ABSTRACT

KIANFAR, KIAVASH. Generalized Mixed Integer Rounding Valid Inequalities for Mixed Integer Programming Problems. (Under the direction of Dr. Yahya Fathi).

Many decision-making problems in practice can be formulated as Mixed Integer Programming (MIP) problems, which are NP-hard in their general form. Over the past few decades, an enormous amount of research has been carried out to develop the theory and algorithms for solving MIP problems. Valid inequalities are a crucial part of these developments since they can be added to the MIP problem as cutting planes to tighten the feasible region of its linear programming relaxation toward the convex hull of its MIP solutions.

Mixed Integer Rounding (MIR) is a fundamental approach to generating cutting planes for general MIP problems. Recently, MIR has received special attention from several researchers. MIR inequalities are obtained from facets of certain simple mixed integer sets (MIR facets). A recent contribution in this context has been the work by Dash and Günlük (2006) who introduced the 2-step MIR inequalities. The work of Dash and Günlük is also one of the recent advancements in the area of valid inequalities related to Gomory's group problems. These problems are of special significance in the context of MIP because facets of their corresponding polyhedra are sources for generating valid inequalities for MIP problems.

In this dissertation, we generalize the concept of MIR valid inequalities. Based on this generalization, we develop new families of MIR inequalities for general MIP problems and show that they define (new) facets for the finite and infinite group polyhedra, and hence are potentially strong cuts. More specifically, the contributions of this research are as follows:

First, we show that MIR facets are not limited to 1-step or 2-step facets, but for any positive integer n, n facets of a certain $(n+1)$-dimensional mixed integer set can be obtained through a process which includes n consecutive applications of MIR. The last of these facets is of special importance and we call it the *n-step MIR facet*. As a result, we generate an infinite number of MIR facets (one for each n), which we then use to generate valid inequalities for MIP problems.

Second, we develop a procedure which, for any n, uses the n-step MIR facet to generate a family of valid inequalities for the feasible set of a general MIP constraint. We refer to these as the *n-step MIR inequalities*. The well-known Gomory Mixed Integer Cut and the 2-step MIR inequality of Dash and Günlük are simply the first two families corresponding to n=1,2, respectively. The n-step MIR inequalities are easily produced using closed-form periodic functions, which we call the *n-step MIR functions*. None of these functions dominates the other on its whole period.

Third, we establish a significant connection between the n-step MIR functions and facets of Gomory's group polyhedra. We prove that for any n, the n-step MIR inequalities define new families of facets for the finite and the infinite group polyhedra, and hence are potentially strong cuts. Many of these facets are new facets that have not been introduced in the literature before.

GENERALIZED MIXED INTEGER ROUNDING VALID INEQUALITIES FOR MIXED INTEGER PROGRAMMING PROBLEMS

by

KIAVASH KIANFAR

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Approved By:

Dr. Henry L. W. Nuttle	Dr. Bibhuti Bhattacharyya
Dr. Yahya Fathi Chair of Advisory Committee	Dr. Shu-Cherng Fang

Dedication

To the suns of my life…

Biography

Kiavash Kianfar received his B.S. and M.S. in Industrial Engineering as the topmost graduate from Sharif University of Technology, Iran, in 1998 and 2000, respectively. He entered the Ph.D. program of Industrial and Systems Engineering at North Carolina State University in 2003. During his study, he has won several awards and produced a number of publications. He has also worked on many research projects and obtained several semesters of teaching experience as an instructor. His primary research area of interest is theory and application of mathematical programming. One of the papers composed based on his dissertation was recognized as a Finalist in 2006 INFORMS George Nicholson Student Paper Competition. He is a member of Phi Kappa Phi, Alpha Pi Mu, INFORMS, IIE and SIAM.

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Chapter 1

Introduction and Background

Integer Programming as a special class of mathematical programming is a well-known optimization technique. The goal is to find values for a set of variables so as to minimize or maximize an objective function subject to a set of constraints, where at least one of the variables is restricted to be integer. The integer programming problem in its general form is NP-hard. Relaxing the integrality restriction on variables reduces the integer programming problem to a linear programming problem for which there exist very efficient algorithms. Therefore, one technique to tackle an integer programming problem is to start with its linear programming relaxation and optimize it, and then cut portions of the feasible region of the linear programming relaxation (including its current optimal solution) which do not contain any integer solution by adding cutting planes, re-optimize the new linear programming problem and repeat this procedure. This gradually tightens the feasible region of the linear programming problem until its solution will eventually satisfy the integrality constraints. Therefore cutting planes can be described as valid inequalities for the feasible region of the integer programming problem, which are not valid for the optimal solution of its current linear programming relaxation.

Cutting planes are usually embedded into the branch-and-bound approach to accelerate the discovery of the optimal solution by them. The result is generally known as the branch-and-cut method, which have been very successful in practice [7]. In short, as a useful tool in solving integer programming problems, the theory of cutting planes or valid inequalities have been investigated for a relatively long period of time and there exists a rich literature on different ways of generating valid inequalities for different integer programming problems using polyhedral theory [33].

A class of valid inequalities for a general (mixed) integer programming problem is the so-called

Mixed Integer Rounding (MIR) inequalities. These inequalities are essentially the facets of very simple mixed integer sets; however, the key point is that they can be used to generate valid inequalities for a general integer programming problem. If the MIR facet of the two-dimensional simple set is used for this purpose the result will be the well-known Gomory mixed integer cut, which has been in use for a long time. Therefore derivation of MIR facets for simple sets, and the corresponding use of these facets to generate MIR valid inequalities for a general integer programming problem, are both important subjects in this context because the former provides the tool for the latter.

Recently, Dash and Günlük [11] extended the idea to one dimension higher. They derived the 2-step MIR facet for a three-dimensional simple mixed integer set and used it to generate valid inequalities, called 2-step MIR inequalities, for the feasible set of a general integer programming constraint. The result of this process is what they call the 2-step MIR functions, which are easy tools for generating valid inequalities. Another important property of MIR functions proved by Dash and Günlük is that they generate facets for Gomory's group polyhedra. This shows that 2-step MIR inequality are potentially strong valid inequalities. Additionally, Gomory [23] showed that facets of group polyhedra are sources for generating valid inequalities for integer programming problems. Therefore, this property of MIR functions is of great value in generating even more valid inequalities.

The major contributions of this dissertation can be summarized as follows: First, we show that MIR facets are not limited to two- or three-dimensional simple sets, but can be derived for n-dimensional simple sets for arbitrary $n \in \mathbb{N}$. Therefore, we introduce an infinite number of (n-step) MIR facets, which can be used to generate valid inequalities for integer programming problems. This also provides a description of the facial structure of the simple sets considered. Second, using these new facets of the higher-dimensional simple sets, we present a generalized procedure for generating n-step MIR inequalities for the feasible set of a general (mixed) integer programming constraint. The result will be an infinite collection of n-step MIR functions with nice properties as easy tools for generating n-step MIR inequalities. Third, we will show that, similar to 1-step and 2-step cases, n-step MIR functions generate facets for the finite and infinite group polyhedra. This shows that n-step MIR inequalities are potentially strong, and also provides us with a powerful tool to generate new facets for group polyhedra, which are themselves sources for generating yet more valid inequalities. Many facets easily generated by n-step MIR functions for group polyhedra are new facets that do not exist in the literature and no way of construction has been proposed for them.

1.1 Dissertation Structure

In the remainder of this chapter, we formally define (mixed) integer programming problem and reproduce some basic definitions from polyhedral theory, which will be used later on. Then, we will review the background of some principal methodologies for solving integer programming problems by referring to the literature. The main focus will be on cutting plane methods.

In chapter 2 we present the theory of MIR inequalities, developed up to date, in our own particular way. This includes 1-step and 2-step MIR inequalities. Our treatment of the theory of 1-step and 2-step MIR inequalities is based on the previous work in the literature, however, it is novel in many aspects. We redevelop the theory in a new format. In particular, our development is for more general sets, uses a new consistent notation, follows a new systematic structure for proofs, presents a new and more general form of MIR functions and introduces some important properties of these new form. All these new aspects will then allow us to generalize the MIR inequalities in an intact framework in chapters 3 and 4. Derivation of MIR facets for simple sets, as well as use of them to generate MIR inequalities for the feasible set of a general constraint, will be discussed.

In chapter 3 we introduce the basic ideas of the generalization by proving that the MIR inequalities studied in chapter 2 can be generalized to 3-step MIR inequalities. This fact, as a main contribution, illuminates the path to extend this generalization to n-step MIR inequalities for arbitrary $n \in \mathbb{N}$ later in the dissertation. We present the 3-step MIR facet for a four-dimensional simple mixed integer set. Then, we use this facet to generate the 3-step MIR inequality for the feasible set of a general constraint, resulting in the introduction of the 3-step MIR functions.

In chapter 4 we present the complete generalization to n-step MIR for any $n \in \mathbb{N}$. We show that for any positive integer n, n facets of a certain $(n+1)$-dimensional mixed integer set can be obtained. The last of these facets is of special importance and we call it the *n-step MIR facet*. As a result, we generate an infinite number of MIR facets (one for each n). Then, we develop a procedure which, for any n, uses the n-step MIR facet to generate a family of valid inequalities for the feasible set of a general (mixed) integer programming constraint. We refer to these as the *n-step MIR inequalities*. The n-step MIR inequalities are easily produced using closed-form periodic functions, which we call the *n-step MIR functions*. None of these functions dominates the other on its whole period.

In chapter 5, we study the relation of MIR inequalities and facets of Gomory's finite and infinite group polyhedra. We review some of the basic concepts form the group theory and introduce corner polyhedra, and the finite and infinite group problems. Then, we show that n-step MIR inequalities are facet-defining for the finite and infinite group polyhedra. We first prove the n-step MIR function, for any positive integer n, generates two-slope facets for the infinite group polyhedron, and then show under appropriate conditions on parameters, these functions also generate facets for the finite master cyclic group problem. We discuss that similar results are true for the group problems with continuous variables.

We conclude in chapter 6 by a summary and a discussion of several paths for future research.

1.2 Integer Programming

Integer Programming, a branch of mathematical programming, is a tool for optimization. Integer programming is finding values for a set of variables so as to minimize or maximize an objective function subject to a set of constraints, where at least one of the variables is integer. In this dissertation, we are focused on linear integer programming problems. Many real world problems can be formulated as integer programming problems. Production planning and scheduling, facility location, airline crew scheduling, train scheduling, vehicle routing, cutting stock, network design and telecommunication are a few areas where integer programming problems arise. The integer programming problem in its general form is NP-hard. Extensive effort has been made in recent decades to develop and improve methodologies for solving integer programming problems. In this section, we formally define the integer programming problem and its varieties, reproduce some useful concepts from the polyhedral theory, and briefly review some of the important general methodologies for solving them.

1.2.1 Integer Programming Problems

A **Mixed Integer Programming (MIP) problem** is the problem defined as

$$\begin{aligned} \min \bar{c}_x \bar{x} + \bar{c}_v \bar{v} & \\ \mathbf{A}_x \bar{x} + \mathbf{A}_v \bar{v} \geq \bar{b} & \\ \bar{x}, \bar{v} \geq 0 & \\ \bar{x} \text{ integer}, & \end{aligned} \tag{MIP}$$

where $\bar{x} = (x_1, \ldots, x_n)^T$ and $\bar{v} = (v_1, \ldots, v_p)^T$ are the decision variables. $\bar{c}_x$, $\bar{c}_v$, $\bar{b}$, $\mathbf{A}_x$ and $\mathbf{A}_v$ are the given data. If there are m constraints, these entities are $1 \times n$, $1 \times p$, $m \times 1$, $m \times n$, and $m \times p$ matrices, respectively. A **pure Integer Programming (IP) problem** is an MIP where all variables are integer:

$$\begin{aligned} \min \bar{c}_x \bar{x} & \\ \mathbf{A}_x \bar{x} \geq \bar{b} & \\ \bar{x} \geq 0 & \\ \bar{x} \text{ integer.} & \end{aligned} \tag{IP}$$

A **Binary Integer Programming (BIP) problem** is an MIP where all variables are required to be 0 or 1:

$$\begin{aligned} \min \bar{c}_x \bar{x} & \\ \mathbf{A}_x \bar{x} \geq \bar{b} & \\ \bar{x} \geq 0 & \\ \bar{x} \in \{0,1\}^n. & \end{aligned} \tag{BIP}$$

The **Linear Programming Relaxation (LPR)** of an integer program is the linear programming problem resulted from removing all integrality constraints. The linear programming relaxation of (MIP) is

$$\begin{aligned} \min \bar{c}_x \bar{x} + \bar{c}_v \bar{v} & \\ \mathbf{A}_x \bar{x} + \mathbf{A}_v \bar{v} \geq \bar{b} & \\ \bar{x}, \bar{v} \geq 0. & \end{aligned} \tag{LPR}$$

1.2.2 Polyhedral Definitions

Some fundamental concepts about polyhedral properties of integer programming problems are reproduced in this section, most of them taken from [43].

Definition 1.1. *The **feasible region** of* (MIP)*, P_{MIP} is the set of points $(\bar{x}^T, \bar{v}^T)$ which satisfy its constraints:*

$$P_{MIP} = \{(\bar{x}^T, \bar{v}^T) \in \mathbb{Z}^n \times \mathbb{R}^p : \mathbf{A}_x \bar{x} + \mathbf{A}_v \bar{v} \geq \bar{b}\}.$$

□

Definition 1.2. *A subset of* $\mathbb{R}^n$ *described by a set of linear constraints* $P = \{\bar{x} \in \mathbb{R}^n : \mathbf{A}\bar{x} \geq \bar{b}\}$ *is a* ***polyhedron****.* □

Therefore the feasible region of (LPR) is a polyhedron.

Definition 1.3. *Given a set* $X \subseteq \mathbb{R}^n$*, the* ***convex hull*** *of* X*, is defined as:* $conv(X) = \{\bar{x} \in \mathbb{R}^n : \bar{x} = \sum_{i=1}^{t} \lambda_i \bar{x}^i, \sum_{i=1}^{t} \lambda_i = 1, \lambda_i \geq 0$ *for* $i = 1, \ldots, t$ *over all finite subsets* $\{\bar{x}^1, \ldots, \bar{x}^t\}$ *of* $X\}$. □

Theorem 1.4. $conv(P_{MIP})$ *is a polyhedron if* $\mathbf{A}_x$*,* $\mathbf{A}_v$ *and* $\bar{b}$ *are rational matrices.*

The proof is given in [38].

Definition 1.5. *A* ***valid inequality*** *for polyhedron* $P = \{\bar{x} \in \mathbb{R}^n : \mathbf{A}\bar{x} \geq \bar{b}\}$ *is an inequality*

$$\bar{\pi}\bar{x} \geq \pi_0,$$

which is satisfied by all points $x \in P$*, where* $\bar{\pi}$ *is a* $1 \times n$ *vector.* □

Definition 1.6. *A* ***valid inequality for*** (MIP) *is a valid inequality for the polyhedron* $conv(P_{MIP})$*.* □

Definition 1.7. *The points* $\bar{x}^1, \ldots, \bar{x}^k \in \mathbb{R}^n$ *are* ***affinely independent*** *if the* $k-1$ *directions* $\bar{x}^2 - \bar{x}^1, \ldots, \bar{x}^k - \bar{x}^1$ *are linearly independent, or alternatively the* k *vectors* $(\bar{x}^1, 1), \ldots, (\bar{x}^k, 1) \in \mathbb{R}^{n+1}$ *are linearly independent.* □

Definition 1.8. *If* P *is a polyhedron, the* ***dimension*** *of* P*,* $dim(P)$ *is one less than the maximum number of affinely independent points in* P*.* □

Definition 1.9. $F = \{\bar{x} \in P : \bar{\pi}\bar{x} = \pi_0\}$ *defines a* ***face*** *of polyhedron* P *if* $\bar{\pi}\bar{x} \geq \pi_0$ *is a valid inequality for* P*. In this case, the valid inequality* $\bar{\pi}\bar{x} \geq \pi_0$ *is said to* ***represent*** *or* ***define*** *the face* F*. A face* F *is said to be proper if* $F \neq \emptyset$ *and* $F \neq P$*.* □

Definition 1.10. $F = \{\bar{x} \in P : \bar{\pi}\bar{x} = \pi_0\}$ *is a* ***facet*** *of polyhedron* P *if* F *is a face of* P *and* $\dim(F) = \dim(P) - 1$*. In this case, the valid inequality* $\bar{\pi}\bar{x} \geq \pi_0$ *is said to* ***represent*** *or* ***define*** *the facet* F*.* □

In this dissertation, we use the term *facet* to refer to the facet itself, the facet-defining hyperplane and the facet-defining inequality. The intended meaning will be clear from the context.

1.3 General Methodologies for Solving MIP Problems

In this section, we review some of the important general methodologies for solving (MIP) problems, which are related to the topic of this proposal. These include branch-and-bound, cutting plane method, and branch-and-cut. These methodologies are among the most common ones for solving (MIP). They have been studied substantially in theory and have been applied extensively in practice. They are also *general*, meaning that they do not require any special structure in the problem and hence, they are applicable to the most general form of (MIP).

There are several other significant approaches which are not directly related to this research and will not be discussed here. These include column generation, branch-and-price, constraint programming, Lagrangian duality, semidefinite programming, basis reduction, approximation algorithms, and heuristics such as simulated annealing, TABU search, and genetic algorithms. References on these methodologies can be found in [38] and [43].

1.3.1 Branch-and-Bound

A branch-and-bound algorithm was first used for integer programming in [32]. The algorithm performs an *implicit enumeration* on the feasible region of the MIP. This is done on a tree structure. Each node of this tree represents a relaxed subproblem. The algorithm starts at the root node by solving a relaxation of the problem. This is commonly the linear programming relaxation, i.e. (LPR); however, it can be a Lagrangian relaxation [18] or any other relaxation. Assume (LPR) is used as the relaxation. If the optimal solution to (LPR) satisfies all integrality constraints, it is also optimal for (MIP). If (LPR) is infeasible, (MIP) is also infeasible. If neither condition is true, a *branching* is performed leading to typically two child nodes. For instance, branching can be done on one of the variables $x_1, \ldots, x_n$ which has non-integer value in the optimal solution of the linear programming, say x_k. If the LP optimal value of x_k in the parent node is x_k^*, then the problem in one of the child nodes is obtained by adding the constraint $x_k \leq \lfloor x_k^* \rfloor$ to the LP problem of the parent node, and the problem in the other child node is obtained by adding $x_k \geq \lceil x_k^* \rceil$. The linear programming problems at these nodes are solved and the process is repeated. Different policies exist for choosing the branching variable in a node and also for choosing the next node problem to solve.

