize the proposed nonlinear output feedback law in the form of a quasi-LPV system. Conditions that ensure the system stabilizability (stabilization) within a Lyapunov level set as well as achieve disturbance attenuation capabilities (performance) are then established in terms of the coefficient matrices of the controller in the quasi-LPV form. Determination of the controller coefficient matrices is formulated and solved as linear matrix inequalities (LMIs) problem with a variable to be optimized. For stabilization problem the optimization variable is the largest stability region could be obtained; while for performance problem it refers to the smallest $\mathcal{L}_2$ gain or $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain for magnitude and energy bounded disturbances. The proposed method applies to general linear systems including strictly unstable ones and is presented in both the continuous-time and discrete-time settings, whenever it is possible.

The proposed saturation control approach is further generalized from two aspects: The first one is about the extension of the Lyapunov function from constant to parameter-dependent. Both of the stabilization and disturbance attenuation problems are reformulated within the framework of discrete-time systems. Synthesis conditions can be casted into LMI optimization problems by introducing an extra intermediate matrix. This extra degree of freedom is used to synthesize the parameter-dependent Lyapunov functions. Numerical examples showed that the approach based on parameter-dependent Lyapunov functions indeed provide less conservative results

in terms of larger domain of attraction and better disturbance attenuation characteristics than constant Lyapunov functions.

The second generalization is about the form of the saturation functions. We generalize the proposed design approach, which is discussed in the context of standard saturation inputs, to a class of saturation-like inputs described by multiple bends piecewise-linear functions. The form of the saturation-like input is represented by fixing a set of tunable parameters including input slopes and bend points, to certain values. Its control synthesis condition can also be solved as LMI optimization problems. The modified saturation control approach is applicable to a wider class of saturation functions, even the class of continuously nonlinear inputs that could be approximated by piecewise concave/convex linear functions.

SATURATION CONTROL OF LTI SYSTEMS

BY

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Dedicated to...

my parents *Yunqiang Zheng and Anping Chen*
my sister *Lian Zheng*
and
my beloved *Qing Zhang*

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Chapter 1

Introduction

1.1 Saturation control problem

Designing a control system when the actuators are subject to hard constraints is a fundamental problem for control engineers. As it is well known, all real-world electrical/mechanical devices have amplitude and/or rate limitations. The control design techniques that ignore these actuator limits may cause undesirable transient response, degrade the closed-loop performance, even the closed-loop instability, which will have fatal consequences in many situations.

The block diagram for a control system with saturating input is shown in Fig. 1.1. In prior research, the saturation control problem has been examined via the extensions of optimal control theory, anti-windup compensation, supervisory error governor approach, Riccati and Lyapunov-based local and semi-global stabilization, and absolute stability frameworks. Tremendous strides are currently being made to advance the saturation control design techniques to address important issues of performance degradation, disturbance attenuation robustness to uncertainty/time delays, domain of attraction estimation, and control rate saturation. All the prior research literature and the currently developing research directions provide a rich variety of techniques to account for actuator saturation.

Classified from the stability point of view, the most fundamental problem for a

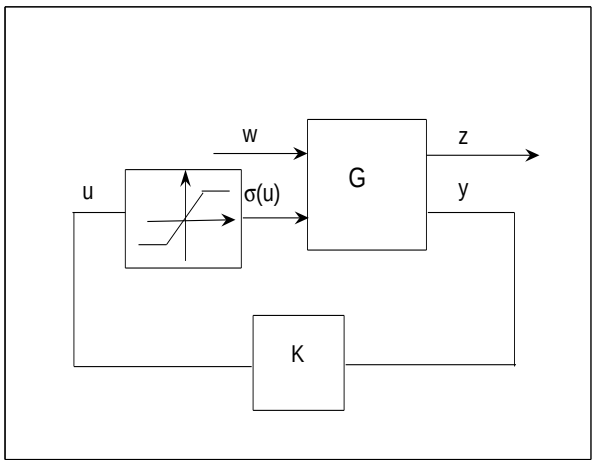


Figure 1.1: System with saturation nonlinearity.

system with saturation is a prediction of regions of attraction. Early work applied absolute stability theory to address this question by isolating the nonlinearity and casting the problem in the Lur'e-Postnikov framework [12, 70]. As is typical in these works, a Lyapunov function is used to define regions in the state space in which energy is guaranteed to decrease, while the nonlinearity is bounded by a prescribed amplitude. With these conditions satisfied, it is then shown that initial conditions in these regions will result in state trajectories that converge to the origin. When the input to the nonlinearity remains within certain bounds, the saturation can then be treated as a memoryless and locally sector-bounded nonlinearity. Directly accounting for this nonlinearity, early analysis was extended to produce better stability region estimates by for example, state/output feedback [55, 56, 54] based on small-gain type conditions. Along this direction, the classical absolute stability analysis tools such as circle criterion, Popov criterion and the vertex criterion have been used to obtain an estimate of the domain of attraction [20, 35]. Typically, the domain of attraction is estimated by using invariant ellipsoids.

Other work on saturation control takes system performance into consideration. Related performance analysis include disturbance attenuation in terms of certain energy gain [22, 31, 61, 62], transient response achievement by state/output feedback

[57, 41, 38, 39, 69], output regulation, robustness to uncertainties [15, 46, 68, 25].

There has been a long history of exploring global or semiglobal stabilizability for linear systems with saturating actuators. In 1969, Fuller [18] studied global stabilizability of a chain of integrators of length greater than two by saturated linear feedback and obtained a negative result. This important problem also attracted the attention of Sussmann and Yang [63]. They obtained similar results independently in 1991. Because of the negative result on global stabilizability with saturated linear feedback, the only choice is to use general nonlinear feedback. In 1992, Teel [66] proposed a nested feedback design technique for designing nonlinear globally asymptotically stabilizing feedback laws for a chain of integrators. This technique was fully generalized by Sussmann, Sontag and Yang [64] in 1994. Alternative solutions to global stabilization problem consisting of scheduling a parameter in an algebraic Riccati equation according to the size of the state vector were later proposed in [49, 61].

Another trend in the development, motivated by the objective of designing simple controllers, is semiglobal stabilizability with saturated linear feedback laws. The semiglobal framework for stabilization requires feedback laws that yield a closed-loop system which has an asymptotically stable equilibrium whose domain of attraction includes an a priori given(arbitrarily large) bounded set. In [44, 42], a low-gain design technology was proposed to construct semiglobal stabilizing controllers both for discrete time and continuous time linear systems. This method is based on the eigenstructure assignment and is referred to as a direct method design for low-gain controllers. Algebraic Riccati equation(ARE)-based methods utilizing H_2 and H_∞ optimal control theory for designing low-gain controllers were also proposed independently in [46, 67].

Later on, the so-called low-and-high gain design technique, for a chain of integrators subject to input saturation was introduced in [43]. And the continuation work in [57] established the low-and-high gain design technique for general linear systems subject to input saturation and utilized this design technique to provide a solution to problems of semiglobal asymptotic stabilization, robust semiglobal stabilization for a

class of matched uncertainties, and semiglobal disturbance rejection.

[28, 30] discussed the stabilizability of systems whose open-loop poles are not all in the closed left-half plane and generalized the notion of global/semiglobal stabilization by characterizing the asymptotically null controllable region in [28]. In [30], a family of saturated linear feedback laws were developed for second-order or higher order systems with two antistable poles.

Another branch in the development of saturation control approach included anti-windup technique. Linear control methods are used initially to obtain desirable nominal controllers, while in the second step anti-windup techniques are developed that reduce the controller gain so that saturation is avoided with only a graceful degradation of performance, representative work included [7, 33, 36].

1.2 Motivations and Objectives

In this thesis, we first consider the basic problem of stabilizing a linear system subject to actuator saturation by use of output feedback. The stabilization of a linear system subject to actuator saturation has been widely reported on in the literature. The works on this topic can be divided into two categories, those that deal with open-loop systems that are not exponentially unstable (or simply called semi-stable) and those that are exponentially unstable. The stabilization of semi-stable systems is now well-understood. Various global and semi-global stabilizing feedback laws, both of state feedback type and output feedback type, have been reported (see,*e.g.*, [42, 44, 61, 64, 66]).

The stabilization of exponentially unstable systems subject to actuator saturation involves subtler issues and is more difficult to deal with. The fundamental difference between a semi-stable system under actuator saturation and an exponentially unstable linear system under actuator saturation lies in their respective null controllable region, the set of all states that can be driven to the origin by a bounded control (provided by the saturating actuator). In particular, the null controllable region of a semi-stable

linear controllable system is the whole state space, while the null controllable region of an exponentially unstable linear controllable system is not (see, *e.g.*, [25]). Therefore, in controlling exponentially unstable linear systems under actuator saturation, one can only expect local stabilization. The objective is then naturally to achieve a domain of attraction that is as large as possible and, ideally, as large as the null controllable region of the system. It is however known that a domain of attraction as large as the null controllable region in general cannot be achieved with linear feedback [27]. As a result, many methods exist in the literature for the synthesis of linear state feedback laws that result in large domains of attraction (see, *e.g.*, [23, 34, 55]).

More recently, an LMI based approach for the design of stabilizing linear state feedback laws was proposed in [28]. This method was shown to result in larger domains of attraction than other existing methods. The key novelties of this method include putting the saturating linear feedback law on the convex hull involving an auxiliary feedback gain matrix and the establishment of set invariant conditions that are equivalent to linear matrix inequalities. This approach is attractive from the theoretical point of view, however, the proposed control technique is not applicable to the situation when not all of states are measurable.

To this end, we propose an output feedback control law in this thesis. Though the domain of attraction achieved by this method is not as large as the one by state feedback, it solves a class of problem in which only part of the states information about the plant are known. Comparing with state feedback, the dynamics of the output feedback controller will be involved into the control scheme, which leads to a more complicated procedure. The feedback law that we will arrive at will be nonlinear in nature. However, we solve it by turning the nonlinear controller to be parameter-dependent and of quasi-LPV form. The measurement of the parameters provides real-time information on the variation of the plant's characteristics. Hence, a set of controllers are constructed, each one of which is scheduled based on this information at the operating point. Therefore, from control point of view, the parameter is used to schedule the controller gains, while from saturation point of view, it indicates the

level of saturation. It turns out that conditions under which the closed-loop system is locally asymptotically stable at the origin with a Lyapunov level set included in the domain of attraction can be established in terms of the coefficient matrices of the controller in the quasi-LPV form and be solved as an LMI optimization problem.

As a natural step beyond stabilization, the problem of disturbance rejection for linear systems subject to actuator saturation has also been addressed by many authors. The results on this topic can be divided into two categories according to the way the disturbances enter the system. One is the works on systems where disturbances are input additive. Examples include [11, 29, 40, 45, 47]. Because of the input additive nature of the disturbances, very strong results such as global stabilization can be established. For neutrally stable open-loop systems, it was shown that a simple linear feedback law render the closed-loop system finite gain $\mathcal{L}_{\mathrm{p}}$-stable [47]. Various continuity and incremental-gain properties of the closed-loop system were discussed in detail in [11]. For a general open-loop system, it was shown that the $\mathcal{L}_{\mathrm{p}}$ gain from the disturbance to the state can be made arbitrarily small by linear feedback if the disturbances are assumed to be bounded in magnitude [45]. This boundedness assumption on the disturbances can be removed if nonlinear feedback is allowed [40]. Also under the boundedness assumption on the magnitude of the disturbances, semi-global practical stabilization on the null controllable region is possible [29].

The second category of the works are those on systems where disturbances are not input additive (see, for example, [23, 28, 49, 50, 51, 53, 59, 62]. Since the disturbances enter the system independently from the bounded control inputs, the results established are often conservative. What can be expected is only regional stabilization and a certain degree of disturbance tolerance of the closed-loop system. Under the boundedness assumption on the magnitude of the disturbances and in the absence of initial condition, the $\mathcal{L}_2$ gain analysis and minimization in the context of both state and output feedback were carried out in [50, 51]. The work of [28] proposed a method for analysis and maximization of an ellipsoid which is invariant under magnitude bounded, but persistent disturbances. The works of [23, 49, 53, 59,

62] all consider the situation where disturbances are bounded in energy. The work of [49] leads to a gain scheduled feedback law that guarantees both closed-loop stability and bounded $\mathcal{L}_2$ gain from the disturbance to the state. The works of [23, 53, 59] formulated and solved the problem of stability analysis and design as optimization problems with LMI or BMI constraints, with the former two papers considering linear feedback laws and the latter using hybrid state feedback laws.

In this thesis, we focus on a class of disturbances with boundedness on both magnitude and energy and they enter the system not input additive. Adopt the design approach proposed in the first part, we will design output feedback laws that attenuate the effect on the system output of the disturbance. The level of disturbance attenuation is measured in terms of the restricted $\mathcal{L}_2$ (ℓ_2) gain and the restricted $\mathcal{L}_2$ to $\mathcal{L}_\infty$ (ℓ_2 to ℓ_∞) gain over this set of disturbances.

Summarizing the above work, it is recognized that Lyapunov functions play an important role in the study of saturated control systems. Namely, the existence of a symmetric positive definite Lyapunov matrix P such that the resulting Lyapunov function x^TPx satisfies certain analysis conditions form the basis for the derivation of synthesis conditions in both stabilization problem and performance problem. Moreover, it is Lyapunov level set $\left\{x : x^TPx \leq \eta\right\}$ that characterizes the domain of attraction where any point starting from it converges to the origin.

However, since P typically be a constant matrix no matter how the dynamics of the closed-loop system changes, it could lead to some conservatism in the control of saturated systems. For this reason, we consider the generalization of Lyapunov functions from constant to be parameter-dependent form $x^TP(\rho)x$ for discrete-time linear systems. The parameter ρ is chosen as the same one that used to schedule the controller. Consequently, the saturation information as well as the controller information are involved into the construction of the parameter-dependent Lyapunov function. By introducing an extra variable, [13] showed that a parameter dependent Lyapunov function leads to a modified analysis condition that features a symmetric variable related to Lyapunov functions as well as an extra instrumental variable.

Although the new analysis condition is not simultaneously convex (affine) on the instrumental Lyapunov variable and the controller parameters, congruent transformation of variables exist that are able to convert it into an LMI form [14]. Thus we establish and solve the saturation control problem as an convex optimization problem by introducing a general-formed intermediate matrix. Constant Lyapunov function is a special case of parameter-dependent Lyapunov function. The advantage of using parameter-dependent Lyapunov method is manifested by less conservative results in terms of larger domain of attraction and better disturbance attenuation characteristics.

Another direction of the research is motivated by the fact that standard saturation input with magnitude 1 and bend points $(\pm 1, 0)$ is a special saturation model that only covers a small part of the saturation nonlinearities. For this reason, we will try to extend the system with standard saturation input to the one that actuated by a more generalized saturation functions. This generalization is based on the concept of piecewise linear concave/convex functions. As a result, the generalized saturation input is expressed by several linear functions, each describes a linear input on a certain section and with certain slope. Thus the model is more flexible and is expected to cover a larger part of the saturation like inputs that may appear in the physical world. We could even characterize some kind of continuously nonlinear input by choosing large enough number of piecewise linear functions to approximate it. The design approach is decomposed into two parts to discuss in terms of the bends of piecewise linear functions that compose the generalized saturation function. It turns out that the stabilization control synthesis conditions could be derived by generalizing those formulated in the standard saturation input case.

1.3 Thesis Outline

The detailed outline of this thesis is as follows:

Chapter 1 introduces the importance of saturation control problem and reviews

the development of research work in this field. This chapter also emphasizes related literature that motivates our work and describes the objectives of the thesis.

Chapter 2 contains definitions, concepts and information about the mathematical background and related tools. They are useful throughout the work and provide foundations for the derivation of theoretical results in the later chapters.

Chapter 3 discusses the stabilization problem for LTI systems subject to actuator saturations using output feedback control. The key is Lemma 3.1 that put the saturation function into a convex hull, which is fundamental and provides basis for the whole thesis. The synthesized controller is nonlinear in nature and is parameterized in a quasi-LPV control form, as detailed in the chapter. Conditions under which the closed-loop system is asymptotically stable is established within an LMI framework and can be easily solved as a convex optimization problem. The objective function to be optimized is the domain of attraction. Therefore, the optimal solution denotes the largest stability region that the system can be achieved.

Chapter 4 focuses on the issue of optimizing saturated control system performances in the presence of external disturbances. Specifically, we aim to construct gain-scheduled output feedback control laws such that the linear saturating system attenuates the disturbances with amplification from disturbances to the controlled outputs less than γ. γ is measured in terms of restricted $\mathcal{L}_2$ gain and the restricted $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain. The smaller γ is, more effectively the disturbance attenuation is achieved. The control synthesis conditions are provided for the class of disturbances that is magnitude and energy bounded. This is done through the use of the ideas contained in previous chapters. The construction procedure for the gain-scheduling output feedback controller is also derived in a similar fashion.

In Chapter 5, we reformulate the output feedback stabilization and disturbance attenuation problems by extending the constant Lyapunov function to be parameter-dependent in discrete-time system setting. As a result, both of the system dynamics and the saturation level information will be involved into the construction of the Lyapunov function. Therefore, less conservative results are expected, as shown in terms

of larger domain of attraction and better disturbance attenuation characteristics.

In Chapter 6, we extend the standard saturation input to a generalized form which is represented by N bends piecewise convex/concave functions. The control schemes are developed first for the case of $N = 1$, then to the case $N > 1$. For $N = 1$ case, stabilization control synthesis conditions could be directly derived by generalizing those formulated in previous chapters; while for $N > 1$ case, a set of conditions drawn from $N = 1$ case need to be satisfied simultaneously based on convex combination argument. With these results, we would expect to approximate a continuously nonlinear input by piecewise linear functions with multiple bands. Hence, the proposed control laws are applicable for a wide class of nonlinear saturation-like input systems.

Chapter 7 concludes the research contribution in this thesis as well as provides a remark on the future research work.

Chapter 2

Mathematical Preliminaries

In this chapter, we provide some background knowledge which forms the basis for the later study. Convex analysis approach, especially Linear Matrix Inequalities, are discussed in Sections 2.1 and 2.2, which will be used to formulate the saturation control problem into mathematical form. They provide an efficient and reliable way to cast and solve the saturation control problem. In Section 2.3, we introduce the linear parameter-varying (LPV) methodology for saturated control system analysis. Section 2.4 provide definitions of two performance indices for saturated control systems. which will be used to evaluate the properties of the systems. Lastly, some relevent concepts to saturated control systems will be discussed in Section 2.5.

2.1 Convex Analysis

In this thesis, the design problem will finally be formulated as a convex optimization problem. We are interested in finding conditions for optimal solutions to exist by using convex analysis. Following are some definitions and elementary properties which make the convexity of sets and functions is a desirable property for optimization.

1. **Convex Set**

   A set Θ in a linear vector space is said to be convex if for all $x_1, x_2 \in \Theta$, we

have

$$\alpha x_1 + (1-\alpha)x_2 \in \Theta, \text{ for all } \alpha \in (0,1)$$

In geometric terms, this states that for any two points of a convex set also the line segment connecting these two points belongs to the set.

2. **Convex Combination**

   Let Θ be a set of a vector space, $x_1, x_2, \ldots, x_n \in \Theta$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are non-negative real numbers with $\sum_{i=1}^{n} \alpha_i = 1$, then

$$x := \sum_{i=1}^{n} \alpha_i x_i$$

   is called a convex combination of $x_1, x_2, \ldots, x_n$.

3. **Convex Function**

   A function $f : \Theta \to \mathbb{R}$ is called convex if Θ is convex and for all $x_1, x_2 \in \Theta$ and $\alpha \in (0,1)$, we have

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2)$$

4. **Affine Function**

   A function $f : \Theta \to \Psi$ is affine if $f(x) = f_0 + T(x)$ where $f_0 \in \Psi$ and $T : \Theta \to \Psi$ is a linear map, i.e,

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2), \text{ for all } x_1, x_2 \in \Theta, \alpha_1, \alpha_2 \in \mathbb{R}$$

5. **Local and Global Minima of Convex Function**

   A function $f : \Theta \to \mathbb{R}$ is said to have a local minimum at $x_0 \in \Theta$ if there exists $\epsilon > 0$ such that

$$f(x_0) \leq f(x), \text{ for all } x \in \Theta \text{ with } \|x - x_0\| < \epsilon$$

   If it is hold for all $x \in \Theta$, then $f(x_0)$ is a global minimum.

   Suppose that f is convex, then the local minimum at $x_0 \in \Theta$ is also the global minimum.

6. **Convex Optimization**

   As we will see, many optimization problems in control can be formulated (or reformulated) using linear matrix inequalities and then be solved in an efficient and reliable way. Since the linear matrix inequality defines a convex constraint on the variable vector, optimization problems involving the minimization (or maximization) of a performance function with the convex constraint belongs to the class of convex optimization problems.

2.2 Linear Matrix Inequality (LMI)

In recent years, Linear Matrix Inequality(LMI) has emerged as a powerful tool to approach control problems that appear hard to solve. This is because in many situations, control synthesis and analysis requirements can be (and will be) written as LMIs and powerful numerical interior point techniques have been developed to solve LMIs in a practically efficient manner. Several Matlab software packages are available that allow a simple coding of general LMI problems and provide efficient tools to solve typical control problems. Here we give out the definition of LMIs and provide several important facts which form the basis for the system analysis of the thesis.

1. **Linear Matrix Inequality (LMI)**

   A linear matrix inequality is an inequality that defines a convex constraint on a vector $x \in \mathbb{R}^n$, i.e,

   $$F(x) = F_0 + \sum_{i=1}^{n} x_i F_i > 0$$

   where F is an affine function mapping an n dimensional vector space x to the set of $n \times n$ real symmetric matrices.

2. **Schur Complement**

   Schur complement is widely used to convert nonlinear (convex) inequalities to

linear matrix inequalities, i.e, a block matrix

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix} < 0$$

if and only if

$$\begin{cases} M_{11} < 0 \\ M_{22} - M_{12}^T M_{11}^{-1} M_{12} < 0 \end{cases}$$

3. **Lyapunov Stability**

   Lyapunov stability concerns the asymptotic behavior of the state of an autonomous dynamical system. The stability of such systems can be verified in terms of the existence of functions, called Lyapunov functions. For linear systems the problem of finding Lyapunov functions turns can be solved adequately as a feasibility test of a linear matrix inequality, i.e,
   Consider a linear time invariant system

   $$\dot{x} = Ax$$

   and a quadratic Lyapunov function $V(x) = x^T P x$, where P is symmetric positive definite. The system is asymptotically stable at the origin if and only if

   $$A^T P + PA < 0$$

2.3 LPV methodology

Linear Parameter Variant(LPV) systems are systems that depend on unknown but measurable time-varying parameters. They are motivated by the gain scheduling control design methodology. Investigation on the analysis and synthesis of LPV systems has shown that the gain-scheduled control synthesis problem can be solved using LMI optimization techniques. The approach involves the design of several linear time-invariant(LTI) controllers for a parameterized family of linear time-invariant system

models and the interpolation of these controller gains.
In this thesis, we consider the objective output feedback controller of LPV form

$$\dot{x}_k(t) = A_k(\rho(t))x_k(t) + B_k(\rho(t))y(t)$$
$$u(t) = C_k(\rho(t))x_k(t) + D_k(\rho(t))y(t)$$

where $\rho(t)$ is a vector of time-varying parameters. Then the state-space matrices of the controller become functions of ρ. During the control synthesis, they are real-time available and used to schedule the controller gains. In saturation control study, the LPV approach is based on first generating a set of linear matrix inequalities (LMIs) over the parameter set, solutions of the LMIs together with the gain-scheduling parameters are used to construct the LPV controller at each operating point. Since this process explicitly takes into account the relationship between real-time parameters and performance requirements, it enables controllers to be designed for whole ranges of operating conditions with stability and performance guaranteed.

2.4 Signal Norm

The most important objective of a control system is to achieve certain performance specifications in addition to providing stability. One way to describe the performance specifications of a control system is in terms of the size of certain signals. In this section, we will introduce several ways of defining a signal's size. Actually, there are many types of signals we may simultaneously interested with in the physical systems. By choosing one way to explore the norms of these signals, we can recast our performance objectives so that they all meet a single type of criterion.

2.4.1 Definitions of Induced $\mathcal{L}_2$ norm and $\mathcal{L}_2$ to $\mathcal{L}_\infty$ norm

Consider a system block diagram as shown in Fig. 1.1 with input w and output z, the two kinds of norm measurements we are interested with in this thesis are defined as following :

1. **Induced $\mathcal{L}_2$ norm**

   The induced $\mathcal{L}_2$ norm from w to z is defined as:

$$\min_{x(0),w} \frac{\|z\|_2}{\|w\|_2}$$
$$\|z\|_2^2 := \int_{-\infty}^{\infty} trace[z^*(t)z(t)]dt$$
$$\|w\|_2^2 := \int_{-\infty}^{\infty} trace[w^*(t)w(t)]dt$$

2. **Induced $\mathcal{L}_2$ to $\mathcal{L}_\infty$ norm**

   The induced $\mathcal{L}_2$ to $\mathcal{L}_\infty$ norm from w to z is defined as:

$$\min_{x(0),w} \frac{\|z\|_\infty}{\|w\|_2}$$
$$\|w\|_2^2 : = \int_{-\infty}^{\infty} trace[w^*(t)w(t)]dt$$
$$\|z\|_\infty : = ess\ sup\ \bar{\sigma}[z(j\omega)]$$

2.4.2 Induced $\mathcal{L}_2$ and $\mathcal{L}_2$ to $\mathcal{L}_\infty$ Norm based on Quadratic Lyapunov function

Based on the above definitions, we will give out two important descriptions of the stability and performance properties for system design in the sense of Lyapunov function. They form the theoretical basis for the performance analysis in the later chapters.

1. Induced $\mathcal{L}_2/(H_\infty)$ Norm

   Given a linear plant

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \tag{2.1}$$

   if there exists a quadratic Lyapunov function $V(x) = x^T P x$ with $P = P^T > 0$ such that its time derivative $\dot{V}$ along the dynamics (2.1) satisfies

$$\dot{V} < -\frac{1}{\gamma} z^T z + \gamma w^T w \tag{2.2}$$

the system (2.1) is quadratically stable and the $\mathcal{L}_2$ gain from w to z is smaller than γ, i.e., $\|z\|_2 < \gamma\|w\|_2$.

Plug the expression of $\dot{V}, w$ and z in, one can get

$$\begin{aligned}
&\dot{V} + \frac{1}{\gamma}z^T z - \gamma w^T w \\
&= (Ax + Bw)^T Px + x^T P(Ax + Bw) + \frac{1}{\gamma}(Cx + Dw)^T(Cx + Dw) - \gamma w^T w \\
&= \begin{bmatrix} x^T & w^T \end{bmatrix} \begin{bmatrix} A^T P + PA + \frac{1}{\gamma}C^T C & PB + \frac{1}{\gamma}C^T D \\ B^T P + \frac{1}{\gamma}D^T C & \frac{1}{\gamma}D^T D - \gamma I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \\
&< 0
\end{aligned}$$

which implies

$$\begin{bmatrix} A^T P + PA + \frac{1}{\gamma}C^T C & PB + \frac{1}{\gamma}C^T D \\ B^T P + \frac{1}{\gamma}D^T C & \frac{1}{\gamma}D^T D - \gamma I \end{bmatrix} < 0$$

By using Schur Complement, it is equivalent to

$$\begin{bmatrix} A^{\mathrm{T}} P + PA & PB & C^{\mathrm{T}} \\ B^{\mathrm{T}} P & -\gamma I & D^{\mathrm{T}} \\ C & D & -\gamma I \end{bmatrix} < 0 \tag{2.3}$$

2. $\mathcal{L}_2$ to $\mathcal{L}_\infty$ / (H_2) Norm

   Consider the linear plant (2.1), if there exists a quadratic Lyapunov function $V(x) = x^T Px$ with $P = P^T > 0$ to satisfy the following conditions

$$D = 0 \tag{2.4}$$

$$A^T P + PA + PBB^T P < 0 \tag{2.5}$$

$$CP^{-1}C^T \leq Z \tag{2.6}$$

$$trace(Z) \leq \gamma^2 \tag{2.7}$$

Then system (2.1) is quadratically stable and the $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain from w to z is smaller than γ, i.e., $\|z\|_\infty < \gamma\|w\|_2$.

By Schur Complement, (2.5) and (2.6) are equivalent to

$$\begin{bmatrix} A^TP + PA & PB \\ B^TP & -I \end{bmatrix} < 0 \tag{2.8}$$

$$\begin{bmatrix} Z & C \\ C^T & P \end{bmatrix} \geq 0 \tag{2.9}$$

Some other instrumental lemmas used in the above theoretical development are given in [6, 19, 80] .

2.5 Some Concepts for Saturation Control

During the rest of the thesis, some technical terms will be used to evaluate or describe the properties of the system in saturation control. Here, we first give out the definitions of these terms for easy understanding.

1. **Invariant Set**

   For an differential equation which characterizes the dynamics of a mechanical system, an invariant set is that a solution of the equation remains in the set once it started there.

2. **Null Controllable Region**

   Null controllable region is the set of all states that can be driven to the origin by a bounded control provided by the saturating actuator.

3. **Domain of Attraction**

   Domain of attraction is defined as a set of states that if any state trajectory starting from a point in this set, it will converge to the origin as the time goes to infinity.

Chapter 3

Stabilization of Linear Systems subject to Actuator Saturations

In this chapter,we study the stabilization of linear systems subject to actuator saturations by designing output feedback laws. Our main objective is to achieve a domain of attraction as large as possible. It consists of a natural sequence of steps. Firstly, a nonlinear output feedback controller is expressed in the form of a quasi-LPV system. Conditions under which the closed-loop system is locally asymptotically stable are then established in terms of the coefficient matrices of the controller. The design of the controller (coefficient matrices) that achieves a large domain of attraction is then formulated and solved as an optimization problem with LMI constraints. The composed approach is presented in both the continuous-time and discrete-time setting and can be applied to general linear systems including strictly unstable ones.

In Section 3.1, we describe the problem and design objectives. After demonstrating some preliminary results, the main results on the synthesis conditions for LTI systems with full order and reduced order output feedback control laws are presented, and a numerical example will be used to illustrate the design procedure and effectiveness of the resulting controllers. The corresponding results for the linear discrete-time systems are presented in Section 3.2. This chapter concludes with a summary in Section 6.4.

3.1 Continuous Linear Time-Invariant Systems

3.1.1 Problem Statement

Consider a linear time-invariant (LTI) plant subject to actuator saturation

$$\begin{cases} \dot{x}_{\rm p} &= A_{\rm p}x_{\rm p} + B_{\rm p}\sigma(u), \\ y &= C_{\rm p}x_{\rm p}, \end{cases} \tag{3.1}$$

where $x_{\rm p} \in \mathbf{R}^{n_x}$, $y \in \mathbf{R}^{n_y}$, $u \in \mathbf{R}^{n_u}$, and the matrix triple $(A_{\rm p}, B_{\rm p}, C_{\rm p})$ are stabilizable and detectable. Also, the function $\sigma : \mathbf{R}^{n_u} \to \mathbf{R}^{n_u}$ is a vector valued standard saturation function, *i.e.*,

$$\sigma(u) = \begin{bmatrix} \sigma(u_1) & \sigma(u_2) & \cdots & \sigma(u_{n_u}) \end{bmatrix}^{\rm T}, \tag{3.2}$$

with

$$\sigma(u_i) = \mathrm{sgn}(u_i)\ \min\{1, |u_i|\}.$$

Here, we have slightly abused the notation by using σ to denote both the scalar valued and vector valued saturation function. We note that it is without loss of generality to assume a unity saturation level, as level of saturation can always be scaled to unity by scaling $B_{\rm p}$ and u.

We will consider a dynamic output feedback law of the form,

$$\begin{cases} \dot{x}_k &= f_k(x_k, y), \quad x_k \in \mathbf{R}^{n_k}, \\ u &= C_k x_k + D_k y, \end{cases} \tag{3.3}$$

where n_k is dimension of the controller, C_k and D_k are constant matrices of appropriate dimensions, and f_k is a function that is Lipschitz in x_k and y. We will consider both the full order controllers with $n_k = n_x$ and reduced order controller with $n_k = n_x - r$, where r is the number of measurable states. The use of this controller is motivated by the nonlinear gain scheduling control.

Our design objective is to construct a dynamic output feedback law of the form (3.3) that asymptotically stabilizes the plant (3.1) at the origin with a domain of

attraction as large as possible. This objective is achieved as follows. We will first parameterize the controller in a quasi-LPV form and then establish conditions on the coefficient matrices of the parameterized controller under which the closed-loop system is asymptotically stable at the origin with an ellipsoid of the form

$$\Omega(P,\eta)=\left\{x\in\mathbf{R}^{n_x+n_k}:\ x^{\mathrm{T}}Px\le\eta\right\},\qquad P\in\mathbf{S}_{+}^{(n_x+n_k)\times(n_x+n_k)},\ \eta>0, \tag{3.4}$$

contained in the domain of attraction. The determination of the controller coefficient matrices is then formulated and solved as an optimization problem with LMI constraints and the objective of maximizing the ellipsoid $\Omega(P,\eta)$.

3.1.2 Some Preliminary Results

To achieve the design objective mentioned above, we will need to use a tool from [25] for expressing the saturated linear feedback $\sigma(u)=\sigma(C_kx_k+D_ky)$ on a convex hull.

For $H_{_C}\in\mathbf{R}^{n_u\times n_x}$ and $H_{_D}\in\mathbf{R}^{n_u\times n_y}$, define

$$\mathcal{L}(H_{_C},H_{_D})=\left\{(x_k,y)\in\mathbf{R}^{n_k+n_y}:\ |H_{_{C_i}}x_k+H_{_{D_i}}y|\le 1,\ i\in\mathbf{I}[1,n_u]\right\}, \tag{3.5}$$

where $H_{_{C_i}}$ and $H_{_{D_i}}$ represent the ith row of matrices $H_{_C}$ and $H_{_D}$ respectively. We note that $\mathcal{L}(H_{_C},H_{_D})$ represents the region in $\mathbf{R}^{n_k+n_y}$ where the auxiliary feedback $H_{_C}x_{\mathrm{k}}+H_{_D}y$ does not saturate.

Also, let $\mathcal{V}$ be the set of $n_u\times n_u$ diagonal matrices whose diagonal elements are either 1 or 0. There are 2^{n_u} elements in $\mathcal{V}$. Suppose these elements of $\mathcal{V}$ are labeled as $E_j,\ j\in\mathbf{I}[0,2^{n_u}-1]$ and denote $E_j^-=I-E_j$. Clearly, $E_j^-\in\mathcal{V}$ if $E_j\in\mathcal{V}$.

The following lemma is adopted from [25].

Lemma 1 *For any* $(x_{\mathrm{k}},y)\in\mathcal{L}(H_{_C},H_{_D})$,

$$\sigma(C_kx_k+D_ky)\in\mathrm{co}\left\{E_j(C_kx_k+D_ky)+E_j^-(H_{_C}x_k+H_{_D}y),\ j\in\mathbf{I}[0,2^{n_u}-1]\right\},$$

where co *stands for the convex hull.*

We note here that the rows of the matrix $E_j C_k + E_j^- H_{C}$ are a mixture of those of C_k and H_{C}. A "1" in the ith element of E_j indicates that the ith row of $E_j C_k + E_j^- H_{C}$ is the ith row of C_k, while a "0" in the ith element of E_j indicates that the ith row of $E_j C_k + E_j^- H_{C}$ is the ith row of H_{C}. Similarly, the rows of $E_j D_k + E_j^- H_{D}$ are a mixture of those of D_k and H_{D}.