At every node, if the optimal solution satisfies the integrality constraints, the corresponding value of the objective function will be an upper bound for the optimal value of (MIP)'s objective function. A node is *fathomed* (will not be branched anymore), if the optimal value of the LP problem on that node is greater than the best upper bound found so far, or if it has an infeasible LP problem, or if its optimal solution satisfies the integrality constraints. The problem is solved when all nodes are fathomed. More details and references can be found in [38] and [43].

In practice, a pure branch-and-bound algorithm for a general (MIP) could result in a huge number of node problems and becomes computationally intractable. This is especially the case when the LP relaxation is not a very tight approximation of the convex hull of the integer solutions. It turns out that this difficulty can be resolved by using strong cutting planes in solving the node problems, which will be discussed in section 1.3.3.

1.3.2 Cutting Plane Method

Cutting plane method was first presented by Gomory in [19]. In this method, we are interested in valid inequalities for (MIP) that are not satisfied by all feasible points to its LP relaxation including its current optimal solution. Such valid inequalities are called *cuts*. There is a notion of strength associated with cuts meaning that the more they cut off from the feasible region of the LP relaxation, the stronger they are. Facets of the convex hull of integer solutions are the strongest possible cuts.

Cuts are added to (LPR) and tighten its feasible region without changing the feasible region of (MIP). Then we can re-solve (LPR) and repeat the process until all integer constraints are satisfied. Gomory [20] and later Chvátal [8] proved that by this procedure a pure integer programming problem can be solved in a finite number of steps. In spite of the initial excitement, it turned out that this approach on its own is not very effective in practice because of the so-called *tailing-off* phenomenon [7]. Tailing-off is the situation where the progress toward the integer solutions obtained by each cut becomes very small. However, when embedded into a branch-and-bound tree, cutting planes play a very important role in accelerating discovery of the optimal solution as discussed in section 1.3.3. Therefore, development of strong cutting planes based on the polyhedral theory has been the subject of a huge volume of research in recent years [33].

There are several methods for generating cuts for an MIP problem. It is important to note that the problem for which a cut is generated may well be a relaxation of the original problem. As discussed in [30] and [33], at a very high level, valid inequalities used to cut the feasible region of (LPR) towards the feasible region of (MIP) can be classified into three categories:

1. **Problem-structure Cuts:** these valid inequalities are typically derived based on the properties of the full problem structure or a substantial part of it. They are usually very strong in that they may come from known classes of facets of the convex hull of feasible solutions. Their

application is limited to the particular problem class.

Special valid inequalities for the Traveling Salesman Problem (TSP) [28] and the set packing problem [40] are examples of this type of valid inequalities.

2. **Simple-structure Cuts:** These inequalities are derived based on simple polyhedral structures obtained from a substantial relaxation of the problem. Therefore, they can at best only separate fractional points that are infeasible to the convex hull of the relaxation. However, these inequalities are frequently facets of the convex hull of the relaxation.

   Knapsack relaxations with binary variables were used to derive *cover inequalities* and *lifted* cover inequalities [10]. *Flow cover inequalities* are derived from a system consisting of a flow balance equation which can be obtained from relaxation of a mixed integer programming constraint with binary variables [41]. Also, cuts obtained using single-constraint *lifting* belong to this category [3, 4].

3. **No-structure Cuts:** these are valid inequalities which are based only on variables being integral or binary in the problem, without exploiting any special structure of the problem. Therefore, these cuts can be generated for any (MIP), regardless of its structure. Gomory's fractional cuts [19, 21], Gomory mixed integer cuts (GMIC) [20], disjunctive cuts [5], and split cuts [9] are in this category and are based only on the integrality of variables. Lift-and-project cuts from this category are based only on variables being binary and can be used for any (BIP) [6].

   Mixed Integer Rounding (MIR) inequalities [34, 38, 43] and valid inequalities based on facets of group polyhedra, which are the core concepts in this dissertation, are also in the no-structure category and can be used to generate cuts for any general (MIP). MIR inequalities are obtained for a general MIP constraint using facets of simple mixed integer sets. The well-known Gomory mixed integer cut (GMIC) is obtainable using a facet of a two-dimensional mixed integer set. As mentioned at the beginning of this chapter, a primary contribution of this research is generalization of MIR inequalities. As a result, their background and theory will be carefully studied in chapter 2 in order to pave the way for our generalization.

1.3.3 Branch-and-Cut

Branch-and-cut is a branch-and-bound algorithm in which cutting planes are generated in the node problems throughout the branch-and-bound tree. After solving the (LPR) corresponding to a certain node, if the node is not fathomed based on the (LPR) solution, we try to add cuts. If one or more cuts are found, they are added to the formulation and the new (LPR) is solved again. If no cuts are found, we branch. The philosophy is to do as much work as necessary to get a tight bound at the node rather than branch and re-optimize quickly.

As intended today, branch-and-cut was first introduced by Padberg and Rinaldi [42]. Branch-and-cut combines the advantages of both branch-and-bound and cutting planes and overcomes the problems associated with each of those approaches. The cutting planes provide a tight approximation of the convex hull of integer solutions and whenever the tailing-off starts to happen branching creates new node problems.

Branch-and-cut is now one of the most widespread and successful tools for solving mixed integer programs. A large number of references on branch-and-cut can be found in [7]. There are several excellent surveys on different aspects of branch-and-cut such as [31, 16, 35, 37]. Most of today's commercial and non-commercial MIP solvers use branch-and-cut algorithms.

The effectiveness of the branch-and-cut approach heavily depends on the quality of the cuts produced. This fact is the major motivation for the research in the area of cutting planes. Moreover, strong cuts not only help branch-and-cut algorithms run faster but also can be used to improve the formulation of the problem regardless of the algorithm used to solve it.

Chapter 2

Mixed Integer Rounding

Mixed Integer Rounding (MIR) is an approach to generating valid inequalities for the feasible set of a general (MIP) or (IP). In MIR valid inequalities are derived using facets of *simple* polyhedral sets, which we refer to as *MIR facets*. MIR facets of two- and three-dimensional sets with only one defining inequality (other than the sign restrictions) have been used to generate MIR inequalities for the feasible set of a general MIP constraint. Gomory mixed integer cut [20] for a general MIP or IP constraint can be derived using MIR facet of a two-dimensional simple mixed integer set.

MIR was first proposed by Nemhauser and Wolsey in [38]. Later in [39], they proved that all facets of any 0-1 mixed integer polyhedron can be described using MIR. Marchand and Wolsey in [34] showed that MIR inequalities can be used to derive several strong simple-structure cuts for MIP problems. Günlük and Pochet [29] proposed the *mixing* of MIR inequalities to derive new strong valid inequalities. Recently, Dash and Günlük [11] presented the extension to the *2-step MIR*. They developed an MIR facet of a three-dimensional simple mixed integer set, called the *2-step MIR* facet, and used it to derive the 2-step MIR valid inequalities for a general MIP constraint. The computational effectiveness of MIR inequalities has been justified in several works [34, 29, 12, 13, 14] and they are being used in many MIP solvers today.

In this chapter, we present the MIR approach as developed before this work. This includes the 1-step and 2-step MIR. The 1-step MIR is discussed in [38], [43], and [34] and the 2-step MIR was introduced by Dash and Günlük in [11]. Our treatment of the 1-step and 2-step MIR is based on the work in these references, but it is novel in many aspects. We redevelop the concepts in a new format. In particular, our development is for more general sets, uses a new consistent notation, and follows a new systematic structure for proofs. It also presents a new and more general form of MIR functions and introduces some important properties of this new form. All these new aspects will then allow us to generalize the MIR approach in an intact framework in chapters 3 and 4.

In section 2.1, we develop the 1-step MIR facet of a two-dimensional simple mixed integer set. In section 2.2, we use this facet to derive the 1-step MIR inequality for a general IP constraint and we introduce our form of the 1-step MIR functions. We also prove some of the important properties of our 1-step MIR functions. A special case will be GMIC. In section 2.3, we present the development of the 2-step MIR facet for a three-dimensional simple mixed integer set. Finally, in section 2.4, we use the 2-step MIR facet to derive the 2-step MIR inequality for a general IP constraint and we introduce our form of the 2-step MIR functions. Again we prove some of the important properties of our 2-step MIR functions. Throughout this chapter, all such derivations will be for general IP constraints. These derivations can be easily extended to the general MIP constraints as discussed in section 4.3.

2.1 1-step MIR Facet for a Two-dimensional Simple Set

In this section, we study the inequality which is conventionally known as the MIR facet. The set for which this facet is derived is the simplest form of a mixed integer set, i.e.

$$Q^{\alpha_1,\beta} = \{(y_1, v) \in \mathbb{Z} \times \mathbb{R}_+ : \alpha_1 y_1 + v \geq \beta\},$$

where $\alpha_1, \beta \in \mathbb{R}, \alpha_1 > 0$. In the literature, the traditional MIR facet for this set is usually derived for the case $\alpha_1 = 1$; see [11, 43] for example. However, here, we use the more general set $Q^{\alpha_1,\beta}$ for future use. Also, since we are going to introduce other MIR facets, we call the traditional MIR facet the *1-step MIR facet* to make it distinct from other types.

The set $Q^{\alpha_1,\beta}$ is displayed in Figure 2.1. This set has a trivial facet, i.e. $v \geq 0$. The following theorem gives its other facet.

Theorem 2.1. *Let $\alpha_1, \beta \in \mathbb{R}, \alpha_1 > 0$. If $\beta/\alpha_1 < \lceil \beta/\alpha_1 \rceil$, then inequality*

$$y_1 + \frac{1}{\beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor} v \geq \lceil \beta/\alpha_1 \rceil. \tag{2.1}$$

is valid and facet-defining for $Q^{\alpha_1,\beta}$.

Proof. If we subtract $\alpha_1 \lfloor \beta/\alpha_1 \rfloor$ from both sides of the defining inequality of $Q^{\alpha_1,\beta}$, $\alpha_1 y_1 + v \geq \beta$, we get

$$\alpha_1 (y_1 - \lfloor \beta/\alpha_1 \rfloor) + v \geq \beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor. \tag{2.2}$$

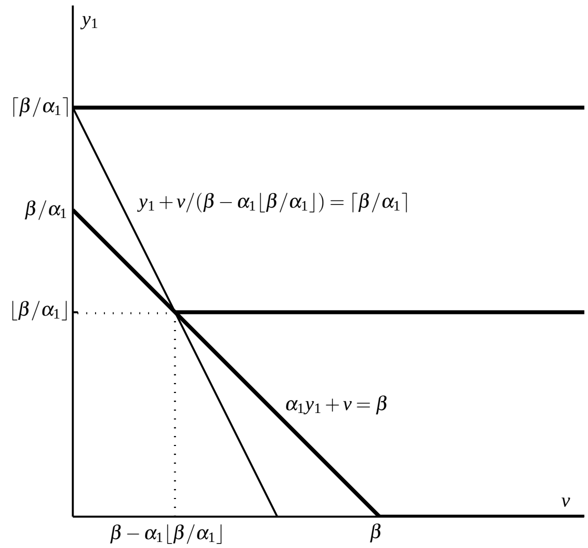


Figure 2.1: 1-step MIR facet for $Q^{\alpha_1,\beta}$

Now

$$(\beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor)(y_1 - \lfloor \beta/\alpha_1 \rfloor) + v \geq \beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor . \tag{2.3}$$

is valid for $Q^{\alpha_1,\beta}$. To see this, note that since $y_1 - \lfloor \beta/\alpha_1 \rfloor \in \mathbb{Z}$, either $y_1 - \lfloor \beta/\alpha_1 \rfloor \geq 1$ or $y_1 - \lfloor \beta/\alpha_1 \rfloor \leq 0$. If $y_1 - \lfloor \beta/\alpha_1 \rfloor \geq 1$, then (2.3) is obvious because $\alpha_1 > 0$ and $v \geq 0$. On the other hand, if $y_1 - \lfloor \beta/\alpha_1 \rfloor \leq 0$, (2.3) is a relaxation of (2.2) because $\beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor < \alpha_1$. Therefore in both cases, (2.3) is valid for $Q^{\alpha_1,\beta}$. Inequality (2.1) is easily obtained from (2.3) by some algebraic manipulation. Notice that the affinely independent points $q_1^1 = (\lfloor \beta/\alpha_1 \rfloor, \beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor)$ and $p_1^1 = (\lceil \beta/\alpha_1 \rceil, 0)$ are in $Q^{\alpha_1,\beta}$ and lie on the line $y_1 + v/\beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor = \lceil \beta/\alpha_1 \rceil$. Therefore (2.1) is also facet-defining for $Q^{\alpha_1,\beta}$. $\square$

Based on theorem 2.1, we define the 1-step MIR facet for $Q^{\alpha_1,\beta}$:

Definition 2.2. *Let* $\alpha_1, \beta \in \mathbb{R}, \alpha_1 > 0$ *such that* $\beta/\alpha_1 < \lceil \beta/\alpha_1 \rceil$. *Then facet* (2.1) *is called the* ***1-step MIR facet*** *for* $Q^{\alpha_1,\beta}$. $\square$

The 1-step MIR facet of $Q^{\alpha_1,\beta}$ is shown in Figure 2.1. One can observe that the 1-step MIR facet is obtained by folding the line $\alpha_1 y_1 + v = \beta$ at point $(\lfloor \beta/\alpha_1 \rfloor, \beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor)$ and passing it through the integer $\lceil \beta/\alpha_1 \rceil$ instead of β/α_1 on the y_1-axis.

2.2 1-step MIR Inequality for a General IP Constraint

The 1-step MIR facet (2.1) can be used to generate valid inequalities for the feasible set of a general IP constraint. We define this set as follows:

Definition 2.3. *Let $J = \{1,\dots,N_J\}$, where $N_J \in \mathbb{N}$. For any $\alpha_1, b \in \mathbb{R}$ and $\overline{a} = (a_1,\dots,a_{N_J}) \in \mathbb{R}^{N_J}$, the set $Y_{\overline{a}}^{\alpha_1,b}$ is defined as*

$$Y_{\overline{a}}^{\alpha_1,b} = \left\{ (x_1,\dots,x_{N_J}) \in \mathbb{Z}_+^{N_J} : \sum_{j\in J} a_j x_j + \alpha_1 z = b, z \in \mathbb{Z} \right\}.$$

□

We define an *MIR procedure* as follows:

Definition 2.4. *Any mathematical procedure which uses the MIR facet of a simple polyhedron P_1 to generate valid inequalities for a more complex polyhedron P_2 is called an **MIR procedure**. The valid inequalities for P_2 resulted from an MIR procedure are called MIR inequalities for P_2.* □

The set $Y_{\overline{a}}^{\alpha_1,b}$ can be a relaxation of an arbitrary equation for which we are interested to develop a valid inequality. In particular, where $\alpha_1 = 1$, it represents the relaxation of a row of the optimal simplex tableau of an (IP) problem in which $\overline{x} = (x_1,\dots,x_{N_J})^T$ is the vector of non-basic variables with values equal to zero. The MIR inequalities we are going to generate for $Y_{\overline{a}}^{\alpha_1,b}$ have the form $\overline{\pi}\overline{x} \geq 1$, where $\overline{\pi} = (\pi_1,\dots,\pi_{N_J})$. Thus they are violated by the current basic solution, and, hence, can be added to the problem as cutting planes.

The 1-step MIR inequality for $Y_{\overline{a}}^{\alpha_1,b}$ can be expressed in terms of the *1-step MIR function*. We first define 1-step MIR function and prove its properties. Then in Theorem 2.8, we present the MIR procedure leading to this function. This MIR procedure derives the 1-step MIR inequality for $Y_{\overline{a}}^{\alpha_1,b}$ based on the 1-step MIR facet of $Q^{\alpha_1,\beta}$. Our 1-step MIR function is defined in a new organized format and for the more general set $Y_{\overline{a}}^{\alpha_1,b}$ and, hence, has new special properties which will be discussed.

Definition 2.5. *Let $\alpha_1, b \in \mathbb{R}$, $\alpha_1 > 0$. If $b/\alpha_1 < \lceil b/\alpha_1 \rceil$, **the positive 1-step MIR function** for the right hand side b with parameter α_1, $g_+^{\alpha_1,b}(u)$, is defined as follows*

$$g_+^{\alpha_1,b}(u) = \frac{\alpha_1 \delta^{\alpha_1,b}(u) - (u - \alpha_1 \lfloor u/\alpha_1 \rfloor)}{\alpha_1 - (b - \alpha_1 \lfloor b/\alpha_1 \rfloor)} \tag{2.4}$$

where

$$\delta^{\alpha_1,b}(u) = \begin{cases} 1 & \text{if } u \in I_0^{\alpha_1,b} \\ \dfrac{u-\alpha_1\lfloor u/\alpha_1\rfloor}{b-\alpha_1\lfloor b/\alpha_1\rfloor} & \text{if } u \in I_1^{\alpha_1,b} \end{cases}$$

and

$$\begin{aligned} I_0^{\alpha_1,b} &= \{u \in \mathbb{R} : u-\alpha_1\lfloor u/\alpha_1\rfloor \geq b-\alpha_1\lfloor b/\alpha_1\rfloor\} \\ I_1^{\alpha_1,b} &= \{u \in \mathbb{R} : u-\alpha_1\lfloor u/\alpha_1\rfloor < b-\alpha_1\lfloor b/\alpha_1\rfloor\}. \end{aligned}$$

*Also if $-b/\alpha_1 < \lceil -b/\alpha_1 \rceil$, **the negative 1-step MIR function** for the right hand side b with parameter α_1, $g_-^{\alpha_1,b}(u)$, is defined as follows*

$$g_-^{\alpha_1,b}(u) = g_+^{\alpha_1,-b}(-u). \tag{2.5}$$

□

The 1-step MIR function as defined above has some nice periodic properties stated in Theorem 2.6. As we will see, the MIR functions that we define later have similar properties too.

Theorem 2.6. *Let $\alpha_1, b \in \mathbb{R}$, $\alpha_1 > 0$. If $b/\alpha_1 < \lceil b/\alpha_1 \rceil$, the following statements are true for any $k_b, k_u \in \mathbb{Z}$:*

(i). $g_+^{\alpha_1,b}(u) = g_+^{\alpha_1,k_b\alpha_1+b}(k_u\alpha_1+u)$,

(ii). $g_-^{\alpha_1,b}(u) = g_-^{\alpha_1,k_b\alpha_1+b}(k_u\alpha_1+u)$ *and* $g_-^{\alpha_1,b}(u) = g_+^{\alpha_1,k_b\alpha_1-b}(k_u\alpha_1-u)$.

Proof. *(i).* We have

$$\begin{aligned} k_u\alpha_1+u-\alpha_1\lfloor (k_u\alpha_1+u)/\alpha_1\rfloor &= k_u\alpha_1+u-k_u\alpha_1-\alpha_1\lfloor u/\alpha_1\rfloor \\ &= u-\alpha_1\lfloor u/\alpha_1\rfloor. \end{aligned} \tag{2.6}$$

Similarly

$$k_b\alpha_1+b-\alpha_1\lfloor (k_b\alpha_1+b)/\alpha_1\rfloor = b-\alpha_1\lfloor b/\alpha_1\rfloor. \tag{2.7}$$

From (2.4), it can be seen that $g_+^{\alpha_1,b}(u)$ is a function of only $u-\alpha_1\lfloor u/\alpha_1\rfloor$ and $b-\alpha_1\lfloor b/\alpha_1\rfloor$, which themselves are periodic with period α_1 according to (2.6) and (2.7). Therefore $g_+^{\alpha_1,b}(u) =$

$g_+^{\alpha_1,k_b\alpha_1+b}(k_u\alpha_1+u)$.

(ii). We have

$$g_-^{\alpha_1,b}(u)=g_+^{\alpha_1,-b}(-u)=g_+^{\alpha_1,-k_b\alpha_1-b}(-k_u\alpha_1-u)=g_-^{\alpha_1,k_b\alpha_1+b}(k_u\alpha_1+u),$$

and

$$g_-^{\alpha_1,b}(u)=g_+^{\alpha_1,-b}(-u)=g_+^{\alpha_1,k_b\alpha_1-b}(k_u\alpha_1-u).$$

□

Theorem 2.6 implies that the 1-step MIR functions $g_+^{\alpha_1,b}(u)$ and $g_-^{\alpha_1,b}(u)$ are periodic in b and u with period α_1.

For the 1-step MIR functions, the positive and negative versions of the function are identical. This is not the case for higher-step MIR functions which will be introduced later.

Theorem 2.7. *Let $\alpha_1, b\in\mathbb{R}$, $\alpha_1>0$ and $b/\alpha_1<\lceil b/\alpha_1\rceil$. Then $g_+^{\alpha_1,b}(u)=g_-^{\alpha_1,b}(u)$.*

Proof. Let $I_e^{\alpha_1,b}=\{u\in I_0^{\alpha_1,b}: u-\alpha_1\lfloor u/\alpha_1\rfloor=b-\alpha_1\lfloor b/\alpha_1\rfloor\}$. There are three possibilities for u: $u\in I_e^{\alpha_1,b}$ or $u\in I_0^{\alpha_1,b}-I_e^{\alpha_1,b}$ or $u\in I_1^{\alpha_1,b}$. We show the identity for each case:

Case 1 ($u\in I_e^{\alpha_1,b}$)**:** If $u\in I_e^{\alpha_1,b}$, then from (2.4), we have $g_+^{\alpha_1,b}(u)=1$. We show that if $u\in I_e^{\alpha_1,b}$ then $-u\in I_e^{\alpha_1,-b}$. $u\in I_e^{\alpha_1,b}$ means

$$u-\alpha_1\lfloor u/\alpha_1\rfloor=b-\alpha_1\lfloor b/\alpha_1\rfloor,$$

or

$$-u+\alpha_1\lfloor u/\alpha_1\rfloor=-b+\alpha_1\lfloor b/\alpha_1\rfloor.$$

This is equivalent to

$$-u+\alpha_1\lceil u/\alpha_1\rceil=-b+\alpha_1\lceil b/\alpha_1\rceil.$$

But we have $\lceil u/\alpha_1\rceil=-\lfloor -u/\alpha_1\rfloor$. This is true not only for u/α_1 but for any number. Therefore the identity above can be written as

$$-u-\alpha_1\lfloor -u/\alpha_1\rfloor=-b-\alpha_1\lfloor -b/\alpha_1\rfloor,$$

which means $-u\in I_e^{\alpha_1,-b}$. Therefore $g_+^{\alpha_1,-b}(-u)=1$; that is $g_-^{\alpha_1,b}(u)=1$, which means $g_+^{\alpha_1,b}(u)=g_-^{\alpha_1,b}(u)$, since we had $g_+^{\alpha_1,b}(u)=1$.

Case 2 ($u \in I_0^{\alpha_1,b} - I_e^{\alpha_1,b}$)**:** Very similar to the argument above, one can prove $u \in I_0^{\alpha_1,b} - I_e^{\alpha_1,b}$ if and only if $-u \in I_1^{\alpha_1,-b}$. Now assume $u \in I_0^{\alpha_1,b} - I_e^{\alpha_1,b}$, then from (2.4), we have

$$
\begin{aligned}
g_+^{\alpha_1,b}(u) &= \frac{\alpha_1 - (u - \alpha_1 \lfloor u/\alpha_1 \rfloor)}{\alpha_1 - (b - \alpha_1 \lfloor b/\alpha_1 \rfloor)} \\
&= \frac{-u - \alpha_1 \lfloor -u/\alpha_1 \rfloor}{-b - \alpha_1 \lfloor -b/\alpha_1 \rfloor} \\
&= \frac{(-u - \alpha_1 \lfloor -u/\alpha_1 \rfloor)(\alpha_1 - (-b - \alpha_1 \lfloor -b/\alpha_1 \rfloor))}{(-b - \alpha_1 \lfloor -b/\alpha_1 \rfloor)(\alpha_1 - (-b - \alpha_1 \lfloor -b/\alpha_1 \rfloor))} \\
&= \frac{\alpha_1 \dfrac{-u - \alpha_1 \lfloor -u/\alpha_1 \rfloor}{-b - \alpha_1 \lfloor -b/\alpha_1 \rfloor} - (-u - \alpha_1 \lfloor -u/\alpha_1 \rfloor)}{\alpha_1 - (-b - \alpha_1 \lfloor -b/\alpha_1 \rfloor)} \\
&= \frac{\alpha_1 \delta^{\alpha_1,-b}(-u) - (-u - \alpha_1 \lfloor -u/\alpha_1 \rfloor)}{\alpha_1 - (-b - \alpha_1 \lfloor -b/\alpha_1 \rfloor)} \\
&= g_+^{\alpha_1,-b}(-u) \\
&= g_-^{\alpha_1,b}(u).
\end{aligned}
$$

The second to last identity is true because $-u \in I_1^{\alpha_1,-b}$.

Case 3 ($u \in I_1^{\alpha_1,b}$)**:** By symmetry, the reverse of the argument in case 2 proves the identity in this case. □

Based on Theorem 2.7, we will use $g_+^{\alpha_1,b}(u)$ in presenting the remaining material about the 1-step MIR inequalities. Everywhere in the discussion, $g_+^{\alpha_1,b}(u)$ can be replaced by $g_-^{\alpha_1,b}(u)$.

Theorem 2.8. *Let $\alpha_1, b \in \mathbb{R}$ and $\alpha_1 > 0$. If $b/\alpha_1 < \lceil b/\alpha_1 \rceil$,* ***the positive (or negative) 1-step MIR inequality***

$$\sum_{j \in J} g_+^{\alpha_1,b}(a_j) x_j \geq 1 \tag{2.8}$$

is valid for $Y_{\overline{a}}^{\alpha_1,b}$.

Proof. We start from the defining equality of $Y_{\overline{a}}^{\alpha_1,b}$, i.e.

$$\sum_{j \in J} a_j x_j + \alpha_1 z = b. \tag{2.9}$$

Assume that J_0 and J_1 form a partition of J. We can relax (2.9) as

$$\alpha_1 z + \sum_{j \in J_0} \alpha_1 \left\lceil \frac{a_j}{\alpha_1} \right\rceil x_j + \sum_{j \in J_1} \left[\alpha_1 \left\lfloor \frac{a_j}{\alpha_1} \right\rfloor + a_j - \alpha_1 \left\lfloor \frac{a_j}{\alpha_1} \right\rfloor \right] x_j \geq b.$$

This is a relaxation because, for $j \in J_0$, the coefficient a_j has been replaced by $\alpha_1 \lceil a_j/\alpha_1 \rceil$ and, obviously, $\alpha_1 \lceil a_j/\alpha_1 \rceil \geq a_j$. For $j \in J_1$, nothing has changed and only the constant $\alpha_1 \lfloor a_j/\alpha_1 \rfloor$ is added to and subtracted from a_j. The equation above can be written as

$$\alpha_1 \left[z + \sum_{j \in J_0} \left\lceil \frac{a_j}{\alpha_1} \right\rceil x_j + \sum_{j \in J_1} \left\lfloor \frac{a_j}{\alpha_1} \right\rfloor x_j \right] + \left[\sum_{j \in J_1} \left(a_j - \alpha_1 \left\lfloor \frac{a_j}{\alpha_1} \right\rfloor \right) x_j \right] \geq b \tag{2.10}$$

The expressions in the first and second brackets in (2.10) can play the roles of y_1 and v in the defining inequality of $Q^{\alpha_1,b}$, respectively. The sign and integrality conditions match. The first expression is an integer and the second one is a non-negative real number. Since it is assumed $b/\alpha_1 < \lceil b/\alpha_1 \rceil$, if y_1 and v in the 1-step MIR facet for $Q^{\alpha_1,b}$ are replaced with the corresponding expressions in (2.10), by Theorem 2.1, the inequality obtained will be valid for $Y_{\overline{a}}^{\alpha_1,b}$. If we do so, we get

$$z + \sum_{j \in J_0} \left\lceil \frac{a_j}{\alpha_1} \right\rceil x_j + \sum_{j \in J_1} \left\lfloor \frac{a_j}{\alpha_1} \right\rfloor x_j + \frac{1}{b - \alpha_1 \lfloor b/\alpha_1 \rfloor} \sum_{j \in J_1} \left(a_j - \alpha_1 \left\lfloor \frac{a_j}{\alpha_1} \right\rfloor \right) x_j \geq \left\lceil \frac{b}{\alpha_1} \right\rceil.$$

Now if we multiply both sides by α_1 and substitute for $\alpha_1 z$ from (2.9), we get

$$b - \sum_{j \in J} a_j x_j + \alpha_1 \left[\sum_{j \in J_0} \left\lceil \frac{a_j}{\alpha_1} \right\rceil x_j + \sum_{j \in J_1} \left\lfloor \frac{a_j}{\alpha_1} \right\rfloor x_j \right.$$
$$\left. + \frac{1}{b - \alpha_1 \lfloor b/\alpha_1 \rfloor} \sum_{j \in J_1} \left(a_j - \alpha_1 \left\lfloor \frac{a_j}{\alpha_1} \right\rfloor \right) x_j \right] \geq \alpha_1 \left\lceil \frac{b}{\alpha_1} \right\rceil.$$

After rearranging the terms, this inequality can be written as

$$\sum_{j \in J} \left[\alpha_1 \delta^{\alpha_1,b}(a_j) - (a_j - \alpha_1 \lfloor a_j/\alpha_1 \rfloor) \right] x_j \geq \alpha_1 - (b - \alpha_1 \lfloor b/\alpha_1 \rfloor), \tag{2.11}$$

where

$$\delta^{\alpha_1,b}(a_j) = \begin{cases} 1 & \text{if } a_j \in J_0 \\ \dfrac{a_j - \alpha_1 \lfloor a_j/\alpha_1 \rfloor}{b - \alpha_1 \lfloor b/\alpha_1 \rfloor} & \text{if } a_j \in J_1. \end{cases}$$

To obtain the strongest inequality the coefficients of x_j's should be minimized. In other words, the partitioning of J into J_0 and J_1 should be determined such that $\delta^{\alpha_1,b}(a_j)$ gets the minimum of the two values above. It is easy to see that the partitioning should be as

$$\begin{aligned} J_0 &= \{ j \in J : a_j - \alpha_1 \lfloor a_j/\alpha_1 \rfloor \geq b - \alpha_1 \lfloor b/\alpha_1 \rfloor \} \\ J_1 &= \{ j \in J : a_j - \alpha_1 \lfloor a_j/\alpha_1 \rfloor < b - \alpha_1 \lfloor b/\alpha_1 \rfloor \}. \end{aligned}$$

This partitioning along with (2.11) and Definition 2.5 gives (2.8). Thus (2.8) is valid for $Y_{\overline{a}}^{\alpha_1,b}$. □

Among all possibilities for the set $Y_{\overline{a}}^{\alpha_1,b}$, the set $Y_{\overline{a}}^{1,b}$ is of special interest because the group polyhedra, discussed in chapter 5, can be expressed in this form, and as we will see, valid inequalities for the group polyhedra are of particular interest. The following corollary to Theorem 2.8 is about MIR inequalities for $Y_{\overline{a}}^{1,b}$.

Corollary 2.9. *Let $t \in \mathbb{N}$, $b \in \mathbb{R}$, $tb < \lceil tb \rceil$ and $Y_{\overline{a}}^{1,b} \neq \emptyset$. Then the positive (or negative) 1-step MIR inequality*

$$\sum_{j \in J} g_+^{1/t,b}(a_j)x_j \geq 1 \tag{2.12}$$

is valid for $Y_{\overline{a}}^{1,b}$.

Proof. By Definition 2.3, $(x_1,\ldots,x_{N_J}) \in Y_{\overline{a}}^{1,b}$ means $b - \sum_{j\in J} a_j x_j \in \mathbb{Z}$. Also we have $\mathbb{Z} \in \frac{1}{t}\mathbb{Z}$, for $t \in \mathbb{N}$. Therefore $b - \sum_{j\in J} a_j x_j \in \frac{1}{t}\mathbb{Z}$, which means $b - \sum_{j\in J} a_j x_j = \frac{1}{t}z$ for some $z \in \mathbb{Z}$. Thus $(x_1,\ldots,x_{N_J}) \in Y_{\overline{a}}^{1/t,b}$. This means $Y_{\overline{a}}^{1,b} \subseteq Y_{\overline{a}}^{1/t,b}$ or $Y_{\overline{a}}^{1/t,b}$ is a relaxation of $Y_{\overline{a}}^{1,b}$, where $t \in \mathbb{N}$. Therefore any valid inequality for $Y_{\overline{a}}^{1/t,b}$ is also valid for $Y_{\overline{a}}^{1,b}$. By Theorem 2.8, (2.12) is valid for $Y_{\overline{a}}^{1/t,b}$ if $tb < \lceil tb \rceil$. Hence it is valid for $Y_{\overline{a}}^{1,b}$ if the same condition holds true. □

The Gomory mixed integer cut (GMIC) is a well-known valid inequality for $Y_{\overline{a}}^{1,b}$. The GMIC is, in fact, the 1-step MIR inequality for $Y_{\overline{a}}^{1,b}$ with parameter $\alpha_1 = 1$. Below, we first define the GMIC and then show this fact.

Definition 2.10. *Let $b < \lceil b \rceil$. The **Gomory mixed integer cut** for $Y_{\overline{a}}^{1,b}$ is the following inequality:*

$$\sum_{j \in J_0} \frac{1-(a_j - \lfloor a_j \rfloor)}{1-(b-\lfloor b \rfloor)} x_j + \sum_{j\in J_1} \frac{a_j - \lfloor a_j \rfloor}{b - \lfloor b \rfloor} x_j \geq 1, \tag{2.13}$$

where

$$\begin{aligned} J_0 &= \{j \in J : a_j - \lfloor a_j \rfloor \geq b - \lfloor b \rfloor\} \\ J_1 &= \{j \in J : a_j - \lfloor a_j \rfloor < b - \lfloor b \rfloor\}. \end{aligned}$$

□

Theorem 2.11. *Let $b < \lceil b \rceil$. The GMIC is valid for $Y_{\overline{a}}^{1,b}$.*

Proof. From Definition 2.5, we have

$$g_+^{1,b}(u) = \frac{\delta^1(u) - (u - \lfloor u \rfloor)}{1-(b-\lfloor b \rfloor)}$$

where

$$\delta^1(u) = \begin{cases} 1 & \text{if } u \in I_0^{1,b} \\ \dfrac{u - \lfloor u \rfloor}{b - \lfloor b \rfloor} & \text{if } u \in I_1^{1,b} \end{cases}$$

and

$$I_0^{1,b} = \{u \in \mathbb{R} : u - \lfloor u \rfloor \geq b - \lfloor b \rfloor\} \quad (2.14)$$

$$I_1^{1,b} = \{u \in \mathbb{R} : u - \lfloor u \rfloor < b - \lfloor b \rfloor\}. \quad (2.15)$$

By substituting the value of $\delta^1(u)$, we get

$$g_+^{1,b}(u) = \begin{cases} \dfrac{1 - (u - \lfloor u \rfloor)}{1 - (b - \lfloor b \rfloor)} & \text{if } u \in I_0^{1,b} \\ \dfrac{u - \lfloor u \rfloor}{b - \lfloor b \rfloor} & \text{if } u \in I_1^{1,b}. \end{cases} \quad (2.16)$$

Now, since $b < \lceil b \rceil$, by corollary 2.9,

$$\sum_{j \in J} g_+^{1,b}(a_j) x_j \geq 1$$

is valid for $Y_{\overline{a}}^{1,b}$. Using 2.16, 2.14 and 2.15, this is exactly the GMIC for $Y_{\overline{a}}^{1,b}$. □

Example 2.12. Figures 2.2 and 2.3 display the graphs of MIR functions $g_+^{1,k_1+0.8}(u)$ and $g_+^{0.25,k_2+0.8}(u)$ for $u \in [0,1]$, respectively. As an example, using these two functions, we can generate 1-step MIR inequalities for the set

$$\left\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{Z}_+^5 : 0.1x_1 + 0.28x_2 + 0.45x_3 + 0.6x_4 + 0.95x_5 + z = 2.8; z \in \mathbb{Z}\right\}$$