For single saturation case (*i.e.*, $n_u = 1$), the convex covering in the above lemma is non-conservative as shown in [25]. In general, Lemma 1 provides more stringent convex covering of saturation nonlinearities than other existing approaches [23, **?**]. As a result, the stability conditions derived from this lemma will be less conservative.

By Lemma 1, the saturated linear feedback, with $(x_k, y) \in \mathcal{L}(H_{C}, H_{D})$, can be expressed as

$$\sigma(C_k x_k + D_k y) = \sum_{j=0}^{2^{n_u}-1} \rho_j \left[E_j (C_k x_k + D_k y) + E_j^- (H_{C} x_k + H_{D} y) \right], \qquad (3.6)$$

for some scalars $0 \le \rho_j \le 1,\ j \in \mathbf{I}[0, 2^{n_u} - 1]$, such that $\sum_{j=0}^{2^{n_u}-1} \rho_j = 1$. We note that the values of the parameters $\rho = [\rho_0, \quad \rho_1, \cdots, \rho_{2^{n_u}-1}]^{\mathrm{T}}$ are dependent on x_k and y and are available for real-time use in gain-scheduling control. They reflect in a way the severity of control saturation. For example, for the case of $n_u = 2$, the values of ρ_is can be determined as follows. Let

$$\lambda_i(x_k, y) = \begin{cases} 1, & \text{if} \quad C_{ki} x_k + D_{ki} y = H_{C_i} x_k + H_{D_i} y, \\ \frac{\sigma(u_i) - (H_{C_i} x_k + H_{D_i} y)}{(C_{ki} - H_{C_i}) x_k + (D_{ki} - H_{D_i}) y}, & \text{otherwise}, \end{cases} \qquad (3.7)$$

for $j = 1, 2$.

Note that if the denominator is non-zero, the expression of λ_i is equivalent to

$$\lambda_i = \frac{\sigma(u_i) - (H_{C_i} x_k + H_{D_i} y)}{(C_{ki} - H_{C_i}) x_k + (D_{ki} - H_{D_i}) y}$$

From the definition of $\sigma(u)$ and $\mathcal{L}(H_{C}, H_{D})$, we know that

$$0 < \frac{\sigma(u_i)}{u_i} \le 1, \quad -1 \le H_{C_i} x_k + H_{D_i} y \le 1$$

Clearly, when no saturation occurs :

$$\lambda_i = 1, \qquad \sigma(u_i) = u_i$$

When saturation occurs:

$$\begin{cases} 0 \leq \lambda_i < 1 & u_i > 1, \sigma(u_i) = 1 \\ 0 \leq \lambda_i < 1 & u_i < -1, \sigma(u_i) = -1 \end{cases}$$

To sum up, all of the λ_is satisfy $\lambda_i \in [0, 1]$, which indicate the saturation severity of the ith input channel.

By choosing $\rho_j, \jmath \in [1, 2^{n_u}]$ as following, we can get a set of applicable ρ_j:

$$\begin{cases} \rho_1 &= \lambda_1\lambda_2 \\ \rho_2 &= (1-\lambda_1)\lambda_2 \\ \rho_3 &= \lambda_1(1-\lambda_2) \\ \rho_4 &= (1-\lambda_1)(1-\lambda_2). \end{cases} \tag{3.8}$$

Clearly, each $\rho_j \in [0, 1]$ and $\sum_{j=1}^{4} \rho_j = 1$. This is the way we used to choose ρ_j in this paper. In general, there are many other choices of ρ_j satisfying the same constraint, leading to non-unique representation of (3.6). For higher order systems with input more than two, the choice of ρ_j can be defined by the similar way.

3.1.3 Full Order Output Feedback Law

We will use the functions $\rho_j(x_k, y)$'s to parameterize the output feedback control (3.3) into the following quasi-LPV system,

$$\begin{cases} \dot{x}_k &= \left(\sum_{j=0}^{2^{n_u}-1} \rho_j(x_k, y) A_{kj}\right) x_k + \left(\sum_{j=0}^{2^{n_u}-1} \rho_j(x_k, y) B_{kj}\right) y, \\ u &= C_k x_k + D_k y, \end{cases} \tag{3.9}$$

where the coefficient matrices A_{kj}'s, B_{kj}'s, C_k and D_k are to be designed. Since ρ_j is a real-time changing parameter which is used to schedule the controller gain, the resulting controller is called a gain scheduling controller .

We can also write the plant (3.1) in a quasi-LPV form as follows,

$$\left\{\begin{array}{rcl} \dot{x}_{\rm p} &=& A_{\rm p}x_{\rm p} + B_{\rm p}\displaystyle\sum_{j=0}^{2^{n_u}-1}\rho_j(x_k,y)\left[E_j(C_kx_k+D_ky)+E_j^-(H_{_C}x_k+H_{_D}y)\right], \\ y &=& C_{\rm p}x_{\rm p}. \end{array}\right. \tag{3.10}$$

Motivated by the quasi-LPV structure of both the plant and the controller, we consider the following auxiliary LPV system, of which the closed-loop system comprising of (3.10) and (4.2) is a special case, if $\Omega(P,\eta)\subset\mathcal{L}(H_{_D}C_{\rm p},H_{_C})$ is an invariant set,

$$\dot{x} = A(\rho)x, \qquad \rho\in\mathbf{\Gamma}, \tag{3.11}$$

where

$$x=\begin{bmatrix}x_{\rm p}\\ x_k\end{bmatrix}, \qquad \rho=[\rho_0,\rho_1,\cdots,\rho_{2^{n_u}-1}],$$
$$\mathbf{\Gamma}=\left\{\rho\in\mathbf{R}^{2^{n_u}}:\ \sum_{j=0}^{2^{n_u}-1}\rho_j=1,\ 0\le\rho_j\le 1,\ j\in\mathbf{I}[0,2^{n_u}-1]\right\},$$

and

$$A(\rho)=\begin{bmatrix}A_{\rm p}&0\\0&0\end{bmatrix}+\begin{bmatrix}0&B_{\rm p}\\I&0\end{bmatrix}\left(\sum_{j=0}^{2^{n_u}-1}\rho_j\begin{bmatrix}A_{kj}&B_{kj}\\E_jC_k+E_j^-H_{_C}&E_jD_k+E_j^-H_{_D}\end{bmatrix}\right)\begin{bmatrix}0&I\\C_{\rm p}&0\end{bmatrix}.$$

By defining

$$A_j=\begin{bmatrix}A_{\rm p}&0\\0&0\end{bmatrix}+\begin{bmatrix}0&B_{\rm p}\\I&0\end{bmatrix}\begin{bmatrix}A_{kj}&B_{kj}\\E_jC_k+E_j^-H_{_C}&E_jD_k+E_j^-H_{_D}\end{bmatrix}\begin{bmatrix}0&I\\C_{\rm p}&0\end{bmatrix}, \tag{3.12}$$

the matrix $A(\rho)$ can be written as

$$A(\rho)=\sum_{j=0}^{2^{n_u}-1}\rho_jA_j.$$

The following theorem establishes conditions on the full order controller coefficient matrices under which the LPV system (3.11) is asymptotically stable with a quadratic Lyapunov function.

Theorem 1 *Consider the closed-loop system (3.11). If there exist positive definite matrices* $R, S \in \mathbf{S}_+^{n_x \times n_x}$ *and matrices* $(\bar{A}_{kj}, \bar{B}_{kj}) \in \mathbf{R}^{n_x \times n_x} \times \mathbf{R}^{n_x \times n_y}, j \in \mathbf{I}[0, 2^{n_u} - 1]$, $(\bar{C}_k, \bar{D}_k) \in \mathbf{R}^{n_u \times n_x} \times \mathbf{R}^{n_u \times n_y}$ *and* $(\bar{H}_{_C}, \bar{H}_{_D}) \in \mathbf{R}^{n_u \times n_x} \times \mathbf{R}^{n_u \times n_y}$ *such that*

$$\begin{bmatrix} A_{\rm p}R + B_{\rm p}\bar{U}_{kj} + RA_{\rm p}^{\rm T} + \bar{U}_{kj}^{\rm T}B_{\rm p}^{\rm T} & \star \\ \bar{A}_{kj} + A_{\rm p}^{\rm T} + C_{\rm p}^{\rm T}\bar{V}_{kj}^{\rm T}B_{\rm p}^{\rm T} & A_{\rm p}^{\rm T}S + C_{\rm p}^{\rm T}\bar{B}_{kj}^{\rm T} + SA_{\rm p} + \bar{B}_{kj}C_{\rm p} \end{bmatrix} < 0,$$

$$j \in \mathbf{I}[0, 2^{n_u} - 1], \tag{3.13}$$

$$\begin{bmatrix} R & I_{n_x} \\ I_{n_x} & S \end{bmatrix} > 0, \tag{3.14}$$

where $\bar{U}_{kj} = E_j\bar{C}_k + E_j^-\bar{H}_{_C}$ *and* $\bar{V}_{kj} = E_j\bar{D}_k + E_j^-\bar{H}_{_D}$*, then, with the following controller coefficient matrices for each* $j \in \mathbf{I}[0, 2^{n_u} - 1]$*,*

$$\begin{bmatrix} A_{kj} & B_{kj} \\ C_k & D_k \\ H_{_C} & H_{_D} \end{bmatrix} = \begin{bmatrix} N & SB_{\rm p}E_j & SB_{\rm p}E_j^- \\ 0 & I_{n_u} & 0 \\ 0 & 0 & I_{n_u} \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \bar{A}_{kj} & \bar{B}_{kj} \\ \bar{C}_k & \bar{D}_k \\ \bar{H}_{_C} & \bar{H}_{_D} \end{bmatrix} - \begin{bmatrix} SA_{\rm p}R & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \begin{bmatrix} M^{\rm T} & 0 \\ C_{\rm p}R & I_{n_y} \end{bmatrix}^{-1} \tag{3.15}$$

where $M, N \in \mathbf{R}^{n_x \times n_x}$ *are such that* $MN^{\rm T} = I_{n_x} - RS$*, the closed-loop system (3.11) is asymptotically stable. Moreover,*

$$A^{\rm T}(\rho)P + PA(\rho) < 0, \qquad \forall \rho \in \mathbf{\Gamma}, \tag{3.16}$$

where $P > 0$ *is given by*

$$P = \begin{bmatrix} I_{n_x} & S \\ 0 & N^{\rm T} \end{bmatrix} \begin{bmatrix} R & I_{n_x} \\ M^{\rm T} & 0 \end{bmatrix}^{-1}. \tag{3.17}$$

Proof. First, define two transformation matrices Z_1 and Z_2 as

$$Z_1 = \begin{bmatrix} R & I \\ M^{\rm T} & 0 \end{bmatrix}, \qquad Z_2 = \begin{bmatrix} I & S \\ 0 & N^{\rm T} \end{bmatrix},$$

which are both nonsingular, then it is easy to show that $PZ_1 = Z_2$ and

$$Z_1^{\mathrm{T}} P Z_1 = Z_2^{\mathrm{T}} Z_1 = \begin{bmatrix} R & I \\ I & S \end{bmatrix} > 0,$$

It implies that

$$P = \left(Z_1^{-1}\right)^{\mathrm{T}} \begin{bmatrix} R & I \\ I & S \end{bmatrix} Z_1^{-1} > 0,$$

P is positive definite and (3.17) is verified.

Using the fact that $PZ_1 = Z_2$ again, we have that, for each $j \in \mathbf{I}[0, 2^{n_u} - 1]$,

$$\begin{aligned} & Z_1^{\mathrm{T}} \left(P A_j\right) Z_1 = Z_2^{\mathrm{T}} A_j Z_1 \\ = & \begin{bmatrix} A_{\mathrm{p}} R & A_{\mathrm{p}} \\ 0 & S A_{\mathrm{p}} \end{bmatrix} + \begin{bmatrix} 0 & B_{\mathrm{p}} \\ I & 0 \end{bmatrix} \begin{bmatrix} \bar{A}_{kj} & \bar{B}_{kj} \\ E_j \bar{C}_k + E_j^- \bar{H}_C & E_j \bar{D}_k + E_j^- \bar{H}_D \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & C_{\mathrm{p}} \end{bmatrix}, \end{aligned} \tag{3.18}$$

where the transformed controller data $(\bar{A}_{kj}, \bar{B}_{kj}, \bar{C}_k, \bar{D}_k, \bar{H}_C, \bar{H}_D)$ relates to the original controller data $(A_{kj}, B_{kj}, C_k, D_k, H_C, H_D)$ by

$$\begin{bmatrix} \bar{A}_{kj} & \bar{B}_{kj} \\ \bar{C}_k & \bar{D}_k \\ \bar{H}_C & \bar{H}_D \end{bmatrix} = \begin{bmatrix} S A_{\mathrm{p}} R & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} N & S B_{\mathrm{p}} E_j & S B_{\mathrm{p}} E_j^- \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_{kj} & B_{kj} \\ C_k & D_k \\ H_C & H_D \end{bmatrix} \begin{bmatrix} M^{\mathrm{T}} & 0 \\ C_{\mathrm{p}} R & I \end{bmatrix}. \tag{3.19}$$

We note here that both matrices

$$\begin{bmatrix} N & S B_{\mathrm{p}} E_j & S B_{\mathrm{p}} E_j^- \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \qquad \begin{bmatrix} M^{\mathrm{T}} & 0 \\ C_{\mathrm{p}} R & I \end{bmatrix}$$

are nonsingular.

Now we consider the quadratic Lyapunov function

$$V(x) = x^{\mathrm{T}} P x.$$

Taking derivative of $V(x)$ along the trajectories of (3.11) yields

$$\dot{V} = x^{\mathrm{T}} \left[A^{\mathrm{T}}(\rho) P + P A(\rho)\right] x, \qquad \forall x \in \mathbf{R}^{n_x + n_k} \setminus \{0\},\ \forall \rho \in \mathbf{\Gamma},$$

The negative definiteness of $\dot{V}$ directly leads to condition (3.16), which implies the asymptotic stability of the closed-loop system.

It follows from (3.18) and (3.13) that, for each $j \in \mathbf{I}[0, 2^{n_u}-1]$,

$$Z_1^{\mathrm{T}} \left(A_j^{\mathrm{T}} P + P A_j\right) Z_1 = \begin{bmatrix} A_{\mathrm{p}} R + B_{\mathrm{p}} \bar{U}_{kj} + R A_{\mathrm{p}}^{\mathrm{T}} + \bar{U}_{kj}^{\mathrm{T}} B_{\mathrm{p}}^{\mathrm{T}} & \star \\ \bar{A}_{kj} + A_{\mathrm{p}}^{\mathrm{T}} + C_{\mathrm{p}}^{\mathrm{T}} \bar{V}_{kj}^{\mathrm{T}} B_{\mathrm{p}}^{\mathrm{T}} & A_{\mathrm{p}}^{\mathrm{T}} S + C_{\mathrm{p}}^{\mathrm{T}} \bar{B}_{kj}^{\mathrm{T}} + S A_{\mathrm{p}} + \bar{B}_{kj} C_{\mathrm{p}} \end{bmatrix} < 0,$$

which implies that

$$A_j^{\mathrm{T}} P + P A_j < 0. \tag{3.20}$$

Taking linear combination of (3.20) over $j \in \mathbf{I}[0, 2^{n_u}-1]$ with coefficients ρ_j yields (3.16), *i.e.*,

$$A^{\mathrm{T}}(\rho) P + P A(\rho) < 0, \qquad \forall \rho \in \mathbf{\Gamma}.$$

Then the asymptotic stability of the closed-loop system (3.11) is verified. Q.E.D.

Recalling that the quasi-LPV representation of the plant (3.10) and controller (4.2) was obtained under the assumption that $(x_k, y) \in \mathcal{L}(H_{_C}, H_{_D})$, thus we have the following result on output stabilization of the plant (3.1) using a full order controller (3.3).

Theorem 2 *Consider the plant (3.1). If there exist positive definite matrices $R, S \in S_+^{n_x \times n_x}$ and matrices $(\bar{A}_{kj}, \bar{B}_{kj}) \in \mathbf{R}^{n_x \times n_x} \times \mathbf{R}^{n_x \times n_y}, j \in \mathbf{I}[0, 2^{n_u}-1]$, $(\bar{C}_k, \bar{D}_k) \in \mathbf{R}^{n_u \times n_x} \times \mathbf{R}^{n_u \times n_y}$ and $(\bar{H}_{_C}, \bar{H}_{_D}) \in \mathbf{R}^{n_u \times n_x} \times \mathbf{R}^{n_u \times n_y}$ that satisfy (3.13)-(3.14) and*

$$\begin{bmatrix} \frac{1}{\eta} & \bar{H}_{_{C i}} & \bar{H}_{_{D i}} C_{\mathrm{p}} \\ \bar{H}_{_{C i}}^{\mathrm{T}} & R & I_{n_x} \\ C_{\mathrm{p}}^{\mathrm{T}} \bar{H}_{_{D i}}^{\mathrm{T}} & I_{n_x} & S \end{bmatrix} \geq 0, \quad i \in \mathbf{I}[1, n_u], \tag{3.21}$$

then, the output feedback controller (4.2) with the controller coefficient matrices given by (3.15) asymptotically stabilizes the plant (3.1) at the origin with the ellipsoid $\Omega(P, \eta)$ contained in the domain of attraction.

Proof. We first note that,

$$\begin{aligned}\mathcal{L}(H_{_C}, H_{_D}) :&= \{(x_k, y) : |H_{_{Ci}} x_k + H_{_{Di}} y| \le 1,\ i \in \mathbf{I}[1, n_u]\} \\ &= \{(x_p, x_k) : |H_{_{Di}} C_p x_p + H_{_{Ci}} x_k| \le 1,\ i \in \mathbf{I}[1, n_u]\} \\ &:= \mathcal{L}(H_{_D} C_p, H_c)\end{aligned} \tag{3.22}$$

For saturation control, the closed-loop system is asymptotically stable at the origin with $\Omega(P, \eta)$ contained in its domain of attraction if

$$\Omega(P, \eta) \subset \mathcal{L}(H_{_D} C_{\rm p}, H_{_C}),$$

which is equivalent to [25]

$$\eta \begin{bmatrix} H_{_{Di}} C_{\rm p} & H_{_{Ci}} \end{bmatrix} P^{-1} \begin{bmatrix} C_{\rm p}^{\rm T} H_{_{Di}}^{\ \rm T} \\ H_{_{Ci}}^{\ \rm T} \end{bmatrix} \le 1, \qquad i \in \mathbf{I}[1, n_u].$$

By Schur complement, the above inequality can be rewritten as

$$\begin{bmatrix} \frac{1}{\eta} & \begin{bmatrix} H_{_{Di}} C_{\rm p} & H_{_{Ci}} \end{bmatrix} \\ \begin{bmatrix} C_{\rm p}^{\rm T} H_{_{Di}}^{\ \rm T} \\ H_{_{Ci}}^{\ \rm T} \end{bmatrix} & P \end{bmatrix} \ge 0, \tag{3.23}$$

Note that

$$\begin{bmatrix} H_{_D} C_{\rm p} & H_{_C} \end{bmatrix} Z_1 = \begin{bmatrix} \bar{H}_{_C} & \bar{H}_{_D} C_{\rm p} \end{bmatrix},$$

As a result, multiplying diag $\{1, Z_1^{\rm T}\}$ from the left-hand side and its transpose from the right-hand side of the inequality (3.23) directly leads to condition (3.21). Q.E.D.

Theorem 2 provides conditions on the coefficient matrices of the full order controller under which the ellipsoid $\Omega(P, \eta)$ is inside the domain of attraction. To maximize the cross-section of the ellipsoid in the plant state-space, we introduce a shape reference ellipsoid $\Omega(P_{\rm R}, 1)$ with a given positive definite matrix $P_{\rm R} \in \mathbf{S}_+^{n_x \times n_x}$ and maximize α such that

$$\alpha\Omega(P_{\rm R}, 1) \subset \Omega(P, \eta)|_{(x_{\rm p}, 0)},$$

which can be converted into an LMI constraint:

$$\frac{1}{\alpha^2} P_{\mathrm{R}} - \frac{S}{\eta} \geq 0. \tag{3.24}$$

Thus, the determination of the controller coefficient matrices can be formulated and solved as an LMI optimization problem:

$$\begin{aligned} &\max \alpha, \\ \text{s.t.} \quad &(3.13) - (3.14), (3.21), (3.24). \end{aligned} \tag{3.25}$$

3.1.4 Reduced Order Output Feedback Law

The dynamical order of output feedback law synthesized in Subsection 3.1.3 is the same as that of the plant. However, in practice, we usually encounter the case that only part of the information about plant is known. Therefore, a lower order controller is desired to facilitate control implementation. The stabilization problem using a reduced order output feedback controller is generally non-convex. However, if part of the state is measurable, then one can synthesize a lower order nonlinear output feedback controller. For this purpose, let us consider the situation where partial state measurement is available,

$$\left\{ \begin{array}{rcl} \dot{x}_{\mathrm{p}} &=& A_{\mathrm{p}} x_{\mathrm{p}} + B_{\mathrm{p}} \sigma(u), \\ y &=& C_{\mathrm{p}} x_{\mathrm{p}} = \left[\begin{array}{c} \left[0_{r \times (n_x - r)} \quad I_r \right] \\ \bar{C}_{\mathrm{p}} \end{array} \right] x_{\mathrm{p}}. \end{array} \right. \tag{3.26}$$

where r indicates the number of measurable states.

Following theorem provides conditions under which the closed-loop system of form (3.26) is stabilized by constructing a nonlinear output feedback controller with order $(n_x - r)$ in the form of

$$\left\{ \begin{array}{rcl} \dot{x}_k &=& \left(\sum_{j=0}^{2^{n_u}-1} \rho_j A_{kj} \right) x_k + \left(\sum_{j=0}^{2^{n_u}-1} \rho_j B_{kj} \right) y, \\ u &=& C_k x_k + D_k y, \end{array} \right. \tag{3.27}$$

where $x_{\mathrm{k}} \in \mathbf{R}^{n_x - r}$, $y \in \mathbf{R}^{n_y}$, $u \in \mathbf{R}^{n_u}$.

Theorem 3 *Consider the plant (3.26). Suppose that there exist positive definite matrices* $R \in \mathbf{S}_+^{n_x \times n_x}, S \in \mathbf{S}_+^{n_x \times n_x}$, $(\bar{A}_{kj}, \bar{B}_{kj}) \in \mathbf{R}^{(n_x-r)\times n_x} \times \mathbf{R}^{(n_x-r)\times(n_y-r)}, j \in \mathbf{I}[0, 2^{n_u}-1]$, $(\bar{C}_k, \bar{D}_k) \in \mathbf{R}^{n_u \times n_x} \times \mathbf{R}^{n_u \times (n_y-r)}$ *and* $(\bar{H}_{C}, \bar{H}_{D}) \in \mathbf{R}^{n_u \times n_x} \times \mathbf{R}^{n_u \times (n_y-r)}$ *such that*

$$\begin{bmatrix} \left\{ \begin{array}{c} A_{\rm p}R + B_{\rm p}\bar{U}_{kj} \\ +RA_{\rm p}^{\rm T} + \bar{U}_{kj}^{\rm T}B_{\rm p}^{\rm T} \end{array} \right\} & \star \\ \left\{ \begin{array}{c} \bar{A}_{kj} + H^{\rm T}A_{\rm p}^{\rm T} \\ +H^{\rm T}\bar{C}_{\rm p}^{\rm T}\bar{V}_{kj}^{\rm T}B_{\rm p}^{\rm T} \end{array} \right\} & \left\{ \begin{array}{c} H^{\rm T}A_{\rm p}^{\rm T}S^{\rm T}H + H^{\rm T}\bar{C}_{\rm p}^{\rm T}\bar{B}_{kj}^{\rm T} \\ +H^{\rm T}SA_{\rm p}H + \bar{B}_{kj}\bar{C}_{\rm p}H \end{array} \right\} \end{bmatrix} < 0, \ j \in \mathbf{I}[0, 2^{n_u}-1], \tag{3.28}$$

$$\begin{bmatrix} \frac{1}{\eta} & \bar{H}_{C_i} & \bar{H}_{D_i}\bar{C}_{\rm p}H \\ \bar{H}_{C_i}^{\rm T} & R & H \\ H^{\rm T}\bar{C}_{\rm p}^{\rm T}\bar{H}_{D_i}^{\rm T} & H^{\rm T} & S_{11} \end{bmatrix} \geq 0, \quad i \in \mathbf{I}[1, n_u], \tag{3.29}$$

$$\begin{bmatrix} R & H \\ H^{\rm T} & S_{11} \end{bmatrix} > 0, \tag{3.30}$$

where $H = \begin{bmatrix} I_{n_x-r} \\ 0_{r\times(n_x-r)} \end{bmatrix}$, $S_{11} = H^{\rm T}SH$, $\bar{U}_{kj}$ *and* $\bar{V}_{kj}$ *are defined similarly as in Theorem 1.*

Then, with the following controller coefficient matrices

$$\begin{bmatrix} A_{kj} & B_{kj} \\ C_k & D_k \\ H_{C} & H_{D} \end{bmatrix} = \begin{bmatrix} H^{\rm T}N & H^{\rm T}SB_{\rm p}E_j & H^{\rm T}SB_{\rm p}E_j^{-} \\ 0 & I_{n_u} & 0 \\ 0 & 0 & I_{n_u} \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \bar{A}_{kj} & \bar{B}_{kj} \\ \bar{C}_k & \bar{D}_k \\ \bar{H}_{C} & \bar{H}_{D} \end{bmatrix} - \begin{bmatrix} H^{\rm T}SA_{\rm p}R & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$
$$\times \left[\begin{array}{cc|c} M^{\rm T} & 0 & \\ \hline \begin{bmatrix} 0 & I_r \end{bmatrix} R & 0 & \\ \bar{C}_{\rm p}R & I_{n_y-r} & \end{array} \right]^{-1}, \quad j \in \mathbf{I}[0, 2^{n_u}-1], \tag{3.31}$$

where $M, N \in \mathbf{R}^{n_x \times (n_x-r)}$ *are such that* $MN^{\rm T} = I_{n_x} - RS$, *the closed-loop system composing of the plant (3.26) and the* $(n_x - r)$*th order output feedback law (3.27) is*

asymptotically stable at the origin with the ellipsoid $\Omega(P,\eta)$ *contained in the domain of attraction, where*

$$P = \begin{bmatrix} I_{n_x} & SH \\ 0 & N^{\mathrm{T}}H \end{bmatrix} \begin{bmatrix} R & H \\ M^{\mathrm{T}} & 0 \end{bmatrix}^{-1}. \tag{3.32}$$

Proof. Let

$$Z_1 = \begin{bmatrix} R & H \\ M^{\mathrm{T}} & 0 \end{bmatrix}, \qquad Z_2 = \begin{bmatrix} I & SH \\ 0 & N^{\mathrm{T}}H \end{bmatrix}.$$

Then $PZ_1 = Z_2$ and

$$Z_1^{\mathrm{T}} P Z_1 = \begin{bmatrix} R & H \\ H^{\mathrm{T}} & H^{\mathrm{T}}SH \end{bmatrix} = \begin{bmatrix} R & H \\ H^{\mathrm{T}} & S_{11} \end{bmatrix} > 0,$$

which implies that Z_1 is nonsingular. Thus,

$$X = (Z_1^{\mathrm{T}})^{-1} \begin{bmatrix} R & H \\ H^{\mathrm{T}} & S_{11} \end{bmatrix} Z_1^{-1} > 0.$$

Similarly, we have that, for all $j \in \mathbf{I}[0, 2^{n_u}-1]$,

$$Z_1^{\mathrm{T}} (PA_j) Z_1 = \begin{bmatrix} A_{\mathrm{p}}R & A_{\mathrm{p}}H \\ 0 & H^{\mathrm{T}}SA_{\mathrm{p}}H \end{bmatrix} + \begin{bmatrix} 0 & B_{\mathrm{p}} \\ I & 0 \end{bmatrix} \begin{bmatrix} \bar{A}_{kj} & \bar{B}_{kj} \\ \bar{U}_{kj} & \bar{V}_{kj} \end{bmatrix} \left[\begin{array}{cc} I & 0 \\ \hline 0 & \begin{bmatrix} 0 & I \end{bmatrix} H \\ 0 & \bar{C}_{\mathrm{p}}H \end{array}\right],$$

where

$$\begin{bmatrix} \bar{A}_{kj} & \bar{B}_{kj} \\ \bar{C}_k & \bar{D}_k \\ \bar{H}_C & \bar{H}_D \end{bmatrix} = \begin{bmatrix} H^{\mathrm{T}}SA_{\mathrm{p}}R & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} H^{\mathrm{T}}N & H^{\mathrm{T}}SB_{\mathrm{p}}E_j & H^{\mathrm{T}}SB_{\mathrm{p}}E_j^{-} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \times \begin{bmatrix} A_{kj} & B_{kj} \\ C_k & D_k \\ H_C & H_D \end{bmatrix} \left[\begin{array}{cc} M^{\mathrm{T}} & 0 \\ \hline \begin{bmatrix} 0 & I \end{bmatrix} R & 0 \\ \bar{C}_{\mathrm{p}}R & I \end{array}\right]. \tag{3.33}$$

The fact Z_1 and P are nonsingular implies that $Z_2 = PZ_1$ is nonsingular, which result in the nonsingularity of NH^{T} and matrix

$$\begin{bmatrix} H^{\mathrm{T}}N & H^{\mathrm{T}}SB_{\mathrm{p}}E_j & H^{\mathrm{T}}SB_{\mathrm{p}}E_j^{-} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

Also,the nonsingularity of Z_1 implies that the matrix $\begin{bmatrix} R \\ M^{\mathrm{T}} \end{bmatrix}$ is of rank n_x and, as a result, $\begin{bmatrix} M^{\mathrm{T}} \\ \begin{bmatrix} 0 & I \end{bmatrix} R \end{bmatrix}$ is of rank n_x. Thus, the matrix

$$\begin{bmatrix} M^{\mathrm{T}} & 0 \\ \begin{bmatrix} 0 & I \end{bmatrix} R & 0 \\ \bar{C}_{\mathrm{p}}R & I \end{bmatrix}$$

is nonsingular. Therefore the controller coefficients (3.31) can be obtained from (3.33). The rest of proof is similar to the proof of Theorem 2. Q.E.D.

Remark 1 *Consider the following two extreme cases for r:*
1. When $r = 0$, no state is measurable from the output, Theorem 3 is converted to Theorem 2.
2. When $r = n_x$, i.e. *$C_{\mathrm{p}} = I_{n_x}$, the resulting controller is a zero dynamical controller. Theorem 3 recovers the state feedback stabilization result in [25] as its special case.*

3.1.5 Example

In this section, we will use an example to demonstrate the effectiveness of the proposed output feedback stabilization approach. This example focuses on the design procedure for an LTI system based on Theorem 2 and Theorem 3. For discrete-time systems, one can follow the similar way by applying Theorem 4.

Example 1. Consider the LTI plant (3.1) with

$$A_{\rm p} = \begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix}, \qquad B_{\rm p} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad C_{\rm p} = \begin{bmatrix} -2 & 2 \end{bmatrix}.$$

A state feedback law for this system was designed in [25]. Solving the optimization problem (5.16) using a shape reference $\Omega(P_{\rm R}, 1) = \left\{x \in \mathbf{R}^2 : x^T x \leq 1\right\}$, we obtain a quasi-LPV controller in the form of (4.2) with $\rho_0(x_k, y) = \lambda_1$ and $\rho_1(x_k, y) = 1 - \lambda_1$ and the coefficient matrices given by

$$\begin{bmatrix} A_{k0} \\ A_{k1} \end{bmatrix} = \left[\begin{array}{cc} -11.974 & 0.03796 \\ 0.07215 & -11.77 \\ \hline 4.7407 & 17.155 \\ -5.0864 & -17.03 \end{array}\right], \qquad \begin{bmatrix} B_{k0} \\ B_{k1} \end{bmatrix} = \left[\begin{array}{c} 0.16538 \\ -0.05341 \\ \hline 7.1291 \times 10^{-3} \\ -4.4428 \times 10^{-3} \end{array}\right],$$

$$C_k = \begin{bmatrix} 10.010 & 10.323 \end{bmatrix}, \qquad D_k = -0.73286.$$

The resulting maximum value of α is 1.11, with auxiliary matrices $H_{_C}$ and $H_{_D}$ given by

$$H_{_C} = \begin{bmatrix} 0.38143 & 0.41273 \end{bmatrix}, \qquad H_{_D} = -0.64196,$$

and the resulting ellipsoid $\Omega(P, 1)$ specified by

$$P = \begin{bmatrix} 3.9649 & -2.1876 & 1.0 & 0 \\ -2.1876 & 1.7747 & 0 & 1.0 \\ 1.0 & 0 & 5.2856 \times 10^2 & 1.3519 \times 10^3 \\ 0 & 1.0 & 1.3519 \times 10^3 & 3.9675 \times 10^3 \end{bmatrix}.$$

To visualize how our output feedback controller compare with the state feedback law of [25], we plot the cross-section of $\Omega(P, \eta)$ (solid line) and compare it with the two dimensional ellipsoid resulting from the state feedback law (dot line in Fig. 3.1). As can be seen, the output feedback law recovers a significant portion of the stability region resulting from the state feedback law of [25].

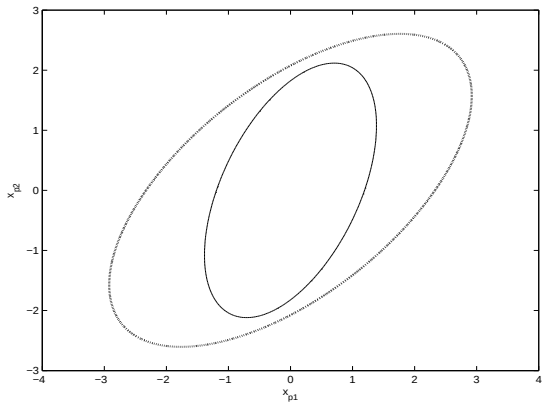
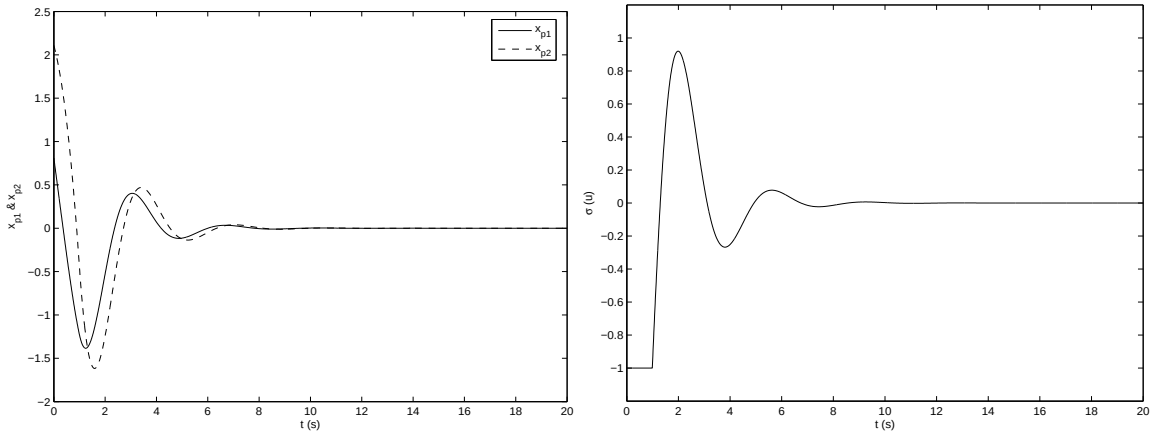


Figure 3.1: A cross-section of $\Omega(P, \eta)$ at $x_k = 0$, under a full order output feedback law.