If we use $g_+^{1,2.8}(u)$, the valid inequality would be the GMIC. We have $g_+^{1,2.8}(0.1) = 0.125$, $g_+^{1,2.8}(0.28) = 0.35$, $g_+^{1,2.8}(0.45) = 0.5625$, $g_+^{1,2.8}(0.6) = 0.75$ and $g_+^{1,2.8}(0.95) = 0.25$. Therefore, by Theorem 2.11, the following inequality is valid for the above set:

$$0.125x_1 + 0.35x_2 + 0.5625x_3 + 0.75x_4 + 0.25x_5 \geq 1.$$

We can also use $g_+^{0.25,2.8}(u)$. We have $g_+^{0.25,2.8}(0.1) = 0.75$, $g_+^{0.25,2.8}(0.28) = 0.6$, $g_+^{0.25,2.8}(0.45) = 0.25$, $g_+^{0.25,2.8}(0.6) = 0.75$ and $g_+^{0.25,2.8}(0.95) = 0.25$. Therefore, by corollary 2.9, the following inequality is valid for the set above:

$$0.75x_1 + 0.6x_2 + 0.25x_3 + 0.75x_4 + 0.25x_5 \geq 1.$$

□

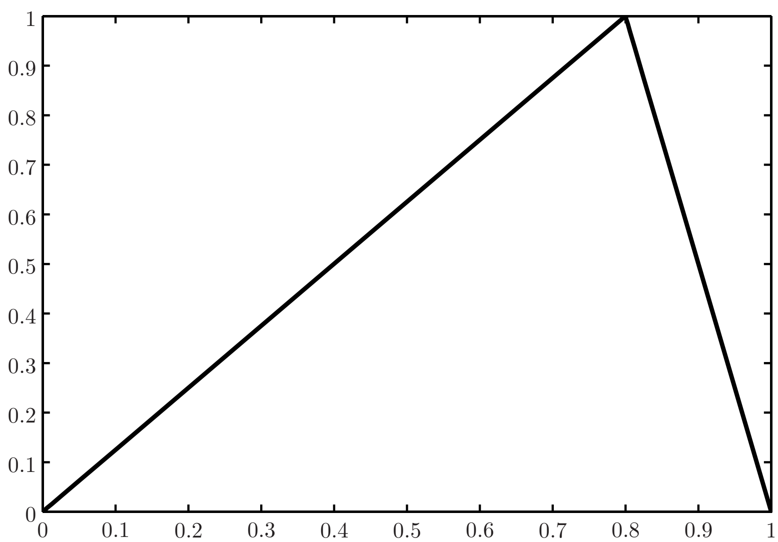


Figure 2.2: Graph of $g_+^{1,k_1+0.8}(u)$ for $u \in [0,1]$

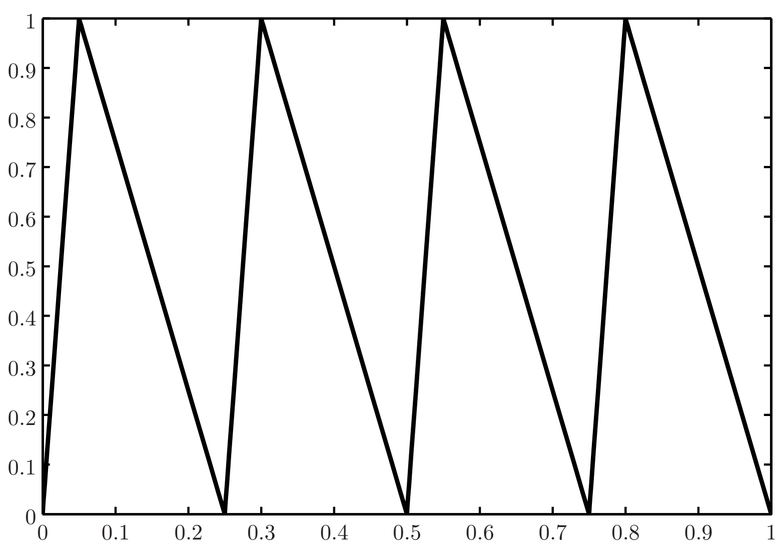


Figure 2.3: Graph of $g_+^{0.25,k_2+0.8}(u)$ for $u \in [0,1]$

2.3 2-step MIR Facet for a Three-dimensional Simple Set

Dash and Günlük in [11] extended the idea of MIR facets to a simple mixed integer set of one dimension higher. They developed the 2-step MIR facet for a three-dimensional simple mixed integer set. Then they used this facet to generate 2-step MIR inequalities for a general IP constraint. In this section, we redevelop the 2-step MIR facet of the three-dimensional simple mixed integer set. Similar to the previous section, our treatment is for a more general form of these sets and follows the systematic path we started at the 1-step MIR inequalities. Later we will continue this path to the generalization of MIR inequalities to higher dimensions. As a result, compared with the development in [11], our development is different in some aspects in order for it to be appropriate for a smooth generalization.

The three-dimensional simple mixed integer set that we are interested in its facets is

$$Q^{(\alpha_1,\alpha_2),\beta} = \{(y_1,y_2,v) \in \mathbb{Z}\times\mathbb{Z}_+\times\mathbb{R}_+ : \alpha_1 y_1 + \alpha_2 y_2 + v \geq \beta\},$$

where $\alpha_1,\alpha_2,\beta \in \mathbb{R}, \alpha_1,\alpha_2 > 0$. It can be observed that the dimension of $Q^{(\alpha_1,\alpha_2),\beta}$ is one more that that of $Q^{\alpha_1,\beta}$ by introduction of the non-negative integer variable y_2. The special case of this set which Dash and Günlük [11] consider is when $\alpha_1 = 1$ and $0 < \alpha_2 < \beta < 1$. As mentioned earlier, we do not limit ourselves to this special case; hence our results are generalization of the results in [11] to the set $Q^{(\alpha_1,\alpha_2),\beta}$.

Before we address the facets of $Q^{(\alpha_1,\alpha_2),\beta}$, it is useful if we limit ourselves to a restriction of this set where $y_1 \geq \lfloor \beta/\alpha_1 \rfloor$, i.e.

$$Q_0^{(\alpha_1,\alpha_2),\beta} = \{(y_1,y_2,v) \in Q^{(\alpha_1,\alpha_2),\beta} : y_1 \geq \lfloor \beta/\alpha_1 \rfloor\}.$$

Lemma 2.13 about the facets of $Q_0^{(\alpha_1,\alpha_2),\beta}$ is a generalization of Lemma 3 in [11].

Lemma 2.13. *Let* $\alpha_1,\alpha_2,\beta \in \mathbb{R}$, $\alpha_1,\alpha_2 > 0$ *such that* $\beta/\alpha_1 < \lceil \beta/\alpha_1 \rceil$ *and* $\frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} < \left\lceil \frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} \right\rceil$. *Inequalities*

$$y_1 + \frac{\alpha_2}{\beta - \alpha_1\lfloor\beta/\alpha_1\rfloor} y_2 + \frac{1}{\beta - \alpha_1\lfloor\beta/\alpha_1\rfloor} v \geq \lceil \beta/\alpha_1 \rceil, \tag{2.17}$$

and

$$\left\lceil \frac{\beta - \alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} \right\rceil y_1 + y_2 + \frac{1}{\beta - \alpha_1\lfloor\beta/\alpha_1\rfloor - \alpha_2\left\lfloor \frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} \right\rfloor} v \geq \left\lceil \frac{\beta}{\alpha_1} \right\rceil \left\lceil \frac{\beta - \alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} \right\rceil \tag{2.18}$$

are valid for $Q_0^{(\alpha_1,\alpha_2),\beta}$. *Inequality* (2.18) *is facet-defining and inequality* (2.17) *is facet-defining if* $\left\lfloor \frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} \right\rfloor \geq 1$.

Proof. We start with the inequality

$$\alpha_1 y_1 + \alpha_2 y_2 + v \geq \beta. \tag{2.19}$$

In $Q_0^{(\alpha_1,\alpha_2),\beta}$, we have $y_1 \in \mathbb{Z}$ and $\alpha_1 y_2 + v \in \mathbb{R}_+$. Therefore it is possible to treat y_1 and $\alpha_1 y_2 + v$ in $Q_0^{(\alpha_1,\alpha_2),\beta}$ as y_1 and v in $Q^{\alpha_1,\beta}$, respectively. Therefore, by Theorem 2.1, if we replace v in the 1-step MIR facet (2.1) with $\alpha_1 y_2 + v$ the resulting inequality is valid for $Q_0^{(\alpha_1,\alpha_2),\beta}$. This is exactly inequality (2.17).

To show that (2.17) is a facet, we need to find three affinely independent points in $Q_0^{(\alpha_1,\alpha_2),\beta}$ which satisfy (2.17) at equality. The points $q_1^2 = (\lfloor\beta/\alpha_1\rfloor, 0, \beta - \alpha_1\lfloor\beta/\alpha_1\rfloor)$, $q_2^2 = (\lfloor\beta/\alpha_1\rfloor, \left\lfloor \frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} \right\rfloor, \beta - \alpha_1\lfloor\beta/\alpha_1\rfloor - \alpha_2\left\lfloor \frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} \right\rfloor)$, and $p_1^2 = (\lceil\beta/\alpha_1\rceil, 0, 0)$ are all in $Q_0^{(\alpha_1,\alpha_2),\beta}$ and satisfy (2.17) at equality. If $\left\lfloor \frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} \right\rfloor \geq 1$, these points are distinct and also affinely independent. To see that they are affinely independent, notice that if we set up a 3 by 3 matrix whose rows are p_1^2, q_1^2 and q_2^2, respectively and rearrange the columns as (y_1, v, y_2), we get

$$\begin{bmatrix} \lceil\beta/\alpha_1\rceil & 0 & 0 \\ \lfloor\beta/\alpha_1\rfloor & \beta - \alpha_1\lfloor\beta/\alpha_1\rfloor & 0 \\ \lfloor\beta/\alpha_1\rfloor & \beta - \alpha_1\lfloor\beta/\alpha_1\rfloor - \alpha_2\left\lfloor \frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} \right\rfloor & \left\lfloor \frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} \right\rfloor \end{bmatrix}.$$

Since it is assumed $\beta/\alpha_1 < \lceil\beta/\alpha_1\rceil$, we have $\beta - \alpha_1\lfloor\beta/\alpha_1\rfloor > 0$. It is also assumed that $\left\lfloor \frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} \right\rfloor \geq 1$. Therefore none of the diagonal entries of this lower-triangular matrix can be zero except the first one, i.e. $\lceil\beta/\alpha_1\rceil$. Therefore either the matrix has a full rank or it has a rank of 2 with a zero first row. Both cases imply that the points p_1^2, q_1^2 and q_2^2 are affinely independent. Thus (2.17) is a facet of $Q_0^{(\alpha_1,\alpha_2),\beta}$.

Now by subtracting $\lfloor\beta/\alpha_1\rfloor$ from both sides of (2.17) and multiplying by $\beta - \alpha_1\lfloor\beta/\alpha_1\rfloor$, we get

$$(\beta - \alpha_1\lfloor\beta/\alpha_1\rfloor)(y_1 - \lfloor\beta/\alpha_1\rfloor) + \alpha_2 y_2 + v \geq \beta - \alpha_1\lfloor\beta/\alpha_1\rfloor,$$

which can be written as

$$\alpha_2\left[\frac{\beta - \alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2}(y_1 - \lceil\beta/\alpha_1\rceil + 1) + y_2\right] + v \geq \beta - \alpha_1\lfloor\beta/\alpha_1\rfloor.$$

In $Q_0^{(\alpha_1,\alpha_2),\beta}$ we have $y_1 - \lfloor \beta/\alpha_1 \rfloor \geq 0$ or $y_1 - \lceil \beta/\alpha_1 \rceil + 1 \geq 0$. Therefore the inequality above can be relaxed to

$$\alpha_2 \left[\left\lceil \frac{\beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor}{\alpha_2} \right\rceil (y_1 - \lceil \beta/\alpha_1 \rceil + 1) + y_2 \right] + v \geq \beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor, \qquad (2.20)$$

as $\frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} < \left\lceil \frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} \right\rceil$ and $\alpha_2 > 0$. This inequality can be written as

$$\alpha_2 \left[\left\lceil \frac{\beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor}{\alpha_2} \right\rceil y_1 - \left\lceil \frac{\beta}{\alpha_1} \right\rceil \left\lceil \frac{\beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor}{\alpha_2} \right\rceil + \left\lceil \frac{\beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor}{\alpha_2} \right\rceil + y_2 \right] + v$$
$$\geq \beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor,$$

By subtracting $\alpha_2 \left\lfloor \frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} \right\rfloor$ from both sides, we get

$$\alpha_2 \left[\left\lceil \frac{\beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor}{\alpha_2} \right\rceil y_1 + y_2 - \left\lceil \frac{\beta}{\alpha_1} \right\rceil \left\lceil \frac{\beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor}{\alpha_2} \right\rceil + 1 \right] + v$$
$$\geq \beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor - \alpha_2 \left\lfloor \frac{\beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor}{\alpha_2} \right\rfloor.$$

Now the expression in brackets which is multiplied by α_2 is an integer and v is a non-negative real variable. Therefore they can be treated as y_1 and v in $Q^{\alpha_2,\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor-\alpha_2\left\lfloor\frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2}\right\rfloor}$. By Theorem 2.1, if we write the 1-step MIR facet replacing y_1 and v with their respective expressions, the result will be valid for $Q_0^{(\alpha_1,\alpha_2),\beta}$. Doing so we get

$$\left\lceil \frac{\beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor}{\alpha_2} \right\rceil y_1 + y_2 - \left\lceil \frac{\beta}{\alpha_1} \right\rceil \left\lceil \frac{\beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor}{\alpha_2} \right\rceil + 1 + \frac{1}{\beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor - \alpha_2 \left\lfloor \frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} \right\rfloor} v \geq 1,$$

because $\left\lfloor \frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor-\alpha_2\left\lfloor\frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2}\right\rfloor}{\alpha_2} \right\rfloor = 0$ and $\left\lceil \frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor-\alpha_2\left\lfloor\frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2}\right\rfloor}{\alpha_2} \right\rceil = 1$. Canceling 1's and rearranging the terms yields (2.18). Therefore (2.18) is valid for $Q_0^{(\alpha_1,\alpha_2),\beta}$.

To show that (2.18) is a facet, again we need to find three affinely independent points in $Q^{(\alpha_1,\alpha_2),\beta}$ which satisfy (2.18) at equality. The points q_2^2, p_1^2 and $p_2^2 = \left(\lfloor \beta/\alpha_1 \rfloor, \left\lceil \frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} \right\rceil, 0\right)$ are all in $Q_0^{(\alpha_1,\alpha_2),\beta}$ and satisfy (2.18) at equality. They are also affinely independent. To see this, notice that the matrix whose rows are p_1^2, p_2^2 and q_2^2 is

$$\begin{bmatrix} \lceil \beta/\alpha_1 \rceil & 0 & 0 \\ \lfloor \beta/\alpha_1 \rfloor & \left\lceil \frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} \right\rceil & 0 \\ \lfloor \beta/\alpha_1 \rfloor & \left\lfloor \frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} \right\rfloor & \beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor - \alpha_2 \left\lfloor \frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} \right\rfloor \end{bmatrix}.$$

Since it is assumed $\beta/\alpha_1 < \lceil \beta/\alpha_1 \rceil$, we have $\beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor > 0$ and hence $\left\lceil \frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} \right\rceil > 0$. Since it is assumed that $\frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} < \left\lceil \frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} \right\rceil$, we have $\beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor - \alpha_2 \left\lfloor \frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} \right\rfloor > 0$. Therefore none of the diagonal entries of this lower-triangular matrix can be zero except the first one, i.e. $\lceil \beta/\alpha_1 \rceil$. Therefore either the matrix has a full rank or it has a rank of 2 with a zero first row. Both cases imply that the points q_2^2, p_1^2 and p_2^2 are affinely independent. Thus (2.18) is a facet of $Q_0^{(\alpha_1,\alpha_2),\beta}$. □

Figure 2.4 illustrates the relative positions of the defining plane of $Q_0^{(\alpha_1,\alpha_2),\beta}$ and the planes of inequalities (2.17) and (2.18), i.e. $r_1^2 r_2^2 q_1^2$, $q_1^2 q_2^2 p_1^2$ and $q_2^2 p_1^2 p_2^2$, respectively.

Now if we return to $Q^{(\alpha_1,\alpha_2),\beta}$ by removing the condition $y_1 \geq \lfloor \beta/\alpha_1 \rfloor$, inequality (2.17) will still be a facet for $Q^{(\alpha_1,\alpha_2),\beta}$. However inequality (2.18) will not necessarily be valid and facet-defining for $Q^{(\alpha_1,\alpha_2),\beta}$ unless an extra condition holds. This fact is presented in Theorem 2.14. This theorem is the generalization of Lemma 5 in [11].

Theorem 2.14. *Let* $\alpha_1, \alpha_2, \beta \in \mathbb{R}$, $\alpha_1, \alpha_2 > 0$ *such that* $\beta/\alpha_1 < \lceil \beta/\alpha_1 \rceil$ *and* $\frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} < \left\lceil \frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} \right\rceil$. *The following statements are true:*

- *Inequality* (2.17) *is valid for* $Q^{(\alpha_1,\alpha_2),\beta}$ *and it is facet-defining if* $\left\lfloor \frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} \right\rfloor \geq 1$.
- *Inequality* (2.18) *is valid and facet-defining for* $Q^{(\alpha_1,\alpha_2),\beta}$ *if* $\left\lceil \frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} \right\rceil \leq \frac{\alpha_1}{\alpha_2}$.

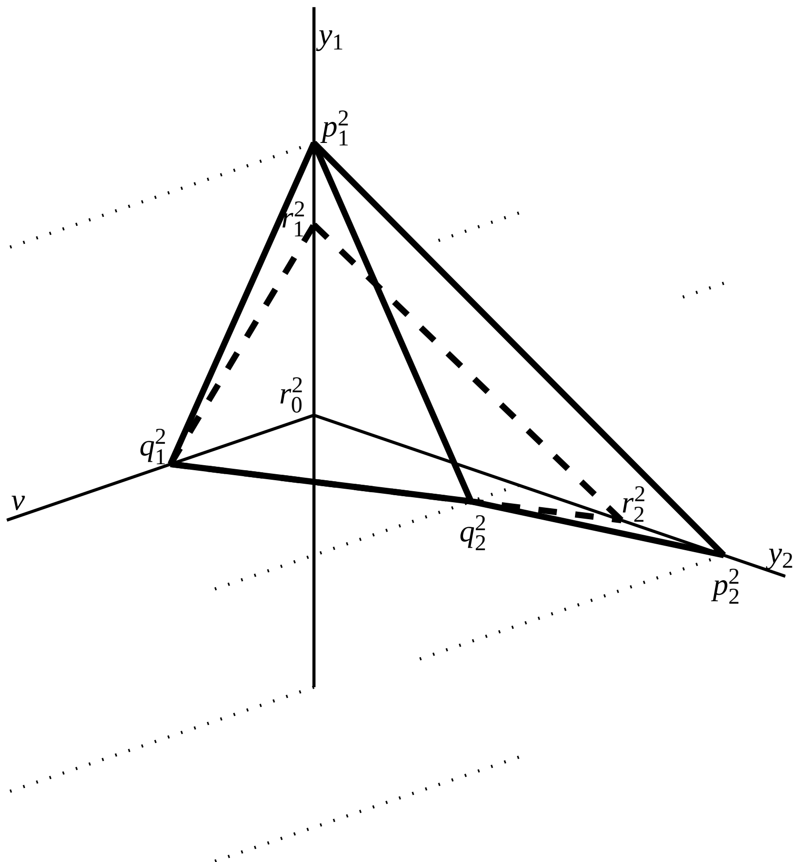
Proof. The first statement is true by exactly the same argument as in the proof of Lemma 2.13, because that argument does not require the extra condition which holds for $Q_0^{(\alpha_1,\alpha_2),\beta}$, i.e. $y_1 \geq \lfloor \beta/\alpha_1 \rfloor$.

This is not the case for the second statement. For inequality (2.18), the relaxation leading to (2.20) can not be done without the condition $y_1 \geq \lfloor \beta/\alpha_1 \rfloor$. However, we show that (2.20) is still valid for $Q^{(\alpha_1,\alpha_2),\beta}$ if $\left\lceil \frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} \right\rceil \leq \frac{\alpha_1}{\alpha_2}$. Inequality (2.17), which is valid for $Q^{(\alpha_1,\alpha_2),\beta}$, can be written as

$$\frac{\beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor}{\alpha_2} \alpha_2 (y_1 - \lceil \beta/\alpha_1 \rceil + 1) + \alpha_2 y_2 + v \geq \beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor, \tag{2.21}$$

as we saw in Lemma 2.13. On the other hand, if we subtract $\alpha_1 \lfloor \beta/\alpha_1 \rfloor$ from both sides of inequality (2.19), we get

$$\alpha_1 (y_1 - \lfloor \beta/\alpha_1 \rfloor) + \alpha_2 y_2 + v \geq \beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor,$$



$$r_0^2 = (\lfloor \beta/\alpha_1 \rfloor, 0, 0)$$
$$r_1^2 = (\beta/\alpha_1, 0, 0)$$
$$r_2^2 = (\lfloor \beta/\alpha_1 \rfloor, \frac{\beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor}{\alpha_2}, 0)$$
$$q_1^2 = (\lfloor \beta/\alpha_1 \rfloor, 0, \beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor)$$
$$q_2^2 = (\lfloor \beta/\alpha_1 \rfloor, \lfloor \frac{\beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor}{\alpha_2} \rfloor, \beta - \lfloor \beta/\alpha_1 \rfloor - \alpha_2 \lfloor \frac{\beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor}{\alpha_2} \rfloor)$$
$$p_1^2 = (\lceil \beta/\alpha_1 \rceil, 0, 0)$$
$$p_2^2 = (\lfloor \beta/\alpha_1 \rfloor, \lceil \frac{\beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor}{\alpha_2} \rceil, 0)$$

Figure 2.4: Facets of $Q_0^{(\alpha_1,\alpha_2),\beta}$

which can be written as

$$\frac{\alpha_1}{\alpha_2}\alpha_2(y_1 - \lceil \beta/\alpha_1 \rceil + 1) + \alpha_2 y_2 + v \geq \beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor . \tag{2.22}$$

Thus the inequality

$$\gamma\alpha_2(y_1 - \lceil \beta/\alpha_1 \rceil + 1) + \alpha_2 y_2 + v \geq \beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor$$

is valid for any $\frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} \leq \gamma \leq \frac{\alpha_1}{\alpha_2}$, as it can be stated as a convex combination of the two valid inequalities (2.21) and (2.22). Since we assumed $\frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} < \left\lceil \frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} \right\rceil \leq \frac{\alpha_1}{\alpha_2}$, we can set $\gamma = \left\lceil \frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} \right\rceil$, which results in inequality (2.20). The rest is exactly like the argument in the proof of Lemma 2.13. The argument about being facet is also exactly the same. □

Based on Theorem 2.14, we define the 2-step MIR facet for $Q^{(\alpha_1,\alpha_2),\beta}$:

Definition 2.15. *Let* $\alpha_1, \alpha_2, \beta \in \mathbb{R}$, $\alpha_1, \alpha_2 > 0$ *such that* $\beta/\alpha_1 < \lceil \beta/\alpha_1 \rceil$ *and* $\frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} < \left\lceil \frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} \right\rceil \leq \frac{\alpha_1}{\alpha_2}$. *Then facet* (2.18) *is called the* ***2-step MIR facet*** *for* $Q^{(\alpha_1,\alpha_2),\beta}$. □

Example 2.16. Let $\alpha_1 = 1$ and $\alpha_2 = 0.3$ and $\beta = 2.8$. Then we have $Q^{(\alpha_1,\alpha_2),\beta} = Q^{(1,0.3),2.8} = \{(y_1, y_2, v) \in \mathbb{Z} \times \mathbb{Z}_+ \times \mathbb{R}_+ : y_1 + 0.3y_2 + v \geq 2.8\}$. Notice that $\beta/\alpha_1 = 2.8 < 3 = \lceil \beta/\alpha_1 \rceil$, $\beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor / \alpha_2 = 8/3 < 3 = \left\lceil \frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} \right\rceil$. Since $\left\lfloor \frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} \right\rfloor = 2 \geq 1$, by Theorem 2.14, inequality (2.17) will give a facet of $Q^{(1,0.3),2.8}$, which is

$$y_1 + 0.375y_2 + 1.25v \geq 3.$$

We observe that the points $q_1^2 = (2,0,0.8)$, $q_2^2 = (2,2,0.2)$ and $p_1^2 = (3,0,0)$ are three affinely independent points in $Q^{(1,0.3),2.8}$ which are on the plane of this facet. Also since $\left\lceil \frac{\beta-\alpha_1\lfloor\beta/\alpha_1\rfloor}{\alpha_2} \right\rceil = 3 \leq \frac{10}{3} = \frac{\alpha_1}{\alpha_2}$, again by Theorem 2.14, inequality (2.18) will give the 2-step MIR facet of $Q^{(1,0.3),2.8}$, which is

$$3y_1 + y_2 + 5v \geq 9.$$

The points $q_2^2 = (2,2,0.2)$, $p_1^2 = (3,0,0)$ and $p_2^2 = (2,3,0)$ are three affinely independent points in $Q^{(1,0.3),2.8}$ which are on the plane of this facet. □

2.4 2-step MIR Inequality for a General IP Constraint

The 2-step MIR inequality of $Q^{(\alpha_1,\alpha_2),\beta}$, i.e. (2.18), can be used to generate valid inequalities for the set $Y_{\overline{a}}^{\alpha_1,b}$. Dash and Günlük in [11] have introduced an MIR procedure which does this for the special case where $\alpha_1 = 1$ and $0 < \alpha_2 < \beta < 1$. Here, we present an MIR procedure which uses the general form of $Q^{(\alpha_1,\alpha_2),\beta}$ and generates the 2-step MIR inequality for the more general set $Y_{\overline{a}}^{\alpha_1,b}$. This procedure results in the 2-step MIR function which is the tool for generating 2-step MIR inequalities for $Y_{\overline{a}}^{\alpha_1,b}$. By setting $\alpha_1 = 1$ or, in general $\alpha_1 = 1/t$ for $t \in \mathbb{N}$, the 2-step MIR inequalities for $Y_{\overline{a}}^{1,b}$ are generated. Similar to 1-step MIR functions of section 2.2, our 2-step MIR function is defined in a new organized format and for the more general set $Y_{\overline{a}}^{\alpha_1,b}$ and, hence, has new special properties which will be discussed.

We first define the 2-step MIR function and prove some of its properties. Then, in Theorem 2.19, we present the MIR procedure leading to these functions. This MIR procedure derives the 2-step MIR inequalities for $Y_{\overline{a}}^{\alpha_1,b}$ based on the 2-step MIR facet of $Q^{(\alpha_1,\alpha_2),\beta}$.

Definition 2.17. *Let $\alpha_1,\alpha_2,b \in \mathbb{R}$, $\alpha_1,\alpha_2 > 0$. If $b/\alpha_1 < \lceil b/\alpha_1 \rceil$ and $\frac{b-\alpha_1\lfloor b/\alpha_1\rfloor}{\alpha_2} < \left\lceil \frac{b-\alpha_1\lfloor b/\alpha_1\rfloor}{\alpha_2} \right\rceil \leq \frac{\alpha_1}{\alpha_2}$, **the positive 2-step MIR function** for the right hand side b with the parameters α_1 and α_2, $g_+^{(\alpha_1,\alpha_2),b}(u)$, is defined as follows*

$$g_+^{(\alpha_1,\alpha_2),b}(u) = \frac{\alpha_1 \delta^{(\alpha_1,\alpha_2),b}(u) - (u - \alpha_1 \lfloor u/\alpha_1 \rfloor) \left\lceil \frac{b-\alpha_1\lfloor b/\alpha_1\rfloor}{\alpha_2} \right\rceil}{[\alpha_1 - (b - \alpha_1 \lfloor b/\alpha_1 \rfloor)] \left\lceil \frac{b-\alpha_1\lfloor b/\alpha_1\rfloor}{\alpha_2} \right\rceil} \tag{2.23}$$

where

$$\delta^{(\alpha_1,\alpha_2),b}(u) = \begin{cases} \left\lceil \frac{b-\alpha_1\lfloor b/\alpha_1\rfloor}{\alpha_2} \right\rceil & \text{if } u \in I_0^{(\alpha_1,\alpha_2),b} \\ \left\lceil \frac{u-\alpha_1\lfloor u/\alpha_1\rfloor}{\alpha_2} \right\rceil & \text{if } u \in I_1^{(\alpha_1,\alpha_2),b} \\ \left\lfloor \frac{u-\alpha_1\lfloor u/\alpha_1\rfloor}{\alpha_2} \right\rfloor + \frac{u-\alpha_1\lfloor u/\alpha_1\rfloor - \alpha_2 \left\lfloor \frac{u-\alpha_1\lfloor u/\alpha_1\rfloor}{\alpha_2} \right\rfloor}{b-\alpha_1\lfloor b/\alpha_1\rfloor - \alpha_2 \left\lfloor \frac{b-\alpha_1\lfloor b/\alpha_1\rfloor}{\alpha_2} \right\rfloor} & \text{if } u \in I_2^{(\alpha_1,\alpha_2),b} \end{cases}$$

and

$$\begin{aligned} I_0^{(\alpha_1,\alpha_2),b} &= \left\{ u \in \mathbb{R} : u - \alpha_1 \left\lfloor \frac{u}{\alpha_1} \right\rfloor \geq b - \alpha_1 \left\lfloor \frac{b}{\alpha_1} \right\rfloor \right\} \\ I_1^{(\alpha_1,\alpha_2),b} &= \left\{ u \in \mathbb{R} : u - \alpha_1 \left\lfloor \frac{u}{\alpha_1} \right\rfloor < b - \alpha_1 \left\lfloor \frac{b}{\alpha_1} \right\rfloor, \right. \\ & \qquad \left. u - \alpha_1 \lfloor u/\alpha_1 \rfloor - \alpha_2 \left\lfloor \frac{u - \alpha_1 \lfloor u/\alpha_1 \rfloor}{\alpha_2} \right\rfloor \geq b - \alpha_1 \lfloor b/\alpha_1 \rfloor - \alpha_2 \left\lfloor \frac{b - \alpha_1 \lfloor b/\alpha_1 \rfloor}{\alpha_2} \right\rfloor \right\} \end{aligned}$$

$$I_2^{(\alpha_1,\alpha_2),b} = \Bigg\{ u \in \mathbb{R} : u - \alpha_1 \left\lfloor \frac{u}{\alpha_1} \right\rfloor < b - \alpha_1 \left\lfloor \frac{b}{\alpha_1} \right\rfloor,$$
$$u - \alpha_1 \lfloor u/\alpha_1 \rfloor - \alpha_2 \left\lfloor \frac{u - \alpha_1 \lfloor u/\alpha_1 \rfloor}{\alpha_2} \right\rfloor < b - \alpha_1 \lfloor b/\alpha_1 \rfloor - \alpha_2 \left\lfloor \frac{b - \alpha_1 \lfloor b/\alpha_1 \rfloor}{\alpha_2} \right\rfloor \Bigg\}.$$

Also if $-b/\alpha_1 < \lceil -b/\alpha_1 \rceil$, $\frac{-b-\alpha_1 \lfloor -b/\alpha_1 \rfloor}{\alpha_2} < \left\lceil \frac{-b-\alpha_1 \lfloor -b/\alpha_1 \rfloor}{\alpha_2} \right\rceil \leq \frac{\alpha_1}{\alpha_2}$, ***the negative 2-step MIR function*** *for the right hand side b with the parameters* α_1 *and* α_2, $g_-^{(\alpha_1,\alpha_2),b}(u)$, *is defined as follows*

$$g_-^{(\alpha_1,\alpha_2),b}(u) = g_+^{(\alpha_1,\alpha_2),-b}(-u). \tag{2.24}$$

□

Like 1-step MIR functions, the 2-step MIR functions are also periodic in b and u with period α_1. This fact is stated in Theorem 2.18.

Theorem 2.18. *Let* $\alpha_1, \alpha_2, b \in \mathbb{R}$, $\alpha_1, \alpha_2 > 0$. *The following statements are true for any* $k_b, k_u \in \mathbb{Z}$:

(i). if $b/\alpha_1 < \lceil b/\alpha_1 \rceil$ *and* $\frac{b-\alpha_1 \lfloor b/\alpha_1 \rfloor}{\alpha_2} < \left\lceil \frac{b-\alpha_1 \lfloor b/\alpha_1 \rfloor}{\alpha_2} \right\rceil \leq \frac{\alpha_1}{\alpha_2}$, *then* $g_+^{(\alpha_1,\alpha_2),b}(u) = g_+^{(\alpha_1,\alpha_2),k_b\alpha_1+b}(k_u\alpha_1 + u)$,

(ii). if $-b/\alpha_1 < \lceil -b/\alpha_1 \rceil$ *and* $\frac{-b-\alpha_1 \lfloor -b/\alpha_1 \rfloor}{\alpha_2} < \left\lceil \frac{-b-\alpha_1 \lfloor -b/\alpha_1 \rfloor}{\alpha_2} \right\rceil \leq \frac{\alpha_1}{\alpha_2}$, *then* $g_-^{(\alpha_1,\alpha_2),b}(u) = g_-^{(\alpha_1,\alpha_2),k_b\alpha_1+b}(k_u\alpha_1 + u)$ *and* $g_-^{(\alpha_1,\alpha_2),b}(u) = g_+^{(\alpha_1,\alpha_2),k_b\alpha_1-b}(k_u\alpha_1 - u)$.

Proof. From (2.23), it can be seen that $g_+^{(\alpha_1,\alpha_2),b}(u)$ is a function of only $u - \alpha_1 \lfloor u/\alpha_1 \rfloor$ and $b - \alpha_1 \lfloor b/\alpha_1 \rfloor$, which are periodic with period α_1 as shown in the proof of Theorem 2.6. Therefore *(i)* is true. The argument for *(ii)* is very similar to the argument for *(ii)* in the proof of Theorem 2.6. □

Theorem 2.19 presents the 2-step MIR inequality:

Theorem 2.19. *Let* $\alpha_1, \alpha_2, b \in \mathbb{R}$, $\alpha_1, \alpha_2 > 0$. *The* ***positive 2-step MIR inequality***

$$\sum_{j \in J} g_+^{(\alpha_1,\alpha_2),b}(a_j) x_j \geq 1 \tag{2.25}$$

is valid for $Y_{\overline{a}}^{\alpha_1,b}$ *if* $b/\alpha_1 < \lceil b/\alpha_1 \rceil$ *and* $\frac{b-\alpha_1 \lfloor b/\alpha_1 \rfloor}{\alpha_2} < \left\lceil \frac{b-\alpha_1 \lfloor b/\alpha_1 \rfloor}{\alpha_2} \right\rceil \leq \frac{\alpha_1}{\alpha_2}$, *and the* ***negative 2-step MIR inequality***

$$\sum_{j \in J} g_-^{(\alpha_1,\alpha_2),b}(a_j) x_j \geq 1 \tag{2.26}$$

is valid for $Y_{\overline{a}}^{\alpha_1,b}$ *if* $-b/\alpha_1 < \lceil -b/\alpha_1 \rceil$ *and* $\frac{-b-\alpha_1 \lfloor -b/\alpha_1 \rfloor}{\alpha_2} < \left\lceil \frac{-b-\alpha_1 \lfloor -b/\alpha_1 \rfloor}{\alpha_2} \right\rceil \leq \frac{\alpha_1}{\alpha_2}$.