Shown in Figs. 3.2 are the responses of the closed-loop system and the control input under a full order output feedback law. In the simulation, the initial conditions are $x_{\rm p}(0) = [0.81 \;\; 2.11]^{\rm T}$, for which states converge towards the origin and the control saturates at -1 during initial time.



(a) State Trajectory. (b) Control Input.

Figure 3.2: State trajectory and control input under a full order output feedback law.

We next modify $C_{\rm p}$ matrix to $[0 \;\; 1]$, *i.e.*, the second state is available for feedback. In this case, one can apply Theorem 3 to design a reduced order output feedback

law to achieve stabilization. The resulting $\Omega(P,1)$ is shown in Fig. 3.3, which is also smaller than the one using state feedback control and the maximum value of α is 1.94.

The state convergence from an initial condition $x_{\rm p}(0) = [-2 \quad -1.5]^{\rm T}$ is shown in Fig. 3.4(a) and the control input is saturated at $+1$ during the initial time as we can see from Fig. 3.4(b).

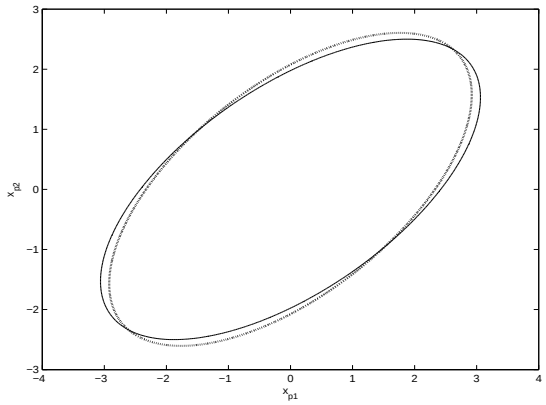


Figure 3.3: A cross-section of $\Omega(P,\eta)$ at $x_k = 0$, under a reduced order output feedback law.

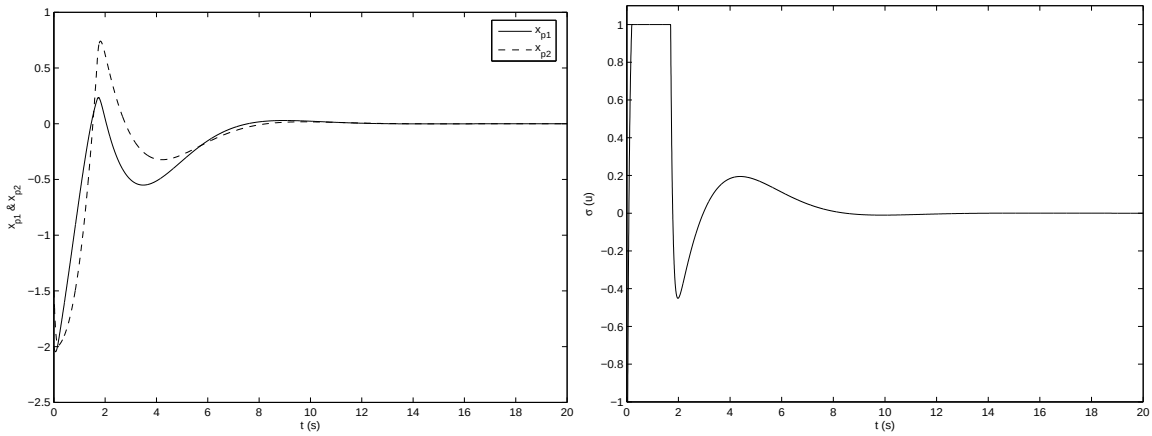
3.2 Discrete-Time Systems

Parallel with the continuous case, we consider a discrete-time linear system subject to actuator saturation

$$\left\{ \begin{array}{rcl} x_{\rm p}(k+1) & = & A_{\rm p}x(k) + B_{\rm p}\sigma(u(k)) \\ y(k) & = & C_{\rm p}x(k). \end{array} \right. \tag{3.34}$$

The discrete-time output feedback controller is of the form

$$\left\{ \begin{array}{rcl} \dot{x}_k(k+1) & = & f_k(x_k(k), y(k)), \quad x_k(k) \in \mathbf{R}^{n_k}, \\ u(k) & = & C_k x_k(k) + D_k y(k), \end{array} \right. \tag{3.35}$$



(a) State Trajectory. (b) Control Input.

Figure 3.4: State trajectory and control input under a reduced order output feedback law.

Use functions $\rho_j(x_k, y)$s to parameterize the controller into quasi-LPV form

$$\begin{cases} \dot{x}_k(k+1) &= \left(\sum_{j=0}^{2^{nu}-1} \rho_j A_{kj}\right) x_k(k) + \left(\sum_{j=0}^{2^{nu}-1} \rho_j B_{kj}\right) y(k) \\ u(k) &= C_k x_k(k) + D_k y(k), \end{cases} \tag{3.36}$$

Derivation of the controller form and the choice of functions $\rho_j(x_k, y)$s are similar to those in Subsection 3.1.2.

Following similar development as in Subsection 3.1.2, we have the following results on asymptotic stabilization of the plant (3.34).

Theorem 4 *Consider the plant (3.34). If there exist positive definite matrices* $R, S \in$

$\mathbf{S}_+^{n_x \times n_x}$ *and matrices* $\bar{A}_{kj}, \bar{B}_{kj}, \bar{C}_k, \bar{D}_k$ *and* $\bar{H}_{C}, \bar{H}_{D}$ *of appropriate dimensions such that*

$$\begin{bmatrix} -R & \star & \star & \star \\ -I & -S & \star & \star \\ A_{\mathrm{p}}R + B_{\mathrm{p}}\bar{U}_{kj} & A_{\mathrm{p}} + B_{\mathrm{p}}\bar{V}_{kj}C_{\mathrm{p}} & -R & \star \\ \bar{A}_{kj} & SA_{\mathrm{p}} + \bar{B}_{kj}C_{\mathrm{p}} & -I & -S \end{bmatrix} < 0, \quad j \in \mathbf{I}[0, 2^{n_u} - 1] \tag{3.37}$$

$$\begin{bmatrix} \frac{1}{\eta} & \bar{H}_{C_i} & \bar{H}_{D_i}C_{\mathrm{p}} \\ \bar{H}_{C_i}^{\mathrm{T}} & R & I \\ C_{\mathrm{p}}^{\mathrm{T}}\bar{H}_{D_i}^{\mathrm{T}} & I & S \end{bmatrix} \geq 0, \quad i \in \mathbf{I}[1, n_u] \tag{3.38}$$

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} > 0, \tag{3.39}$$

where $\bar{U}_{kj}, \bar{V}_{kj}$ *are as defined in Theorem 1, then, the controller (3.36) asymptotically stabilizes the plant (3.34) at the origin with the ellipsoid* $\Omega(P, \eta)$ *contained in the domain of attraction. The controller coefficient matrices and the Lyapunov matrix* P *are of the same form as (3.15) and (3.17) in coutinuous case.*

Proof: Basically, the proof follows the development in Section 1. Given a Lyapunov function $V(x) = x^T P x$, the stabilization condition of the discrete closed-loop system by output feedback control is

$$(A_j)^{\mathrm{T}} P A_j - P < 0 \tag{3.40}$$

$$\Omega(P, \eta) \subset \mathcal{L}(H_{D}C_{\mathrm{p}}, H_{C}) \tag{3.41}$$

$$P > 0 \tag{3.42}$$

By Schur Complement, the first condition is equivalent to

$$\begin{bmatrix} -P & (A_j)^{\mathrm{T}}P \\ PA_j & -P \end{bmatrix} < 0 \tag{3.43}$$

Taking linear combinations of equation (3.43), we have

$$\begin{bmatrix} -P & A^{\mathrm{T}}(\rho)P \\ PA_(\rho) & -P \end{bmatrix} < 0$$

The above condition combined with equation (3.41) proves the stability of saturated control system.

Multiplying condition (3.43) from the left-hand side by diag $\{Z_1^{\mathrm{T}}, Z_1^{\mathrm{T}}\}$ and its transpose from the right, we get constraint (5.8) through the congruent transformation. The rest of proof is the same as that of Theorem 2. *Q.E.D.*

3.3 Summary

In this chapter, we developed a method for synthesizing output feedback laws that asymptotically stabilize linear time systems subject to actuator saturation. The resulting output feedback laws are nonlinear in nature, and are parameterized in a quasi-LPV form. The determination of the controller coefficient matrices was formulated and solved as an LMI optimization problem, with the objective of enlarging the stability regions. A numerical example is presented to demonstrate the effectiveness of the proposed design method.

Note that the output feedback synthesis conditions in Theorems 1-4 involve only finite number of LMIs. This is due to the specific forms of output feedback control laws and the use of a single quadratic Lyapunov function. It is also possible to consider more general type of output feedback LPV controllers and parameter-dependent Lyapunov functions. However, the resulting synthesis conditions will be infinite-dimensional and the LMI optimization problem requires functional space approximation and gridding approach [1, 4, 71, 72].

Chapter 4

Disturbance Attenuation for Saturated Linear Systems

The result of actuator saturation that caused by the differences between the control input demanded by the controller and the actual output of the actuator often leads to instability of the control system and in many cases, the degradation of the system performance. After discussing the stability problems in Chapter 3, we will furtherly move to the issue of disturbance attenuation for linear systems subject to actuator saturation by using output feedback. In this chapter, a nonlinear output feedback law that attenuates the effect of the disturbance on the output of the system is constructed. The level of disturbance attenuation is measured in terms of the restricted $\mathcal{L}_2$ (ℓ_2) gain and the restricted $\mathcal{L}_2$ to $\mathcal{L}_\infty$ (ℓ_2 to ℓ_∞) gain over a class of bounded disturbances. Both continuous-time and discrete-time systems are considered.

The chapter is organized as follows. In Section 4.1, we will establish the mathematical model, compose the design objectives and provide some preliminary materials for the analysis of the saturated linear systems. Subsection 4.1.2 contains the disturbance tolerance conditions under which the trajectory boundedness of the closed-loop system under magnitude and energy bounded disturbances is guaranteed. Section 4.2 presents the output feedback laws for disturbance attenuation in terms of restricted

$\mathcal{L}_2$ gain and restricted $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain, respectively. Section 4.2 deals with continuous-time systems. Parallel results for discrete-time systems are presented in Section 4.3. In both Sections 4.2 and 4.3, numerical examples are used to illustrate our design procedure and demonstrate the effectiveness of the resulting controllers. Section 4.4 draws the conclusion to this section.

4.1 Quasi-LPV Output Feedback Laws

Given a continuous-time linear plant subject to actuator saturation,

$$\begin{cases} \dot{x}_{\rm p} &= A_{\rm p}x_{\rm p} + B_{\rm p1}w + B_{\rm p2}\sigma(u), \\ z &= C_{\rm p1}x_{\rm p} + D_{\rm p11}w + D_{\rm p12}\sigma(u), \\ y &= C_{\rm p2}x_{\rm p} + D_{\rm p21}w, \end{cases} \tag{4.1}$$

where $x_{\rm p} \in \mathbf{R}^{n_x}$ is the state, $u \in \mathbf{R}^{n_u}$ is the control input, $z \in \mathbf{R}^{n_z}$ is the controlled output, $w \in \mathbf{R}^{n_w}$ is the disturbance, $y \in \mathbf{R}^{n_y}$ is the measurement output. The definition of $\sigma(u)$ is the same as in (3.2). Here, we will assume that the triple $(A_{\rm p}, B_{\rm p2}, C_{\rm p2})$ is stabilizable and detectable.

4.1.1 Design Objectives

As we discussed in Chapter 3, the saturation input $\sigma(u)$ can be put on a convex hull and represented as (3.6). Assume the dynamic output feedback law is of the form (3.3). The linear subspace $\mathcal{L}(H_{_C}, H_{_D})$ where the auxiliary feedback $H_{_C}x_k + H_{_D}y$ does not saturate is defined as (3.5). Then when $(x_k, y) \in \mathcal{L}(H_{_C}, H_{_D})$, we can follow the similar way as in Subsection 3.1.3 to parameterize the controller and plant into the following quasi-LPV form:

$$\begin{cases} \dot{x}_k &= \left(\sum_{j=0}^{2^{n_u}-1} \rho_j(x_k, y) A_{kj}\right) x_k + \left(\sum_{j=0}^{2^{n_u}-1} \rho_j(x_k, y) B_{kj}\right) y, \\ u &= C_k x_k + D_k y, \end{cases} \tag{4.2}$$

and

$$\dot{x}_{\rm p} = A_{\rm p}x_{\rm p} + B_{\rm p1}w + B_{\rm p2}\left\{\sum_{j=0}^{2^{n_u}-1}\rho_j\left[E_j(C_kx_k + D_ky) + E_j^-(H_{_C}x_k + H_{_D}y)\right]\right\}, \quad (4.3)$$

$$z = C_{\rm p1}x_{\rm p} + D_{\rm p11}w + D_{\rm p12}\left\{\sum_{j=0}^{2^{n_u}-1}\rho_j\left[E_j(C_kx_c + D_ky) + E_j^-(H_{_C}x_k + H_{_D}y)\right]\right\}, \quad (4.4)$$

$$y = C_{\rm p2}x + D_{\rm p21}w. \quad (4.5)$$

Combining the dynamics of the plant and the output feedback law, we obtain,

$$\begin{bmatrix}\dot{x}\\ z\end{bmatrix} = \begin{bmatrix}A(\rho) & B(\rho)\\ C(\rho) & D(\rho)\end{bmatrix}\begin{bmatrix}x\\ w\end{bmatrix}, \quad \rho \in \mathbf{\Gamma}, \quad (4.6)$$

where

$$x = \begin{bmatrix}x_{\rm p}\\ x_k\end{bmatrix},$$

and

$$\mathbf{\Gamma} = \left\{\rho \in \mathbf{R}^{2^{n_u}} : \sum_{j=0}^{2^{n_u}-1}\rho_j = 1,\ 0 \le \rho_j \le 1,\ j \in \mathbf{I}[0, 2^{n_u}-1]\right\}.$$

The state space matrices of the closed-loop system is given by

$$\begin{aligned}\begin{bmatrix}A(\rho) & B(\rho)\\ C(\rho) & D(\rho)\end{bmatrix} &= \left[\begin{array}{cc|c} A_{\rm p} & 0 & B_{\rm p1}\\ 0 & 0 & 0\\ \hline C_{\rm p1} & 0 & D_{\rm p11}\end{array}\right] + \left[\begin{array}{cc} 0 & B_{\rm p2}\\ I & 0\\ \hline 0 & D_{\rm p12}\end{array}\right]\\ &\quad\times\left(\sum_{j=0}^{2^{n_u}-1}\rho_j\begin{bmatrix}A_{kj} & B_{kj}\\ E_jC_k + E_j^-H_{_C} & E_jD_k + E_j^-H_{_D}\end{bmatrix}\right)\left[\begin{array}{cc|c} 0 & I & 0\\ C_{\rm p2} & 0 & D_{\rm p21}\end{array}\right]\\ &:= \sum_{j=0}^{2^{n_u}-1}\rho_j\begin{bmatrix}A_j & B_j\\ C_j & D_j\end{bmatrix}, \end{aligned} \quad (4.7)$$

which are affine functions of scheduling parameter ρ.

Our objective is to design a dynamic output feedback law of the form (4.2) that asymptotically stabilizes the plant (4.1) and minimizes the gain from the disturbance w to the controlled output z over a class of bounded disturbances. This objective is achieved as follows. We will first establish conditions under which the zero state response of the closed-loop system under a given set of bounded disturbances remains in an ellipsoid of the form

$$\Omega(P,\eta) = \left\{x \in \mathbf{R}^{n_x+n_{x_k}} : \ x^{\mathrm{T}}Px \le \eta\right\}, \quad P \in \mathbf{S}^{(n_x+n_{x_k})\times(n_x+n_{x_k})}, \ \eta > 0.$$

The controller coefficient matrices are then optimized by solving an optimization problem with LMI constraints. We are particularly interested in two measures of the level of disturbance attenuation:

1. the restricted $\mathcal{L}_2$ gain, the $\mathcal{L}_2$ gain defined over a class of bounded disturbances, and,

2. the restricted $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain, also defined over a class of bounded disturbances.

4.1.2 Disturbance Tolerance

We will consider a class of energy and magnitude bounded disturbances

$$\mathcal{W}_\eta = \left\{w : \ \mathbf{R}_+ \to \mathbf{R}^{n_w}, \ \int_0^\infty w^{\mathrm{T}}(t)w(t)dt \le \eta, \ |w_i(t)| \le \bar{w}_i, \forall t \ge 0, i \in \mathbf{I}[1,n_w]\right\},$$

where η and $\bar{w}_i$'s are some positive scalars whose values are known.

For a given $\Lambda > 0$, with $\Lambda \le \alpha I$ for some $\alpha > 0$, we define

$$\Omega\left(P,\Lambda,\eta+\alpha\sum_{i=1}^{n_w}\bar{w}_i^2\right) = \left\{(x,w) : \ x^{\mathrm{T}}Px + w^{\mathrm{T}}\Lambda w \le \eta + \alpha\sum_{i=1}^{n_w}\bar{w}_i^2\right\}. \tag{4.8}$$

The purpose of introducing $\Omega(P,\Lambda,\eta+\alpha\sum_{i=1}^{n_w}\bar{w}_i^2)$ is to overbound the set $\Omega(P,\eta)$. In particular, we have the following relation between these two sets.

Lemma 2 *For any $x \in \Omega(P,\eta)$ and any $w \in \mathcal{W}_\eta$, $(x,w) \in \Omega\left(P,\Lambda,\eta+\alpha\sum_{i=1}^{n_w}\bar{w}_i^2\right)$.*

Proof. If $x \in \Omega(P,\eta)$, then $x^{\mathrm{T}}Px \le \eta$. On the other hand, it is clear that

$$\alpha \sum_{i=1}^{n_w} \bar{w}_i^2 - w^{\mathrm{T}}\Lambda w \ge 0, \quad \forall\, w \in \mathcal{W}_\eta.$$

We thus have,

$$x^{\mathrm{T}}Px \le \eta + \left(\alpha \sum_{i=1}^{n_w} \bar{w}_i^2 - w^{\mathrm{T}}\Lambda w \right),$$

which implies that

$$(x,w) \in \Omega\left(P, \Lambda, \eta + \alpha \sum_{i=1}^{n_w} \bar{w}_i^2 \right).$$

Q.E.D.

Note that

$$\begin{aligned}
\mathcal{L}(H_{_C}, H_{_D}) &= \{(x_k, y) :\ |H_{_{C_i}} x_k + H_{_{D_i}} y| \le 1,\ i \in \mathbf{I}[1, n_u]\} \\
&= \{(x_{\mathrm{p}}, x_k, w) :\ |H_{_{D_i}} C_{\mathrm{p}2} x_{\mathrm{p}} + H_{_{C_i}} x_k + H_{_{D_i}} D_{\mathrm{p}21} w| \le 1,\ i \in \mathbf{I}[1, n_u]\} \\
&:= \mathcal{L}(H_{_D} C_{\mathrm{p}2}, H_{_C}, H_{_D} D_{\mathrm{p}21}).
\end{aligned}$$

In the sequel, whenever necessary, we will replace the following relation

$$(x_k, y) \in \mathcal{L}(H_{_C}, H_{_D}), \quad \forall (x,w) \in \Omega(P,\eta) \times \mathcal{W}_\eta,$$

with a more stringent condition

$$\Omega\left(P, \Lambda, \eta + \alpha \sum_{i=1}^{m} \bar{w}_i^2 \right) \subset \mathcal{L}(H_{_D} C_{\mathrm{p}2}, H_{_C}, H_{_D} D_{\mathrm{p}21}).$$

Although this will introduce some conservatism, the latter condition can be readily converted into an LMI constraint. The following theorem provides a quantification of the disturbance tolerance ability of the closed-loop system.

Theorem 5 *If there exist positive definite matrices* $P \in \mathcal{S}_+^{(n_x+n_{x_k})\times(n_x+n_{x_k})}$, $\Lambda \in \mathcal{S}_+^{n_w \times n_w}$, *matrices* $(A_{kj}, B_{kj}) \in \mathbf{R}^{n_x \times n_x} \times \mathbf{R}^{n_x \times n_y}$, $j \in \mathbf{I}[0, 2^{n_u}-1]$, $(C_k, D_k) \in \mathbf{R}^{n_u \times n_x} \times$

$\mathbf{R}^{n_u \times n_y}$, $(H_{_C}, H_{_D}) \in \mathbf{R}^{n_u \times n_x} \times \mathbf{R}^{n_u \times n_y}$, *and a scalar* $\alpha > 0$ *such that*

$$\begin{bmatrix} A_j^{\mathrm{T}} P + P A_j & P B_j \\ B_j^{\mathrm{T}} P & -I \end{bmatrix} < 0, \quad j \in \mathbf{I}[0, 2^{n_u} - 1], \tag{4.9}$$

$$\Omega\left(P, \Lambda, \eta + \alpha \sum_{i=1}^{n_w} \bar{w}_i^2\right) \subset \mathcal{L}(H_{_D} C_{\mathrm{p}2}, H_{_C}, H_{_D} D_{\mathrm{p}21}), \tag{4.10}$$

$$P > 0, \tag{4.11}$$

$$\Lambda \leq \alpha I, \tag{4.12}$$

then the trajectory of the closed-loop system that starts from the origin will remain inside the ellipsoid $\Omega(P, \eta)$ *for every* $w \in \mathcal{W}_\eta$.

Proof. From Lemma 2, we know that conditions (4.10) and (4.12) imply

$$(x_k, y) \in \mathcal{L}(H_{_C}, H_{_D})$$

when $(x, w) \in \Omega(P, \eta) \times \mathcal{W}_\eta$. Therefore, the LPV representation (4.6) is valid.

Using a quadratic Lyapunov function $V(x) = x^{\mathrm{T}} P x$, we have

$$\begin{aligned} \dot{V} - w^{\mathrm{T}} w &= x^{\mathrm{T}}(A^{\mathrm{T}}(\rho)P + PA(\rho))x + w^{\mathrm{T}}(B^{\mathrm{T}}(\rho)P + PB(\rho))w - w^{\mathrm{T}} w \\ &= \begin{bmatrix} x^{\mathrm{T}} & w^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} A^{\mathrm{T}}(\rho)P + PA(\rho) & PB(\rho) \\ B^{\mathrm{T}}(\rho)P & -I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} < 0, \end{aligned}$$

where the last inequality comes from (4.9) and a linear combination argument. Integration of both sides of the above inequality from 0 to T with $x(0) = 0$ leads to

$$V(x(T)) - V(x(0)) = V(x(T)) < \int_0^T w^{\mathrm{T}}(t) w(t) dt \leq \eta.$$

Therefore $x(T) \in \Omega(P, \eta)$ for any $T \geq 0$. Q.E.D.

Theorem 5 provides a bounded state region $\Omega(P, \eta)$ for the closed-loop system under an output feedback law of the form (4.2). Moreover, one can also find the maximum disturbance tolerance level $\eta_{\max}$ by solving the following optimization problem:

$$\begin{aligned} &\max_{P, \Lambda, A_{kj}, B_{kj}, C_k, D_k, H_{_C}, H_{_D}, \alpha} \eta, \\ &\text{s.t. } (4.9) - (4.12). \end{aligned} \tag{4.13}$$

This determines the maximum energy $\eta_{\max}$ that the closed-loop system can tolerate, without driving the trajectory from the origin to unboundedness.

4.2 Disturbance Attenuation

4.2.1 Restricted $\mathcal{L}_2$ Gain Synthesis

Under the actuator saturation, the $\mathcal{L}_2$ gain may not be well defined for sufficiently large disturbances. With such large disturbances, the state may go unbounded under any control input. This is what motivated us to consider only energy bounded disturbances, and determine the minimum energy amplification from disturbance to output. In this section, we will design an output feedback law that minimizes the restricted $\mathcal{L}_2$ gain

$$\min_{x(0)=0, w\in\mathcal{W}_\eta} \frac{\|z\|_2}{\|w\|_2}.$$

The synthesis condition for a full order output feedback law is given in the following theorem.

Theorem 6 *Given scalars $\eta \le \eta_{\max}$ and $\gamma > 0$, the restricted $\mathcal{L}_2$ gain of the closed-loop system from the disturbance w to the output z is rendered less than or equal to γ if there exist positive definite matrices $R, S \in \mathcal{S}_+^{n_x \times n_x}$, $\Lambda \in \mathcal{S}_+^{n_w \times n_w}$, and matrices $(\bar{A}_{kj}, \bar{B}_{kj}) \in \mathbf{R}^{n_x \times n_x} \times \mathbf{R}^{n_x \times n_y}, j \in \mathbf{I}[0, 2^{n_u} - 1]$, $(\bar{C}_k, \bar{D}_k) \in \mathbf{R}^{n_u \times n_x} \times \mathbf{R}^{n_u \times n_y}$,*

$(\bar{H}_{C}, \bar{H}_{D}) \in \mathbf{R}^{n_u \times n_x} \times \mathbf{R}^{n_u \times n_y}$, *and a scalar* $\alpha > 0$ *such that*

$$\begin{bmatrix} A_{\mathrm{p}}R + B_{\mathrm{p}2}\bar{U}_{kj} + RA_{\mathrm{p}}^{\mathrm{T}} + \bar{U}_{kj}^{\mathrm{T}}B_{\mathrm{p}2}^{\mathrm{T}} & \star \\ \bar{A}_{kj} + A_{\mathrm{p}}^{\mathrm{T}} + C_{\mathrm{p}2}^{\mathrm{T}}\bar{V}_{kj}^{\mathrm{T}}B_{\mathrm{p}2}^{\mathrm{T}} & A_{\mathrm{p}}^{\mathrm{T}}S + C_{\mathrm{p}2}^{\mathrm{T}}\bar{B}_{kj}^{\mathrm{T}} + SA_{\mathrm{p}} + \bar{B}_{kj}C_{\mathrm{p}2} \\ B_{\mathrm{p}1}^{\mathrm{T}} + D_{\mathrm{p}21}^{\mathrm{T}}\bar{V}_{kj}^{\mathrm{T}}B_{\mathrm{p}2}^{\mathrm{T}} & B_{\mathrm{p}1}^{\mathrm{T}}S + D_{\mathrm{p}21}^{\mathrm{T}}\bar{B}_{kj}^{\mathrm{T}} \\ C_{\mathrm{p}1}R + D_{\mathrm{p}12}\bar{U}_{kj} & C_{\mathrm{p}1} + D_{\mathrm{p}12}\bar{V}_{kj}C_{\mathrm{p}2} \end{bmatrix}$$

$$\begin{bmatrix} \star & \star \\ \star & \star \\ -I & \star \\ D_{\mathrm{p}11} + D_{\mathrm{p}12}\bar{V}_{kj}D_{\mathrm{p}21} & -\gamma^2 I \end{bmatrix} < 0, \quad j \in \mathbf{I}[0, 2^{n_u} - 1], \tag{4.14}$$

$$\begin{bmatrix} \frac{1}{\eta + \alpha \sum_{j=1}^{n_w} \bar{w}_j^2} & \star & \star & \star \\ \bar{H}_{C_i}^{\mathrm{T}} & R & \star & \star \\ C_{\mathrm{p}2}^{\mathrm{T}}\bar{H}_{D_i}^{\mathrm{T}} & I & S & \star \\ D_{\mathrm{p}21}^{\mathrm{T}}\bar{H}_{D_i}^{\mathrm{T}} & 0 & 0 & \Lambda \end{bmatrix} \geq 0, \quad i \in \mathbf{I}[1, n_u], \tag{4.15}$$

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} > 0, \tag{4.16}$$

$$\Lambda \leq \alpha I, \tag{4.17}$$

where $\bar{U}_{kj} = E_j\bar{C}_k + E_j^{-}\bar{H}_{C}$ *and* $\bar{V}_{kj} = E_j\bar{D}_k + E_j^{-}\bar{H}_{D}$. *Moreover, the coefficient matrices of an* n_x*th order output feedback LPV controller are given by*

$$\begin{bmatrix} A_{kj} & B_{kj} \\ C_k & D_k \\ H_{C} & H_{D} \end{bmatrix} = \begin{bmatrix} N & SB_{\mathrm{p}2}E_j & SB_{\mathrm{p}2}E_j^{-} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^{-1} \times \left\{ \begin{bmatrix} \bar{A}_{kj} & \bar{B}_{kj} \\ \bar{C}_k & \bar{D}_k \\ \bar{H}_{C} & \bar{H}_{D} \end{bmatrix} - \begin{bmatrix} SA_{\mathrm{p}}R & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \begin{bmatrix} M^{\mathrm{T}} & 0 \\ C_{\mathrm{p}2}R & I \end{bmatrix}^{-1}, \tag{4.18}$$

where $M, N \in \mathbf{R}^{n_x \times n_x}$ *are such that* $MN^{\mathrm{T}} = I_{n_x} - RS$.

Proof. The restricted $\mathcal{L}_2$ gain of the closed-loop system is less than or equal to γ if

there exists a quadratic Lyapunov function $V(x) = x^{\mathrm{T}}Px$ such that

$$\begin{bmatrix} A^{\mathrm{T}}(\rho)P + PA(\rho) & PB(\rho) & C^{\mathrm{T}}(\rho) \\ B^{\mathrm{T}}(\rho)P & -I & D^{\mathrm{T}}(\rho) \\ C(\rho) & D(\rho) & -\gamma^2 I \end{bmatrix} < 0, \quad \forall\, \rho \in \mathbf{\Gamma}, \tag{4.19}$$

$$\Omega\left(P, \Lambda, \eta + \alpha \sum_{i=1}^{n_w} \bar{w}_i^2\right) \subset \mathcal{L}\left(H_{_D} C_{\mathrm{p}2}, H_{_C}, H_{_D} D_{\mathrm{p}21}\right), \tag{4.20}$$

$$P > 0, \tag{4.21}$$

$$\Lambda \leq \alpha I. \tag{4.22}$$

The theoretical basis for condition (4.19) comes from (2.3). Note that condition (4.19) implies the condition (4.9). Therefore the state trajectory of the closed-loop system starting from $x(0) = 0$ remains inside the ellipsoid $\Omega(P, \eta)$ for any bounded disturbance $w \in \mathcal{W}_\eta$.

Let

$$Z_1 = \begin{bmatrix} R & I \\ M^{\mathrm{T}} & 0 \end{bmatrix}, \qquad Z_2 = \begin{bmatrix} I & S \\ 0 & N^{\mathrm{T}} \end{bmatrix}$$

and specify $P = Z_2 Z_1^{-1}$. Then $PZ_1 = Z_2$ and

$$\begin{aligned} Z_1^{\mathrm{T}} P Z_1 &= \begin{bmatrix} R & I \\ I & S \end{bmatrix} > 0, \\ Z_1^{\mathrm{T}}\left(PA_j\right) Z_1 &= \begin{bmatrix} A_{\mathrm{p}}R + B_{\mathrm{p}2}\bar{U}_{kj} & A_{\mathrm{p}} + B_{\mathrm{p}2}\bar{V}_{kj}C_{\mathrm{p}2} \\ \bar{A}_{kj} & SA_{\mathrm{p}} + \bar{B}_{kj}C_{\mathrm{p}2} \end{bmatrix}, \\ Z_1^{\mathrm{T}}\left(PB_j\right) &= \begin{bmatrix} B_{\mathrm{p}1} + B_{\mathrm{p}2}\bar{V}_{kj}D_{\mathrm{p}21} \\ SB_{\mathrm{p}1} + \bar{B}_{kj}D_{\mathrm{p}21} \end{bmatrix}, \\ C_j Z_1 &= \begin{bmatrix} C_{\mathrm{p}1}R + D_{\mathrm{p}12}\bar{U}_{kj} & C_{\mathrm{p}1} + D_{\mathrm{p}12}\bar{V}_{kj}C_{\mathrm{p}2} \end{bmatrix}, \\ D_j &= D_{\mathrm{p}11} + D_{\mathrm{p}12}\bar{V}_{kj}D_{\mathrm{p}21}, \end{aligned}$$

where the transformed controller data relates to the original $(A_{kj}, B_{kj}, C_k, D_k, H_{_C}, H_{_D})$

by

$$\begin{bmatrix} \bar{A}_{kj} & \bar{B}_{kj} \\ \bar{C}_k & \bar{D}_k \\ \bar{H}_{C} & \bar{H}_{D} \end{bmatrix} = \begin{bmatrix} SA_{\mathrm{p}}R & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} N & SB_{\mathrm{p}2}E_j & SB_{\mathrm{p}2}E_j^{-} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_{kj} & B_{kj} \\ C_k & D_k \\ H_{C} & H_{D} \end{bmatrix} \begin{bmatrix} M^{\mathrm{T}} & 0 \\ C_{\mathrm{p}2}R & I \end{bmatrix}. \tag{4.23}$$

Then, multiplying $\mathrm{diag}\{Z_1^{\mathrm{T}}, I, I\}$ to the left and its transpose from the right of inequality (4.19), we obtain with inequality (4.14).

Also we have

$$\begin{bmatrix} H_{D}C_{\mathrm{p}2} & H_{C} & H_{D}D_{\mathrm{p}21} \end{bmatrix} Z_1 = \begin{bmatrix} \bar{H}_{C} & \bar{H}_{D}C_{\mathrm{p}2} & \bar{H}_{D}D_{\mathrm{p}21} \end{bmatrix}.$$

Note that condition (4.20) is the same as

$$\left(\eta + \alpha \sum_{j=1}^{n_w} \bar{w}_j^2\right) \begin{bmatrix} H_{D_i}C_{\mathrm{p}2} & H_{C_i} & H_{D_i}D_{\mathrm{p}21} \end{bmatrix} \mathrm{diag}\left\{P^{-1}, \Lambda^{-1}\right\} \begin{bmatrix} C_{\mathrm{p}2}^{\mathrm{T}}H_{D_i}^{\mathrm{T}} \\ H_{C_i}^{\mathrm{T}} \\ D_{\mathrm{p}21}^{\mathrm{T}}H_{D_i}^{\mathrm{T}} \end{bmatrix} \leq 1.$$

By Schur complement, it can be rewritten as

$$\begin{bmatrix} \frac{1}{\eta+\alpha\sum_{j=1}^{n_w}\bar{w}_j^2} & \begin{bmatrix} H_{D_i}C_{\mathrm{p}2} & H_{C_i} & H_{D_i}D_{\mathrm{p}21} \end{bmatrix} \\ \begin{bmatrix} C_{\mathrm{p}2}^{\mathrm{T}}H_{D_i}^{\mathrm{T}} \\ H_{C_i}^{\mathrm{T}} \\ D_{\mathrm{p}21}^{\mathrm{T}}H_{D_i}^{\mathrm{T}} \end{bmatrix} & \mathrm{diag}\{P, \Lambda\} \end{bmatrix} \geq 0,$$

and the above equation can be shown equivalent to (4.15) using congruent transformation.

Finally, the output feedback controller formula can be verified by inverting the equation (4.23). The linear subspace can be determined simultaneously by solving H_{C}, H_{D} matrices. Q.E.D.

Theorem 6 provides an upper bound on the restricted $\mathcal{L}_2$ gain for energy and magnitude bounded disturbances. In order to determine the minimal restricted $\mathcal{L}_2$

gain, we will choose an $\eta \leq \eta_{\max}$ and solve the optimization problem

$$\begin{aligned} \min_{R,S,\Lambda,\bar{A}_{kj},\bar{B}_{kj},\bar{C}_k,\bar{D}_k,\bar{H}_C,\bar{H}_D,\alpha} & \gamma^2, \\ \text{s.t. } & (4.14) - (4.17). \end{aligned} \tag{4.24}$$

In the above optimization problem, the synthesis condition (4.14)-(4.17) becomes a set of LMIs for each fixed value of α. Thus, this optimization problem can be solved by sweeping over the values of α.

4.2.2 Restricted $\mathcal{L}_2$ to $\mathcal{L}_\infty$ Gain Synthesis

We next consider a different measure of the level of disturbance attenuation, the restricted $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain, which is defined as

$$\min_{x(0)=0, w\in\mathcal{W}_\eta} \frac{\|z\|_\infty}{\|w\|_2}.$$

For simplicity, we further assumed that $D_{\text{p}11} = 0$, either $D_{\text{p}12}$ or $D_{\text{p}21}$ is also zero in plant dynamics (4.1). This assumption renders $D(\rho) = 0$ as desired. The control synthesis condition under this performance measure is then given as follows.

Theorem 7 *For given scalars $\eta \leq \eta_{\max}$ and $\gamma > 0$, the restricted $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain of the closed-loop system from the disturbance w to the output z is less than or equal to γ if there exist positive definite matrices $R, S \in \mathcal{S}_+^{n_x \times n_x}$, $\Lambda \in \mathcal{S}_+^{n_w \times n_w}$, matrices $(\bar{A}_{kj}, \bar{B}_{kj}) \in \mathbf{R}^{n_x \times n_x} \times \mathbf{R}^{n_x \times n_y}, j \in \mathbf{I}[0, 2^{n_u} - 1]$, $(\bar{C}_k, \bar{D}_k) \in \mathbf{R}^{n_u \times n_x} \times \mathbf{R}^{n_u \times n_y}$,*

$(\bar{H}_{C}, \bar{H}_{D}) \in \mathbf{R}^{n_u \times n_x} \times \mathbf{R}^{n_u \times n_y}$, and a scalar $\alpha > 0$ such that

$$\begin{bmatrix} A_{\rm p}R + B_{\rm p2}\bar{U}_{kj} + RA_{\rm p}^{\rm T} + \bar{U}_{kj}^{\rm T}B_{\rm p2}^{\rm T} & \star & \star \\ \bar{A}_{kj} + A_{\rm p}^{\rm T} + C_{\rm p2}^{\rm T}\bar{V}_{kj}^{\rm T}B_{\rm p2}^{\rm T} & A_{\rm p}^{\rm T}S + C_{\rm p2}^{\rm T}\bar{B}_{kj}^{\rm T} + SA_{\rm p} + \bar{B}_{kj}C_{\rm p2} & \star \\ B_{\rm p1}^{\rm T} + D_{\rm p21}^{\rm T}\bar{V}_{kj}^{\rm T}B_{\rm p2}^{\rm T} & B_{\rm p1}^{\rm T}S + D_{\rm p21}^{\rm T}\bar{B}_{kj}^{\rm T} & -I \end{bmatrix} < 0,$$

$$j \in \mathbf{I}[0, 2^{n_u} - 1], \quad (4.25)$$

$$\begin{bmatrix} \frac{1}{\eta + \alpha\sum_{j=1}^{n_w}\bar{w}_j^2} & \star & \star & \star \\ \bar{H}_{Ci}^{\rm T} & R & \star & \star \\ C_{\rm p2}^{\rm T}\bar{H}_{Di}^{\rm T} & I & S & \star \\ D_{\rm p21}^{\rm T}\bar{H}_{Di}^{\rm T} & 0 & 0 & \Lambda \end{bmatrix} > 0, \quad i \in \mathbf{I}[1, n_u], \quad (4.26)$$

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} > 0, \quad (4.27)$$

$$\Lambda \leq \alpha I, \quad (4.28)$$

$$\begin{bmatrix} \gamma^2 I & \star & \star \\ RC_{\rm p1}^{\rm T} + \bar{U}_{kj}^{\rm T}D_{\rm p12}^{\rm T} & R & \star \\ C_{\rm p1}^{\rm T} + C_{\rm p2}^{\rm T}\bar{V}_{kj}^{\rm T}D_{\rm p12}^{\rm T} & I & S \end{bmatrix} \geq 0, \quad j \in \mathbf{I}[0, 2^{n_u} - 1], \quad (4.29)$$

where $\bar{U}_{kj} = E_j\bar{C}_k + E_j^{-}\bar{H}_{C}$ and $\bar{V}_{kj} = E_j\bar{D}_k + E_j^{-}\bar{H}_{D}$. Moreover, the coefficient matrices of an n_xth order output feedback controller (4.2) are determined by (4.18).

Proof. Based on $\mathcal{L}_2$ to L_∞ gain design approach introduced in Chapter 2 Section 2.4, we can conclude that for the closed-loop system with system matrices $(A(\rho), B(\rho), C(\rho), D(\rho))$ and Lyapunov function $V(x) = x^{\rm T}Px$, its $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain

is no higher than γ if

$$\begin{bmatrix} A^{\mathrm{T}}(\rho)P + PA(\rho) & PB(\rho) \\ B^{\mathrm{T}}(\rho)P & -I \end{bmatrix} < 0, \tag{4.30}$$

$$\Omega\left(P, \Lambda, \eta + \alpha \sum_{i=1}^{n_w} \bar{w}_i^2\right) \subset \mathcal{L}(H_{_D} C_{\mathrm{p2}}, H_{_C}, H_{_D} D_{\mathrm{p21}}), \tag{4.31}$$

$$P > 0, \tag{4.32}$$

$$\Lambda \leq \alpha I, \tag{4.33}$$

$$C(\rho)P^{-1}C^{\mathrm{T}}(\rho) \leq \gamma^2 I, \tag{4.34}$$

for any $\rho \in \mathbf{\Gamma}$. It can be seen that conditions (4.31) and (4.33) validate the closed-loop LPV representation within the state region $\Omega(P, \eta)$. Moreover, from conditions (4.30) and (4.32), we get

$$\dot{V} < w^{\mathrm{T}} w,$$

integration of both sides of the above inequality from 0 to T with $x(0) = 0$ yields,

$$x^{\mathrm{T}}(T)Px(T) < \|w\|_2^2.$$

Also note that condition (4.34) implies

$$C^{\mathrm{T}}(\rho)C(\rho) \leq \gamma^2 P, \quad \forall \rho \in \mathbf{\Gamma}.$$

Therefore,

$$\begin{aligned} z^{\mathrm{T}}(T)z(T) &= x^{\mathrm{T}}(T)C^{\mathrm{T}}(\rho(T))C(\rho(T))x(T) \\ &\leq \gamma^2 x^{\mathrm{T}}(T)Px(T) \\ &< \gamma^2 \|w\|_2^2. \end{aligned}$$

This confirms that the restricted $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain is less than γ.

To show the equivalence between (4.30)-(4.32) and (4.25)-(4.27), we will define matrices Z_1, Z_2 and P as in the proof of Theorem 6, and apply congruent transformations.

For condition (4.34), first we convert it to

$$\begin{bmatrix} \gamma^2 I & C(\rho) \\ C^{\mathrm{T}}(\rho) & P \end{bmatrix} \geq 0$$

by Schur complement. Then multiplication of diag $\{1, Z_1^T\}$ to the left and its transpose from the right of the above equation leads to condition (4.29). Q.E.D.

Similar to the restricted $\mathcal{L}_2$ gain synthesis procedure, one can design an output feedback law to minimize the restricted $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain using LMI techniques.

4.2.3 Numerical Examples

Consider a second order continuous-time system (4.1) with

$$A_{\mathrm{p}} = \begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix}, \quad B_{\mathrm{p1}} = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \quad B_{\mathrm{p2}} = \begin{bmatrix} -3.8 \\ 0.9 \end{bmatrix},$$
$$C_{\mathrm{p1}} = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix}, \quad D_{\mathrm{p11}} = 0.1, \quad D_{\mathrm{p12}} = 0.01,$$
$$C_{\mathrm{p2}} = \begin{bmatrix} -2.2 & 2 \end{bmatrix}, \quad D_{\mathrm{p21}} = -0.2.$$

We set the magnitude bound of the disturbance as 3.4636, and choose the value of α to be 0.1. Applying Theorem 5, we obtain $\eta_{\max} = 12.0$. This is the maximal disturbance energy the system can tolerate. Using a smaller $\eta = 11.997$, we solve the $\mathcal{L}_2$ gain control synthesis by Theorem 6 and the resulting restricted $\mathcal{L}_2$ gain is estimated at $\gamma^* = 0.433$. Therefore we obtain an output feedback controller that guarantees the system stability and performance within a feasible region. Shown in Figs. 4.1-4.3 are some simulation results of the resulting closed-loop system under the influence of the disturbance w,

$$w(t) = \begin{cases} 3.464, & 0 \leq t \leq 1 \text{ sec}, \\ 0.0, & t > 1 \text{ sec}. \end{cases}$$

In particular, the state trajectory and control input are shown in Figs. 4.1 and 4.2, respectively. As can be seen in Fig. 4.3, the ratio between the truncated norms $\|z\|_{2,T}$ and $\|w\|_{2,T}$ is always less than γ^*.

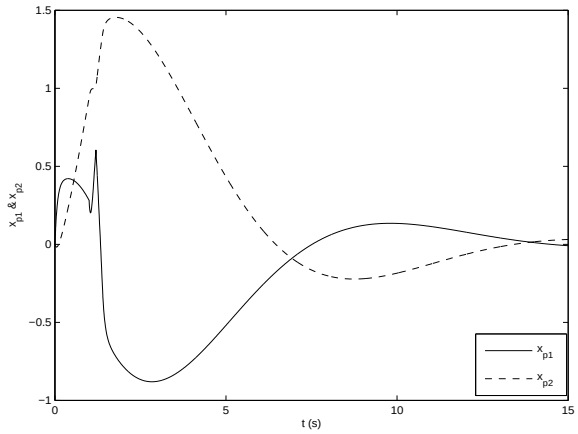


Figure 4.1: The closed-loop system: a state trajectory.

Similarly, we apply Theorem 7 to study the disturbance attenuation in terms of the restricted $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain for the following second order system,

$$
\begin{aligned}
&A_{\rm p} = \begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix}, \quad B_{\rm p1} = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \quad B_{\rm p2} = \begin{bmatrix} 1 \\ 3.8 \end{bmatrix}, \\
&C_{\rm p1} = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix}, \quad D_{\rm p11} = 0, \quad D_{\rm p12} = 0, \\
&C_{\rm p2} = \begin{bmatrix} -6 & 6 \end{bmatrix}, \quad D_{\rm p21} = 0.1.
\end{aligned}
$$

Selecting the magnitude bound of the disturbance as 7.7492 and α as 0.1, we obtain $\eta_{\max} = 60.061$. For $\eta = 60.051 < \eta_{\max}$, the solution of the optimization problem based on Theorem 7 results in an estimate of the restricted $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain $\gamma^* = 0.1$. Shown in Figs. 4.4-4.5 are some simulation results of the closed-loop system under the influence of the disturbance

$$
w(t) = \begin{cases} 7.7492, & 0 \le t \le 1 \text{ sec}, \\ 0.0, & t > 1 \text{ sec}. \end{cases}
$$

From Fig. 4.6, it is easy to observe that the magnitude of output z is always less than $\gamma^* ||w||_2 = 0.77492$.

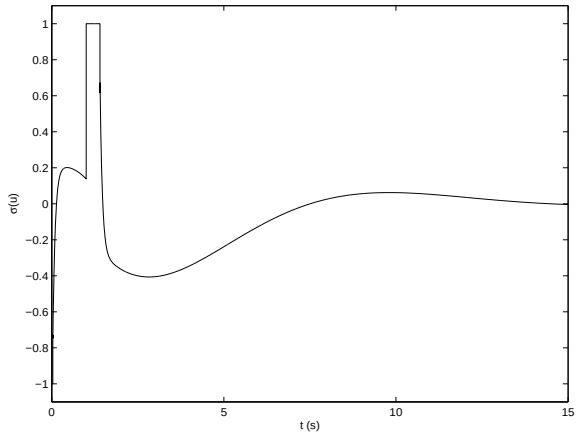


Figure 4.2: The closed-loop system: the saturated control input.

4.3 Discrete-Time Systems

We consider a discrete-time plant subject to actuator saturation,

$$\begin{cases} x_{\rm p}(k+1) &= A_{\rm p}x_{\rm p}(k) + B_{\rm p1}w(k) + B_{\rm p2}\sigma(u(k)), \\ z(k) &= C_{\rm p1}x_{\rm p}(k) + D_{\rm p11}w(k) + D_{\rm p12}\sigma(u(k)), \\ y(k) &= C_{\rm p2}x_{\rm p}(k) + D_{\rm p21}w(k). \end{cases} \tag{4.35}$$

As with continuous-time systems, we will consider the following discrete-time output feedback law in the form of a quasi-LPV system,

$$\begin{cases} x_k(k+1) &= \left(\sum_{j=0}^{2^{n_u}-1} \rho_j A_{kj}\right) x_k(k) + \left(\sum_{j=0}^{2^{n_u}-1} \rho_j B_{kj}\right) y(k), \\ u(k) &= C_k x_k(k) + D_k y(k), \end{cases} \tag{4.36}$$

where $x_k \in \mathbf{R}^{n_{x_k}}$ and the functions $\rho_j(x_k, y)$'s are defined by the same way as before. In $\Omega(P, \eta)$ of the state space, the discrete-time closed-loop system has its state space data given by (4.7).

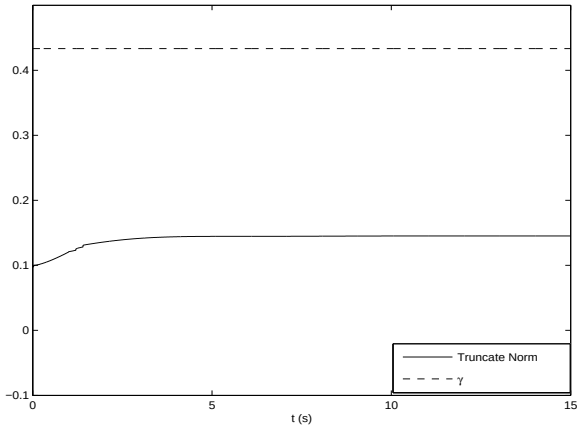


Figure 4.3: The closed-loop system: the truncated $\mathcal{L}_2$ gain.

4.3.1 Disturbance Tolerance

In parallel to the development in Section 4.1, we will define the set of energy and magnitude bounded disturbances in the discrete-time setting,

$$\mathcal{W}_\eta = \left\{ w : \mathbf{Z}_+ \to \mathbf{R}^{n_w},\ \sum_{k=0}^{\infty} w^{\mathrm{T}}(k) w(k) \le \eta,\ |w_i| \le \bar{w}_i, i \in [1, n_w] \right\},$$

where η and $\bar{w}_i$'s are known positive scalars.

Theorem 8 *If there exist positive definite matrices* $P \in \mathcal{S}_+^{(n_x+n_{x_k})\times(n_x+n_{x_k})}$, $\Lambda \in \mathcal{S}_+^{n_w \times n_w}$, *matrices* (A_{kj}, B_{kj}), $j \in \mathbf{I}[0, 2^{n_u} - 1]$, (C_k, D_k), $(H_{_C}, H_{_D})$, *and a positive scalar* α *such that*

$$\begin{bmatrix} -P & PA_j & PB_j \\ A_j^{\mathrm{T}} P & -P & 0 \\ B_j^{\mathrm{T}} P & 0 & -I \end{bmatrix} < 0, \quad j \in \mathbf{I}[0, 2^{n_u} - 1], \tag{4.37}$$

$$\Omega\left(P, \Lambda, \eta + \alpha \sum_{i=1}^{n_w} \bar{w}_i^2\right) \subset \mathcal{L}(H_{_D} C_{\mathrm{p}2}, H_{_C}, H_{_D} D_{\mathrm{p}21}), \tag{4.38}$$

$$P > 0, \tag{4.39}$$

$$\Lambda \le \alpha I, \tag{4.40}$$

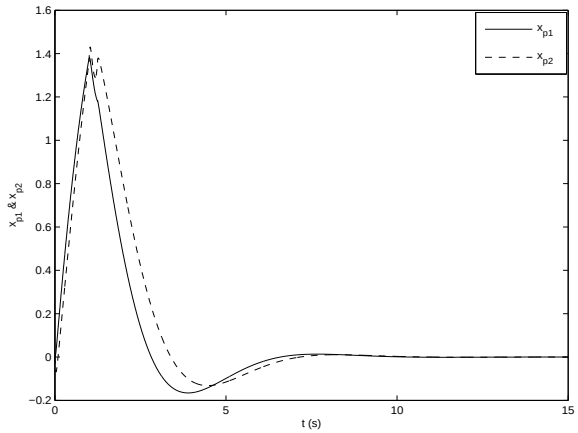


Figure 4.4: The closed-loop system: a state trajectory.

then the trajectory of the closed-loop system that starts from the origin will remain inside the ellipsoid $\Omega(P,\eta)$ *for every* $w \in \mathcal{W}_\eta$.

Proof. Again, conditions (4.38) and (4.40) validate the LPV representation of the closed-loop system within the region $\Omega(P,\eta)$. To show the boundedness of state trajectories, we choose a Lyapunov function $V(x) = x^{\mathrm{T}}Px$, then $\forall \rho(k) \in \mathbf{\Gamma}$

$$
\begin{aligned}
& V(x(k+1)) - V(x(k)) - w^{\mathrm{T}}(k)w(k) \\
&= (x^{\mathrm{T}}(k)A^{\mathrm{T}}(\rho(k)) + w^{\mathrm{T}}(k)B^{\mathrm{T}}(\rho(k))P \\
&\qquad \times (A(\rho(k))x(k) + B(\rho(k))w(k)) - x^{\mathrm{T}}(k)Px(k) - w^{\mathrm{T}}(k)w(k) \\
&= \begin{bmatrix} x^{\mathrm{T}}(k) & w^{\mathrm{T}}(k) \end{bmatrix} \begin{bmatrix} A^{\mathrm{T}}(\rho(k))PA(\rho(k)) - P & A^{\mathrm{T}}(\rho(k))PB(\rho(k)) \\ B^{\mathrm{T}}(\rho(k))PA(\rho(k)) & B^{\mathrm{T}}(\rho(k))PB(\rho(k)) - I \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} < 0.
\end{aligned}
$$

The last inequality is due to condition (4.37). Summing the above inequality over k from 0 to N, we get

$$V(x(N+1)) - V(x(0)) < \sum_{k=0}^{N} w^{\mathrm{T}}(k)w(k) \leq \eta$$

Therefore, for $x(0) = 0$, $x(N+1) \in \Omega(P,\eta)$ for any integer $N \geq 0$. Q.E.D.

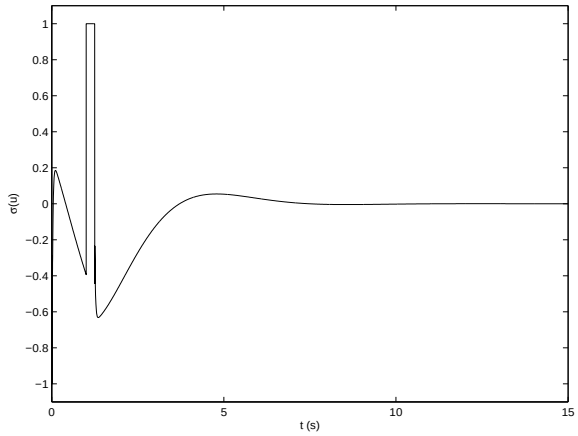


Figure 4.5: The closed-loop system: the saturated control input.

4.3.2 Restricted ℓ_2 and Restricted ℓ_2 to ℓ_∞ Gain Synthesis

As with continuous-time systems, the level of disturbance attenuation for discrete-time systems will be measured by one of the following two performance indices

$$\min_{x(0)=0, w\in\mathcal{W}_\eta} \frac{\|z\|_2}{\|w\|_2},$$

and

$$\min_{x(0)=0, w\in\mathcal{W}_\eta} \frac{\|z\|_\infty}{\|w\|_2},$$

where $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are defined over the set Z_+ of non-negative integers. The former performance index specifies restricted ℓ_2 gain, and the latter restricted ℓ_2 to ℓ_∞ gain. Note that we have removed the assumption $D(\rho) = 0$ from the restricted ℓ_2 to ℓ_∞ gain synthesis condition in the discrete-time setting.

Theorem 9 *Given scalars $\eta \le \eta_{\max}$ and $\gamma > 0$, we have the following two statements on the level of disturbance attenuation:*

1. *The restricted ℓ_2 gain of the closed-loop system from the disturbance w to the output z is less than or equal to γ if there exist positive definite matrices $R, S > 0$,*

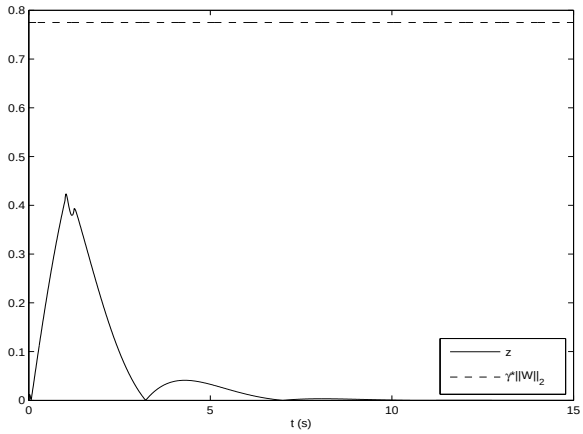


Figure 4.6: The closed-loop system: disturbance attenuation from w to z.

$\Lambda > 0$, *matrices* $(\bar{A}_{kj}, \bar{B}_{kj})$, $j \in \mathbf{I}[0, 2^{n_u} - 1]$, $(\bar{C}_k, \bar{D}_k)$, $(\bar{H}_C, \bar{H}_D)$ *of appropriate*

dimensions, and a scalar $\alpha > 0$ *such that*

$$\begin{bmatrix} -R & \star & \star & \star \\ -I & -S & \star & \star \\ A_{\mathrm{p}}R + B_{\mathrm{p}2}\bar{U}_{kj} & A_{\mathrm{p}} + B_{\mathrm{p}2}\bar{V}_{kj}C_{\mathrm{p}2} & -R & \star \\ \bar{A}_c^j & SA_{\mathrm{p}} + \bar{B}_{kj}C_{\mathrm{p}2} & -I & -S \\ B_{\mathrm{p}1}^{\mathrm{T}} + D_{\mathrm{p}21}^{\mathrm{T}}\bar{V}_{kj}^{\mathrm{T}}B_{\mathrm{p}2}^{\mathrm{T}} & B_{\mathrm{p}1}^{\mathrm{T}}S + D_{\mathrm{p}21}^{\mathrm{T}}\bar{B}_{kj}^{\mathrm{T}} & 0 & 0 \\ 0 & 0 & C_{\mathrm{p}1}R + D_{\mathrm{p}12}\bar{U}_{kj} & C_{\mathrm{p}1} + D_{\mathrm{p}12}\bar{V}_{kj}C_{\mathrm{p}2} \end{bmatrix}$$

$$\begin{bmatrix} \star & \star \\ \star & \star \\ \star & \star \\ \star & \star \\ -I & \star \\ D_{\mathrm{p}11} + D_{\mathrm{p}12}\bar{V}_{kj}D_{\mathrm{p}21} & -\gamma^2 I \end{bmatrix} < 0, \quad j \in \mathbf{I}[0, 2^{n_u} - 1], \tag{4.41}$$

$$\begin{bmatrix} \frac{1}{\eta + \alpha\sum_{j=1}^{n_w}\bar{w}_j^2} & \star & \star & \star \\ \bar{H}_{C_i}^{\mathrm{T}} & R & \star & \star \\ C_{\mathrm{p}2}^{\mathrm{T}}\bar{H}_{D_i}^{\mathrm{T}} & I & S & \star \\ D_{\mathrm{p}21}^{\mathrm{T}}\bar{H}_{D_i}^{\mathrm{T}} & 0 & 0 & \Lambda \end{bmatrix} \geq 0, \quad i \in \mathbf{I}[1, n_u], \tag{4.42}$$

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} > 0, \tag{4.43}$$

$$\Lambda \leq \alpha I, \tag{4.44}$$

where $\bar{U}_{kj} = E_j\bar{C}_k + E_j^-\bar{H}_{C}$ *and* $\bar{V}_{kj} = E_j\bar{D}_k + E_j^-\bar{H}_{D}$.

2. *The restricted* ℓ_2 *to* ℓ_∞ *gain of the closed-loop system from the disturbance* w *to the output* z *is less than or equal to* γ *if there exist positive definite matrices* $R, S > 0$, $\Lambda > 0$, *matrices* $(\bar{A}_{kj}, \bar{B}_{kj})$, $j \in \mathbf{I}[0, 2^{n_u} - 1]$, $(\bar{C}_k, \bar{D}_k)$, $(\bar{H}_{C}, \bar{H}_{D})$ *of*

appropriate dimensions, and a scalar $\alpha > 0$ *such that*

$$\begin{bmatrix} -R & \star & \star & \star & \star \\ -I & -S & \star & \star & \star \\ A_{\rm p}R + B_{\rm p2}\bar{U}_{kj} & A_{\rm p} + B_{\rm p2}\bar{V}_{kj}C_{\rm p2} & -R & \star & \star \\ \bar{A}_{kj} & SA_{\rm p} + \bar{B}_{kj}C_{\rm p2} & -I & -S & \star \\ B_{\rm p1}^{\rm T} + D_{\rm p21}^{\rm T}\bar{V}_{kj}^{\rm T}B_{\rm p2}^{\rm T} & B_{\rm p1}^{\rm T}S + D_{\rm p21}^{\rm T}\bar{B}_{kj}^{\rm T} & 0 & 0 & -I \end{bmatrix} < 0, j \in \mathbf{I}[0, 2^{n_u} - 1], \tag{4.45}$$

$$\begin{bmatrix} \frac{1}{\eta + \alpha\sum_{j=1}^{n_w}\bar{w}_j^2} & \star & \star & \star \\ \bar{H}_{Ci}^{\rm T} & R & \star & \star \\ C_{\rm p2}^{\rm T}\bar{H}_{Di}^{\rm T} & I & S & \star \\ D_{\rm p21}^{\rm T}\bar{H}_{Di}^{\rm T} & 0 & 0 & \Lambda \end{bmatrix} > 0, \quad i \in \mathbf{I}[1, n_u], \tag{4.46}$$

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} > 0, \tag{4.47}$$

$$\Lambda \leq \alpha I, \tag{4.48}$$

$$\begin{bmatrix} \gamma^2 I & \star & \star & \star \\ RC_{\rm p1}^{\rm T} + \bar{U}_{kj}^{\rm T}D_{\rm p12}^{\rm T} & R & \star & \star \\ C_{\rm p1}^{\rm T} + C_{\rm p2}^{\rm T}\bar{V}_{kj}^{\rm T}D_{\rm p12}^{\rm T} & I & S & \star \\ D_{\rm p11}^{\rm T} + D_{\rm p21}^{\rm T}\bar{V}_{kj}^{\rm T}D_{\rm p12}^{\rm T} & 0 & 0 & I \end{bmatrix} \geq 0, \quad j \in \mathbf{I}[0, 2^{n_u} - 1], \tag{4.49}$$

where $\bar{U}_{kj}$ *and* $\bar{V}_{kj}$ *are as defined in Item 1 of the theorem.*

Moreover, the coefficient matrices of an n_x*th order output feedback controller are given by (4.18).*

Proof. If there exists positive definite matrix P and Λ, such that

$$\begin{bmatrix} A^{\mathrm{T}}(\rho)PA(\rho) - P & A^{\mathrm{T}}(\rho)PB(\rho) & C^{\mathrm{T}}(\rho) \\ B^{\mathrm{T}}(\rho)PA(\rho) & B^{\mathrm{T}}(\rho)PB(\rho) - I & D^{\mathrm{T}}(\rho) \\ C(\rho) & D(\rho) & -\gamma^2 I \end{bmatrix} < 0, \quad \forall \rho \in \mathbf{\Gamma}, \tag{4.50}$$

$$\Omega\left(P, \Lambda + \alpha \sum_{i=1}^{n_w} \bar{w}_i^2\right) \subset \mathcal{L}(H_{_D} C_{\mathrm{p2}}, H_{_C}, H_{_D} D_{\mathrm{p21}}), \tag{4.51}$$

$$P > 0, \tag{4.52}$$

$$\Lambda \leq \alpha I, \tag{4.53}$$

then the closed-loop saturated system has restricted ℓ_2 gain less than or equal to γ. Derivation of (4.50) is similar to the derivation of (4.19) in continuous case. By Schur complement and linear combination argument, it is clear that (4.50) is equivalent to

$$\begin{bmatrix} -P^{-1} & A_j & B_j & 0 \\ A_j^{\mathrm{T}} & -P & 0 & C_j^{\mathrm{T}} \\ B_j^{\mathrm{T}} & 0 & -I & D_j^{\mathrm{T}} \\ 0 & C_j & D_j & -\gamma^2 I \end{bmatrix} < 0, \quad j \in \mathbf{I}[0, 2^{n_u} - 1].$$

Now, multiplying diag $\{Z_1^{\mathrm{T}} P, Z_1^{\mathrm{T}}, I, I\}$ to the left and its transpose to the right of the above inequality, and using the congruent transformation (4.23) as the continuous-time case, one can show the equivalence between (4.41) and (4.50). The rest of the proof follows closely to that of Theorem 6.

To show the restricted ℓ_2 to ℓ_∞ gain is less than or equal to γ, we need

$$\begin{bmatrix} A^{\mathrm{T}}(\rho)PA(\rho) - P & A^{\mathrm{T}}(\rho)PB(\rho) \\ B^{\mathrm{T}}(\rho)PA(\rho) & B^{\mathrm{T}}(\rho)PB(\rho) - I \end{bmatrix} < 0, \tag{4.54}$$

$$\Omega\left(P, \Lambda + \alpha \sum_{i=1}^{n_w} \bar{w}_i^2\right) \subset \mathcal{L}(H_{_D} C_{\mathrm{p}2}, H_{_C}, H_{_D} D_{\mathrm{p}21}), \tag{4.55}$$

$$P > 0, \tag{4.56}$$

$$\Lambda \leq \alpha I, \tag{4.57}$$

$$\begin{bmatrix} C(\rho) & D(\rho) \end{bmatrix} \mathrm{diag}\left\{P^{-1}, I\right\} \begin{bmatrix} C^{\mathrm{T}}(\rho) \\ D^{\mathrm{T}}(\rho) \end{bmatrix} \leq \gamma^2 I, \tag{4.58}$$

for any $\rho \in \mathbf{\Gamma}$. Again, from inequality (4.54), we have

$$V(x(k+1)) - V(x(k)) < w^{\mathrm{T}}(k)w(k).$$

Summing both sides of the above inequality from 0 to N and assuming $x(0) = 0$ lead to

$$V(x(N+1)) < \sum_{k=0}^{N} w^{\mathrm{T}}(k)w(k).$$

Also, condition (4.58) implies that

$$\begin{bmatrix} C^{\mathrm{T}}(\rho(N+1)) \\ D^{\mathrm{T}}(\rho(N+1)) \end{bmatrix} \begin{bmatrix} C(\rho(N+1)) & D(\rho(N+1)) \end{bmatrix} \leq \gamma^2 \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix}.$$

Therefore,

$$\begin{aligned}
& z^{\mathrm{T}}(N+1)z(N+1) \\
&= \begin{bmatrix} x^{\mathrm{T}}(N+1) & w^{\mathrm{T}}(N+1) \end{bmatrix} \begin{bmatrix} C^{\mathrm{T}}(\rho(N+1)) \\ D^{\mathrm{T}}(\rho(N+1)) \end{bmatrix} \begin{bmatrix} C(\rho(N+1)) & D(\rho(N+1)) \end{bmatrix} \begin{bmatrix} x(N+1) \\ w(N+1) \end{bmatrix} \\
&\leq \gamma^2 \begin{bmatrix} x^{\mathrm{T}}(N+1) & w^{\mathrm{T}}(N+1) \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x(N+1) \\ w(N+1) \end{bmatrix} \\
&< \gamma^2 \left(\sum_{k=0}^{N} w^{\mathrm{T}}(k)w(k) + w^{\mathrm{T}}(N+1)w(N+1) \right) \\
&\leq \gamma^2 \|w\|_2^2,
\end{aligned}$$

which means that the restricted ℓ_2 to ℓ_∞ gain is less than γ.

Note that due to the special structure of the discrete-time system, we can remove the restriction of $D(\rho) = 0$, which result in the relaxation from condition (4.34) to condition (4.58). The equivalence between (4.58) and (4.49) can be achieved by first converting condition (4.58) to

$$\begin{bmatrix} \gamma^2 I & C(\rho) & D(\rho) \\ C^{\mathrm{T}}(\rho) & P & 0 \\ D^{\mathrm{T}}(\rho) & 0 & I \end{bmatrix} \geq 0$$

by Schur complement, then multiplying diag $\left\{1, Z_1^T, I\right\}$ to the left and its transpose from the right of the above equation.

The equivalence between (4.54)-(4.57) and (4.45)-(4.48) can be shown as in the proof of Theorem 7. Q.E.D.

4.3.3 Numerical Examples

In the first example, the discrete-time system (4.35) has its state space matrices given by

$$\begin{aligned} A_{\mathrm{p}} &= \begin{bmatrix} 0.8 & -0.9 \\ 0.8 & 0.9 \end{bmatrix}, \quad B_{\mathrm{p1}} = \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix}, \quad B_{\mathrm{p2}} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \\ C_{\mathrm{p1}} &= \begin{bmatrix} 0.4 & 0.1 \end{bmatrix}, \quad D_{\mathrm{p11}} = -0.3, \quad D_{\mathrm{p12}} = -1.1, \\ C_{\mathrm{p2}} &= \begin{bmatrix} -0.5 & 2 \end{bmatrix}, \quad D_{\mathrm{p21}} = 1. \end{aligned}$$

The magnitude of the disturbance is assumed to be bounded by $\bar{w} = 1.421$. For $\alpha = 0.1$, the solution of the optimization problem based on Theorem 8 results in an estimated the maximum energy bound $\eta_{\max} = 8.087$. For a given $\eta = 8.077 < \eta_{\max}$, we obtain $\gamma^* = 0.873$ by solving the optimization problem based on the synthesis condition in Theorem 9, Item 1. Shown in Figs. 4.7-4.9 are the simulation results of

the closed-loop system with

$$w(k) = \begin{cases} 1.421, & 0 \le k \le 4, \\ 0.0, & k > 4. \end{cases}$$

Note that in Fig.4.9, the ratio between the truncated norms $||z||_{2,T}$ and $||w||_{2,T}$ is less than γ, indicating that the energy amplification from disturbance w to output z is less than 0.873.

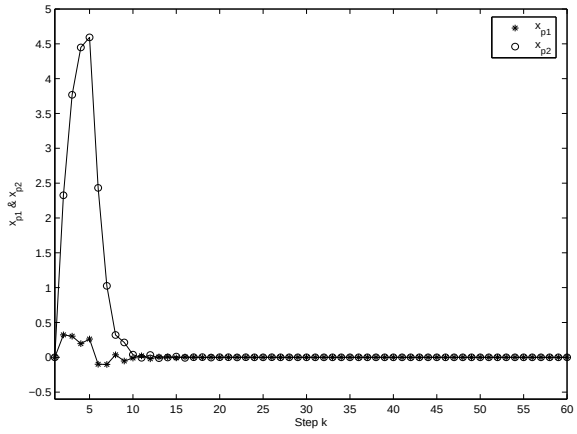


Figure 4.7: The closed-loop system: a state trajectory.