Proof. First we prove (2.25) assuming the required conditions hold. Similar to the proof of Theorem 2.8, we start from equation (2.9), i.e.

$$\sum_{j\in J} a_j x_j + \alpha_1 z = b.$$

Assume that J_0, J_1 and J_2 form a partition of J. We can relax (2.9) as

$$\alpha_1 z + \sum_{j\in J_0} \alpha_1 \left\lceil \frac{a_j}{\alpha_1} \right\rceil x_j + \sum_{j\in J_1} \left[\alpha_1 \left\lfloor \frac{a_j}{\alpha_1} \right\rfloor + \alpha_2 \left\lceil \frac{a_j - \alpha_1 \lfloor a_j/\alpha_1 \rfloor}{\alpha_2} \right\rceil \right] x_j$$
$$+ \sum_{j\in J_2} \left[\alpha_1 \left\lfloor \frac{a_j}{\alpha_1} \right\rfloor + \alpha_2 \left\lfloor \frac{a_j - \alpha_1 \lfloor a_j/\alpha_1 \rfloor}{\alpha_2} \right\rfloor + a_j - \alpha_1 \left\lfloor \frac{a_j}{\alpha_1} \right\rfloor - \alpha_2 \left\lfloor \frac{a_j - \alpha_1 \lfloor a_j/\alpha_1 \rfloor}{\alpha_2} \right\rfloor \right] x_j \geq b$$

This is a relaxation because, for $j \in J_0$, the coefficient a_j has been replaced by $\alpha_1 \lceil a_j/\alpha_1 \rceil$ and we have $\alpha_1 \lceil a_j/\alpha_1 \rceil \geq a_j$. Also, for $j \in J_1$, a_j has been replaced by $\alpha_1 \lfloor a_j/\alpha_1 \rfloor + \alpha_2 \left\lceil \frac{a_j - \alpha_1 \lfloor a_j/\alpha_1 \rfloor}{\alpha_2} \right\rceil$ and we have $\alpha_1 \lfloor a_j/\alpha_1 \rfloor + \alpha_2 \left\lceil \frac{a_j - \alpha_1 \lfloor a_j/\alpha_1 \rfloor}{\alpha_2} \right\rceil \geq a_j$ because $\alpha_2 \left\lceil \frac{a_j - \alpha_1 \lfloor a_j/\alpha_1 \rfloor}{\alpha_2} \right\rceil \geq a_j - \alpha_1 \lfloor a_j/\alpha_1 \rfloor$. For $j \in J_2$, nothing has changed and only the constant term $\alpha_1 \left\lfloor \frac{a_j}{\alpha_1} \right\rfloor + \alpha_2 \left\lfloor \frac{a_j - \alpha_1 \lfloor a_j/\alpha_1 \rfloor}{\alpha_2} \right\rfloor$ is added to and subtracted from a_j. The equation above can be written as

$$\alpha_1 \left[z + \sum_{j\in J_0} \left\lceil \frac{a_j}{\alpha_1} \right\rceil x_j + \sum_{j\in J_1} \left\lfloor \frac{a_j}{\alpha_1} \right\rfloor x_j + \sum_{j\in J_2} \left\lfloor \frac{a_j}{\alpha_1} \right\rfloor x_j \right]$$
$$+ \alpha_2 \left[\sum_{j\in J_1} \left\lceil \frac{a_j - \alpha_1 \lfloor a_j/\alpha_1 \rfloor}{\alpha_2} \right\rceil x_j + \sum_{j\in J_2} \left\lfloor \frac{a_j - \alpha_1 \lfloor a_j/\alpha_1 \rfloor}{\alpha_2} \right\rfloor x_j \right]$$
$$+ \left[\sum_{j\in J_2} \left[a_j - \alpha_1 \left\lfloor \frac{a_j}{\alpha_1} \right\rfloor - \alpha_2 \left\lfloor \frac{a_j - \alpha_1 \lfloor a_j/\alpha_1 \rfloor}{\alpha_2} \right\rfloor \right] x_j \right] \geq b \quad (2.27)$$

The expressions in the first, second and third brackets in (2.27) can play the roles of y_1, y_2 and v in the defining inequality of $Q^{(\alpha_1,\alpha_2),b}$, respectively. The sign and integrality conditions match. The first expression is an integer, the second one a non-negative integer and the third one a non-negative real number. Since it is assumed $b/\alpha_1 < \lceil b/\alpha_1 \rceil$ and $\frac{-b-\alpha_1\lfloor -b/\alpha_1 \rfloor}{\alpha_2} < \left\lceil \frac{-b-\alpha_1\lfloor -b/\alpha_1 \rfloor}{\alpha_2} \right\rceil \leq \frac{\alpha_1}{\alpha_2}$, if y_1, y_2 and v in the 2-step MIR inequality for $Q^{(\alpha_1,\alpha_2),b}$ are replaced by the corresponding expressions in

(2.27), by Theorem 2.14, the inequality obtained will be valid for $Y_{\overline{a}}^{\alpha_1,b}$. Doing so, we get

$$\left\lceil \frac{b-\alpha_1\lfloor b/\alpha_1\rfloor}{\alpha_2}\right\rceil \left[z+\sum_{j\in J_0}\left\lceil\frac{a_j}{\alpha_1}\right\rceil x_j+\sum_{j\in J_1}\left\lfloor\frac{a_j}{\alpha_1}\right\rfloor x_j+\sum_{j\in J_2}\left\lfloor\frac{a_j}{\alpha_1}\right\rfloor x_j\right]$$
$$+\left[\sum_{j\in J_1}\left\lceil\frac{a_j-\alpha_1\lfloor a_j/\alpha_1\rfloor}{\alpha_2}\right\rceil x_j+\sum_{j\in J_2}\left\lfloor\frac{a_j-\alpha_1\lfloor a_j/\alpha_1\rfloor}{\alpha_2}\right\rfloor x_j\right]$$
$$+\frac{1}{b-\alpha_1\lfloor b/\alpha_1\rfloor-\alpha_2\left\lfloor\frac{b-\alpha_1\lfloor b/\alpha_1\rfloor}{\alpha_2}\right\rfloor}\left[\sum_{j\in J_2}\left[a_j-\alpha_1\left\lfloor\frac{a_j}{\alpha_1}\right\rfloor-\alpha_2\left\lfloor\frac{a_j-\alpha_1\lfloor a_j/\alpha_1\rfloor}{\alpha_2}\right\rfloor\right]x_j\right]$$
$$\geq\left\lceil\frac{b}{\alpha_1}\right\rceil\left\lceil\frac{b-\alpha_1\lfloor b/\alpha_1\rfloor}{\alpha_2}\right\rceil$$

If we multiply both sides by α_1 and replace for $\alpha_1 z$ from (2.9), we get

$$\left\lceil \frac{b-\alpha_1\lfloor b/\alpha_1\rfloor}{\alpha_2}\right\rceil \left[b-\sum_{j\in J}a_jx_j\right]$$
$$+\alpha_1\left\lceil \frac{b-\alpha_1\lfloor b/\alpha_1\rfloor}{\alpha_2}\right\rceil \left[\sum_{j\in J_0}\left\lceil\frac{a_j}{\alpha_1}\right\rceil x_j+\sum_{j\in J_1}\left\lfloor\frac{a_j}{\alpha_1}\right\rfloor x_j+\sum_{j\in J_2}\left\lfloor\frac{a_j}{\alpha_1}\right\rfloor x_j\right]$$
$$+\alpha_1\left[\sum_{j\in J_1}\left\lceil\frac{a_j-\alpha_1\lfloor a_j/\alpha_1\rfloor}{\alpha_2}\right\rceil x_j+\sum_{j\in J_2}\left\lfloor\frac{a_j-\alpha_1\lfloor a_j/\alpha_1\rfloor}{\alpha_2}\right\rfloor x_j\right]$$
$$+\frac{\alpha_1}{b-\alpha_1\lfloor b/\alpha_1\rfloor-\alpha_2\left\lfloor\frac{b-\alpha_1\lfloor b/\alpha_1\rfloor}{\alpha_2}\right\rfloor}\left[\sum_{j\in J_2}\left[a_j-\alpha_1\left\lfloor\frac{a_j}{\alpha_1}\right\rfloor-\alpha_2\left\lfloor\frac{a_j-\alpha_1\lfloor a_j/\alpha_1\rfloor}{\alpha_2}\right\rfloor\right]x_j\right]$$
$$\geq\alpha_1\left\lceil\frac{b}{\alpha_1}\right\rceil\left\lceil\frac{b-\alpha_1\lfloor b/\alpha_1\rfloor}{\alpha_2}\right\rceil.$$

By rearranging some terms, we will have

$$\sum_{j\in J}\left(\alpha_1\delta^{(\alpha_1,\alpha_2),b}(a_j)-\left[a_j-\alpha_1\lfloor a_j/\alpha_1\rfloor\right]\left\lceil\frac{b-\alpha_1\lfloor b/\alpha_1\rfloor}{\alpha_2}\right\rceil\right)x_j$$
$$\geq\left[\alpha_1-(b-\alpha_1\lfloor b/\alpha_1\rfloor)\right]\left\lceil\frac{b-\alpha_1\lfloor b/\alpha_1\rfloor}{\alpha_2}\right\rceil \quad (2.28)$$

where

$$\delta^{(\alpha_1,\alpha_2),b}(a_j)=\begin{cases}\left\lceil\frac{b-\alpha_1\lfloor b/\alpha_1\rfloor}{\alpha_2}\right\rceil & \text{if } j\in J_0\\ \left\lceil\frac{a_j-\alpha_1\lfloor a_j/\alpha_1\rfloor}{\alpha_2}\right\rceil & \text{if } j\in J_1\\ \left\lfloor\frac{a_j-\alpha_1\lfloor a_j/\alpha_1\rfloor}{\alpha_2}\right\rfloor+\frac{a_j-\alpha_1\lfloor a_j/\alpha_1\rfloor-\alpha_2\left\lfloor\frac{a_j-\alpha_1\lfloor a_j/\alpha_1\rfloor}{\alpha_2}\right\rfloor}{b-\alpha_1\lfloor b/\alpha_1\rfloor-\alpha_2\left\lfloor\frac{b-\alpha_1\lfloor b/\alpha_1\rfloor}{\alpha_2}\right\rfloor} & \text{if } j\in J_2\end{cases}$$

To obtain the strongest inequality the coefficients of x_j's should be minimized. In other words, the partitioning of J into J_0, J_1 and J_2 should be determined such that $\delta^{(\alpha_1,\alpha_2),b}(a_j)$ gets the minimum of the three values above. It is not difficult to verify that the partitioning should be as follows

$$J_0 = \left\{ j \in J : a_j - \alpha_1 \left\lfloor \frac{a_j}{\alpha_1} \right\rfloor \geq b - \alpha_1 \left\lfloor \frac{b}{\alpha_1} \right\rfloor \right\}$$

$$J_1 = \left\{ j \in J : a_j - \alpha_1 \left\lfloor \frac{a_j}{\alpha_1} \right\rfloor < b - \alpha_1 \left\lfloor \frac{b}{\alpha_1} \right\rfloor, \right.$$
$$\left. a_j - \alpha_1 \lfloor a_j/\alpha_1 \rfloor - \alpha_2 \left\lfloor \frac{a_j - \alpha_1 \lfloor a_j/\alpha_1 \rfloor}{\alpha_2} \right\rfloor \geq b - \alpha_1 \lfloor b/\alpha_1 \rfloor - \alpha_2 \left\lfloor \frac{b - \alpha_1 \lfloor b/\alpha_1 \rfloor}{\alpha_2} \right\rfloor \right\}$$

$$J_2 = \left\{ j \in J : a_j - \alpha_1 \left\lfloor \frac{a_j}{\alpha_1} \right\rfloor < b - \alpha_1 \left\lfloor \frac{b}{\alpha_1} \right\rfloor, \right.$$
$$\left. a_j - \alpha_1 \lfloor a_j/\alpha_1 \rfloor - \alpha_2 \left\lfloor \frac{a_j - \alpha_1 \lfloor a_j/\alpha_1 \rfloor}{\alpha_2} \right\rfloor < b - \alpha_1 \lfloor b/\alpha_1 \rfloor - \alpha_2 \left\lfloor \frac{b - \alpha_1 \lfloor b/\alpha_1 \rfloor}{\alpha_2} \right\rfloor \right\}.$$

This partitioning along with (2.28) and Definition 2.17 gives (2.25). Thus (2.25) is valid for $Y_{\overline{a}}^{\alpha_1,b}$.

Now to prove validity of (2.26) assuming $-b/\alpha_1 < \lceil -b/\alpha_1 \rceil$ and $\frac{-b-\alpha_1\lfloor -b/\alpha_1 \rfloor}{\alpha_2} < \left\lceil \frac{-b-\alpha_1\lfloor -b/\alpha_1 \rfloor}{\alpha_2} \right\rceil \leq \frac{\alpha_1}{\alpha_2}$, notice that since $z \in \mathbb{Z}$, the set $Y_{\overline{a}}^{\alpha_1,b}$ can also be written as

$$Y_{\overline{a}}^{\alpha_1,b} = \left\{ (x_1,\ldots,x_{N_J}) \in \mathbb{Z}_+^{N_J} : \sum_{j\in J} -a_j x_j + \alpha_1 z = -b, z \in \mathbb{Z} \right\}. \tag{2.29}$$

Therefore the above argument can be repeated exactly as above, this time starting with the alternative defining equation of $Y_{\overline{a}}^{\alpha_1,b}$, i.e.

$$\sum_{j\in J} -a_j x_j + \alpha_1 z = -b.$$

and with a_j and b replaced by $-a_j$ and $-b$. Since $-b/\alpha_1 < \lceil -b/\alpha_1 \rceil$ and $\frac{-b-\alpha_1\lfloor -b/\alpha_1 \rfloor}{\alpha_2} < \left\lceil \frac{-b-\alpha_1\lfloor -b/\alpha_1 \rfloor}{\alpha_2} \right\rceil \leq \frac{\alpha_1}{\alpha_2}$, the argument leads to

$$\sum_{j\in J} g_+^{\alpha_1,-b}(-a_j)x_j \geq 1$$

which, by Definition 2.17, is the same as (2.26). □

As in the case of 1-step MIR inequalities, the special case of $Y_{\overline{a}}^{1,b}$ is of particular interest:

Corollary 2.20. *Let $t \in \mathbb{N}$, $\alpha_2, b \in \mathbb{R}$, $\alpha_2 > 0$. The positive 2-step MIR inequality*

$$\sum_{j\in J} g_+^{(1/t,\alpha_2),b}(a_j)x_j \geq 1 \tag{2.30}$$

is valid for $Y_{\overline{a}}^{1,b}$ *if* $tb < \lceil tb \rceil$ *and* $\frac{b-(1/t)\lfloor tb \rfloor}{\alpha_2} < \left\lceil \frac{b-(1/t)\lfloor tb \rfloor}{\alpha_2} \right\rceil \leq \frac{1}{t\alpha_2}$ *and the negative 2-step MIR inequality*

$$\sum_{j \in J} g_{-}^{(1/t,\alpha_2),b}(a_j)x_j \geq 1 \tag{2.31}$$

is valid for $Y_{\overline{a}}^{1,b}$ *if* $-tb < \lceil -tb \rceil$ *and* $\frac{-b-(1/t)\lfloor -tb \rfloor}{\alpha_2} < \left\lceil \frac{-b-(1/t)\lfloor -tb \rfloor}{\alpha_2} \right\rceil \leq \frac{1}{t\alpha_2}$.

Proof. As argued in the proof of Corollary 2.9, $Y_{\overline{a}}^{1/t,b}$, where $t \in \mathbb{N}$, is a relaxation of $Y_{\overline{a}}^{1,b}$. Therefore any valid inequality for $Y_{\overline{a}}^{1/t,b}$ is also valid for $Y_{\overline{a}}^{1,b}$. By Theorem 2.19, (2.30) and (2.31) are valid for $Y_{\overline{a}}^{1/t,b}$ if the respective conditions are satisfied. Hence they are valid for $Y_{\overline{a}}^{1,b}$ if the same conditions hold true. □

$g_{+}^{(1,\alpha_2),b}$ is the same as the 2-step MIR function developed in [11]. Also $g_{+}^{(1/t,\alpha_2),b}$ and $g_{-}^{(1/t,\alpha_2),b}$ are the same as the 't-scaled and $(-t)$-scaled 2-step MIR functions' in [11]. Note that by using the general form of the set $Q^{(\alpha_1,\alpha_2),\beta}$, in which α_1 can take any positive value, we do not need to define 'scaled' MIR functions separately, as defined in [11]. The concept of scaling is included in the 2-step MIR functions presented here and is determined by α_1, the first parameter of the function.

Example 2.21. Let $\alpha_1 = 1$ and $\alpha_2 = 0.3$ and $b = 0.8$. In Example 2.16, we observed that these values satisfy the conditions of Theorem 2.19 for the positive 2-step MIR inequality. Figures 2.5 and 2.6 show the graphs of $g_{+}^{(1,0.3),k_1+0.8}(u)$ and $g_{-}^{(1,0.3),k_2+0.2}(u)$ for $u \in [0,1]$. These functions are periodic with period $\alpha_1 = 1$. Also observe that $g_{-}^{(1,0.3),k_1+0.2}(u) = g_{+}^{(1,0.3),k_2+0.8}(1-u)$, as stated in Theorem 2.18. By Theorem 2.19, using $g_{+}^{(1,0.3),2.8}(u)$ a 2-step MIR valid inequality for the set

$$\left\{(x_1,x_2,x_3,x_4,x_5) \in \mathbb{Z}_{+}^{5} : 0.1x_1 + 0.28x_2 + 0.45x_3 + 0.6x_4 + 0.95x_5 + z = 2.8; z \in \mathbb{Z}\right\}$$

is

$$g_{+}^{(1,0.3),2.8}(0.1)x_1 + g_{+}^{(1,0.3),2.8}(0.28)x_2 + g_{+}^{(1,0.3),2.8}(0.45)x_3 + g_{+}^{(1,0.3),2.8}(0.6)x_4 + g_{+}^{(1,0.3),2.8}(0.95)x_5 \geq 1,$$

or

$$0.333x_1 + 0.267x_2 + 0.667x_3 + 0.333x_4 + 0.25x_5 \geq 1.$$

Another example for the right hand side $b = k_1 + 0.8$ could be obtained by setting $\alpha_1 = 0.5$ and