In the next example, we consider the restricted ℓ_2 to ℓ_∞ gain. The discrete-time system is given by

$$A_{\rm p} = \begin{bmatrix} 0.6 & -1 \\ 0.6 & 1.1 \end{bmatrix}, \quad B_{\rm p1} = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \quad B_{\rm p2} = \begin{bmatrix} 3 \\ 3.8 \end{bmatrix},$$
$$C_{\rm p1} = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix}, \quad D_{\rm p11} = 0.3, \quad D_{\rm p12} = 0,$$
$$C_{\rm p2} = \begin{bmatrix} -6 & 4 \end{bmatrix}, \quad D_{\rm p21} = 0.1.$$

Here, we assume that the magnitude of the disturbance is bounded by $\bar{w} = 12.925$ and estimated the maximum energy bound to be $\eta_{\max} = 668.20$. For a given $\eta = 668.17 < \eta_{\max}$, the lowest restricted ℓ_2 to ℓ_∞ gain for a is estimated at $\gamma^* = 0.314$ by solving

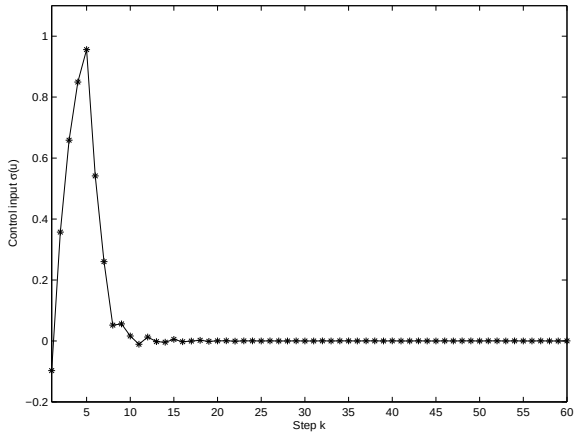


Figure 4.8: The closed-loop system: the saturated control input.

the optimization problem based on the synthesis condition given in Theorem 9, Item 2. Shown in Figs. 4.10-4.12 are some simulation results of the resulting closed-loop system with

$$w(k) = \begin{cases} 12.925, & 0 \leq k \leq 4, \\ 0.0, & k > 4. \end{cases}$$

It is also observed from Fig. 4.12 that the magnitude of output z is always less than $\gamma\|w\|_2 = 8.12$, as expected.

4.4 Conclusion

Continuing with the work in Chapter 3, we furtherly take the system properties into consideration in this chapter. Generally, there are many properties that may characterize a control system. What we are interested here is the capability of disturbance attenuation, which is reflected by the signal amplification from the disturbance to the controlled output. A nonlinear feedback controller that has the same structure as in Chapter 3 is constructed to achieve an optimal $\mathcal{L}_2$ (ℓ_2) gain and $\mathcal{L}_2$ to $\mathcal{L}_\infty$ (ℓ_2 to ℓ_∞) gain performance over a class of bounded disturbances. The synthesis of control laws

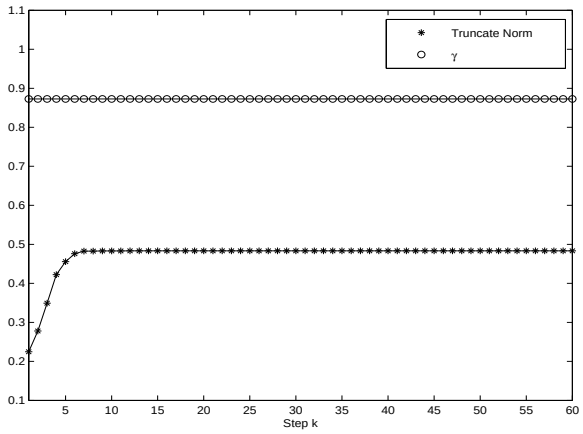


Figure 4.9: The closed-loop system: the truncated ℓ_2 gain.

are formulated into LMI optimization problems, which can be solved efficiently. Both continuous-time and discrete-time systems are considered and several examples are worked out to validate the proposed design approach.

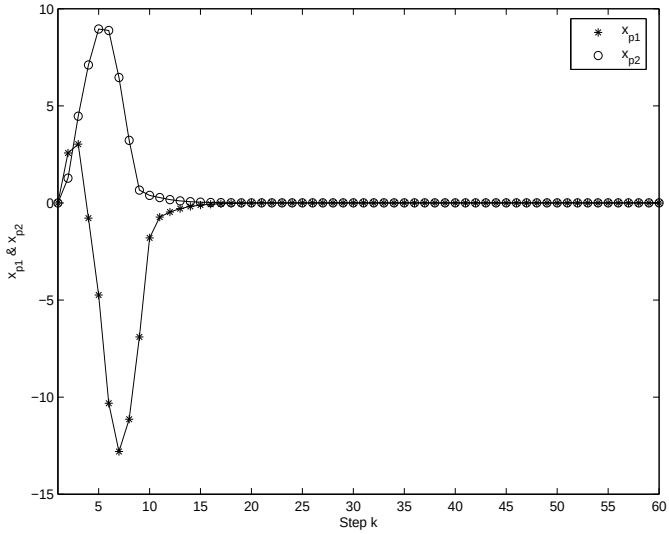


Figure 4.10: The closed-loop system: a state trajectory.

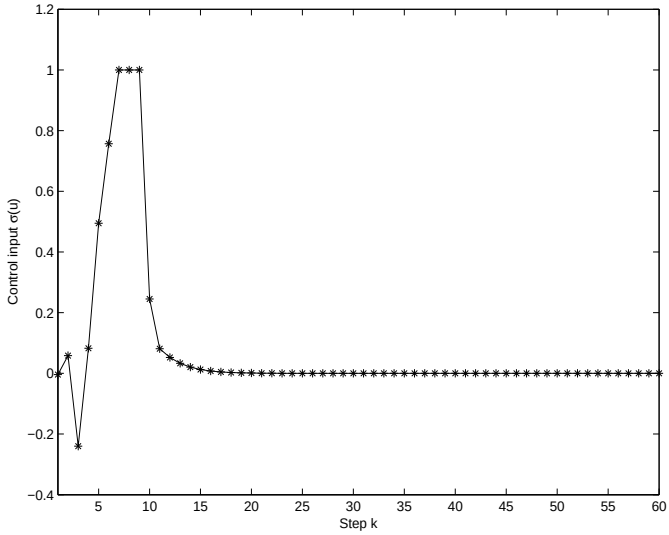


Figure 4.11: The closed-loop system: the saturated control input.

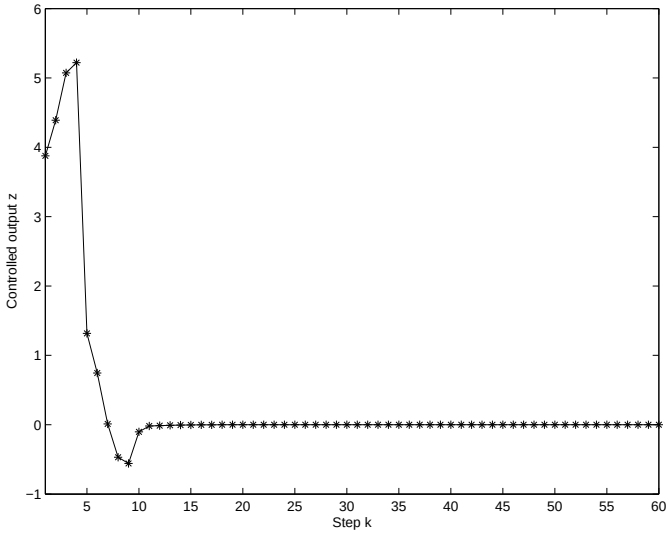


Figure 4.12: The closed-loop system: the output.

Chapter 5

Parameter-Dependent Lyapunov Functions in Saturation Control

In this chapter, we will extend the saturation control approach for discrete-time linear systems developed in Chapter 3 Section 3.2 and Chapter 4 Section 4.3 to a more generalized case in terms of Lyapunov functions. To be specific, the choice of Lyapunov function will depend on a parameter, which is real time scheduled for the controller construction and indicates the level of saturation as well. By introducing a general-formed intermediate matrix, a nonlinear controller with the similar form as before is obtained and the coefficient matrices of the controller are formulated and solved as an optimization problem with LMI constraints. Since constant Lyapunov function is a special case of general parameter-dependent Lyapunov function, the proposed controller is expected to provide less conservative results in terms of larger domain of attraction and better disturbance attenuation characteristics.

The chapter is organized as follows. In Section 5.1, we will reformulate the stabilization and performance problems discussed in previous chapters based on parameter-dependent Lyapunov method. Then the synthesis conditions will be derived in Section 5.2 and Section 5.3 respectively, to guarantee a large stability region and nice disturbance attenuation in terms of ℓ_2 gain and ℓ_2 to ℓ_∞ gain. In Section 5.4, the proposed design procedure is validated by a numerical example and the result by using

parameter-dependent Lyapunov method is compared with that of constant Lyapunov method. Conclusion is drawn in Section 5.5.

5.1 Problem Formulation

All definitions and problems we formulate in Subsection 5.1.1 will follow the same way as in Chapter 3 and Chapter 4. The parameter-dependent Lyapunov function introduced in Subsection 5.1.2 is a new concept that extends the proposed design approach to a more generalized case.

5.1.1 System Modeling

Using Lemma 1, the discrete-time linear system we consider is of the form

$$\begin{aligned} x_{\mathrm{p}}(k+1) &= A_{\mathrm{p}}x_{\mathrm{p}}(k) + B_{\mathrm{p}1}w(k) \\ &\quad + B_{\mathrm{p}2}\left\{\sum_{j=0}^{2^{n_u}-1}\rho_j\left[E_j(C_kx_k(k)+D_ky(k)) + E_j^-(H_{_C}x_k(k)+H_{_D}y(k))\right]\right\} \quad (5.1) \\ z(k) &= C_{\mathrm{p}1}x_{\mathrm{p}}(k) + D_{\mathrm{p}11}w(k) \\ &\quad + D_{\mathrm{p}12}\left\{\sum_{j=0}^{2^{n_u}-1}\rho_j\left[E_j(C_kx_k(k)+D_ky(k)) + E_j^-(H_{_C}x_k(k)+H_{_D}y(k))\right]\right\} \quad (5.2) \\ y(k) &= C_{\mathrm{p}2}x(k) + D_{\mathrm{p}21}w(k) \quad (5.3) \end{aligned}$$

The controller is of the form

$$\left\{\begin{array}{rcl} x_k(k+1) &=& \left(\sum_{j=0}^{2^{n_u}-1}\rho_jA_{kj}\right)x_k(k) + \left(\sum_{j=0}^{2^{n_u}-1}\rho_jB_{kj}\right)y(k), \\ u(k) &=& C_kx_k(k)+D_ky(k), \end{array}\right. \quad (5.4)$$

where $\sum_{j=0}^{2^{n_u}-1}\rho_j = 1,\ 0 \le \rho_j \le 1,\ j \in \mathbf{I}[0, 2^{n_u}-1]$.

Combine the dynamics of the plant and the output-feedback controller to obtain the dynamics of the closed-loop system

$$\begin{bmatrix} x(k+1) \\ z(k) \end{bmatrix} = \begin{bmatrix} A(\rho) & B(\rho) \\ C(\rho) & D(\rho) \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \tag{5.5}$$

where $x^{\mathrm{T}}(k) = \begin{bmatrix} x_{\mathrm{p}}^{\mathrm{T}}(k) & x_k^{\mathrm{T}}(k) \end{bmatrix}$ and $\rho \in \Gamma$ with

$$\Gamma = \left\{ \rho \in \mathbf{R}^{2^{n_u}} : \sum_{j=0}^{2^{n_u}-1} \rho_j = 1,\ 0 \le \rho_j \le 1,\ j \in \mathbf{I}[0, 2^{n_u}-1] \right\}.$$

Note that the quasi-LPV form is a valid representation of the original nonlinear system when $(x_k, y) \in \mathcal{L}(H_{_C}, H_{_D}) = \{(x_k, y) \in \mathbf{R}^{n_x+n_y} : |H_{_{C_i}} x_k + H_{_{D_i}} y| \le 1, i \in [1, n_u]\}$. The state-space data of the closed-loop LPV system (5.5) is given by

$$\begin{aligned} \begin{bmatrix} A(\rho) & B(\rho) \\ C(\rho) & D(\rho) \end{bmatrix} &= \left[\begin{array}{cc|c} A_{\mathrm{p}} & 0 & B_{\mathrm{p1}} \\ 0 & 0 & 0 \\ \hline C_{\mathrm{p1}} & 0 & D_{\mathrm{p11}} \end{array}\right] + \left[\begin{array}{cc} 0 & B_{\mathrm{p2}} \\ I & 0 \\ \hline 0 & D_{\mathrm{p12}} \end{array}\right] \\ &\quad \times \left(\sum_{j=0}^{2^{n_u}-1} \rho_j \begin{bmatrix} A_{kj} & B_{kj} \\ E_j C_k + E_j^- H_{_C} & E_j D_k + E_j^- H_{_D} \end{bmatrix} \right) \left[\begin{array}{cc|c} 0 & I & 0 \\ C_{\mathrm{p2}} & 0 & D_{\mathrm{p21}} \end{array}\right] \\ &:= \sum_{j=0}^{2^{n_u}-1} \rho_j \begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix}. \end{aligned}$$

Note that the closed-loop matrices are affine functions of scheduling parameter ρ_j.

5.1.2 Parameter-Dependent Lyapunov Function

Now we consider a class of parameter-dependent Lyapunov matrix

$$P(\rho) = \sum_{j=0}^{2^{n_u}-1} \rho_j P_j, \quad P_j > 0, \ \sum_{j=0}^{2^{n_u}-1} \rho_j = 1, 0 \le \rho_j \le 1,\ j \in \mathbf{I}[0, 2^{n_u}-1] \tag{5.6}$$

where P_j's are constant matrices involved in the control synthesis process.

Remark 2 *Note that* $\{A(\rho), B(\rho), C(\rho), D(\rho)\}$ *and* $P(\rho)$ *can be represented by the same set of linear combination coefficients* $\rho_j s$. *From saturation point of view,* ρ_j

serves as a saturation level indicator, while from the control point of view, it is a parameter that is real time scheduled for the controller construction. Later we will see that this "same linear combination" property efficiently simplifies the derivation by turning all terms in the objective analysis condition into affine functions of ρ_j.

Remark 3 *Once* $P_j, j \in [0, 2^{n_u}-1]$ *are set to be the same constant value, then* $P(\rho) = P = P_j$. *Parameter-dependent Lyapunov function is converted to a constant Lyapunov function. From this aspect, constant Lyapunov function is a special case of general parameter-dependent Lyapunov function and the proposed controller is expected to be less conservative.*

5.1.3 Design Objectives

One of our design objective is to construct a dynamic output feedback law of the form (5.4) that asymptotically stabilizes the plant (5.1)-(5.3) at the origin with a domain of attraction as large as possible. This is achieved by establishing stabilization conditions within an ellipsoid of the form

$$\Omega(P(\rho), \eta) = \{x : \ x^{\mathrm{T}} P(\rho) x \leq \eta\}, \ P(\rho) = \sum_{i=0}^{2^{n_u}-1} \rho_j P_j, \ P_j > 0, \rho \in \Gamma, \ \eta > 0$$

contained in the domain of attraction.

Secondly, we will develop output feedback laws of the form (5.4) that establish invariance property in an ellipsoid for bounded disturbances, and minimize the gain from disturbance to the controlled output within the ellipsoid. For this purpose, we will define the set of energy and magnitude bounded disturbances as

$$\mathcal{W}_\eta = \left\{ w : \ \mathbf{R}_+ \to \mathbf{R}^{n_w}, \sum_{k=0}^{\infty} w^T(k) w(k) \leq \eta, \ |w_i| \leq \bar{w}_i, i \in [1, n_w] \right\}, \tag{5.7}$$

where η is some positive number and $\bar{w}_i$ is the upper bound of disturbance for each channel.

Remark 4 *Our objectives in this chapter are exactly the same as those in Chapter 4. The difference basically lies in the derivation process of converting the analysis condition into LMI condition which is solvable. Specifically, as we can see later, the direct consequence of applying parameter-dependent Lyapunov functions is that P will be shown up as $P(\rho(k+1))$ and $P(\rho(k))$ instead of a constant P in the analysis condition. As a result, it inevitably introduces a P^{-1} term, which is not easy to deal with into the inequality. Our solution is to replace this term by a new term which excludes the inversion term while inducting a new general-formed intermediate matrix G. Though this replacement will cause certain conservativeness, it makes the condition easy to solve based on congruent transformation with some new transformation matrices. Finally, in both stabilization and disturbance attenuation problems, the determination of the controller coefficient matrices will be formulated as optimization problems with LMI constraints, which can be solved efficiently using interior-point algorithms.*

5.2 Output Feedback Stabilization

For stabilization problem, we assume the disturbance $w = 0$. Our design goal here is to construct a dynamic output feedback law of the form (5.4) that asymptotically stabilizes the plant (4.35) locally.

The following theorem establishes conditions on the controller coefficient matrices under which the LPV system (5.5) is asymptotically stable with a parameter-dependent quadratic Lyapunov function.

Theorem 10 *Consider the plant (4.35). If there exist positive-definite matrices* $\bar{P}_j, \bar{P}_\ell \in \mathbf{S}_+^{2n_x \times 2n_x}$, $j, \ell \in [0, 2^{n_u} - 1]$ *and matrices* $X, Y, W \in \mathbf{R}^{n_x \times n_x}, (\bar{A}_{kj}, \bar{B}_{kj}) \in \mathbf{R}^{n_x \times n_x} \times \mathbf{R}^{n_x \times n_y}, j \in \mathbf{I}[0, 2^{n_u} - 1]$, $(\bar{C}_k, \bar{D}_k) \in \mathbf{R}^{n_u \times n_x} \times \mathbf{R}^{n_u \times n_y}$ *and* $(\bar{H}_{_C}, \bar{H}_{_D}) \in$

$\mathbf{R}^{n_u\times n_x}\times\mathbf{R}^{n_u\times n_y}$ *such that*

$$\begin{bmatrix} -\bar{P}_j & \star \\ \begin{bmatrix} Y^{\mathrm{T}}A_{\mathrm{p}}+\bar{B}_{kj}C_{\mathrm{p}2} & \bar{A}_{kj} \\ A_{\mathrm{p}}+B_{\mathrm{p}2}\bar{V}_{kj}C_{\mathrm{p}2} & A_{\mathrm{p}}X^{\mathrm{T}}+B_{\mathrm{p}2}\bar{U}_{kj} \end{bmatrix} & \bar{P}_\ell-\begin{bmatrix} Y+Y^{\mathrm{T}} & \star \\ I+W & X+X^{\mathrm{T}} \end{bmatrix} \end{bmatrix}<0,$$

$$j,\ell\in\mathbf{I}[0,2^{n_u}-1], \tag{5.8}$$

$$\begin{bmatrix} \frac{1}{\eta} & \star \\ \begin{bmatrix} C_{\mathrm{p}2}^{\mathrm{T}}\bar{H}_{Di}^{\mathrm{T}} \\ \bar{H}_{Ci}^{\mathrm{T}} \end{bmatrix} & \bar{P}_j \end{bmatrix}\geq 0,\quad i\in[1,n_u],\ j\in\mathbf{I}[0,2^{n_u}-1], \tag{5.9}$$

where $\bar{U}_{kj}=E_j\bar{C}_k+E_j^-\bar{H}_C$ *and* $\bar{V}_{kj}=E_j\bar{D}_k+E_j^-\bar{H}_D$.
Then, with the controller coefficient matrices for $j\in\mathbf{I}[0,2^{n_u}-1]$,

$$\begin{bmatrix} A_{kj} & B_{kj} \\ C_k & D_k \\ H_C & H_D \end{bmatrix}=\begin{bmatrix} N^{\mathrm{T}} & Y^{\mathrm{T}}B_{\mathrm{p}2}E_j & Y^{\mathrm{T}}B_{\mathrm{p}2}E_j^- \\ 0 & I_m & 0 \\ 0 & 0 & I_m \end{bmatrix}^{-1}$$

$$\times\left\{\begin{bmatrix} \bar{A}_{kj} & \bar{B}_{kj} \\ \bar{C}_k & \bar{D}_k \\ \bar{H}_C & \bar{H}_D \end{bmatrix}-\begin{bmatrix} Y^{\mathrm{T}}A_{\mathrm{p}}X^{\mathrm{T}} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}\begin{bmatrix} M^{\mathrm{T}} & 0 \\ C_{\mathrm{p}2}X^{\mathrm{T}} & I_p \end{bmatrix}^{-1}, \tag{5.10}$$

where $M,N\in\mathbf{R}^{n_x\times n_x}$ *are such that* $MN=W-XY$, *the output feedback controller (5.4) asymptotically stabilizes the plant (4.35) at the origin with the ellipsoid* $\Omega(P(\rho),\eta)$ *contained in the domain of attraction, where the parameter-dependent Lyapunov matrix function* $P(\rho)$ *is given by*

$$P(\rho)=\sum_{j=0}^{2^{n_u}-1}\rho_jP_j=\sum_{j=0}^{2^{n_u}-1}\rho_j\begin{bmatrix} I & 0 \\ X & M \end{bmatrix}^{-1}\bar{P}_j\begin{bmatrix} I & X^{\mathrm{T}} \\ 0 & M^{\mathrm{T}} \end{bmatrix}^{-1}. \tag{5.11}$$

Proof: We will show that the saturated linear system (4.35) is asymptotically stabilized by the output feedback controller (5.4) using the parameter-dependent Lyapunov function

$$V(x)=x^{\mathrm{T}}P(\rho)x,\quad P(\rho)=\sum_{j=0}^{2^{n_u}-1}\rho_jP_j,\ P_j>0.$$

Given $P(\rho)$ defined over the compact set Γ, its minimum eigenvalue will be strictly larger than 0, *i.e.*,

$$\alpha_1 \leq \lambda_{\min}(P(\rho)) \leq \alpha_2, \qquad \forall \rho \in \Gamma,$$

for some positive scalars α_1, α_2.

First, from the $(2,2)$ element of inequality (5.8), it can be seen that the matrix $\begin{bmatrix} Y & I \\ W & X \end{bmatrix}$ is non-singular. This implies M, N are also non-singular from the fact $MN = W - XY$. Let

$$T_1 = \begin{bmatrix} I & X^{\mathrm{T}} \\ 0 & M^{\mathrm{T}} \end{bmatrix}, \qquad T_2 = \begin{bmatrix} Y & I \\ N & 0 \end{bmatrix},$$

and specify the non-singular matrix $G = T_2 T_1^{-1}$, then we have that $GT_1 = T_2$ and

$$\begin{aligned} T_1^{\mathrm{T}} P_j T_1 &= \bar{P}_j, \\ T_1^{\mathrm{T}} G^{\mathrm{T}} T_1 = T_2^{\mathrm{T}} T_1 &= \begin{bmatrix} Y^{\mathrm{T}} & W^{\mathrm{T}} \\ I & X^{\mathrm{T}} \end{bmatrix}, \\ T_1^{\mathrm{T}} \left(G^{\mathrm{T}} A_j\right) T_1 &= \begin{bmatrix} Y^{\mathrm{T}} A_{\mathrm{p}} + \bar{B}_{kj} C_{\mathrm{p}2} & \bar{A}_{kj} \\ A_{\mathrm{p}} + B_{\mathrm{p}2} \bar{V}_{kj} C_{\mathrm{p}2} & A_{\mathrm{p}} X^{\mathrm{T}} + B_{\mathrm{p}2} \bar{U}_{kj} \end{bmatrix}, \end{aligned}$$

where the transformed controller data relates to the original $(A_{kj}, B_{kj}, C_k, D_k, H_{_C}, H_{_D})$ in the following way

$$\begin{aligned} \begin{bmatrix} \bar{A}_{kj} & \bar{B}_{kj} \\ \bar{C}_k & \bar{D}_k \\ \bar{H}_{_C} & \bar{H}_{_D} \end{bmatrix} &= \begin{bmatrix} Y^{\mathrm{T}} A_{\mathrm{p}} X^{\mathrm{T}} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ &+ \begin{bmatrix} N^{\mathrm{T}} & Y^{\mathrm{T}} B_{\mathrm{p}2} E_j & Y^{\mathrm{T}} B_{\mathrm{p}2} E_j^{-} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_{kj} & B_{kj} \\ C_k & D_k \\ H_{_C} & H_{_D} \end{bmatrix} \begin{bmatrix} M^{\mathrm{T}} & 0 \\ C_{\mathrm{p}2} X^{\mathrm{T}} & I \end{bmatrix}. \end{aligned} \tag{5.12}$$

By multiplying diag $\left\{T_1^{-1}, T_1^{-1}\right\}$ from the right hand side and its transpose from right to eqn. (5.8), we obtain the following condition

$$\begin{bmatrix} -P_j & A_j^{\mathrm{T}} G \\ G^{\mathrm{T}} A_j & P_\ell - G - G^{\mathrm{T}} \end{bmatrix} < 0, \quad j, \ell \in \mathbf{I}[0, 2^{n_u} - 1].$$

Taking linear combinations over j and ℓ, then

$$\begin{bmatrix} -P(\rho) & A^{\mathrm{T}}(\rho)G \\ G^{\mathrm{T}}A(\rho) & P(\bar{\rho}) - G - G^{\mathrm{T}} \end{bmatrix} < 0 \tag{5.13}$$

for any $\rho, \bar{\rho} \in \Gamma$. Since

$$-P^{-1}(\bar{\rho}) \leq -G^{-T}(G + G^{\mathrm{T}} - P(\bar{\rho}))G^{-1}$$

for any non-singular G, it is clear that eqn. (5.13) guarantees that

$$\begin{bmatrix} -P(\rho) & A^{\mathrm{T}}(\rho) \\ A(\rho) & -P^{-1}(\bar{\rho}) \end{bmatrix} < 0,$$

by Schur complement, it is equivalent to

$$A^{\mathrm{T}}(\rho)P(\bar{\rho})A(\rho) - P(\rho) < 0. \tag{5.14}$$

Next, without disturbance w, we have

$$\begin{aligned} \mathcal{L}(H_{_C}, H_{_D}) &= \{(x_k, y) : \ |H_{_{C_i}} x_k + H_{_{D_i}} y| \leq 1, i \in \mathbf{I}[1, n_u]\} \\ &= \{(x_{\mathrm{p}}, x_k) : \ |H_{_{D_i}} C_{\mathrm{p}2} x_{\mathrm{p}} + H_{_{C_i}} x_k| \leq 1, i \in \mathbf{I}[1, n_u]\} \\ &= \mathcal{L}(H_{_D} C_{\mathrm{p}2}, H_{_C}). \end{aligned}$$

Then multiply diag $\left\{1, T_1^{-1}\right\}$ from the right hand side and its transpose from left to (5.9), and note that

$$\begin{bmatrix} \bar{H}_{_D} C_{\mathrm{p}2} & \bar{H}_{_C} \end{bmatrix} T_1^{-1} = \begin{bmatrix} H_{_D} C_{\mathrm{p}2} & H_{_C} \end{bmatrix},$$

we get the following equivalent condition of (5.9) as

$$\begin{bmatrix} \frac{1}{\eta} & \begin{bmatrix} H_{_{D_i}} C_{\mathrm{p}2} & H_{_{C_i}} \end{bmatrix} \\ \begin{bmatrix} C_{\mathrm{p}2}^{\mathrm{T}} H_{_{D_i}}^{\mathrm{T}} \\ H_{_{C_i}}^{\mathrm{T}} \end{bmatrix} & P_j \end{bmatrix} \geq 0, \qquad i \in \mathbf{I}[1, n_u], \ j \in \mathbf{I}[0, 2^{n_u} - 1].$$

By Schur complement and linear combination arguments, the above equation can be rewritten as

$$\eta \begin{bmatrix} H_{D_i} C_{\mathrm{p}2} & H_{C_i} \end{bmatrix} P^{-1}(\rho) \begin{bmatrix} C_{\mathrm{p}2}^{\mathrm{T}} H_{D_i}^{\mathrm{T}} \\ H_{C_i}^{\mathrm{T}} \end{bmatrix} \leq 1, \quad i \in \mathbf{I}[1, n_u]$$

Therefore,

$$\Omega(P(\rho), \eta) \subset \mathcal{L}(H_{D} C_{\mathrm{p}2}, H_{C}) \tag{5.15}$$

for all $\rho \in \Gamma$. Then we conclude that condition (5.14) guarantees the stability of the quasi-LPV system (5.5), while the condition (5.15) validates the LPV representation of nonlinear saturation control systems in $\Omega(P(\rho), \eta)$.

Lastly, the controller formula (5.10) is derived by inverting eqn. (5.12). *Q.E.D.*

In Theorem 10, we use a parameter-dependent Lyapunov function to expand the stability region in construction of a stabilizing output feedback controller. The estimated domain of attraction is given by

$$\mathcal{D} = \left\{ x \in \mathbf{R}^{2n_x} : \; x^{\mathrm{T}} P(\rho) x < \eta, \; \forall \rho \in \Gamma \right\}.$$

Similar with what we discussed in Chapter 3, to maximize the cross-section of the ellipsoid in the plant state-space, one can be formulated and solved as an LMI optimization problem:

$$\begin{aligned} & \max \alpha, \\ \text{s.t.} \quad & \tfrac{1}{\alpha^2} P_{\mathrm{R}} - \begin{bmatrix} I & 0 \end{bmatrix} \tfrac{\bar{P}_j}{\eta} \begin{bmatrix} I \\ 0 \end{bmatrix} \geq 0, \quad j \in \mathbf{I}[0, 2^{n_u} - 1], \\ & (5.8) - (5.9), \end{aligned} \tag{5.16}$$

where $\Omega(P_{\mathrm{R}}, 1)$ is a reference ellipsoid with $P_{\mathrm{R}} \in \mathbf{S}_{+}^{n_x \times n_x}$.

If we set $P_j = P_\ell = P, \; j, \ell \in [0, 2^{n_u} - 1]$ and $G = P$, then conditions (5.8)-(5.9) recover the results of constant Lyapunov functions in Chapter 3 Theorem 1. Therefore, the parameter-dependent Lyapunov method contains constant Lyapunov functions as a special case. It will be less conservative in estimating the domain of attraction.

5.3 Performance Control of Discrete-Time Linear Systems

Now, we will focus on the performance control problems for discrete-time saturated systems under energy and magnitude bounded disturbances by using parameter-dependent Lyapunov method. It is parallel with the content in Chapter 4 Section 4.3, but the derivation process becomes more complicated because of the involvement of the parameter ρ.

5.3.1 Disturbance Tolerance

As we argued before, for energy bounded disturbances, the trajectories of saturated systems can not be invariant for an ellipsoid. Nevertheless, one can find a slightly larger ellipsoid to accommodate disturbance effect. For this purpose, we introduce a set

$$\widetilde{\Omega}(P(\rho),\Lambda,\eta+\alpha\sum_{i=1}^{n_w}\bar{w}_i^2)=\left\{(x,w):\ x^{\mathrm{T}}P(\rho)x+w^{\mathrm{T}}\Lambda w\le\eta+\alpha\sum_{i=1}^{n_w}\bar{w}_i^2,\ \sigma(\Lambda)\le\alpha\right\}$$

to overbound the set $\Omega(P(\rho),\eta)$. The disturbance is defined as (5.7). Based on Lemma 2, we will replace the following relation

$$(x_k,y)\in\mathcal{L}(H_{_C},H_{_D}),\qquad\forall\ (x,w)\in\Omega(P(\rho),\eta)\times\mathcal{W}_\eta$$

by a stronger condition

$$\widetilde{\Omega}(P(\rho),\Lambda,\eta+\alpha\sum_{i=1}^{m}\bar{w}_i^2)\subset\mathcal{L}(H_{_D}C_{\mathrm{p}2},H_{_C},H_{_D}D_{\mathrm{p}21}).$$

Theorem 11 *If there exist positive-definite parameter-dependent matrix function* $P(\rho)\in$

$\mathcal{S}_+^{2n_x\times 2n_x}$, *positive-definite matrix* $\Lambda \in \mathcal{S}_+^{n_w\times n_w}$ *such that*

$$\begin{bmatrix} -P(\rho) & 0 & A^{\mathrm{T}}(\rho) \\ 0 & -I & B^{\mathrm{T}}(\rho) \\ A(\rho) & B(\rho) & -P^{-1}(\bar{\rho}) \end{bmatrix} < 0, \tag{5.17}$$

$$\widetilde{\Omega}(P(\rho), \Lambda, \eta + \alpha \sum_{i=1}^{n_w} \bar{w}_i^2) \subset \mathcal{L}(H_{_D} C_{\mathrm{p}2}, H_{_C}, H_{_D} D_{\mathrm{p}21}), \tag{5.18}$$

$$P > 0, \tag{5.19}$$

$$\Lambda \leq \alpha I, \tag{5.20}$$

where $\rho, \bar{\rho} \in \Gamma$*, then all trajectories of the saturated control system that starts from the origin will remain inside the ellipsoid* $\Omega(P(\rho), \eta)$ *for every* $w \in \mathcal{W}_\eta$*.*

Proof: We neglect the proof of conditions (5.18)-(5.20) since it is exactly the same as before. As for condition (5.17), it comes from the trajectory boundedness condition.

Note that by using a parameter-dependent Lyapunov function $V(x) = x^{\mathrm{T}} P(\rho) x$ for the closed-loop system (5.5), the trajectory boundedness condition is

$$\begin{aligned}
&V(x(k+1)) - V(x(k)) - w^{\mathrm{T}}(k) w(k) \\
&= \left[(A(\rho(k))x(k) + B(\rho(k))w(k) \right]^{\mathrm{T}} P(\rho(k+1)) \left[A(\rho(k))x(k) + B(\rho(k))w(k) \right] \\
&\quad - x^{\mathrm{T}}(k) P(\rho(k)) x(k) - w^{\mathrm{T}}(k) w(k) \\
&= \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} A^{\mathrm{T}}(\rho(k)) P(\rho(k+1)) A(\rho(k)) - P(\rho(k)) & A^{\mathrm{T}}(\rho(k)) P(\rho(k+1)) B(\rho(k)) \\ B^{\mathrm{T}}(\rho(k)) P(\rho(k+1)) A(\rho(k)) & B^{\mathrm{T}}(\rho(k)) P(\rho(k+1)) B(\rho(k)) - I \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} < 0,
\end{aligned}$$

It is exactly condition (5.17) by Schur Complement.

Sum up from 0 to N and assume $x(0) = 0$, it can be seen that

$$V(x(N+1)) < \sum_{k=0}^{N} w^{\mathrm{T}}(k) w(k) < \eta$$

Therefore $x(N+1) \in \Omega(P(\rho), \eta)$ for any integer N. This concludes the proof. *Q.E.D.*

Theorem 11 provides a tolerable region for disturbances. Under any disturbance in this region, the saturated system can be stabilized within a bounded state region by an output feedback controller (5.4). Furthermore, one can find the largest disturbance level η_{max} by solving an LMI optimization problem subject to constraints (5.17)-(5.20).

5.3.2 ℓ_2 and ℓ_2 to ℓ_∞ Gain Control Synthesis

Parallel with the results in Chapter 4 4.3.2, we consider the performance control for discrete-time system subject to actuator saturations based on parameter-dependent Lyapunov method. The performance indices are in terms of ℓ_2 gain and ℓ_2 to ℓ_∞ gain.