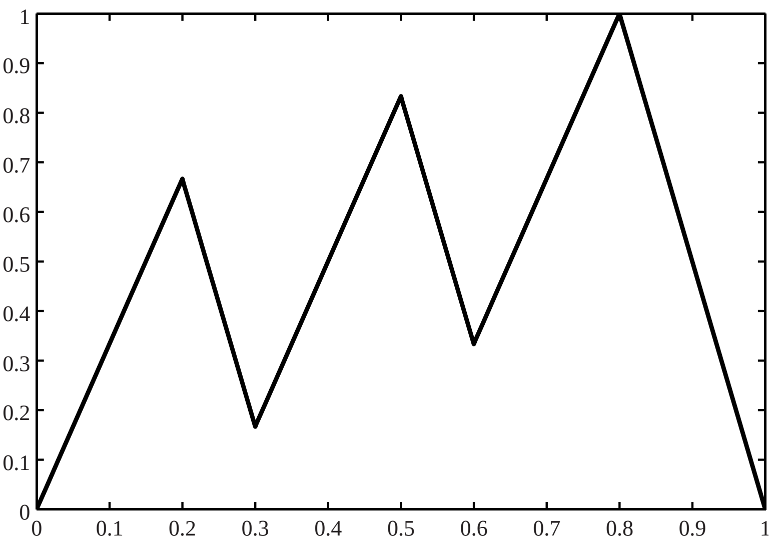


Figure 2.5: Graph of $g_{+}^{(1,0.3),k_1+0.8}(u)$ for $u \in [0,1]$

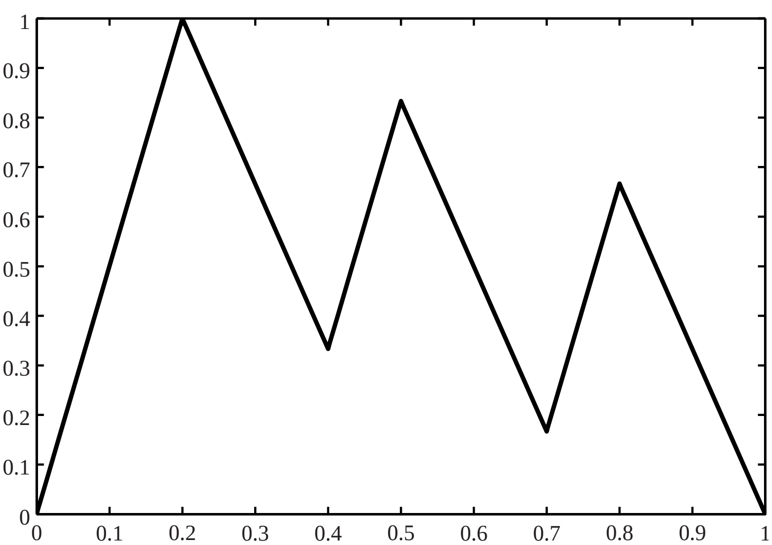


Figure 2.6: Graph of $g_{-}^{(1,0.3),k_2+0.2}(u)$ for $u \in [0,1]$

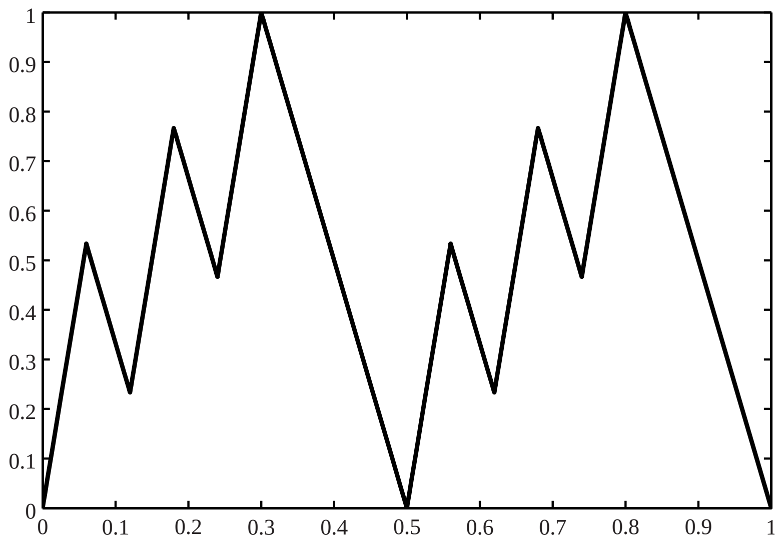


Figure 2.7: Graph of $g_{+}^{(0.5,0.12),k_1+0.8}(u)$ for $u \in [0,1]$

$\alpha_2 = 0.12$. Figure 2.7 shows the graph of $g_{+}^{(0.5,0.12),k_1+0.8}(u)$ for $u \in [0,1]$. The valid inequality obtained for the set above using this 2-step MIR function is

$$g_{+}^{(0.5,0.12),2.8}(0.1)x_1 + g_{+}^{(0.5,0.12),2.8}(0.28)x_2 + g_{+}^{(0.5,0.12),2.8}(0.45)x_3 + \\ g_{+}^{((0.5,0.12),2.8}(0.6)x_4 + g_{+}^{(0.5,0.12),2.8}(0.95)x_5 \geq 1,$$

or

$$0.333x_1 + 0.822x_2 + 0.25x_3 + 0.333x_4 + 0.25x_5 \geq 1.$$

□

Chapter 3

First Step of Generalization: 3-step MIR

In this chapter, we will prove that the MIR approach studied in chapter 2 can be generalized to the 3-step MIR. This fact, as one of the main contributions of this research, illuminates the path to extend this generalization to n-step MIR for arbitrary $n \in \mathbb{N}$, which will be presented in chapter 4.

In section 3.1, we define some efficient closed notations for the expressions that we regularly used in chapter 2 and will continue to use in this chapter. This notation will be very useful in development of different concepts through the rest of the dissertation. In section 3.2, we present the 3-step MIR facet for a four-dimensional simple mixed integer set. Two other facets for this set, obtained from 1-step and 2-step MIR, are also generated along with the 3-step MIR facet. In section 3.3, we use the 3-step MIR facet of section 3.2 to generate the 3-step MIR inequality for the feasible set of a general IP constraint, i.e. $Y_{\overline{a}}^{\alpha_1,b}$. This will result in the introduction of the 3-step MIR function as a new class of MIR functions that can be used to generate valid inequalities for general constraints.

3.1 Efficient Notations

As noticed in sections 2.3 and 2.4, the open notations used there become relatively hard to handle by going from 1-step to 2-step MIR inequalities. This would be worse for the 3-step MIR. To avoid this problem, we introduce some closed notations in this section and will use them in the rest of this chapter. These notations will prove very helpful in development of different concepts related to MIR as we go forward in this dissertation.

Definition 3.1. *For any vector* $\overline{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$*, we define the vector* $\overline{\alpha}^i \in \mathbb{R}^i$ *as* $\overline{\alpha}^i = (\alpha_1, \ldots, \alpha_i)$ *for* $i = 1, \ldots, n$*. In particular,* $\overline{\alpha}^1 = \alpha_1$ *is a scalar and we will use both of them interchangeably.* □

Definition 3.2. *Let* $\alpha, \beta \in \mathbb{R}$ *and* $\alpha > 0$*. Then there exist unique numbers* $q \in \mathbb{Z}$ *and* $r \in \mathbb{R}_+$ *such that* $\beta = \alpha q + r$ *and* $r < \alpha$*. We define* $\lambda^{\alpha}(\beta) = r = \beta - \alpha \lfloor \beta/\alpha \rfloor$*. In other words* $\lambda^{\alpha}(\beta)$ *is the remainder when the maximum integer multiple of* α *less than* β *is taken away from* β*.* □

As a special case, $\lambda^1(\beta)$ is the fractional part of β, i.e. $\beta - \lfloor \beta \rfloor$.

Definition 3.3. *Let* $\beta \in \mathbb{R}$ *and* $\overline{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ *where* $n \in \mathbb{N}$ *and* $\alpha_i > 0$ *for* $i = 1, \ldots, n$*. We define* $\lambda^{\overline{\alpha}}(\beta) = \lambda^{\overline{\alpha}^n}(\beta)$ *recursively as follows*

$$\lambda^{\overline{\alpha}}(\beta) = \lambda^{\overline{\alpha}^n}(\beta) = \lambda^{\alpha_n}(\lambda^{\overline{\alpha}^{n-1}}(\beta)),$$

where $\lambda^{\overline{\alpha}^1}(\beta) = \lambda^{\alpha_1}(\beta)$ *is as defined in Definition 3.2.* □

According to the definitions above, we have $\lambda^{\overline{\alpha}^1}(\beta) = \lambda^{\alpha_1}(\beta) = \beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor$, and

$$\lambda^{\overline{\alpha}^2}(\beta) = \lambda^{\alpha_2}(\lambda^{\overline{\alpha}^1}(\beta)) = \lambda^{\overline{\alpha}^1}(\beta) - \alpha_2 \left\lfloor \lambda^{\overline{\alpha}^1}(\beta)/\alpha_2 \right\rfloor = \beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor - \alpha_2 \left\lfloor \frac{\beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor}{\alpha_2} \right\rfloor.$$

Definition 3.4. *Let* $\beta \in \mathbb{R}$ *and* $\overline{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ *such that* $\alpha_i > 0$ *for* $i = 1, \ldots, n$*. We define functions* $\tau^{\overline{\alpha}^i}(\beta)$ *and* $\sigma^{\overline{\alpha}^i}(\beta)$ *as follows:*

$$\tau^{\overline{\alpha}^1}(\beta) = \tau^{\alpha_1}(\beta) = \left\lceil \frac{\beta}{\alpha_1} \right\rceil, \qquad \tau^{\overline{\alpha}^i}(\beta) = \left\lceil \frac{\lambda^{\overline{\alpha}^{i-1}}(\beta)}{\alpha_i} \right\rceil \text{ for } i = 2, \ldots, n;$$
$$\sigma^{\overline{\alpha}^1}(\beta) = \sigma^{\alpha_1}(\beta) = \left\lfloor \frac{\beta}{\alpha_1} \right\rfloor, \qquad \sigma^{\overline{\alpha}^i}(\beta) = \left\lfloor \frac{\lambda^{\overline{\alpha}^{i-1}}(\beta)}{\alpha_i} \right\rfloor \text{ for } i = 2, \ldots, n.$$

□

A useful relationship between λ and σ functions is

$$\begin{aligned} &\lambda^{\overline{\alpha}^1}(\beta) + \alpha_1 \sigma^{\overline{\alpha}^1}(\beta) = \beta \\ &\lambda^{\overline{\alpha}^i}(\beta) + \alpha_i \sigma^{\overline{\alpha}^i}(\beta) = \lambda^{\overline{\alpha}^{i-1}}(\beta), \text{ for } i = 2, \ldots, n \end{aligned} \tag{3.1}$$

With notations above the MIR facets and inequalities developed in chapter 2 can be written in very compact forms. For the 1-step MIR, we set $\overline{\alpha} = \alpha_1$. Then the 1-step MIR facet (2.1) can be written as

$$y_1 + \frac{1}{\lambda^{\overline{\alpha}^1}(\beta)} v \geq \tau^{\overline{\alpha}^1}(\beta).$$

The positive 1-step MIR function (2.4), $g_+^{\overline{\alpha}^1,b}(u)$, in the new notations will be

$$g_+^{\overline{\alpha}^1,b}(u) = \frac{\alpha_1 \delta^{\overline{\alpha}^1,b}(u) - \lambda^{\overline{\alpha}^1}(u)}{\alpha_1 - \lambda^{\overline{\alpha}^1}(b)}$$

where

$$\delta^{\overline{\alpha}^1,b}(u) = \begin{cases} 1 & \text{if } u \in I_0^{\overline{\alpha}^1,b} \\ \lambda^{\overline{\alpha}^1}(u)/\lambda^{\overline{\alpha}^1}(b) & \text{if } u \in I_1^{\overline{\alpha}^1,b} \end{cases}$$

and

$$\begin{aligned} I_0^{\overline{\alpha}^1,b} &= \{u \in \mathbb{R} : \lambda^{\overline{\alpha}^1}(u) \geq \lambda^{\overline{\alpha}^1}(b)\} \\ I_1^{\overline{\alpha}^2,b} &= \{u \in \mathbb{R} : \lambda^{\overline{\alpha}^1}(u) < \lambda^{\overline{\alpha}^1}(b)\}. \end{aligned}$$

For the 2-step MIR, we set $\overline{\alpha} = (\alpha_1, \alpha_2)$. Then the 2-step MIR facet (2.18) will be

$$\tau^{\overline{\alpha}^2}(\beta) y_1 + y_2 + \frac{1}{\lambda^{\overline{\alpha}^2}(\beta)}\, v \geq \tau^{\overline{\alpha}^1}(\beta)\tau^{\overline{\alpha}^2}(\beta)$$

and the positive 2-step MIR function (2.23) will be written as

$$g_+^{\overline{\alpha}^2,b}(u) = \frac{\alpha_1 \delta^{\overline{\alpha}^2,b}(u) - \lambda^{\overline{\alpha}^1}(u)\tau^{\overline{\alpha}^2}(b)}{[\alpha_1 - \lambda^{\overline{\alpha}^1}(b)]\tau^{\overline{\alpha}^2}(b)}$$

where

$$\delta^{\overline{\alpha}^2,b}(u) = \begin{cases} \tau^{\overline{\alpha}^2}(b) & \text{if } u \in I_0^{\overline{\alpha}^2,b} \\ \tau^{\overline{\alpha}^2}(u) & \text{if } u \in I_1^{\overline{\alpha}^2,b} \\ \sigma^{\overline{\alpha}^2}(u) + \lambda^{\overline{\alpha}^2}(u)/\lambda^{\overline{\alpha}^2}(b) & \text{if } u \in I_2^{\overline{\alpha}^2,b} \end{cases}$$

and

$$\begin{aligned} I_0^{\overline{\alpha}^2,b} &= \{u \in \mathbb{R} : \lambda^{\overline{\alpha}^1}(u) \geq \lambda^{\overline{\alpha}^1}(b)\} \\ I_1^{\overline{\alpha}^2,b} &= \{u \in \mathbb{R} : \lambda^{\overline{\alpha}^1}(u) < \lambda^{\overline{\alpha}^1}(b), \lambda^{\overline{\alpha}^2}(u) \geq \lambda^{\overline{\alpha}^2}(b)\} \\ I_2^{\overline{\alpha}^2,b} &= \{u \in \mathbb{R} : \lambda^{\overline{\alpha}^1}(u) < \lambda^{\overline{\alpha}^1}(b), \lambda^{\overline{\alpha}^2}(u) < \lambda^{\overline{\alpha}^2}(b)\}. \end{aligned}$$

3.2 3-step MIR Facet

In section 2.1, we discussed how the 1-MIR facet can be developed for a two-dimensional simple mixed integer set and, in section 2.3, we showed how the 2-step MIR facet can be developed for a three-dimensional simple mixed integer set. In this section, we are going to prove that this idea can be extended to development of a facet for a four-dimensional simple mixed integer set, called the 3-step MIR facet.

Similar to the extension from $Q^{\alpha_1,\beta}$ to $Q^{(\alpha_1,\alpha_2),\beta}$, here, the simple four-dimensional set of interest is obtained by adding one more non-negative integer variable. Throughout this section, we define $\overline{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ where $\alpha_i \in \mathbb{R}$, $\alpha_i > 0$ for $i = 1,2,3$. Then our simple set is

$$Q^{\overline{\alpha}^3,\beta} = \left\{(y_1, y_2, y_3, v) \in \mathbb{Z} \times \mathbb{Z}_+^2 \times \mathbb{R}_+ : \alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 + v \geq \beta \right\},$$

where $\beta \in \mathbb{R}$.

If certain conditions hold, two facets for $Q^{\overline{\alpha}^3,\beta}$ can be obtained using the 1-step and 2-step MIR facets of $Q^{\alpha_1,\beta}$ and $Q^{(\alpha_1,\alpha_2),\beta}$. Lemmas 3.5 and 3.6 present these facets.

Lemma 3.5. *Let* $\beta, \alpha_i \in \mathbb{R}$, $\alpha_i > 0$ *for* $i = 1,2,3$ *and* $\overline{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ *such that* $\beta/\alpha_1 < \tau^{\overline{\alpha}^1}(\beta)$. *Then the inequality*

$$y_1 + \frac{\alpha_2}{\lambda^{\overline{\alpha}^1}(\beta)} y_2 + \frac{\alpha_3}{\lambda^{\overline{\alpha}^1}(\beta)} y_3 + \frac{1}{\lambda^{\overline{\alpha}^1}(\beta)} v \geq \tau^{\overline{\alpha}^1}(\beta) \tag{3.2}$$

is valid for $Q^{\overline{\alpha}^3,\beta}$. *It is facet-defining if* $\sigma^{\overline{\alpha}^2}(\beta) \geq 1$ *and* $\sigma^{\overline{\alpha}^3}(\beta) \geq 1$.

Proof. By definition

$$\alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 + v \geq \beta \tag{3.3}$$

is valid for $Q^{\overline{\alpha}^3,\beta}$. In $Q^{\overline{\alpha}^3,\beta}$, we have $y_1 \in \mathbb{Z}$ and $\alpha_2 y_2 + \alpha_3 y_3 + v \in \mathbb{R}_+$. Thus we can treat y_1 and $\alpha_2 y_2 + \alpha_3 y_3 + v$ in $Q^{\overline{\alpha}^3,\beta}$ as y_1 and v in $Q^{\overline{\alpha}^1,\beta}$, respectively and substitute into (2.1). This yields (3.2). Therefore (3.2) is valid for $Q^{\overline{\alpha}^3,\beta}$.

To show that (3.2) is a facet, we need to find four affinely independent points in $Q^{\overline{\alpha}^3,\beta}$ which satisfy (3.2) at equality. The points $q_1^3 = (\sigma^{\overline{\alpha}^1}(\beta), 0, 0, \lambda^{\overline{\alpha}^1}(\beta))$, $q_2^3 = (\sigma^{\overline{\alpha}^1}(\beta), \sigma^{\overline{\alpha}^2}(\beta), 0, \lambda^{\overline{\alpha}^2}(\beta))$, $q_3^3 = (\sigma^{\overline{\alpha}^1}(\beta), \sigma^{\overline{\alpha}^2}(\beta), \sigma^{\overline{\alpha}^3}(\beta), \lambda^{\overline{\alpha}^3}(\beta))$ and $p_1^3 = (\tau^{\overline{\alpha}^1}(\beta), 0, 0, 0)$ are all in $Q^{\overline{\alpha}^3,\beta}$. This is easy to verify: $y_1 \in \mathbb{Z}$, $y_2, y_3 \in \mathbb{Z}_+$ and $v \in \mathbb{R}_+$ for all four points. If we replace the coordinates of each point into (3.3), for q_1^3, q_2^3 and q_3^3, using identities (3.1), we get $\beta = \beta$ and, for p_1^3, we get $\alpha_1 \tau^{\overline{\alpha}^1}(\beta) \geq \beta$, which is trivial. Thus all four points are in $Q^{\overline{\alpha}^3,\beta}$. They all also satisfy (3.2) at equality. This is trivial for p_1^3 and can be easily verified for q_1^3, q_2^3 and q_3^3 using identities (3.1). If we set up a 4 by 4 matrix whose rows are p_1^3, q_1^3, q_2^3 and q_3^3, respectively and rearrange the columns as (y_1, v, y_2, y_3), we get

$$\begin{bmatrix} \tau^{\overline{\alpha}^1}(\beta) & 0 & 0 & 0 \\ \sigma^{\overline{\alpha}^1}(\beta) & \lambda^{\overline{\alpha}^1}(\beta) & 0 & 0 \\ \sigma^{\overline{\alpha}^1}(\beta) & \lambda^{\overline{\alpha}^2}(\beta) & \sigma^{\overline{\alpha}^2}(\beta) & 0 \\ \sigma^{\overline{\alpha}^1}(\beta) & \lambda^{\overline{\alpha}^3}(\beta) & \sigma^{\overline{\alpha}^2}(\beta) & \sigma^{\overline{\alpha}^3}(\beta) \end{bmatrix}.$$

Since it is assumed $\beta/\alpha_1 < \tau^{\overline{\alpha}^1}(\beta)$, we have $\lambda^{\overline{\alpha}^1}(\beta) > 0$. It is also assumed that $\sigma^{\overline{\alpha}^2}(\beta) \geq 1$ and $\sigma^{\overline{\alpha}^3}(\beta) \geq 1$. Therefore none of the diagonal entries of this lower-triangular matrix can be zero except the first one, i.e. $\tau^{\overline{\alpha}^1}(\beta)$. Therefore either the matrix has a full rank or it has a rank of 3 with a zero first row. Both cases imply that the points q_1^3, q_2^3, q_3^3 and p_1^3 are affinely independent. Thus (3.2) is a facet of $Q^{\overline{\alpha}^3,\beta}$. □

Lemma 3.6. *Let $\beta, \alpha_i \in \mathbb{R}$, $\alpha_i > 0$ for $i = 1,2,3$ and $\overline{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ such that $\beta/\alpha_1 < \tau^{\overline{\alpha}^1}(\beta)$ and $\lambda^{\overline{\alpha}^1}(\beta)/\alpha_2 < \tau^{\overline{\alpha}^2}(\beta)$. Then the inequality*

$$\tau^{\overline{\alpha}^2}(\beta) y_1 + y_2 + \frac{\alpha_3}{\lambda^{\overline{\alpha}^2}(\beta)} y_3 + \frac{1}{\lambda^{\overline{\alpha}^2}(\beta)} v \geq \tau^{\overline{\alpha}^1}(\beta)\tau^{\overline{\alpha}^2}(\beta) \tag{3.4}$$

is valid for $Q^{\overline{\alpha}^3,\beta}$ if $\tau^{\overline{\alpha}^2}(\beta) \leq \alpha_1/\alpha_2$. It is facet-defining if $\sigma^{\overline{\alpha}^3}(\beta) \geq 1$.