Theorem 12 *Given $x(0) = 0$, scalars $\eta < \eta_{max}$ and α, there exists a stabilizing output feedback controller (5.4) rendering ℓ_2 gain of the saturated control system less than γ for all $w \in \mathcal{W}_\eta$ if there exist positive definite matrices $\bar{P}_j, \bar{P}_\ell \in \mathbf{S}_+^{n_x \times n_x}$, $j, \ell \in [0, 2^{n_u} - 1]$, and matrices $X, Y, W \in \mathbf{R}^{n_x \times n_x}$, $\Lambda \in \mathbf{S}_+^{n_w \times n_w}$, $(\bar{A}_{kj}, \bar{B}_{kj}) \in \mathbf{R}^{n_x \times n_x} \times \mathbf{R}^{n_x \times n_y}, j \in \mathbf{I}[0, 2^{n_u} - 1]$, $(\bar{C}_k, \bar{D}_k) \in \mathbf{R}^{n_u \times n_x} \times \mathbf{R}^{n_u \times n_y}$ and $(\bar{H}_{_C}, \bar{H}_{_D}) \in \mathbf{R}^{n_u \times n_x} \times$*

$\mathbf{R}^{n_u \times n_y}$ *such that*

$$\begin{bmatrix} -\bar{P}_j & \star \\ 0 & -I \\ \begin{bmatrix} Y^{\mathrm{T}}A_{\mathrm{p}} + \bar{B}_{kj}C_{\mathrm{p2}} & \bar{A}_{kj} \\ A_{\mathrm{p}} + B_{\mathrm{p2}}\bar{V}_{kj}C_{\mathrm{p2}} & A_{\mathrm{p}}X^{\mathrm{T}} + B_{\mathrm{p2}}\bar{U}_{kj} \end{bmatrix} & \begin{bmatrix} Y^{\mathrm{T}}B_{\mathrm{p1}} + \bar{B}_{kj}D_{\mathrm{p21}} \\ B_{\mathrm{p1}} + B_{\mathrm{p2}}\bar{V}_{kj}D_{\mathrm{p21}} \end{bmatrix} \\ \begin{bmatrix} C_{\mathrm{p1}} + D_{\mathrm{p12}}\bar{V}_{kj}C_{\mathrm{p2}} & C_{\mathrm{p1}}X^{\mathrm{T}} + D_{\mathrm{p12}}\bar{U}_{kj} \end{bmatrix} & D_{\mathrm{p11}} + D_{\mathrm{p12}}\bar{V}_{kj}D_{\mathrm{p21}} \end{bmatrix}$$

$$\begin{bmatrix} \star & \star \\ \star & \star \\ \bar{P}_\ell - \begin{bmatrix} Y + Y^{\mathrm{T}} & \star \\ I + W & X + X^{\mathrm{T}} \end{bmatrix} & \star \\ 0 & -\gamma^2 I \end{bmatrix} < 0, \quad j, \ell \in [0, 2^{n_u} - 1], \tag{5.21}$$

$$\begin{bmatrix} \frac{1}{\eta + \alpha \sum_{k=1}^{n_w} \bar{w}_k^2} & \star & \star \\ \begin{bmatrix} C_{\mathrm{p2}}^{\mathrm{T}}\bar{H}_{D_i}^{\mathrm{T}} \\ \bar{H}_{C_i}^{\mathrm{T}} \end{bmatrix} & \bar{P}_j & \star \\ D_{\mathrm{p21}}^{\mathrm{T}}\bar{H}_{D_i}^{\mathrm{T}} & 0 & \Lambda \end{bmatrix} \geq 0, \qquad i \in \mathbf{I}[1, n_u],\ j \in [0, 2^{n_u} - 1], \tag{5.22}$$

$$\Lambda \leq \alpha I, \tag{5.23}$$

where $\bar{U}_{kj} = E_j\bar{C}_k + E_j^-\bar{H}_{C}, \bar{V}_{kj} = E_j\bar{D}_k + E_j^-\bar{H}_{D}$. *Moreover, the coefficient matrices of such* n_x*th-order output-feedback controller and its associated linear subspace are given by (5.10), and all trajectories of the closed-loop system that starts from the origin will remain inside the ellipsoid* $\Omega(P(\rho), \eta)$ *for every* $w \in \mathcal{W}_\eta$, *where* $P(\rho)$ *is defined by (5.11).*

Proof: Conditions (5.21)-(5.23) ensure that the trajectories of saturated control system contain in $\Omega(P(\rho), \eta)$ for bounded disturbance $w \in \mathcal{W}_\eta$. Within $\Omega(P(\rho), \eta)$, the ℓ_2 gain of the closed-loop system is no more than γ if there exists a parameter-dependent Lyapunov function

$$V(x) = x^{\mathrm{T}}P(\rho)x, \qquad P(\rho) = \sum_{j=0}^{2^{n_u}-1} \rho_j P_j,\ P_j > 0$$

as defined in (5.11), such that

$$V(x(k+1)) - V(x(k)) + \frac{1}{\gamma^2} z^{\mathrm{T}}(k) z(k) - w^{\mathrm{T}}(k) w(k) < 0, \tag{5.24}$$

$$\widetilde{\Omega}(P(\rho), \Lambda, \eta + \alpha \sum_{i=1}^{n_w} \bar{w}_i^2) \subset \mathcal{L}(H_{_D} C_{\mathrm{p}2}, H_{_C}, H_{_D} D_{p21}), \tag{5.25}$$

$$\Lambda \leq \alpha I. \tag{5.26}$$

For (5.24),

$$\begin{aligned}
&V(x(k+1)) - V(x(k)) + \frac{1}{\gamma^2} z^{\mathrm{T}}(k) z(k) - w^{\mathrm{T}}(k) w(k) \\
&= \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}^{\mathrm{T}} \left\{ \begin{bmatrix} A^{\mathrm{T}}(\rho(k)) P(\rho(k+1)) A(\rho(k)) - P(\rho(k)) & A^{\mathrm{T}}(\rho(k)) P(\rho(k+1)) B(\rho(k)) \\ B^{\mathrm{T}}(\rho(k)) P(\rho(k+1)) A(\rho(k)) & B^{\mathrm{T}}(\rho(k)) P(\rho(k+1)) B(\rho(k)) \end{bmatrix} \right. \\
&\qquad \left. + \begin{bmatrix} \frac{1}{\gamma^2} C^{\mathrm{T}}(\rho(k)) C(\rho(k)) & \frac{1}{\gamma^2} C^{\mathrm{T}}(\rho(k)) D(\rho(k)) \\ \frac{1}{\gamma^2} D^{\mathrm{T}}(\rho(k)) C(\rho(k)) & \frac{1}{\gamma^2} D^{\mathrm{T}}(\rho(k)) D(\rho(k)) - I \end{bmatrix} \right\} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \\
&< 0,
\end{aligned}$$

By Schur Complement, it is equivalent to

$$\begin{bmatrix} -P(\rho) & 0 & A^{\mathrm{T}}(\rho) & C^{\mathrm{T}}(\rho) \\ 0 & -I & B^{\mathrm{T}}(\rho) & D^{\mathrm{T}}(\rho) \\ A(\rho) & B(\rho) & -P^{-1}(\bar{\rho}) & 0 \\ C(\rho) & D(\rho) & 0 & -\gamma^2 I \end{bmatrix} < 0$$

for any $\rho, \bar{\rho} \in \Gamma$. Use the relation

$$-P^{-1}(\bar{\rho}) \leq -G^{-T}(G + G^{\mathrm{T}} - P(\bar{\rho}))G^{-1},$$

And note from (5.6) that $P(\rho)$ depends on $P_j, j \in \mathbf{I}[0, 2^{n_u} - 1]$ of affine form. Then our objective is to prove

$$\begin{bmatrix} -P_j & 0 & A_j^{\mathrm{T}} G & C_j^{\mathrm{T}} \\ 0 & -I & B_j^{\mathrm{T}} G & D_j^{\mathrm{T}} \\ G^{\mathrm{T}} A_j & G^{\mathrm{T}} B_j & P_\ell - G - G^{\mathrm{T}} & 0 \\ C_j & D_j & 0 & -\gamma^2 I \end{bmatrix} < 0, \quad j, \ell \in \mathbf{I}[0, 2^{n_u} - 1]. \tag{5.27}$$

To this end, we define the congruent transformation matrices as

$$T_1 = \begin{bmatrix} I & X^{\mathrm{T}} \\ 0 & M^{\mathrm{T}} \end{bmatrix}, \qquad T_2 = \begin{bmatrix} Y & I \\ N & 0 \end{bmatrix}.$$

Then $GT_1 = T_2$ and

$$\begin{aligned}
&T_1^{\mathrm{T}} P_j T_1 = \bar{P}_j, \\
&T_1^{\mathrm{T}} G^{\mathrm{T}} T_1 = \begin{bmatrix} X^{\mathrm{T}} & W^{\mathrm{T}} \\ I & Y^{\mathrm{T}} \end{bmatrix}, \\
&\begin{bmatrix} T_1^{\mathrm{T}} \left(G^{\mathrm{T}} A_j\right) T_1 & T_1^{\mathrm{T}} \left(G^{\mathrm{T}} B_j\right) \\ C_j T_1 & D_j \end{bmatrix} \\
&\qquad = \left[\begin{array}{cc|c} Y^{\mathrm{T}} A_{\mathrm{p}} + \bar{B}_{kj} C_{\mathrm{p2}} & \bar{A}_{kj} & Y^{\mathrm{T}} B_{\mathrm{p1}} + \bar{B}_{kj} D_{\mathrm{p21}} \\ A_{\mathrm{p}} + B_{\mathrm{p2}} \bar{V}_{kj} C_{\mathrm{p2}} & A_{\mathrm{p}} X^{\mathrm{T}} + B_{\mathrm{p2}} \bar{U}_{kj} & B_{\mathrm{p1}} + B_{\mathrm{p2}} \bar{V}_{kj} D_{\mathrm{p21}} \\ \hline C_{\mathrm{p1}} + D_{\mathrm{p12}} \bar{V}_{kj} C_{\mathrm{p2}} & C_{\mathrm{p1}} X^{\mathrm{T}} + D_{\mathrm{p12}} \bar{U}_{kj} & D_{\mathrm{p11}} + D_{\mathrm{p12}} \bar{V}_{kj} D_{\mathrm{p21}} \end{array}\right].
\end{aligned}$$

The relation between transformed controller data and the original ones are given by (5.12). Then (5.21) is verified by multiplying $\mathrm{diag}\left\{T_1^T, I, T_1^T, I\right\}$ from the left hand side and its transpose from right to eqn. (5.27).

For (5.25), we have

$$(\eta + \alpha \sum_{k=1}^{n_w} \bar{w}_k^2) \begin{bmatrix} H_{D_i} C_{\mathrm{p2}} & H_{C_i} & H_{D_i} D_{\mathrm{p21}} \end{bmatrix} \mathrm{diag}\left\{P^{-1}(\rho), \Lambda^{-1}\right\} \begin{bmatrix} C_{\mathrm{p2}}^{\mathrm{T}} H_{D_i}^{\mathrm{T}} \\ H_{C_i}^{\mathrm{T}} \\ D_{\mathrm{p21}}^{\mathrm{T}} H_{D_i}^{\mathrm{T}} \end{bmatrix} \leq 1,$$

$$i \in \mathbf{I}[1, n_u]$$

Based on Schur Complement and linear combination argument, it is the same as

$$\begin{bmatrix} \frac{1}{\eta + \alpha \sum_{k=1}^{n_w} \bar{w}_k^2} & \begin{bmatrix} H_{D_i} C_{\mathrm{p2}} & H_{C_i} & H_{D_i} D_{\mathrm{p21}} \end{bmatrix} \\ \begin{bmatrix} C_{\mathrm{p2}}^{\mathrm{T}} H_{D_i}^{\mathrm{T}} \\ H_{C_i}^{\mathrm{T}} \\ D_{\mathrm{p21}}^{\mathrm{T}} H_{D_i}^{\mathrm{T}} \end{bmatrix} & \mathrm{diag}\left\{P_j, \Lambda\right\} \end{bmatrix} \geq 0, \in \mathbf{I}[1, n_u],\ j \in [0, 2^{n_u} - 1]. \quad (5.28)$$

Note that,

$$\begin{bmatrix} \bar{H}_{_D} C_{\rm p2} & \bar{H}_{_C} & \bar{H}_{_D} D_{\rm p21} \end{bmatrix} = \begin{bmatrix} H_{_D} C_{\rm p2} & H_{_C} & H_{_D} D_{\rm p21} \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ 0 & I \end{bmatrix},$$

Then multiply diag $\{1, T_1^T, I\}$ from the left hand side and its transpose from right on eqn. (5.28) to verify condition (5.22). Therefore the saturated control system has its ℓ_2 gain less than γ. *Q.E.D.*

Theorem 13 *The ℓ_2 to ℓ_∞ gain of the closed-loop system (5.5) is less than γ for all $w \in \mathcal{W}_\eta$ if there exist positive definite matrices $\bar{P}_j, \bar{P}_\ell \in \mathbf{S}_+^{n_x \times n_x}$, $j, \ell \in [0, 2^{n_u} - 1]$, and matrices $X, Y, W \in \mathbf{R}^{n_x \times n_x}$, $\Lambda \in \mathbf{S}_+^{n_w \times n_w}$, $(\bar{A}_{kj}, \bar{B}_{kj}) \in \mathbf{R}^{n_x \times n_x} \times \mathbf{R}^{n_x \times n_y}, j \in \mathbf{I}[0, 2^{n_u} - 1]$, $(\bar{C}_k, \bar{D}_k) \in \mathbf{R}^{n_u \times n_x} \times \mathbf{R}^{n_u \times n_y}$, $(\bar{H}_{_C}, \bar{H}_{_D}) \in \mathbf{R}^{n_u \times n_x} \times \mathbf{R}^{n_u \times n_y}$, and a scalar $\alpha > 0$ such that*

$$\begin{bmatrix} -\bar{P}_j & \star & \star \\ 0 & -I & \star \\ \begin{bmatrix} Y^{\rm T} A_{\rm p} + \bar{B}_{kj} C_{\rm p2} & \bar{A}_{kj} \\ A_{\rm p} + B_{\rm p2} \bar{V}_{kj} C_{\rm p2} & A_{\rm p} X^{\rm T} + B_{\rm p2} \bar{U}_{kj} \end{bmatrix} & \begin{bmatrix} Y^{\rm T} B_{\rm p1} + \bar{B}_{kj} D_{\rm p21} \\ B_{\rm p1} + B_{\rm p2} \bar{V}_{kj} D_{\rm p21} \end{bmatrix} & \bar{P}_\ell - \begin{bmatrix} Y + Y^{\rm T} & \star \\ I + W & X + X^{\rm T} \end{bmatrix} \end{bmatrix} < 0, \ell \in [0, 2^{n_u} - 1], \tag{5.29}$$

$$\begin{bmatrix} \frac{1}{\eta + \alpha \sum_{k=1}^{n_w} \bar{w}_k^2} & \star & \star \\ \begin{bmatrix} C_{\rm p2}^{\rm T} \bar{H}_{Di}^{\rm T} \\ \bar{H}_{Ci}^{\rm T} \end{bmatrix} & \bar{P}_j & \star \\ D_{\rm p21}^{\rm T} \bar{H}_{Di}^{\rm T} & 0 & \Lambda \end{bmatrix} \geq 0, \qquad i \in \mathbf{I}[1, n_u],\ j \in [0, 2^{n_u} - 1], \tag{5.30}$$

$$\Lambda \leq \alpha I, \tag{5.31}$$

$$\begin{bmatrix} \gamma^2 I & \begin{bmatrix} C_{\mathrm{p}1} + D_{\mathrm{p}12}\bar{V}_{kj}C_{\mathrm{p}2} & C_{\mathrm{p}1}X^{\mathrm{T}} + D_{\mathrm{p}12}\bar{U}_{kj} \end{bmatrix} & D_{\mathrm{p}11} + D_{\mathrm{p}12}\bar{V}_{kj}D_{\mathrm{p}21} \\ \star & \bar{P}_j & 0 \\ \star & \star & I \end{bmatrix} \geq 0,$$

$$j \in \mathbf{I}[0, 2^{n_u} - 1], \quad (5.32)$$

where $\bar{U}_{kj}$ and $\bar{V}_{kj}$ are as defined in Theorem 12. The coefficient matrices of such n_xth-order output-feedback controller and its associated linear subspace are given by (5.10), and all trajectories of the closed-loop system that starts from the origin will remain inside the ellipsoid $\Omega(P(\rho), \eta)$ for every $w \in \mathcal{W}_\eta$.

Proof: Note that (5.29) is the first 3×3 elements of (5.21). It reflects condition

$$V(x(k+1) - V(x(k) < w^T(k)w(k),$$

summation of which ensures all the trajectories of saturated control system starting from the origin contain in $\Omega(P(\rho), \eta)$ for bounded disturbance $w \in \mathcal{W}_\eta$.

For ℓ_2 to ℓ_∞ gain control problem, we expect to find a parameter-dependent Lyapunov function of the form (5.11), such that

$$\begin{bmatrix} C(\rho(k+1)) & D(\rho(k+1)) \end{bmatrix} \begin{bmatrix} P^{-1}(\rho(k+1)) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} C^T(\rho(k+1)) \\ D^T(\rho(k+1)) \end{bmatrix} \leq \gamma^2 I \quad (5.33)$$

$$\widetilde{\Omega}(P(\rho), \Lambda, \eta + \alpha \sum_{i=1}^{n_w} \bar{w}_i^2) \subset \mathcal{L}(H_{_D} C_{\mathrm{p}2}, H_{_C}, H_{_D} D_{p21}), \quad (5.34)$$

$$\Lambda \leq \alpha I. \quad (5.35)$$

Equivalence between (5.30),(5.23) and (5.30),(5.31) are the same with that in ℓ_2 gain case.

Condition (5.33) implies

$$\begin{bmatrix} C^{\mathrm{T}}(\rho(k+1)) \\ D^{\mathrm{T}}(\rho(k+1)) \end{bmatrix} \begin{bmatrix} C(\rho(k+1)) & D(\rho(k+1)) \end{bmatrix} \leq \gamma^2 \begin{bmatrix} P(\rho(k+1)) & 0 \\ 0 & I \end{bmatrix}.$$

It guarantees the ℓ_2 to ℓ_∞ norm is less than γ, since

$$\begin{aligned}
& z^T(N+1)z(N+1) \\
&= [C(\rho(N+1))x(N+1) + D(\rho(N+1))w(N+1)]^T [C(\rho(N+1))x(N+1) \\
&\quad + D(\rho(N+1))w(N+1)] \\
&= \begin{bmatrix} x^T(N+1) & w^T(N+1) \end{bmatrix} \begin{bmatrix} C^{\mathrm{T}}(\rho(N+1)) \\ D^{\mathrm{T}}(\rho(N+1)) \end{bmatrix} \\
&\quad \times \begin{bmatrix} C(\rho(N+1)) & D(\rho(N+1)) \end{bmatrix} \begin{bmatrix} x(N+1) \\ w(N+1) \end{bmatrix} \\
&\leq \gamma^2 \begin{bmatrix} x^T(N+1) & w^T(N+1) \end{bmatrix} \begin{bmatrix} P(\rho(N+1)) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x(N+1) \\ w(N+1) \end{bmatrix} \\
&= \gamma^2 \left[x^T(N+1)P(\rho(N+1))x(N+1) + w^T(N+1)w(N+1) \right] \\
&< \gamma^2 \left[\sum_{k=0}^{N} w^T(k)w(k) + w^T(N+1)w(N+1) \right] \\
&\leq \gamma^2 \|w\|_2^2
\end{aligned}$$

Applying Schur Complement and linear combination argument on condition (5.33) leads to

$$\begin{bmatrix} \gamma^2 & C_j & D_j \\ C_j^T & P_j & 0 \\ D_j^T & 0 & I \end{bmatrix} \geq 0, \qquad j \in \mathbf{I}[0, 2_{n_u} - 1] \tag{5.36}$$

Rest of the proof just follow the similar way as in Theorem 12. Condition (5.32) is the result of multiplying diag $\left\{1, T_2^T, I\right\}$ to the left and its transpose to the right on inequality (5.36).

Q.E.D.

Similar to the domain of attraction calculation, it can be shown that Theorem 12 and Theorem 13 cover Theorem 9 as special cases once setting $P_j = P_\ell = P, j, \ell \in \mathbf{I}[0, 2^{n_u} - 1]$ and $G = P$ and they always provide smaller γ value (or better disturbance

attenuation) than the constant Lyapunov function approach in [75].

5.4 Example

Here, we consider a discrete-time linear system subject to input saturation (4.35) with

$$A_{\rm p} = \begin{bmatrix} 0.8 & -0.9 \\ 0.8 & 0.9 \end{bmatrix}, \qquad B_{\rm p1} = \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix}, \qquad B_{\rm p2} = \begin{bmatrix} 3.8 \\ -0.9 \end{bmatrix},$$
$$C_{\rm p1} = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix}, \qquad D_{\rm p11} = -0.3, \qquad D_{\rm p12} = -0.8,$$
$$C_{\rm p2} = \begin{bmatrix} -0.5 & 2 \end{bmatrix}, \qquad D_{\rm p21} = 1.$$

We first compare the domain of attractions predicted using constant and parameter-dependent Lyapunov functions. In Fig. 5.1, the larger ellipsoids are the cross-section of the domain of attraction from parameter-dependent Lyapunov function which clearly contains the ellipsoid achieved by a constant Lyapunov function.

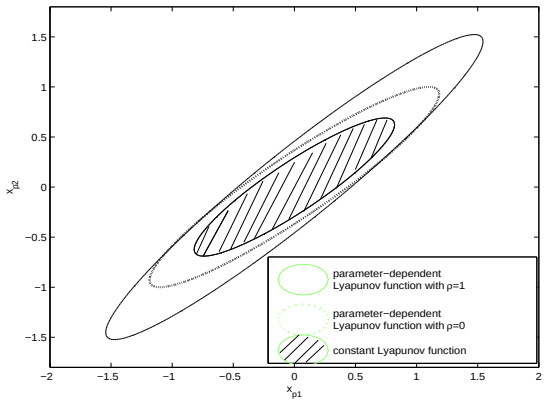


Figure 5.1: Comparison of DOAs using constant and parameter-dependent Lyapunov functions.

Next, we set the magnitude bound of disturbance as 1.186, and restrict the singular value of Λ no more than 0.1. Applying Theorem 11, we obtain the maximal

disturbance energy level $\eta_{max} = 5.638$. Therefore, any trajectory starting from the origin will stay within a bounded state region for the disturbances whose magnitude less than 1.186 and energy less than 5.638. Choosing a smaller $\eta = 5.628$, we then apply Theorem 12 to solve disturbance attenuation problem with minimized ℓ_2 gain $\gamma = 0.5538$. The obtained coefficient matrices of the output feedback controller are

$$A_{k0} = \begin{bmatrix} -0.0201 & 0.1275 \\ 1.7940 & -2.0792 \end{bmatrix}, A_{k1} = \begin{bmatrix} 0.6453 & -1.1160 \\ 1.6907 & -2.2012 \end{bmatrix},$$

$$B_{k0} = \begin{bmatrix} -0.0891 \\ 1.4173 \end{bmatrix}, \qquad B_{k1} = \begin{bmatrix} 0.3652 \\ 1.4879 \end{bmatrix},$$

$$C_k = \begin{bmatrix} -0.2814 & 0.3885 \end{bmatrix}, \qquad D_k = -0.0904,$$

and the auxiliary matrices $H_{_C}$ and $H_{_D}$ are given by

$$H_{_C} = \begin{bmatrix} -0.1205 & 0.1676 \end{bmatrix}, \qquad H_{_D} = -0.0162$$

In simulation study, a disturbance trajectory with its energy of 5.628 is chosen as

$$w(k) = \begin{cases} 1.186, & 1 \leq k < 4 \\ 0.0, & k \geq 4 \end{cases}.$$

Figs. 5.2-5.3 show the state trajectory and control input under the bounded disturbance. As can be seen, the disturbance effect is quickly reduced after step 15. For verification purpose, we also draw the Lyapunov function values and truncated ℓ_2 gain in Figs. 5.4 - 5.5. Note that the $V(x(k))$ is always less than the energy bound η, which means the trajectory of the saturated system starting from the origin remains within the bounded state region $\mathcal{D}$. The second plot shows that the truncated ℓ_2 gain from w to z over the time interval $[0, N]$ is no greater than γ at each step N, as expected.

5.5 Conclusion

In this chapter, We have solved output feedback stabilization and disturbance attenuation problems for saturated control systems using parameter-dependent Lyapunov

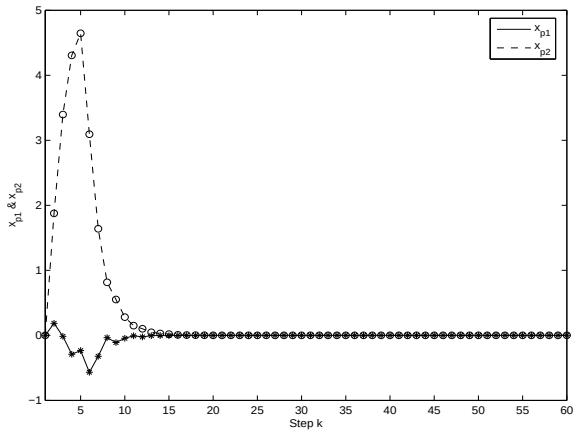


Figure 5.2: State trajectory.

functions. The resulting output feedback controller is nonlinear in nature, and was parameterized in quasi-LPV form. By introducing an intermediate matrix, we formulated the control synthesis problem as LMI optimization problems. The parameter-dependent Lyapunov approach improved our previous work in [73, 75] by enlarging the domain of attraction and reducing disturbance effect on the output.

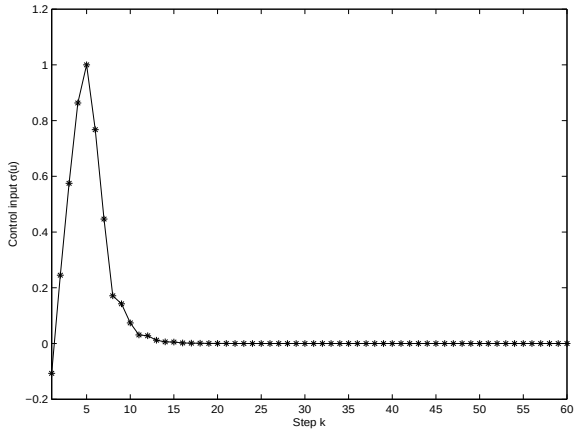


Figure 5.3: Control input.

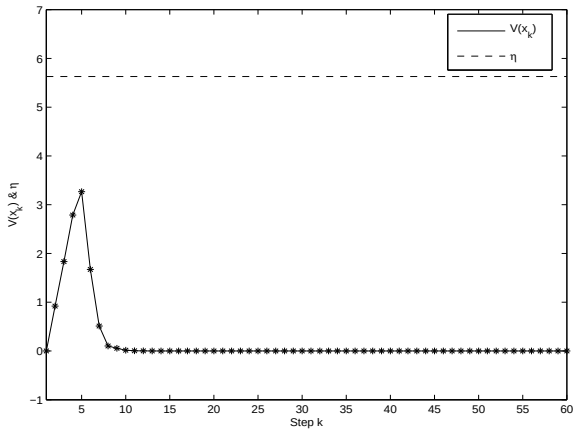


Figure 5.4: Bounded state region.

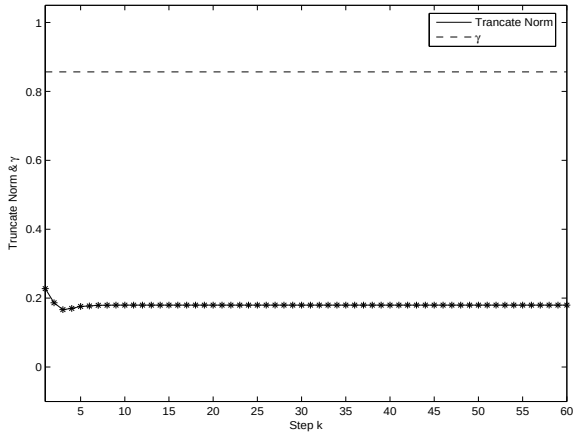


Figure 5.5: Truncated ℓ_2 gain.

Chapter 6

Generalized Saturation Input

In this chapter, we will extend the standard saturation input addressed in the previous chapters to a general type of input described by $N(N \geq 1)$ bends piecewise concave/convex functions. As would be shown later, by fixing the bending points and the slopes of piecewise linear functions to certain values, it would recover the standard saturation function as a special case. For this reason, we call such kind of inputs as generalized saturation inputs. With the number of bends increasing, we would expect to approximate a smoothly nonlinear input specified by piecewise linear functions. Both of stabilization and performance synthesis conditions could be derived by generalizing the results presented in Chapter 3 and 4.

In Section 6.1, we will introduce some basic concepts about concave/convex functions and then provide the formula of N bends piecewise linear functions. In Section 6.2, we will exploit the connection between the standard saturation function and single bend generalized saturation function and then derive three sets of synthesis conditions to achieve stabilization, restricted $\mathcal{L}_2$ gain and $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain minimizations, respectively. Parallel results for generalized saturation input with multiple bends will also be derived in Section 6.3 since a piecewise linear function with multiple bends can be treated as the convex combination of several single bend piecewise linear functions. In Section 6.4, we will provide a summary for the chapter.

6.1 Concave and Convex Functions

The generalized saturation input we are interested in this chapter will be defined by using a class of linear symmetric functions. First, we will provide the definitions of concave and convex functions:

Definition 1 *Given a scalar function* $y = F(u), u \in \mathbb{U}$. *Assume that*

(a) $F(u)$ *is continuous, piecewise differentiable,* $F(0) = 0$ *and* $F^{'}(u)|_{u=0} > 0$

(b) $F(u)$ *is odd symmetric, i.e,* $F(-u) = -F(u)$

Then,

If for any $u_1, u_2 \in \mathbb{U}$ *and any* $\lambda \in [0, 1]$, *we have*

$$F(\lambda u_1 + (1-\lambda)u_2) \geq \lambda F(u_1) + (1-\lambda)F(u_2)$$

$F(u)$ *is called a concave function.*

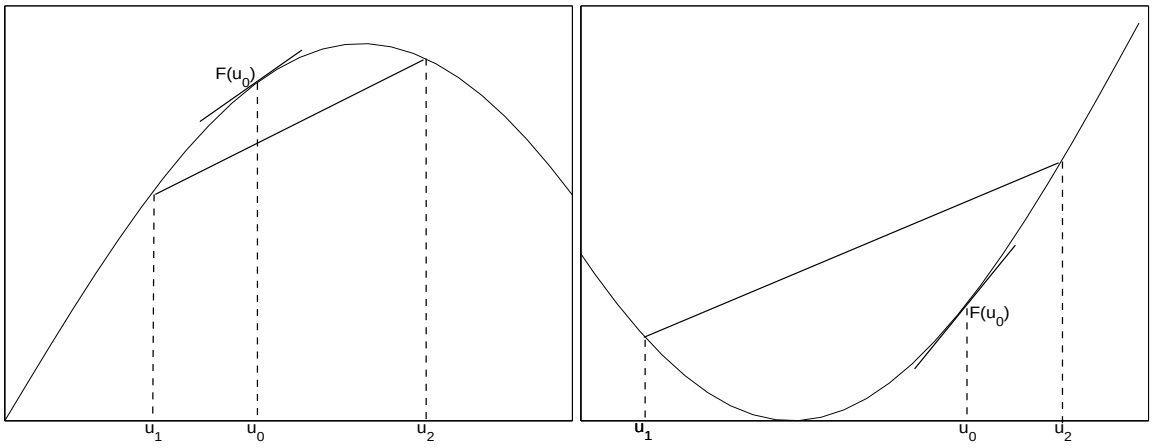
If for any u_1,$u_2 \in \mathbb{U}$ *and any* $\lambda \in [0, 1]$, *we have*

$$F(\lambda u_1 + (1-\lambda)u_2) \leq \lambda F(u_1) + (1-\lambda)F(u_2)$$

$F(u)$ *is called a convex function.*

Fig. 6.1 illustrates a concave function and a convex function. Intuitively, it is clear that for a concave (convex) function $F(u)$, if we draw a straight line that is tangential to $F(u)$ at point $(u_0, F(u_0))$, then the straight line lies above (below) $F(u)$ for all $u \in \mathbb{U}$.

Secondly, we will introduce a class of piecewise linear functions. For simplicity, we take $u \in \mathbb{U} = [0, \infty)$ for example to give the mathematical representation of the N bends piecewise linear function. It can be easily generalized to $(-\infty, \infty)$ by odd



(a) Concave function. (b) Convex function.

Figure 6.1: Concave and Convex Functions.

symmetry.

$$F(u) = \begin{cases} k_0 u, & \text{if} \quad u \in [0, b_1] \\ k_1 u + c_1, & \text{if} \quad u \in [b_1, b_2] \\ \qquad \vdots & \\ k_N u + c_N, & \text{if} \quad u \in (b_N, \infty) \end{cases} \tag{6.1}$$

where $0 < b_1 < b_2 < \ldots < b_N$.From the definitions of concave and convex functions, it is easy to conclude that

$$\begin{cases} k_0 > k_1 > \ldots > k_N, 0 < c_1 < c_2 < \ldots < c_N, & \text{if } F(u) \text{ is concave} \\ k_0 < k_1 < \ldots < k_N, 0 > c_1 > c_2 > \ldots > c_N, & \text{if } F(u) \text{ is convex} \end{cases}$$

Note that the last slope k_N can be positive, negative or zero. In the case that $k_N = 0$, $F(u)$ is a saturation like function with saturation bound c_N at point b_N. Fig. 6.2 shows a piecewise linear concave function with N bends.

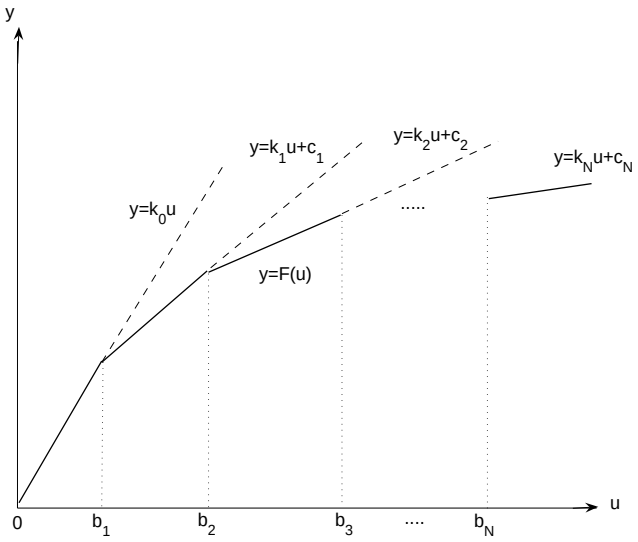


Figure 6.2: Piecewise linear concave function with N bends.

6.2 Output Feedback Control with One Bend Piecewise Linear Saturation Input

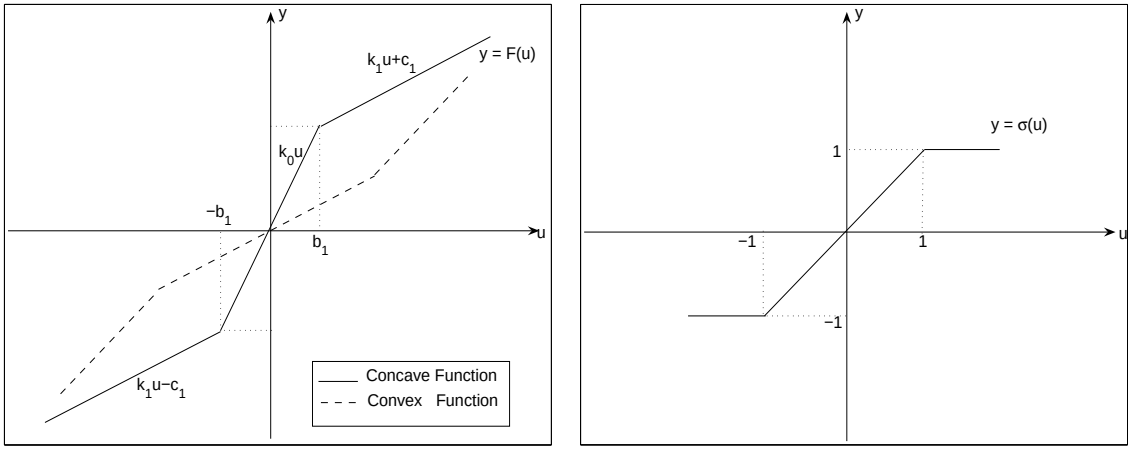
Let us consider a simplest piecewise linear function with only one bend

$$F(u) = \begin{cases} k_0 u, & \text{if } u \in [0, b_1] \\ k_1 u + c_1, & \text{if } u \in (b_1, \infty) \end{cases} \tag{6.2}$$

It is very important to study the piecewise linear functions with one bend since all the results for general concave or convex functions will be based on those for this class of simple piecewise linear functions.

In expression (6.2), note that if $F(u)$ is concave on the set $u \in [0, \infty)$, then $0 < k_1 < k_0, c_1 > 0$ and $b_1 = \frac{c_1}{k_0 - k_1}$. If $F(u)$ is convex on this set, then $k_1 > k_0 > 0, c_1 < 0$ and $b_1 = \frac{c_1}{k_1 - k_0}$. By odd symmetry, it is not difficult to get the corresponding results for $u \in (-\infty, 0]$. Shown in Fig. 6.3(a) are two piecewise linear functions with one bend. The solid one represents a concave $F(u)$ while the dashed one represents a convex $F(u)$. As can be seen, in the case of $k_0 = 1, k_1 = 0$ and $b_1 = 1$, then $c_1 = 1$,

$F(u)$ becomes the standard saturation function $\sigma(u)$ (Fig. 6.3(b)).



(a) Generalized saturation functions. (b) Standard saturation function.

Figure 6.3: Generalization of Standard Saturation Function.

Therefore, one can treat such kind of piecewise linear functions with one bend as a generalized saturation function.

6.2.1 Problem Statement

Now let us substitute the standard saturation input by a generalized saturation input mentioned above and reconsider the stabilization and disturbance attenuation problems addressed in Chapter 3 and Chapter 4.

The linear time-invariant plant subject to the generalized saturation is

$$\begin{cases} \dot{x}_{\rm p} &= A_{\rm p}x_{\rm p} + B_{\rm p1}w + B_{\rm p2}F(u), \\ z &= C_{\rm p1}x_{\rm p} + D_{\rm p11}w + D_{\rm p12}F(u), \\ y &= C_{\rm p2}x_{\rm p} + D_{\rm p21}w, \end{cases} \tag{6.3}$$

where $x_{\rm p} \in \mathbf{R}^{n_x}$, $y \in \mathbf{R}^{n_y}$, $u \in \mathbf{R}^{n_u}$,$z \in \mathbf{R}^{n_z}$, $w \in \mathbf{R}^{n_w}$ and the matrix triple $(A_{\rm p}, B_{\rm p2}, C_{\rm p2})$ are stabilizable and detectable. $F(u)$ is odd symmetric and of the form

$$F(u) = \begin{cases} k_1 u - c_1, & \text{if } u \in (-\infty, -b_1) \\ k_0 u, & \text{if } u \in [-b_1, b_1] \\ k_1 u + c_1, & \text{if } u \in (b_1, \infty) \end{cases} \tag{6.4}$$

$$k_0 > k_1 > 0, \quad b_1 = \frac{c_1}{k_0 - k_1} > 0 \tag{6.5}$$

Note that $F(u)$ is assumed to be a concave piecewise linear function described by the solid line in Fig. 6.3(a). The discussion of the convex case is similar.

Consider an output feedback controller of the form

$$\begin{cases} \dot{x}_k &= f_k(x_k, y), \quad x_k \in \mathbf{R}^{n_k}, \\ u &= C_k x_k + D_k y, \end{cases} \tag{6.6}$$

Our design objective is to construct a dynamic output feedback law of the form (6.6) such that the closed-loop system is asymptotically stable and have good disturbance attenuation capabilities. The objective is achieved as follows:

1. From (6.4) and (6.5), it can be verified that

$$\begin{aligned} F(u) &= k_1 u + (k_0 - k_1) b_1 \sigma\left(\frac{u}{b_1}\right) \\ &= k_1 (C_k x_k + D_k y) + (k_0 - k_1) b_1 \sigma\left(\frac{C_k x_k + D_k y}{b_1}\right) \end{aligned} \tag{6.7}$$

   In this way, we convert the expression of a generalized saturation input to the form that involves the standard saturation input. Thus, we can conveniently use many results from the previous chapters.

2. Based on Lemma 1, we express the standard saturation part in (6.7) as

$$\sigma\left(\frac{C_k x_k + D_k y}{b_1}\right) = \sum_{j=0}^{2^{n_u}-1} \rho_j \left[E_j \left(\frac{C_k x_k + D_k y}{b_1}\right) + E_j^- \left(\frac{H_C x_k + H_D y}{b_1}\right)\right] \tag{6.8}$$

   with

$$(x_k, y) \in \widetilde{\mathcal{L}}\left(\frac{H_C}{b_1}, \frac{H_D}{b_1}\right), \quad \text{i.e} \quad (x_{\mathrm{p}}, x_k) \in \widetilde{\mathcal{L}}\left(\frac{H_D C_{\mathrm{p}}}{b_1}, \frac{H_C}{b_1}\right)$$

$$\widetilde{\mathcal{L}}\left(\frac{H_D C_{\mathrm{p}}}{b_1}, \frac{H_C}{b_1}\right) = \left\{ (x_{\mathrm{p}}, x_k) \in \mathbf{R}^{n_x + n_k} : \left|\frac{H_{Di} C_{\mathrm{p}} x_{\mathrm{p}} + H_{Ci} x_k}{b_1}\right| \le 1, \ i \in \mathbf{I}[1, n_u] \right\},$$

for some scalars $0 \leq \rho_j \leq 1,\ j \in \mathbf{I}[0, 2^{n_u} - 1]$, such that $\sum_{j=0}^{2^{n_u}-1} \rho_j = 1$. Definition of E_j, E_j^- and the choice of ρ_js follow the same way as before.

3. Use the functions $\rho_j(x_k, y)$'s to parameterize the output feedback controller (6.6) into the following quasi-LPV system,

$$\left\{ \begin{array}{rcl} \dot{x}_k &=& \left(\sum_{j=0}^{2^{n_u}-1} \rho_j A_{kj}\right) x_k + \left(\sum_{j=0}^{2^{n_u}-1} \rho_j B_{kj}\right) y, \\ u &=& C_k x_k + D_k y, \end{array} \right. \tag{6.9}$$

where the coefficient matrices A_{kj}'s, B_{kj}'s, C_k and D_k are to be designed.

4. Based on (6.7) and (6.8), write the plant (6.3) in a quasi-LPV form,

$$\left\{ \begin{array}{rcl} \dot{x}_{\rm p} &=& A_{\rm p} x_{\rm p} + B_{\rm p1} w + B_{\rm p2} k_1 (C_k x_k + D_k y) \\ && \quad + B_{\rm p2}(k_0 - k_1) \sum_{j=0}^{2^{n_u}-1} \rho_j \left[E_j (C_k x_k + D_k y) + E_j^- (H_{_C} x_k + H_{_D} y) \right], \\ z &=& C_{\rm p1} x_{\rm p} + D_{\rm p11} w + D_{\rm p12} k_1 (C_k x_k + D_k y) \\ && \quad + D_{\rm p12}(k_0 - k_1) \sum_{j=0}^{2^{n_u}-1} \rho_j \left[E_j (C_k x_k + D_k y) + E_j^- (H_{_C} x_k + H_{_D} y) \right], \\ y &=& C_{\rm p2} x + D_{\rm p21} w. \end{array} \right.$$

5. Combining the dynamics of the plant and the controller law to obtain the representation of the closed-loop system

$$\begin{bmatrix} \dot{x}_{\rm p} \\ \dot{x}_k \\ z \end{bmatrix} = \begin{bmatrix} \left\{ \begin{array}{c} A_{\rm p} + B_{\rm p2}\big[k_1 D_k + \sum_{j=0}^{2^{n_u}-1} \rho_j \\ \times (k_0 - k_1)(E_j D_k + E_j^- H_{_D})\big] C_{\rm p2} \end{array} \right\} & \left\{ \begin{array}{c} B_{\rm p2}\big[k_1 C_k + \sum_{j=0}^{2^{n_u}-1} \rho_j \\ \times (k_0 - k_1)(E_j C_k + E_j^- H_{_C})\big] \end{array} \right\} \\ \sum_{j=0}^{2^{n_u}-1} \rho_j B_{kj} C_{\rm p2} & \sum_{j=0}^{2^{n_u}-1} \rho_j A_{kj} \\ \left\{ \begin{array}{c} C_{\rm p1} + D_{\rm p12}\big[k_1 D_k + \sum_{j=0}^{2^{n_u}-1} \rho_j \\ \times (k_0 - k_1)(E_j D_k + E_j^- H_{_D})\big] C_{\rm p2} \end{array} \right\} & \left\{ \begin{array}{c} D_{\rm p12}\big[k_1 C_k + \sum_{j=0}^{2^{n_u}-1} \rho_j \\ \times (k_0 - k_1)(E_j C_k + E_j^- H_{_C})\big] \end{array} \right\} \end{bmatrix}$$

$$\begin{bmatrix} \left\{ \begin{array}{c} B_{\rm p1} + B_{\rm p2}\big[k_1 D_k + \sum_{j=0}^{2^{n_u}-1} \rho_j \\ \times (k_0 - k_1)(E_j D_k + E_j^- H_{_D})\big] D_{\rm p21} \end{array} \right\} \\ \sum_{j=0}^{2^{n_u}-1} \rho_j B_{kj} D_{\rm p21} \\ \left\{ \begin{array}{c} D_{\rm p11} + D_{\rm p12}\big[k_1 D_k + \sum_{j=0}^{2^{n_u}-1} \rho_j \\ \times (k_0 - k_1)(E_j D_k + E_j^- H_{_D})\big] D_{\rm p21} \end{array} \right\} \end{bmatrix} \begin{bmatrix} x_{\rm p} \\ x_k \\ w \end{bmatrix}$$

$$= \sum_{j=0}^{2^{n_u}-1} \rho_j \begin{bmatrix} \left\{ \begin{array}{r} A_{\rm p} + B_{\rm p2}\big\{\big[k_1 + (k_0-k_1)E_j\big]D_k \\ +(k_0-k_1)E_j^- H_{_D}\big\}C_{\rm p2} \end{array} \right\} & \left\{ \begin{array}{r} B_{\rm p2}\big\{\big[k_1 + (k_0-k_1)E_j\big]C_k \\ +(k_0-k_1)E_j^- H_{_C}\big\} \end{array} \right\} \\ B_{kj}C_{\rm p2} & A_{kj} \\ \left\{ \begin{array}{r} C_{\rm p1} + D_{\rm p12}\big\{\big[k_1 + (k_0-k_1)E_j\big]D_k \\ +(k_0-k_1)E_j^- H_{_D}\big\}C_{\rm p2} \end{array} \right\} & \left\{ \begin{array}{r} D_{\rm p12}\big\{\big[k_1 + (k_0-k_1)E_j\big]C_k \\ +(k_0-k_1)E_j^- H_{_C}\big\} \end{array} \right\} \end{bmatrix}$$

$$\begin{bmatrix} \left\{ \begin{array}{r} B_{\rm p1} + B_{\rm p2}\big\{\big[k_1 + (k_0-k_1)E_j\big]D_k \\ +(k_0-k_1)E_j^- H_{_D}\big\}D_{\rm p21} \end{array} \right\} \\ B_{kj}D_{\rm p21} \\ \left\{ \begin{array}{r} D_{\rm p11} + D_{\rm p12}\big\{\big[k_1 + (k_0-k_1)E_j\big]D_k \\ +(k_0-k_1)E_j^- H_{_D}\big\}D_{\rm p21} \end{array} \right\} \end{bmatrix} \begin{bmatrix} x_{\rm p} \\ x_k \\ w \end{bmatrix}$$

with the definition

$$\widetilde{E}_j = k_1 I + (k_0 - k_1)E_j, \quad \widetilde{E}_j^- = (k_0 - k_1)E_j^- \tag{6.10}$$

Then, the auxiliary LPV system which describes the dynamics of the closed-loop system is

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} = \begin{bmatrix} A(\rho) & B(\rho) \\ C(\rho) & D(\rho) \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = \sum_{j=0}^{2^{n_u}-1} \rho_j \begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}, \qquad \rho \in \mathbf{\Gamma}, \tag{6.11}$$

where

$$x = \begin{bmatrix} x_{\rm p} \\ x_k \end{bmatrix}, \qquad \rho = \left[\rho_0, \rho_1, \cdots, \rho_{2^{n_u}-1}\right],$$

$$\mathbf{\Gamma} = \left\{ \rho \in \mathbf{R}^{2^{n_u}} : \ \sum_{j=0}^{2^{n_u}-1} \rho_j = 1, \ 0 \le \rho_j \le 1, \ j \in \mathbf{I}[0, 2^{n_u}-1] \right\},$$

and

$$\begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix} = \left[\begin{array}{cc|c} A_{\mathrm{p}} & 0 & B_{\mathrm{p1}} \\ 0 & 0 & 0 \\ \hline C_{\mathrm{p1}} & 0 & D_{\mathrm{p11}} \end{array}\right] + \left[\begin{array}{cc} 0 & B_{\mathrm{p2}} \\ I & 0 \\ \hline 0 & D_{\mathrm{p12}} \end{array}\right] \times \begin{bmatrix} A_{kj} & B_{kj} \\ \widetilde{E}_j C_k + \widetilde{E}_j^- H_{_C} & \widetilde{E}_j D_k + \widetilde{E}_j^- H_{_D} \end{bmatrix} \left[\begin{array}{cc|c} 0 & I & 0 \\ C_{\mathrm{p2}} & 0 & D_{\mathrm{p21}} \end{array}\right]$$

Remark 5 *Comparing the above process with its counterpart in Chapter 4, it is clear that the state space description of the closed-loop system under generalized saturation input $F(u)$ is the same as the one under the standard saturation input $\sigma(u)$ except replacing E_j and E_j^- terms by $\widetilde{E}_j$ and $\widetilde{E}_j^-$. Furthermore, the linear subspace restricting the region that (x_{p}, x_k) could lie in varies from $\mathcal{L}\big(H_{_D} C_{\mathrm{p}}, H_{_C}\big)$ to $\widetilde{\mathcal{L}}\left(\frac{H_{_D} C_{\mathrm{p}}}{b_1}, \frac{H_{_C}}{b_1}\right)$.*
Taking a closer look at these two changes, it is not difficult to find that the relation between $\widetilde{E}_j, \widetilde{E}_j^-$ and E_j, E_j^- depend on the slopes of the two piecewise linear functions and $\widetilde{\mathcal{L}}$ relates $\mathcal{L}$ by a factor b_1. When setting $k_0 = 1, k_1 = 0, b_1 = 1$, then $\widetilde{E}_j = E_j$, $\widetilde{E}_j^- = E_j^-$ and $\widetilde{\mathcal{L}} = \mathcal{L}$. Generalized saturation input recovers the standard saturation input, as mentioned at the beginning of this section.
Therefore, we conclude that the generalized saturation input discussed in this section can be treated as a saturation like input sweeping over a bend by tuning the bend points and slopes of the two piecewise linear functions. Standard saturation input is just a special case.

6.2.2 Main Results

The following theorems provide conditions that ensure the system stability and performance properties under one bend generalized saturation input with $F(u)$ of the form (6.7).

1. **Output Feedback Stabilizing Control Synthesis**

Theorem 14 *Consider the closed-loop system $\dot{x} = A(\rho)x$. If there exist positive definite matrices $R, S \in S_+^{n_x \times n_x}$ and matrices $(\widetilde{A}_{kj}, \widetilde{B}_{kj}) \in \mathbf{R}^{n_x \times n_x} \times \mathbf{R}^{n_x \times n_y}$, $(\widetilde{C}_k, \widetilde{D}_k) \in \mathbf{R}^{n_u \times n_x} \times \mathbf{R}^{n_u \times n_y}$ and $(\widetilde{H}_{C}, \widetilde{H}_{D}) \in \mathbf{R}^{n_u \times n_x} \times \mathbf{R}^{n_u \times n_y}$, $j \in \mathbf{I}[0, 2^{n_u} - 1]$ such that*

$$\begin{bmatrix} A_{\mathrm{p}}R + B_{\mathrm{p}2}\widetilde{U}_{kj} + RA_{\mathrm{p}}^{\mathrm{T}} + \widetilde{U}_{kj}^{\mathrm{T}}B_{\mathrm{p}2}^{\mathrm{T}} & \star \\ \widetilde{A}_{kj} + A_{\mathrm{p}}^{\mathrm{T}} + C_{\mathrm{p}2}^{\mathrm{T}}\widetilde{V}_{kj}^{\mathrm{T}}B_{\mathrm{p}2}^{\mathrm{T}} & A_{\mathrm{p}}^{\mathrm{T}}S + C_{\mathrm{p}2}^{\mathrm{T}}\widetilde{B}_{kj}^{\mathrm{T}} + SA_{\mathrm{p}} + \widetilde{B}_{kj}C_{\mathrm{p}2} \end{bmatrix} < 0, \quad (6.12)$$

$$\begin{bmatrix} \frac{b_1^2}{\eta} & \widetilde{H}_{C\,i} & \widetilde{H}_{D\,i}C_{\mathrm{p}2} \\ \widetilde{H}_{C\,i}^{\mathrm{T}} & R & I \\ C_{\mathrm{p}2}^{\mathrm{T}}\widetilde{H}_{D\,i}^{\mathrm{T}} & I & S \end{bmatrix} \geq 0, \quad i \in \mathbf{I}[1, n_u], \quad (6.13)$$

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} > 0, \quad (6.14)$$

where $\widetilde{U}_{kj} = \widetilde{E}_j\widetilde{C}_k + \widetilde{E}_j^{-}\widetilde{H}_{C}$ and $\widetilde{V}_{kj} = \widetilde{E}_j\widetilde{D}_k + \widetilde{E}_j^{-}\widetilde{H}_{D}$, then, with the following controller coefficient matrices for each $j \in \mathbf{I}[0, 2^{n_u} - 1]$,

$$\begin{bmatrix} A_{kj} & B_{kj} \\ C_k & D_k \\ H_{C} & H_{D} \end{bmatrix} = \begin{bmatrix} N & SB_{\mathrm{p}2}\widetilde{E}_j & SB_{\mathrm{p}2}\widetilde{E}_j^{-} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \widetilde{A}_{kj} & \widetilde{B}_{kj} \\ \widetilde{C}_k & \widetilde{D}_k \\ \widetilde{H}_{C} & \widetilde{H}_{D} \end{bmatrix} - \begin{bmatrix} SA_{\mathrm{p}}R & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \times \begin{bmatrix} M^{\mathrm{T}} & 0 \\ C_{\mathrm{p}2}R & I \end{bmatrix}^{-1}, \quad (6.15)$$

where $M, N \in \mathbf{R}^{n_x \times n_x}$ are such that $MN^{\mathrm{T}} = I - RS$, the plant is asymptotically stabilized at the origin with the ellipsoid $\Omega(P, \eta)$ contained in the domain of attraction.

$$P = \begin{bmatrix} I & S \\ 0 & N^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} R & I \\ M^{\mathrm{T}} & 0 \end{bmatrix}^{-1}. \quad (6.16)$$

2. **Restricted $\mathcal{L}_2$ and Restricted $\mathcal{L}_2$ to $\mathcal{L}_\infty$ Gain Synthesis**

   Consider the closed-loop system (6.11) and a class of energy and magnitude

bounded disturbances

$$\mathcal{W}_\eta = \left\{ w : \ \mathbf{R}_+ \to \mathbf{R}^{n_w}, \ \int_0^\infty w^{\mathrm{T}}(t) w(t) dt \le \eta, \ |w_i(t)| \le \bar{w}_i, \forall t \ge 0, i \in \mathbf{I}[1, n_w] \right\}, \tag{6.17}$$

with η and $\bar{w}_i$'s are given.

Theorem 15 *Given scalars $\eta \le \eta_{\max}$ and $\gamma > 0$, the restricted $\mathcal{L}_2$ gain of the closed-loop system from the disturbance w to the output z is rendered less than or equal to γ if there exist positive definite matrices $R, S \in \mathcal{S}_+^{n_x \times n_x}$, $\Lambda \in \mathcal{S}_+^{n_w \times n_w}$, and matrices $(\widetilde{A}_{kj}, \widetilde{B}_{kj}) \in \mathbf{R}^{n_x \times n_x} \times \mathbf{R}^{n_x \times n_y}$, $(\widetilde{C}_k, \widetilde{D}_k) \in \mathbf{R}^{n_u \times n_x} \times \mathbf{R}^{n_u \times n_y}$, $(\widetilde{H}_{C}, \widetilde{H}_{D}) \in \mathbf{R}^{n_u \times n_x} \times \mathbf{R}^{n_u \times n_y}$, $j \in \mathbf{I}[0, 2^{n_u} - 1]$ and a scalar $\alpha > 0$ such that*

$$\begin{bmatrix} A_{\mathrm{p}} R + B_{\mathrm{p2}} \widetilde{U}_{kj} + R A_{\mathrm{p}}^{\mathrm{T}} + \widetilde{U}_{kj}^{\mathrm{T}} B_{\mathrm{p2}}^{\mathrm{T}} & \star \\ \widetilde{A}_{kj} + A_{\mathrm{p}}^{\mathrm{T}} + C_{\mathrm{p2}}^{\mathrm{T}} \widetilde{V}_{kj}^{\mathrm{T}} B_{\mathrm{p2}}^{\mathrm{T}} & A_{\mathrm{p}}^{\mathrm{T}} S + C_{\mathrm{p2}}^{\mathrm{T}} \widetilde{B}_{kj}^{\mathrm{T}} + S A_{\mathrm{p}} + \widetilde{B}_{kj} C_{\mathrm{p2}} \\ B_{\mathrm{p1}}^{\mathrm{T}} + D_{\mathrm{p21}}^{\mathrm{T}} \widetilde{V}_{kj}^{\mathrm{T}} B_{\mathrm{p2}}^{\mathrm{T}} & B_{\mathrm{p1}}^{\mathrm{T}} S + D_{\mathrm{p21}}^{\mathrm{T}} \widetilde{B}_{kj}^{\mathrm{T}} \\ C_{\mathrm{p1}} R + D_{\mathrm{p12}} \widetilde{U}_{kj} & C_{\mathrm{p1}} + D_{\mathrm{p12}} \widetilde{V}_{kj} C_{\mathrm{p2}} \end{bmatrix}$$

$$\begin{bmatrix} \star & \star \\ \star & \star \\ -I & \star \\ D_{\mathrm{p11}} + D_{\mathrm{p12}} \widetilde{V}_{kj} D_{\mathrm{p21}} & -\gamma^2 I \end{bmatrix} < 0, \tag{6.18}$$

$$\begin{bmatrix} \frac{b_1^2}{\eta + \alpha \sum_{j=1}^{n_w} \bar{w}_j^2} & \star & \star & \star \\ \widetilde{H}_{C\,i}^{\mathrm{T}} & R & \star & \star \\ C_{\mathrm{p2}}^{\mathrm{T}} \widetilde{H}_{D\,i}^{\mathrm{T}} & I & S & \star \\ D_{\mathrm{p21}}^{\mathrm{T}} \widetilde{H}_{D\,i}^{\mathrm{T}} & 0 & 0 & \Lambda \end{bmatrix} \ge 0, \quad i \in \mathbf{I}[1, n_u], \tag{6.19}$$

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} > 0, \tag{6.20}$$

$$\Lambda \le \alpha I, \tag{6.21}$$

$\widetilde{U}_{kj}, \widetilde{V}_{kj}, P$ are defined the same as in Theorem 14 and the controller coefficient matrices are given by (6.15).

Theorem 16 *Assume that $D(\rho) = 0$, given scalars $\eta \leq \eta_{\max}$ and $\gamma > 0$, the restricted $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain of the closed-loop system from the disturbance w to the output z is less than or equal to γ if there exist positive definite matrices $R, S \in \mathcal{S}_+^{n_x \times n_x}$, $\Lambda \in \mathcal{S}_+^{n_w \times n_w}$, matrices $(\widetilde{A}_{kj}, \widetilde{B}_{kj}) \in \mathbf{R}^{n_x \times n_x} \times \mathbf{R}^{n_x \times n_y}$, $(\widetilde{C}_k, \widetilde{D}_k) \in \mathbf{R}^{n_u \times n_x} \times \mathbf{R}^{n_u \times n_y}$, $(\widetilde{H}_{C}, \widetilde{H}_{D}) \in \mathbf{R}^{n_u \times n_x} \times \mathbf{R}^{n_u \times n_y}$, $j \in \mathbf{I}[0, 2^{n_u} - 1]$ and a scalar $\alpha > 0$ such that*

$$\begin{bmatrix} A_{\rm p}R + B_{\rm p2}\widetilde{U}_{kj} + RA_{\rm p}^{\rm T} + \widetilde{U}_{kj}^{\rm T}B_{\rm p2}^{\rm T} & \star & \star \\ \widetilde{A}_{kj} + A_{\rm p}^{\rm T} + C_{\rm p2}^{\rm T}\widetilde{V}_{kj}^{\rm T}B_{\rm p2}^{\rm T} & A_{\rm p}^{\rm T}S + C_{\rm p2}^{\rm T}\widetilde{B}_{kj}^{\rm T} + SA_{\rm p} + \widetilde{B}_{kj}C_{\rm p2} & \star \\ B_{\rm p1}^{\rm T} + D_{\rm p21}^{\rm T}\widetilde{V}_{kj}^{\rm T}B_{\rm p2}^{\rm T} & B_{\rm p1}^{\rm T}S + D_{\rm p21}^{\rm T}\widetilde{B}_{kj}^{\rm T} & -I \end{bmatrix} < 0, \tag{6.22}$$

$$\begin{bmatrix} \frac{b_1^2}{\eta + \alpha\sum_{j=1}^{n_w}\widetilde{w}_j^2} & \star & \star & \star \\ \widetilde{H}_{Ci}^{\rm T} & R & \star & \star \\ C_{\rm p2}^{\rm T}\widetilde{H}_{Di}^{\rm T} & I & S & \star \\ D_{\rm p21}^{\rm T}\widetilde{H}_{Di}^{\rm T} & 0 & 0 & \Lambda \end{bmatrix} > 0, \quad i \in \mathbf{I}[1, n_u], \tag{6.23}$$

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} > 0, \tag{6.24}$$

$$\Lambda \leq \alpha I, \tag{6.25}$$

$$\begin{bmatrix} \gamma^2 I & \star & \star \\ RC_{\rm p1}^{\rm T} + \widetilde{U}_{kj}^{\rm T}D_{\rm p12}^{\rm T} & R & \star \\ C_{\rm p1}^{\rm T} + C_{\rm p2}^{\rm T}\widetilde{V}_{kj}^{\rm T}D_{\rm p12}^{\rm T} & I & S \end{bmatrix} \geq 0, \tag{6.26}$$

$\widetilde{U}_{kj}, \widetilde{V}_{kj}, P$ are the same as above. Moreover, the coefficient matrices of an n_xth order output feedback controller (6.9) are determined by (6.15).

Proof. Based on Theorem 2, Theorem 6 and Theorem 7, we could easily list out analysis conditions that guarantee the closed-loop system stability and disturbance attenuation capabilities:

1. **stability condition**

$$A_j^{\mathrm{T}} P + P A_j < 0, \tag{6.27}$$

$$\Omega(P, \eta) \subset \widetilde{\mathcal{L}}\left(\frac{H_{Ci}}{b_1}, \frac{H_{Di}}{b_1}\right), \tag{6.28}$$

$$P > 0. \tag{6.29}$$

2. **Restricted $\mathcal{L}_2$ gain condition**

$$\begin{bmatrix} A_j^{\mathrm{T}} P + P A_j & P B_j & C_j^{\mathrm{T}} \\ B_j^{\mathrm{T}} P & -I & D_j^{\mathrm{T}} \\ C_j & D_j & -\gamma^2 I \end{bmatrix} < 0, \tag{6.30}$$

$$\Omega\left(P, \Lambda, \eta + \alpha \sum_{i=1}^{n_w} \bar{w}_i^2\right) \subset \widetilde{\mathcal{L}}\left(\frac{H_{Di} C_{\mathrm{p}2}}{b_1}, \frac{H_{Ci}}{b_1}, \frac{H_{Di} D_{\mathrm{p}21}}{b_1}\right), \tag{6.31}$$

$$P > 0, \tag{6.32}$$

$$\Lambda \leq \alpha I. \tag{6.33}$$

3. **Restricted $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain conditions**

$$\begin{bmatrix} A_j^{\mathrm{T}} P + P A_j & P B_j \\ B_j^{\mathrm{T}} P & -I \end{bmatrix} < 0, \tag{6.34}$$

$$\Omega\left(P, \Lambda, \eta + \alpha \sum_{i=1}^{n_w} \bar{w}_i^2\right) \subset \widetilde{\mathcal{L}}\left(\frac{H_{Di} C_{\mathrm{p}2}}{b_1}, \frac{H_{Ci}}{b_1}, \frac{H_{Di} D_{\mathrm{p}21}}{b_1}\right), \tag{6.35}$$

$$P > 0, \tag{6.36}$$

$$\Lambda \leq \alpha I, \tag{6.37}$$

$$\begin{bmatrix} \gamma^2 & C_j \\ C_j^{\mathrm{T}} & P \end{bmatrix} > 0. \tag{6.38}$$

where $j \in \mathbf{I}[0, 2^{n_u} - 1], i \in \mathbf{I}[1, n_u]$.

The rest of the proof directly follow the way in previous chapters by replacing E_j, E_j^- terms with $\widetilde{E}_j, \widetilde{E}_j^-$. Define the transformation matrices Z_1 and Z_2 as

$$Z_1 = \begin{bmatrix} R & I \\ M^{\mathrm{T}} & 0 \end{bmatrix}, \qquad Z_2 = \begin{bmatrix} I & S \\ 0 & N^{\mathrm{T}} \end{bmatrix},$$

which are both nonsingular. Choosing P with the form (6.16), we have following relations

$$Z_2 = PZ_1 \tag{6.39}$$

$$Z_1^{\mathrm{T}} P Z_1 = Z_2^{\mathrm{T}} Z_1 = \begin{bmatrix} R & I \\ I & S \end{bmatrix} > 0, \tag{6.40}$$

$$\begin{bmatrix} Z_1^{\mathrm{T}}(PA_j)Z_1 & Z_1^{\mathrm{T}}(PB_j) \\ C_j Z_1 & D_j \end{bmatrix} = \left[\begin{array}{cc|c} A_{\mathrm{p}}R & A_{\mathrm{p}} & B_{\mathrm{p1}} \\ 0 & SA_{\mathrm{p}} & SB_{\mathrm{p1}} \\ \hline C_{\mathrm{p1}}R & C_{\mathrm{p1}} & D_{\mathrm{p11}} \end{array}\right] + \left[\begin{array}{cc} 0 & B_{\mathrm{p2}} \\ I & 0 \\ \hline 0 & D_{\mathrm{p12}} \end{array}\right] \times \begin{bmatrix} \widetilde{A}_{kj} & \widetilde{B}_{kj} \\ \widetilde{U}_{kj} & \widetilde{V}_{kj} \end{bmatrix} \left[\begin{array}{cc|c} I & 0 & 0 \\ 0 & C_{\mathrm{p2}} & D_{\mathrm{p21}} \end{array}\right], \tag{6.41}$$

$$\begin{bmatrix} H_{_D}C_{\mathrm{p2}} & H_{_C} & H_{_D}D_{\mathrm{p21}} \end{bmatrix} Z_1 = \begin{bmatrix} \widetilde{H}_{_C} & \widetilde{H}_{_D}C_{\mathrm{p2}} & \widetilde{H}_{_D}D_{\mathrm{p21}} \end{bmatrix}. \tag{6.42}$$

where the transformed controller data $(\widetilde{A}_{kj}, \widetilde{B}_{kj}, \widetilde{C}_k, \widetilde{D}_k, \widetilde{H}_{_C}, \widetilde{H}_{_D})$ relate to the original controller data $(A_{kj}, B_{kj}, C_k, D_k, H_{_C}, H_{_D})$ by

$$\begin{bmatrix} \widetilde{A}_{kj} & \widetilde{B}_{kj} \\ \widetilde{C}_k & \widetilde{D}_k \\ \widetilde{H}_{_C} & \widetilde{H}_{_D} \end{bmatrix} = \begin{bmatrix} SA_{\mathrm{p}}R & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} N & SB_{\mathrm{p2}}\widetilde{E}_j & SB_{\mathrm{p2}}\widetilde{E}_j^- \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_{kj} & B_{kj} \\ C_k & D_k \\ H_{_C} & H_{_D} \end{bmatrix} \begin{bmatrix} M^{\mathrm{T}} & 0 \\ C_{\mathrm{p2}}R & I \end{bmatrix}.$$

Note that the set inclusion conditions (6.28),(6.31),(6.35) can be rewritten as LMI form by observing that $\widetilde{\mathcal{L}}\left(\frac{H_{_C}}{b_1}, \frac{H_{_D}}{b_1}\right)$ and $\widetilde{\mathcal{L}}\left(\frac{H_{_D}C_{\mathrm{p2}}}{b_1}, \frac{H_{_C}}{b_1}, \frac{H_{_D}D_{\mathrm{p21}}}{b_1}\right)$ are uniquely determined by $\mathcal{L}(H_{_C}, H_{_D})$ and $\mathcal{L}(H_{_D}C_{\mathrm{p2}}, H_{_C}, H_{_D}D_{\mathrm{p21}})$ with a constant factor b_1 and therefore adopting the procedure as before.

Rest of the proof is about doing congruent transformation of (6.40)-(6.42) on (6.27)-(6.38) to get (6.12)-(6.26). For brevity, we omit the detailed calculation. Q.E.D.

Theorem 14 provides conditions under which the plant actuated by a generalized saturation input is asymptotically stabilizable. The stability region is characterized by an ellipsoid $\Omega(P, \eta)$ which is inside the domain of attraction. We may use the similar way introduced in Chapter 3 to maximize the cross-section of the ellipsoid in the plant state-space by a shape reference ellipsoid $\Omega(P_{\mathrm{R}}, 1)$. When the plant is perturbed by a class of magnitude and energy bounded disturbances, determination of the minimized level disturbances could be rejected to are provided in terms of the restricted $\mathcal{L}_2$ gain measurement and $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain measurement, as shown in Theorem 15 and Theorem 16.

It is worthy to mention that once restricting input to a standard saturation input via fixing $k_0 = 1, k_1 = 0$ and $b_1 = 1$, then one can easily verify that Theorem 14- 16 recovers the results in Theorem 2, 6 and 7, which denote the corresponding cases under standard saturation input. This fact is consistent with the conclusion we declared before.

6.3 Generalization to Piecewise Linear Function Input with Multiple Bends

Now, we consider generalizing the piecewise linear function from one bend to multiple bends. The purpose of studying such kind of functions is that in many situations, a smoothly nonlinear function can be approximated by a bunch of piecewise linear functions, as shown in Fig. 6.4. As more bends are specified, more accurately the nonlinearity is characterized.

6.3.1 Preliminary Results and Problem Formulation

Fact 1: Consider a piecewise linear function with N bends as defined in (6.1) and for illustration, we assume that $F(u)$ is concave. The case of convex $F(u)$ is similar.
As shown in Fig. 6.5, let $v_i = \frac{c_i}{k_0 - k_i}$. Then $(v_i, k_0 v_i)$ is the intersection of the straight

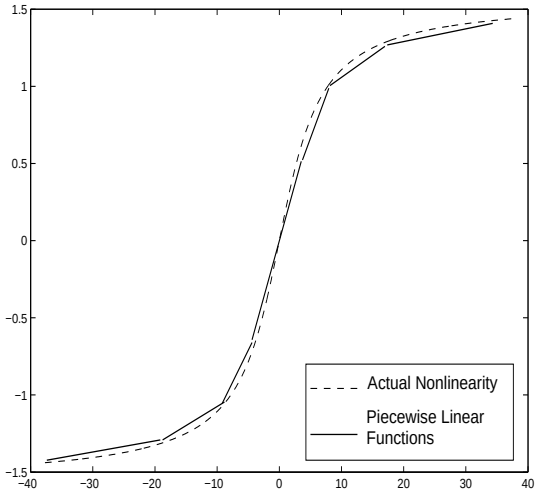


Figure 6.4: Nonlinearity approximation by piecewise linear functions.

line $y = k_0 u$ with the other straight line obtained by extending the $(i+1)$th section of $F(u)$.

Define another set of odd symmetric functions

$$f_i(u) = \begin{cases} k_0 u, & \text{if} \quad u \in [0, v_i] \\ k_i u + c_i, & \text{if} \quad u \in (v_i, \infty) \end{cases} \tag{6.43}$$

Then, for all $u \in \mathbb{U}$,

$$F(u) \in co\{f_1(u), f_2(u), \ldots, f_N(u)\} \tag{6.44}$$

and

$$f_i(u) \in co\{F(u), k_0 u\}, \quad i \in \mathbf{I}[1, N] \tag{6.45}$$

The outline of proof is as follows. First, from the definition of $f_i(u)$, we can rewrite

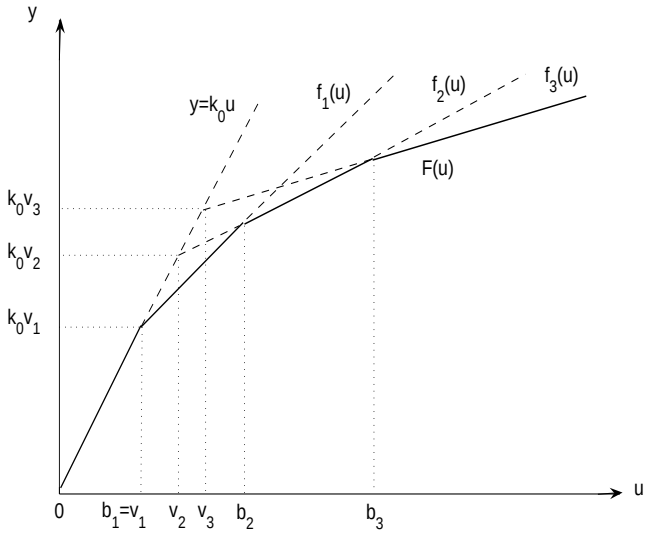


Figure 6.5: Illustration for Fact 1.

$F(u)$ as

$$F(u) = \begin{cases} f_1(u), & \text{if} \quad u \in [0, b_2] \\ f_2(u), & \text{if} \quad u \in (b_2, b_3] \\ \vdots & \\ f_{N-1}(u), & \text{if} \quad u \in (b_{N-1}, b_N] \\ f_N(u), & \text{if} \quad u \in (b_N, \infty) \end{cases} \tag{6.46}$$

It directly leads to

$$F(u) \in co\{f_1(u), f_2(u), \ldots, f_N(u)\}, \forall u \in \mathbb{U}$$

On the other hand, it is clear that for every $i \in \mathbf{I}[1, N]$

$$F(u) \leq f_i(u) \leq k_0 u, \quad \forall u \in \mathbb{U}$$

Therefore,

$$f_i(u) \in co\{F(u), k_0 u\}, \quad \forall u \in \mathbb{U}$$

By odd symmetry, it is easy to extend the construction from $\mathbb{U} = [0, \infty)$ to $\mathbb{U} = (-\infty, \infty)$.

This fact shows that a piecewise linear function with multiple bends can be treated as the convex combination of several one bend piecewise linear functions, i.e,

$$F(u) = \sum_{i=1}^{N} \delta_i f_i(u), \quad \forall u \in \mathbb{U}, \tag{6.47}$$

$$\delta_1 = \begin{cases} 1 & , \text{ if } u \in [0, b_2] \\ 0 & , \text{ else} \end{cases} \tag{6.48}$$

$$\delta_i = \begin{cases} 1 & , \text{ if } u \in (b_i, b_{i+1}], i \in \mathbf{I}[2, N-1] \\ 0 & , \text{ else} \end{cases} \tag{6.49}$$

$$\delta_N = \begin{cases} 1 & , \text{ if } u \in (N, \infty) \\ 0 & , \text{ else} \end{cases} \tag{6.50}$$

$$\delta = [\delta_1, \delta_2, \ldots, \delta_N], \quad \mathbf{\Delta} = \left\{ \delta \in \mathbf{R}^N, \sum_{i=1}^{N} \delta_i = 1, \delta_i = 0 \text{ or } 1, i \in \mathbf{I}[1, N] \right\} \tag{6.51}$$

Now, reconsider system (6.3) with $F(u)$ defined as in (6.47). Note that each $f_i(u)$ in the form of (6.43) is consistent with an one bend piecewise linear function discussed in 6.2. Therefore, we could adopt the procedure in that section to construct an output feedback controller for each $f_i(u)$, i.e, for each generalized saturation input represented by a piecewise linear function with one bend at point $(v_i, k_0 v_i)$.
Moreover,(6.46) shows that $F(u)$ must be consistent with some $f_i(u)$ while u lies in a certain section. Therefore, we may expect to use δ defined in (6.48)-(6.51) to compose a dynamic output feedback control law for $F(u)$ by constructing N sets of controllers, each one of which works for an $f_i(u), i \in \mathbf{I}[1, N]$.

Specific results based on this idea are as follows:

$$\begin{aligned} F(u) &= \sum_{i=1}^{N} \delta_i f_i(u) = \sum_{i=1}^{N} \delta_i \left[k_i u + (k_0 - k_i) v_i \sigma\left(\frac{u}{v_i}\right) \right] \\ &= \sum_{i=1}^{N} \delta_i \left[k_i (C_k x_k + D_k y) + (k_0 - k_i) v_i \sigma\left(\frac{C_k x_k + D_k y}{v_i}\right) \right] \end{aligned} \tag{6.52}$$

where a set of (k_0, k_i, v_i) determines certain one bend generalized saturation input $f_i(u)$.

The composed output feedback controller formula is

$$\left\{\begin{array}{rcl} \dot{x}_k &=& \sum_{i=1}^{N}\delta_i\left[\left(\sum_{j=0}^{2^{n_u}-1}\rho_j A_{kj}^i\right)x_k+\left(\sum_{j=0}^{2^{n_u}-1}\rho_j B_{kj}^i\right)y\right], \\ u &=& \sum_{i=1}^{N}\delta_i\left[C_k^i x_k+D_k^i y\right], \end{array}\right. \tag{6.53}$$

where $\left\{A_{kj}^i, B_{kj}^i, C_k^i, D_k^i, j\in\mathbf{I}[0,2^{n_u}-1]\right\}$ determines a controller that works for the system while $F(u)=f_i(u)$.

The description of the closed-loop system is:

$$\begin{bmatrix}\dot{x}_{\rm p}\\ \dot{x}_k\\ z\end{bmatrix} = \sum_{i=1}^{N}\delta_i\sum_{j=0}^{2^{n_u}-1}\rho_j\left[\begin{array}{cc} A_{\rm p}+B_{\rm p2}\{\widetilde{E}_{ij}D_k^i+\widetilde{E}_{ij}^-H_D^i\}C_{\rm p2} & B_{\rm p2}\{\widetilde{E}_{ij}C_k^i+\widetilde{E}_{ij}^-H_C^i\} \\ B_{kj}^iC_{\rm p2} & A_{kj}^i \\ C_{\rm p1}+D_{\rm p12}\{\widetilde{E}_{ij}D_k^i+\widetilde{E}_{ij}^-H_D^i\}C_{\rm p2} & D_{\rm p12}\{\widetilde{E}_{ij}C_k^i+\widetilde{E}_{ij}^-H_C^i\} \end{array}\right.$$
$$\left.\begin{array}{c} B_{\rm p1}+B_{\rm p2}\{\widetilde{E}_{ij}D_k^i+\widetilde{E}_{ij}^-H_D^i\}D_{\rm p21} \\ B_{kj}^iD_{\rm p21} \\ D_{\rm p11}+D_{\rm p12}\{\widetilde{E}_{ij}D_k^i+\widetilde{E}_{ij}^-H_D^i\}D_{\rm p21} \end{array}\right]\begin{bmatrix}x_{\rm p}\\ x_k\\ w\end{bmatrix}$$

$$=\sum_{i=1}^{N}\delta_i\left\{\left[\begin{array}{cc|c} A_{\rm p} & 0 & B_{\rm p1}\\ 0 & 0 & 0\\ \hline C_{\rm p1} & 0 & D_{\rm p11}\end{array}\right]+\left[\begin{array}{cc} 0 & B_{\rm p2}\\ I & 0\\ \hline 0 & D_{\rm p12}\end{array}\right]\right.$$
$$\left.\times\left(\sum_{j=0}^{2^{n_u}-1}\rho_j\begin{bmatrix}A_{kj}^i & B_{kj}^i\\ \widetilde{E}_{ij}C_k^i+\widetilde{E}_{ij}^-H_C^i & \widetilde{E}_{ij}D_k^i+\widetilde{E}_{ij}^-H_D^i\end{bmatrix}\right)\left[\begin{array}{cc|c} 0 & I & 0\\ C_{\rm p2} & 0 & D_{\rm p21}\end{array}\right]\right\}\begin{bmatrix}x_{\rm p}\\ x_k\\ w\end{bmatrix}$$

where for $i\in\mathbf{I}[1,N],\ j\in\mathbf{I}[0,2^{n_u}-1]$

$$\widetilde{E}_{ij}=k_iI+(k_0-k_i)E_j,\quad \widetilde{E}_{ij}^-=(k_0-k_i)E_j^-, \tag{6.54}$$

Then, the auxiliary LPV system which describes the dynamics of the closed-loop system is

$$\begin{bmatrix}\dot{x}\\ z\end{bmatrix}=\begin{bmatrix}A(\delta,\rho) & B(\delta,\rho)\\ C(\delta,\rho) & D(\delta,\rho)\end{bmatrix}\begin{bmatrix}x\\ w\end{bmatrix}=\sum_{i=1}^{N}\delta_i\begin{bmatrix}A^i(\rho) & B^i(\rho)\\ C^i(\rho) & D^i(\rho)\end{bmatrix}\begin{bmatrix}x\\ w\end{bmatrix}, \tag{6.55}$$

where

$$x = \begin{bmatrix} x_{\mathrm{p}} \\ x_k \end{bmatrix}, \quad \delta \in \mathbf{\Delta}, \quad \rho \in \mathbf{\Gamma}$$

and

$$\begin{aligned}
\begin{bmatrix} A^i(\rho) & B^i(\rho) \\ C^i(\rho) & D^i(\rho) \end{bmatrix} &= \left[\begin{array}{cc|c} A_{\mathrm{p}} & 0 & B_{\mathrm{p1}} \\ 0 & 0 & 0 \\ \hline C_{\mathrm{p1}} & 0 & D_{\mathrm{p11}} \end{array}\right] + \left[\begin{array}{cc} 0 & B_{\mathrm{p2}} \\ I & 0 \\ \hline 0 & D_{\mathrm{p12}} \end{array}\right] \\
&\quad \times \left(\sum_{j=0}^{2^{n_u}-1} \rho_j \begin{bmatrix} A^i_{kj} & B^i_{kj} \\ \widetilde{E}_{ij} C^i_k + \widetilde{E}^-_{ij} H^i_{C} & \widetilde{E}_{ij} D^i_k + \widetilde{E}^-_{ij} H^i_{D} \end{bmatrix} \right) \left[\begin{array}{cc|c} 0 & I & 0 \\ C_{\mathrm{p2}} & 0 & D_{\mathrm{p21}} \end{array}\right] \\
&= \sum_{j=0}^{2^{n_u}-1} \rho_j \begin{bmatrix} A^i_j & B^i_j \\ C^i_j & D^i_j \end{bmatrix}
\end{aligned}$$

We could treat $\{A(\delta,\rho), B(\delta,\rho), C(\delta,\rho), D(\delta,\rho)\}$ as the system matrices of the generalized saturation system actuated by $F(u)$ and as can be seen from (6.55), it has been put on a convex hull of a set of system matrices $\{A^i(\rho), B^i(\rho), C^i(\rho), D^i(\rho)\}$s, each one of which characterizes a sub-system actuated by $f_i(u), i \in \mathbf{I}[1, N]$.

6.3.2 Main Results

Similar to Theorem 14-16, the following theorems present parallel results on stability and performance guarantees in the situation of N bends generalized saturation input with $F(u)$ of the form (6.52).

1. **Output Feedback Stabilizing Control Synthesis**

   Theorem 17 *Consider the closed-loop system $\dot{x} = A(\delta,\rho)x$. If there exist positive definite matrices $R, S \in S_+^{n_x \times n_x}$ and matrices $(\widetilde{A}^i_{kj}, \widetilde{B}^i_{kj}) \in \mathbf{R}^{n_x \times n_x} \times \mathbf{R}^{n_x \times n_y}$, $(\widetilde{C}^i_k, \widetilde{D}^i_k) \in \mathbf{R}^{n_u \times n_x} \times \mathbf{R}^{n_u \times n_y}$ and $(\widetilde{H}^i_{C}, \widetilde{H}^i_{D}) \in \mathbf{R}^{n_u \times n_x} \times \mathbf{R}^{n_u \times n_y}$, $i \in \mathbf{I}[1, N], j \in$*

$\mathbf{I}[0, 2^{n_u} - 1]$ *such that*

$$\begin{bmatrix} A_{\rm p}R + B_{\rm p2}\widetilde{U}^i_{kj} + RA_{\rm p}^{\rm T} + \widetilde{U}^{i^{\rm T}}_{kj}B_{\rm p2}^{\rm T} & \star \\ \widetilde{A}^i_{kj} + A_{\rm p}^{\rm T} + C_{\rm p2}^{\rm T}\widetilde{V}^{i^{\rm T}}_{kj}B_{\rm p2}^{\rm T} & A_{\rm p}^{\rm T}S + C_{\rm p2}^{\rm T}\widetilde{B}^{i^{\rm T}}_{kj} + SA_{\rm p} + \widetilde{B}^i_{kj}C_{\rm p2} \end{bmatrix} < 0, \quad (6.56)$$

$$\begin{bmatrix} \frac{v_i^2}{\eta} & \widetilde{H}^i_{C\ell} & \widetilde{H}^i_{D\ell}C_{\rm p2} \\ \widetilde{H}^{i^{\rm T}}_{C\ell} & R & I \\ C_{\rm p2}^{\rm T}\widetilde{H}^{i^{\rm T}}_{D\ell} & I & S \end{bmatrix} \geq 0, \quad \ell \in \mathbf{I}[1, n_u], \quad (6.57)$$

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} > 0, \quad (6.58)$$

where $\widetilde{U}^i_{kj} = \widetilde{E}_{ij}\widetilde{C}^i_k + \widetilde{E}^-_{ij}\widetilde{H}^i_C$ *and* $\widetilde{V}^i_{kj} = \widetilde{E}_{ij}\widetilde{D}^i_k + \widetilde{E}^-_{ij}\widetilde{H}^i_D$*, then, with the following controller coefficient matrices for each* $i \in \mathbf{I}[1, N], j \in \mathbf{I}[0, 2^{n_u} - 1]$*,*

$$\begin{bmatrix} A^i_{kj} & B^i_{kj} \\ C^i_k & D^i_k \\ H^i_C & H^i_D \end{bmatrix} = \begin{bmatrix} N & SB_{\rm p2}\widetilde{E}_{ij} & SB_{\rm p2}\widetilde{E}^-_{ij} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \widetilde{A}^i_{kj} & \widetilde{B}^i_{kj} \\ \widetilde{C}^i_k & \widetilde{D}^i_k \\ \widetilde{H}^i_C & \widetilde{H}^i_D \end{bmatrix} - \begin{bmatrix} SA_{\rm p}R & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \begin{bmatrix} M^{\rm T} & 0 \\ C_{\rm p2}R & I \end{bmatrix}^{-1}, \quad (6.59)$$

where $M, N \in \mathbf{R}^{n_x \times n_x}$ *are such that* $MN^{\rm T} = I - RS$*, the plant is asymptotically stabilized at the origin with the ellipsoid* $\Omega(P, \eta)$ *contained in the domain of attraction.*

$$P = \begin{bmatrix} I & S \\ 0 & N^{\rm T} \end{bmatrix} \begin{bmatrix} R & I \\ M^{\rm T} & 0 \end{bmatrix}^{-1}. \quad (6.60)$$

2. **Restricted $\mathcal{L}_2$ and Restricted $\mathcal{L}_2$ to $\mathcal{L}_\infty$ Gain Synthesis**

   Consider the closed-loop system (6.55). The class of energy and magnitude bounded disturbances are of the same form as in (6.17).

   Theorem 18 *Given scalars* $\eta \leq \eta_{\max}$ *and* $\gamma > 0$*, the restricted* $\mathcal{L}_2$ *gain of the closed-loop system from the disturbance* w *to the output* z *is rendered less than or equal to* γ *if there exist positive definite matrices* $R, S \in \mathcal{S}_+^{n_x \times n_x}$*,* $\Lambda \in \mathcal{S}_+^{n_w \times n_w}$*,*

and matrices $(\widetilde{A}^i_{kj}, \widetilde{B}^i_{kj}) \in \mathbf{R}^{n_x\times n_x} \times \mathbf{R}^{n_x\times n_y}$,, $(\widetilde{C}^i_k, \widetilde{D}^i_k) \in \mathbf{R}^{n_u\times n_x} \times \mathbf{R}^{n_u\times n_y}$, $(\widetilde{H}^i_{_C}, \widetilde{H}^i_{_D}) \in \mathbf{R}^{n_u\times n_x} \times \mathbf{R}^{n_u\times n_y}$, $i \in \mathbf{I}[1, N], j \in \mathbf{I}[0, 2^{n_u} - 1]$ *and a scalar* $\alpha > 0$ *such that*

$$\begin{bmatrix} A_{\rm p}R + B_{\rm p2}\widetilde{U}^i_{kj} + RA_{\rm p}^{\rm T} + \widetilde{U}^{i\rm T}_{kj}B_{\rm p2}^{\rm T} & \star \\ \widetilde{A}^i_{kj} + A_{\rm p}^{\rm T} + C_{\rm p2}^{\rm T}\widetilde{V}^{i\rm T}_{kj}B_{\rm p2}^{\rm T} & A_{\rm p}^{\rm T}S + C_{\rm p2}^{\rm T}\widetilde{B}^{i\rm T}_{kj} + SA_{\rm p} + \widetilde{B}^i_{kj}C_{\rm p2} \\ B_{\rm p1}^{\rm T} + D_{\rm p21}^{\rm T}\widetilde{V}^{i\rm T}_{kj}B_{\rm p2}^{\rm T} & B_{\rm p1}^{\rm T}S + D_{\rm p21}^{\rm T}\widetilde{B}^{i\rm T}_{kj} \\ C_{\rm p1}R + D_{\rm p12}\widetilde{U}^i_{kj} & C_{\rm p1} + D_{\rm p12}\widetilde{V}^i_{kj}C_{\rm p2} \end{bmatrix}$$

$$\begin{bmatrix} \star & \star \\ \star & \star \\ -I & \star \\ D_{\rm p11} + D_{\rm p12}\widetilde{V}^i_{kj}D_{\rm p21} & -\gamma^2 I \end{bmatrix} < 0, \qquad (6.61)$$

$$\begin{bmatrix} \frac{v_i^2}{\eta+\alpha\sum_{j=1}^{n_w}\widetilde{w}_j^2} & \star & \star & \star \\ \widetilde{H}^{i\rm T}_{C\ell} & R & \star & \star \\ C_{\rm p2}^{\rm T}\widetilde{H}^{i\rm T}_{D\ell} & I & S & \star \\ D_{\rm p21}^{\rm T}\widetilde{H}^{i\rm T}_{D\ell} & 0 & 0 & \Lambda \end{bmatrix} \geq 0, \quad \ell \in \mathbf{I}[1, n_u], \qquad (6.62)$$

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} > 0, \qquad (6.63)$$

$$\Lambda \leq \alpha I, \qquad (6.64)$$

$\widetilde{U}^i_{kj}, \widetilde{V}^i_{kj}, P$ *are defined the same as in Theorem 17 and the controller coefficient matrices are given by (6.59).*

Theorem 19 *Assume that* $D(\delta, \rho) = 0$*, given scalars* $\eta \leq \eta_{\max}$ *and* $\gamma > 0$*, the restricted* $\mathcal{L}_2$ *to* $\mathcal{L}_\infty$ *gain of the closed-loop system from the disturbance* w *to the output* z *is less than or equal to* γ *if there exist positive definite matrices* $R, S \in \mathcal{S}_+^{n_x\times n_x}$, $\Lambda \in \mathcal{S}_+^{n_w\times n_w}$, *matrices* $(\widetilde{A}^i_{kj}, \widetilde{B}^i_{kj}) \in \mathbf{R}^{n_x\times n_x} \times \mathbf{R}^{n_x\times n_y}$, $(\widetilde{C}^i_k, \widetilde{D}^i_k) \in \mathbf{R}^{n_u\times n_x} \times \mathbf{R}^{n_u\times n_y}$, $(\widetilde{H}^i_{_C}, \widetilde{H}^i_{_D}) \in \mathbf{R}^{n_u\times n_x} \times \mathbf{R}^{n_u\times n_y}$,$i \in \mathbf{I}[1, N], j \in \mathbf{I}[0, 2^{n_u} - 1]$ *and*

a scalar $\alpha > 0$ such that

$$\begin{bmatrix} A_{\rm p}R + B_{\rm p2}\widetilde{U}^i_{kj} + RA_{\rm p}^{\rm T} + \widetilde{U}^{i^{\rm T}}_{kj}B_{\rm p2}^{\rm T} & \star & \star \\ \widetilde{A}^i_{kj} + A_{\rm p}^{\rm T} + C_{\rm p2}^{\rm T}\widetilde{V}^{i^{\rm T}}_{kj}B_{\rm p2}^{\rm T} & A_{\rm p}^{\rm T}S + C_{\rm p2}^{\rm T}\widetilde{B}^{i^{\rm T}}_{kj} + SA_{\rm p} + \widetilde{B}^i_{kj}C_{\rm p2} & \star \\ B_{\rm p1}^{\rm T} + D_{\rm p21}^{\rm T}\widetilde{V}^{i^{\rm T}}_{kj}B_{\rm p2}^{\rm T} & B_{\rm p1}^{\rm T}S + D_{\rm p21}^{\rm T}\widetilde{B}^{i^{\rm T}}_{kj} & -I \end{bmatrix} < 0, \tag{6.65}$$

$$\begin{bmatrix} \frac{v_i^2}{\eta+\alpha\sum_{j=1}^{n_w}\widetilde{w}_j^2} & \star & \star & \star \\ \widetilde{H}^{i^{\rm T}}_{C\ell} & R & \star & \star \\ C_{\rm p2}^{\rm T}\widetilde{H}^{i^{\rm T}}_{D\ell} & I & S & \star \\ D_{\rm p21}^{\rm T}\widetilde{H}^{i^{\rm T}}_{D\ell} & 0 & 0 & \Lambda \end{bmatrix} > 0, \quad \ell \in \mathbf{I}[1, n_u], \tag{6.66}$$

$$\begin{bmatrix} R & I \\ I & S \end{bmatrix} > 0, \tag{6.67}$$

$$\Lambda \leq \alpha I, \tag{6.68}$$

$$\begin{bmatrix} \gamma^2 I & \star & \star \\ RC_{\rm p1}^{\rm T} + \widetilde{U}^{i^{\rm T}}_{kj}D_{\rm p12}^{\rm T} & R & \star \\ C_{\rm p1}^{\rm T} + C_{\rm p2}^{\rm T}\widetilde{V}^{i^{\rm T}}_{kj}D_{\rm p12}^{\rm T} & I & S \end{bmatrix} \geq 0, \tag{6.69}$$

$\widetilde{U}^i_{kj}, \widetilde{V}^i_{kj}, P$ are the same as above. Moreover, the coefficient matrices of an n_xth order output feedback controller (6.53) are determined by (6.59).

Proof. Based on Section 6.2, we could list the sufficient condition that guarantee the closed-loop system $\dot{x} = A^i(\rho)x$ is asymptotically stable

$$A^{i^{\rm T}}(\rho)P + PA^i(\rho) = \sum_{j=0}^{2^{n_u}-1} \rho_j(A_j^{i^{\rm T}}P + PA_j^i) < 0, \tag{6.70}$$

$$\Omega(P, \eta) \subset \widetilde{\mathcal{L}}\left(\frac{H^i_C}{v_i}, \frac{H^i_D}{v_i}\right), \tag{6.71}$$

$$P > 0 \tag{6.72}$$

for $i \in \mathbf{I}[1, N]$ and $\rho \in \mathbf{\Gamma}$. Note that these conditions ensure the open-loop system with system matrices $\{A, B_{\rm p2}, C_{\rm p2}\}$ actuated by the generalized saturation input $f_i(u)$ with one bend at point $(v_i, k_0 v_i)$ is stabilizable.

Firstly, we want to prove the relation between the stabilizability of sub-system $\dot{x} = A^i(\rho)x$ and stabilizability of system $\dot{x} = A(\delta, \rho)x$. From Fact 1, we conclude that

$$F(u) \in co\{f_1(u), f_2(u), \ldots, f_N(u)\}$$

and the resulting closed loop system is $\dot{x} = A(\delta, \rho)x = \sum_{i=1}^{N} \delta_i A^i(\rho)x, \delta \in \mathbf{\Delta}$. Obviously, (6.70) and (6.72) ensure that $A^T(\delta, \rho)P + PA(\delta, \rho) < 0$ will be satisfied due to linear combination argument. Furthermore, in order to use the convex combination argument to express the saturation input, we need $(x_k, y) \in \widetilde{\mathcal{L}}\left(\frac{H^i_{C}}{v_i}, \frac{H^i_{D}}{v_i}\right)$ for every $f_i(u), i \in \mathbf{I}[1, N]$. Consequently, every point in the ellipsoid, which describes the stability region under $F(u)$, must inside the intersection of all the linear subspaces $\widetilde{\mathcal{L}}\left(\frac{H^i_{C}}{v_i}, \frac{H^i_{D}}{v_i}\right), i \in \mathbf{I}[1, N]$. Then it leads to the analysis conditions that guarantee the closed-loop system $\dot{x} = A(\delta, \rho)x$ is asymptotically stable

$$A^{i^{\mathrm{T}}}_j P + PA^i_j < 0, \tag{6.73}$$

$$\Omega(P, \eta) \subset \bigcap_{i=1}^{N} \widetilde{\mathcal{L}}\left(\frac{H^i_{C}}{v_i}, \frac{H^i_{D}}{v_i}\right), \tag{6.74}$$

$$P > 0 \tag{6.75}$$

for all $i \in \mathbf{I}[1, N]$ and $j \in \mathbf{I}[0, 2^{n_u} - 1]$. The proof of the equivalence between (6.73)-(6.75) and (6.56)-(6.58) follows the same way as in Section 6.2 by first converting (6.73)-(6.75) into LMIs and then through congruent transformations. The resulting set $\Omega(P, \eta)$ outlines the ellipsoid within which the open-loop system under the generalized saturation input $F(u)$ is stabilizable via an output feedback controller with coefficient matrices $\left\{A^i_{kj}, B^i_{kj}, C^i_k, D^i_k, H^i_{C}, H^i_{D}\right\}$, $i \in \mathbf{I}[1, N], j \in \mathbf{I}[0, 2^{n_u} - 1]$ determined by

$$\begin{bmatrix} \widetilde{A}^i_{kj} & \widetilde{B}^i_{kj} \\ \widetilde{C}^i_k & \widetilde{D}^i_k \\ \widetilde{H}^i_{C} & \widetilde{H}^i_{D} \end{bmatrix} = \begin{bmatrix} SA_{\mathrm{p}}R & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} N & SB_{\mathrm{p2}}\widetilde{E}_{ij} & SB_{\mathrm{p2}}\widetilde{E}^{-}_{ij} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A^i_{kj} & B^i_{kj} \\ C^i_k & D^i_k \\ H^i_{C} & H^i_{D} \end{bmatrix} \begin{bmatrix} M & 0 \\ C_{\mathrm{p2}}R & I \end{bmatrix}. \tag{6.76}$$

where $\widetilde{E}_{ij}, \widetilde{E}_{ij}^{-}$ are defined as in (6.54). As we have discussed, there will be N sets of controllers constructed and each one works on a section of u.

The proof of Theorem 18 and 19 are trivial and omitted here for brevity. Basically the idea is similar with the one used in stabilization problem presented above. Q.E.D.

6.4 Summary

In this chapter, we introduced the concept of piecewise concave/convex linear functions. The purpose is to extend the standard saturation input $\sigma(u)$ to some more generalized cases. We have studied two cases here, one is the saturation like input described by a piecewise linear function with one bend and the other is with multiple bends.

The resulting synthesis conditions for generalized saturation input with one bend is based on those for standard saturation input and it is verified that the latter is covered by the former while fixing the coefficients of the input function to certain values.

Noticing the fact that a piecewise linear function with multiple bends is the convex combination of several piecewise linear functions with one bend, we decomposed the stability and disturbance attenuation problems of the whole system to several corresponding problems of the sub-systems, each one of which can be solved by adopting the results from the one bend case. As a result, for each sub-system, there exists an output feedback controller and a linear subspace where saturation does not occur. The stability region of the whole system lies in the intersection of these N linear subspaces.

The importance of studying the class of generalized saturation inputs is that it makes the exploration of continuously nonlinear inputs possible, since a continuously nonlinear function may be approximated by a set of piecewise concave/convex linear functions. However, it should be noted that since N controllers will be constructed, and each one only serves for one section of u, the applicability of the control law at

the bend points is a potential issue. Especially when N is very large, the stability of the system while switching the control laws is not guaranteed.

The stabilization and performance optimization results can also be developed for discrete-time systems with generalized saturation inputs, which will not be discussed here.

Chapter 7

Conclusion

7.1 Thesis Conclusion and Contribution

In this thesis, we presented a methodology to solve the output feedback stabilization and disturbance attenuation problems for linear systems subject to actuator saturations. This method applied to general linear systems including strictly unstable ones. Both of the continuous-time and discrete-time systems were addressed, if it is possible.

The contributions of this thesis are summarized as following:

- Most of the literature on exponentially unstable systems pertained to state feedback. A few exceptions where output feedback were used include [26, 50, 51]. In this thesis, we further explored the construction of output feedback control laws, since they are applicable to broader class of systems including those that only part of the states information about the plant are known. Both of the stabilization and performance problems were considered and the determination of the controller coefficient matrices was finally formulated and solved as optimization problem with LMI constraints.
- The key point of this approach included putting the saturating linear feedback law on the convex hull involving an auxiliary feedback gain matrix.

Though the resulting output feedback laws were nonlinear in nature, due to this convex combination formulation of the saturating feedback law, we could parameterize them into quasi-LPV forms. As a result, the dynamics of the output feedback controller affinely depended on a scheduling parameter, so did the closed-loop system. Synthesis conditions that ensured the invariance property in an ellipsoid (system stability) as well as the disturbance attenuation capabilities (performance criteria) could be formulated as linear matrix inequalities (LMIs) with a variable to be optimized. For stabilization problem, the optimized variable referred to the largest stability region could be achieved while for performance problem, it indicated the smallest level of a class of magnitude and energy bounded disturbances could be rejected in terms of certain norm measure (induced $\mathcal{L}_2$ gain or $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain).

- A class of generalized Lyapunov functions, called parameter-dependent Lyapunov functions, made the choice of Lyapunov functions to a more flexible form. The parameter-dependent Lyapunov matrix was expressed as the convex combination of several constant Lyapunov matrices, each one of which was involved in the LMI control synthesis process and how much it contributed to the composing of parameter-depend Lyapunov matrix depended on a scheduling parameter. From this point, it was not difficult to see that constant Lyapunov function could be treated as a special case of the parameter-dependent Lyapunov function. Therefore we may get less conservative results by using this generalized Lyapunov function.
- Generalization on the form of saturation functions broaden the applicable scope of the proposed saturation control scheme, which is discussed in the context of standard saturation inputs to a class of saturation-like inputs described by piecewise linear functions with multiple bends. The generalized synthesis conditions could also be solved as LMI problem. Since

the form of the saturation-like input was determined by a set of tunable parameters including input slopes and bend points, the generalized saturation control approach was applicable to a wider class of saturation functions and the standard saturation function was covered as a special case.

In summary, this research has provided a new approach to solve output feedback control problem for saturated linear systems from gain-scheduling control perspective. Other than its theoretical interest, the newly developed output control approach will greatly improve the applicability of saturated control techniques to practical engineering problems.

7.2 Future Work

Besides the results presented in this thesis, there are some remaining issues as well as promising directions that worth to be further studied in the future.

- Simulation tests in stabilization problem showed that directly applying synthesis conditions often pushed the pole locations of the closed-loop system far away from the origin in the left half plane. Thus the state trajectory converged to the equilibrium point (origin) very fast such that input sharply decreased to zero without saturating. It also caused numerical problems in simulation. In the thesis, we adopted the pole placement design approach [9] to restrict the poles of the closed-loop system lie in a certain region that not so far from the origin in the left half plane in order to clearly observe saturation phenomena and improve numerical condition. As a result, the domain of attraction was decreased with pole locations moving closer to the origin. Therefore, properly adjusting the pole locations to balance the requirements on domain of attraction and closed loop behavior is a critical issue from implementation point of view. Future work should address the design trade-off between these two

conflicting requirements.

- As demonstrated in $\mathcal{L}_2$ to $\mathcal{L}_\infty$ performance control, we have considered relaxing the requirement of $D(\rho) = 0$ since the class of disturbances discussed there was assumed to be magnitude and energy bounded. Therefore, a magnitude unbounded controlled output z would not be produced. For discrete-time system, we have successfully removed the condition $D(\rho) = 0$, however, for continuous-time system, the derivation of the synthesis condition while $D(\rho) = 0$ would involve the integration terms which is not easy to deal with. So this problem needed to be carefully examined in the future.
- Anti-windup approach employs a two-step design procedure by firstly designing the linear controller ignoring the saturation nonlinearities and then adding anti-windup compensation to minimize the adverse effects of the saturation on the closed-loop performance. Some anti-windup control schemes were proposed from different perspectives [36, 7] with the aim of allowing for general designs with stability and performance guarantees. Most of previous anti-windup compensator designs are only applicable to open-loop stable LTI systems. Therefore, extending our approach within the framework of anti-windup control is a direction worth to pursue.
- Other than linear time invariant (LTI) open-loop systems, there exist other more complicated systems which are valuable for studying, such as LPV systems. This generalization is well motivated because of the relevance of LPV systems to nonlinear systems. In fact, the LPV model can be thought of as a group of local descriptions of nonlinear dynamics. As a special class of LPV systems, LFT systems have been well understood and in the absence of actuator saturation, LFT control problems have been systematically addressed based on a scaled small gain condition and the existence of such a gain-scheduled controller has been fully characterized by a set of LMIs. It will be of interest to exploit output feedback control

designs for LFT saturating systems.

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