Proof. Inequality (3.3) is valid for $Q^{\overline{\alpha}^3,\beta}$. In $Q^{\overline{\alpha}^3,\beta}$ we have $y_1 \in \mathbb{Z}$, $y_2 \in \mathbb{Z}_+$ and $\alpha_3 y_3 + v \in \mathbb{R}_+$. Thus we can treat y_1, y_2 and $\alpha_3 y_3 + v$ in $Q^{\overline{\alpha}^3,\beta}$ as y_1, y_2 and v in $Q^{\overline{\alpha}^2,\beta}$, respectively. Since $\tau^{\overline{\alpha}^2}(\beta) \leq \alpha_1/\alpha_2$, according to Theorem 2.14 and by substitution in inequality (2.18), we get (3.4). Thus (3.4) is valid for $Q^{\overline{\alpha}^3,\beta}$.

To show that (3.4) is a facet, we need to find four affinely independent points in $Q^{\overline{\alpha}^3,\beta}$ which satisfy (3.4) at equality. The points q_2^3, q_3^3, p_1^3 and $p_2^3 = (\sigma^{\overline{\alpha}^1}(\beta), \tau^{\overline{\alpha}^2}(\beta), 0, 0)$ are all in $Q^{\overline{\alpha}^3,\beta}$. This was verified for q_2^3, q_3^3 and p_1^3 in Lemma 3.5. For p_2^3, we have $y_1 \in \mathbb{Z}$, $y_2, y_3 \in \mathbb{Z}_+$ and $v \in \mathbb{R}_+$ for all four points. If we replace the coordinates of p_2^3 into (3.3), we get $\alpha_1\sigma^{\overline{\alpha}^1}(\beta) + \alpha_2\tau^{\overline{\alpha}^2}(\beta) \geq \beta$, or, using identities (3.1), $\alpha_2\tau^{\overline{\alpha}^2}(\beta) \geq \lambda^{\overline{\alpha}^1}(\beta)$, which is trivial. These four points also satisfy (3.4) at equality. This is easy to verify. If we set up a 4 by 4 matrix whose rows are p_1^3, p_2^3, q_2^3 and q_3^3, respectively and rearrange the columns as (y_1, y_2, v, y_3), we get

$$\begin{bmatrix} \tau^{\overline{\alpha}^1}(\beta) & 0 & 0 & 0 \\ \sigma^{\overline{\alpha}^1}(\beta) & \tau^{\overline{\alpha}^2}(\beta) & 0 & 0 \\ \sigma^{\overline{\alpha}^1}(\beta) & \sigma^{\overline{\alpha}^2}(\beta) & \lambda^{\overline{\alpha}^2}(\beta) & 0 \\ \sigma^{\overline{\alpha}^1}(\beta) & \sigma^{\overline{\alpha}^2}(\beta) & \lambda^{\overline{\alpha}^3}(\beta) & \sigma^{\overline{\alpha}^3}(\beta) \end{bmatrix}.$$

Since it is assumed $\beta/\alpha_1 < \tau^{\overline{\alpha}^1}(\beta)$, we have $\lambda^{\overline{\alpha}^1}(\beta) > 0$ and hence $\tau^{\overline{\alpha}^2}(\beta) > 0$. Also since $\lambda^{\overline{\alpha}^1}(\beta)/\alpha_2 < \tau^{\overline{\alpha}^2}(\beta)$, we have $\lambda^{\overline{\alpha}^2}(\beta) > 0$. It is also assumed that $\sigma^{\overline{\alpha}^3}(\beta) \geq 1$. Therefore none of the diagonal entries of this lower-triangular matrix can be zero except the first one, i.e. $\tau^{\overline{\alpha}^1}(\beta)$. Therefore either the matrix has a full rank or it has a rank of 3 with a zero first row. Both cases imply that the points q_2^3, q_3^3, p_1^3 and p_2^3 are affinely independent. Thus (3.4) is a facet of $Q^{\overline{\alpha}^3,\beta}$. □

Theorem 3.7 presents the 3-step MIR facet of $Q^{\overline{\alpha}^3,\beta}$. The proof uses the facets developed in lemmas 3.5 and 3.6.

Theorem 3.7. *Let $\beta, \alpha_i \in \mathbb{R}$, $\alpha_i > 0$ for $i = 1,2,3$ and $\overline{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ such that $\beta/\alpha_1 < \tau^{\overline{\alpha}^1}(\beta)$, $\lambda^{\overline{\alpha}^1}(\beta)/\alpha_2 < \tau^{\overline{\alpha}^2}(\beta)$ and $\lambda^{\overline{\alpha}^2}(\beta)/\alpha_3 < \tau^{\overline{\alpha}^3}(\beta)$. The inequality*

$$\tau^{\overline{\alpha}^2}(\beta)\tau^{\overline{\alpha}^3}(\beta)y_1 + \tau^{\overline{\alpha}^3}(\beta)y_2 + y_3 + \frac{1}{\lambda^{\overline{\alpha}^3}(\beta)}\, v \geq \tau^{\overline{\alpha}^1}(\beta)\tau^{\overline{\alpha}^2}(\beta)\tau^{\overline{\alpha}^3}(\beta) \tag{3.5}$$

is valid and facet-defining for $Q^{\overline{\alpha}^3,\beta}$ if $\tau^{\overline{\alpha}^2}(\beta) \leq \alpha_1/\alpha_2$ and $\tau^{\overline{\alpha}^3}(\beta) \leq \alpha_2/\alpha_3$.

Proof. By Lemma 3.5, inequality (3.2) is valid for $Q^{\overline{\alpha}^3,\beta}$. If we subtract $\sigma^{\overline{\alpha}^1}(\beta) = \tau^{\overline{\alpha}^1}(\beta) - 1$ from its both sides and multiply both sides by $\lambda^{\overline{\alpha}^1}(\beta)$, we get

$$\lambda^{\overline{\alpha}^1}(\beta)\left[y_1 - \tau^{\overline{\alpha}^1}(\beta) + 1\right] + \alpha_2 y_2 + \alpha_3 y_3 + v \geq \lambda^{\overline{\alpha}^1}(\beta)$$

Multiplying the first term by $1 = \alpha_2/\alpha_2$ gives

$$\frac{\lambda^{\overline{\alpha}^1}(\beta)}{\alpha_2}\alpha_2\left[y_1 - \tau^{\overline{\alpha}^1}(\beta) + 1\right] + \alpha_2 y_2 + \alpha_3 y_3 + v \geq \lambda^{\overline{\alpha}^1}(\beta). \tag{3.6}$$

On the other hand, inequality (3.3) is valid for $Q^{\overline{\alpha}^3,\beta}$ by definition. If we subtract $\alpha_1\sigma^{\overline{\alpha}^1}(\beta) = \alpha_1\left(\tau^{\overline{\alpha}^1}(\beta) - 1\right)$ from its both sides, we get

$$\alpha_1\left[y_1 - \tau^{\overline{\alpha}^1}(\beta) + 1\right] + \alpha_2 y_2 + \alpha_3 y_3 + v \geq \lambda^{\overline{\alpha}^1}(\beta), \tag{3.7}$$

Multiplying the first term by $1 = \alpha_2/\alpha_2$ gives

$$\frac{\alpha_1}{\alpha_2}\alpha_2\left[y_1 - \tau^{\overline{\alpha}^1}(\beta)+1\right] + \alpha_2 y_2 + \alpha_3 y_3 + v \geq \lambda^{\overline{\alpha}^1}(\beta), \tag{3.8}$$

Now inequalities (3.6) and (3.8) imply that

$$\gamma\alpha_2\left[y_1 - \tau^{\overline{\alpha}^1}(\beta)+1\right] + \alpha_2 y_2 + \alpha_3 y_3 + v \geq \lambda^{\overline{\alpha}^1}(\beta)$$

is valid for $Q^{\overline{\alpha}^3,\beta}$ for any $\lambda^{\overline{\alpha}^1}(\beta)/\alpha_2 \leq \gamma \leq \alpha_1/\alpha_2$. This is true because it can be stated as a convex combination of the two valid inequalities (3.6) and (3.8). Now since it is assumed that $\lambda^{\overline{\alpha}^1}(\beta)/\alpha_2 < \tau^{\overline{\alpha}^2}(\beta) \leq \alpha_1/\alpha_2$, we can set $\gamma = \tau^{\overline{\alpha}^2}(\beta)$ which gives

$$\alpha_2\left[\tau^{\overline{\alpha}^2}(\beta)y_1 - \tau^{\overline{\alpha}^1}(\beta)\tau^{\overline{\alpha}^2}(\beta) + \tau^{\overline{\alpha}^2}(\beta)\right] + \alpha_2 y_2 + \alpha_3 y_3 + v \geq \lambda^{\overline{\alpha}^1}(\beta).$$

After subtracting $\alpha_2\sigma^{\overline{\alpha}^2}(\beta)$ from both sides, we get

$$\alpha_2\left[\tau^{\overline{\alpha}^2}(\beta)y_1 + y_2 - \tau^{\overline{\alpha}^1}(\beta)\tau^{\overline{\alpha}^2}(\beta) + 1\right] + \alpha_3 y_3 + v \geq \lambda^{\overline{\alpha}^2}(\beta), \tag{3.9}$$

We multiply the first term by $1 = \alpha_3/\alpha_3$ to get

$$\frac{\alpha_2}{\alpha_3}\alpha_3\left[\tau^{\overline{\alpha}^2}(\beta)y_1 + y_2 - \tau^{\overline{\alpha}^1}(\beta)\tau^{\overline{\alpha}^2}(\beta) + 1\right] + \alpha_3 y_3 + v \geq \lambda^{\overline{\alpha}^2}(\beta), \tag{3.10}$$

Next observe that, by Lemma 3.6, inequality (3.4) is valid for $Q^{\overline{\alpha}^3,\beta}$. If we subtract $\tau^{\overline{\alpha}^1}(\beta)\tau^{\overline{\alpha}^2}(\beta) - 1$ from its both sides and multiply both sides by $\lambda^{\overline{\alpha}^2}(\beta)$, we get

$$\lambda^{\overline{\alpha}^2}(\beta)\left[\tau^{\overline{\alpha}^2}(\beta)y_1 + y_2 - \tau^{\overline{\alpha}^1}(\beta)\tau^{\overline{\alpha}^2}(\beta) + 1\right] + \alpha_3 y_3 + v \geq \lambda^{\overline{\alpha}^2}(\beta)$$

Again we multiply the first term by $1 = \alpha_3/\alpha_3$ to get

$$\frac{\lambda^{\overline{\alpha}^2}(\beta)}{\alpha_3}\alpha_3\left[\tau^{\overline{\alpha}^2}(\beta)y_1 + y_2 - \tau^{\overline{\alpha}^1}(\beta)\tau^{\overline{\alpha}^2}(\beta) + 1\right] + \alpha_3 y_3 + v \geq \lambda^{\overline{\alpha}^2}(\beta) \tag{3.11}$$

Now inequalities (3.10) and (3.11) imply that

$$\gamma\alpha_3\left[\tau^{\overline{\alpha}^2}(\beta)y_1 + y_2 - \tau^{\overline{\alpha}^1}(\beta)\tau^{\overline{\alpha}^2}(\beta) + 1\right] + \alpha_3 y_3 + v \geq \lambda^{\overline{\alpha}^2}(\beta)$$

is valid for $Q^{\overline{\alpha}^3,\beta}$ for any $\lambda^{\overline{\alpha}^2}(\beta)/\alpha_3 \leq \gamma \leq \alpha_2/\alpha_3$ as it can be stated as a convex combination of the two valid inequalities (3.10) and (3.11). Since it is assumed that $\lambda^{\overline{\alpha}^2}(\beta)/\alpha_3 < \tau^{\overline{\alpha}^3}(\beta) \leq \alpha_2/\alpha_3$, we can set $\gamma = \tau^{\overline{\alpha}^3}(\beta)$, which results in

$$\alpha_3\left[\tau^{\overline{\alpha}^2}(\beta)\tau^{\overline{\alpha}^3}(\beta)y_1 + \tau^{\overline{\alpha}^3}(\beta)y_2 - \tau^{\overline{\alpha}^1}(\beta)\tau^{\overline{\alpha}^2}(\beta)\tau^{\overline{\alpha}^3}(\beta) + \tau^{\overline{\alpha}^3}(\beta)\right] + \alpha_3 y_3 + v \geq \lambda^{\overline{\alpha}^2}(\beta),$$

After subtracting $\alpha_3\sigma^{\overline{\alpha}^3}(\beta)$ from both sides, we get

$$\alpha_3\left[\tau^{\overline{\alpha}^2}(\beta)\tau^{\overline{\alpha}^3}(\beta)y_1+\tau^{\overline{\alpha}^3}(\beta)y_2+y_3-\tau^{\overline{\alpha}^1}(\beta)\tau^{\overline{\alpha}^2}(\beta)\tau^{\overline{\alpha}^3}(\beta)+1\right]+v\geq\lambda^{\overline{\alpha}^3}(\beta), \tag{3.12}$$

Now the expression in brackets is in $\mathbb{Z}$ and $v\in\mathbb{R}_+$, thus we can treat these two in $Q^{\overline{\alpha}^3,\beta}$ as y_1 and v in $Q^{\alpha_3,\lambda^{\overline{\alpha}^3}(\beta)}$, respectively. Hence, by Theorem 2.1, the 1-step MIR facet for $Q^{\alpha_3,\lambda^{\overline{\alpha}^3}(\beta)}$, when y_1 is replaced by its substitute, is valid for $Q^{\overline{\alpha}^3,\beta}$:

$$\tau^{\overline{\alpha}^2}(\beta)\tau^{\overline{\alpha}^3}(\beta)y_1+\tau^{\overline{\alpha}^3}(\beta)y_2+y_3-\tau^{\overline{\alpha}^1}(\beta)\tau^{\overline{\alpha}^2}(\beta)\tau^{\overline{\alpha}^3}(\beta)+1+\frac{1}{\lambda^{\alpha_3}(\lambda^{\overline{\alpha}^3}(\beta))}v\geq\tau^{\alpha_3}(\lambda^{\overline{\alpha}^3}(\beta)).$$

But $\lambda^{\overline{\alpha}^3}(\beta)<\alpha_3$; therefore, $\lambda^{\alpha_3}(\lambda^{\overline{\alpha}^3}(\beta))=\lambda^{\overline{\alpha}^3}(\beta)$ and $\tau^{\alpha_3}(\lambda^{\overline{\alpha}^3}(\beta))=1$. Thus the inequality above reduces to (3.5) and the validity of (3.5) is proved.

To show that (3.5) is a facet, we need to find four affinely independent points in $Q^{\overline{\alpha}^3,\beta}$ which satisfy (3.5) at equality. In Lemmas 3.5 and 3.6, we saw that the points q_3^3, p_1^3 and p_2^3 are in $Q^{\overline{\alpha}^3,\beta}$. Now it is easy to verify that the point $p_3^3=(\sigma^{\overline{\alpha}^1}(\beta),\sigma^{\overline{\alpha}^2}(\beta),\tau^{\overline{\alpha}^3}(\beta),0)$ is also in $Q^{\overline{\alpha}^3,\beta}$; because, for this point, we have $y_1\in\mathbb{Z}$, $y_2,y_3\in\mathbb{Z}_+$ and $v\in\mathbb{R}_+$ and if we replace the coordinates of p_3^3 into (3.3), we get $\alpha_1\sigma^{\overline{\alpha}^1}(\beta)+\alpha_2\sigma^{\overline{\alpha}^2}(\beta)+\alpha_3\tau^{\overline{\alpha}^3}(\beta)\geq\beta$, or, using identities (3.1), $\alpha_2\tau^{\overline{\alpha}^3}(\beta)\geq\lambda^{\overline{\alpha}^2}(\beta)$, which is trivial. These four points also satisfy (3.2) at equality, which is easy to verify. If we set up a 4 by 4 matrix whose rows are p_1^3, p_2^3, p_3^3 and q_3^3, respectively, we get

$$\begin{bmatrix}\tau^{\overline{\alpha}^1}(\beta) & 0 & 0 & 0\\ \sigma^{\overline{\alpha}^1}(\beta) & \tau^{\overline{\alpha}^2}(\beta) & 0 & 0\\ \sigma^{\overline{\alpha}^1}(\beta) & \sigma^{\overline{\alpha}^2}(\beta) & \tau^{\overline{\alpha}^3}(\beta) & 0\\ \sigma^{\overline{\alpha}^1}(\beta) & \sigma^{\overline{\alpha}^2}(\beta) & \sigma^{\overline{\alpha}^3}(\beta) & \lambda^{\overline{\alpha}^3}(\beta)\end{bmatrix}.$$

Since it is assumed $\beta/\alpha_1<\tau^{\overline{\alpha}^1}(\beta)$, we have $\lambda^{\overline{\alpha}^1}(\beta)>0$ and hence $\tau^{\overline{\alpha}^2}(\beta)>0$ and, since $\lambda^{\overline{\alpha}^1}(\beta)/\alpha_2<\tau^{\overline{\alpha}^2}(\beta)$, we have $\lambda^{\overline{\alpha}^2}(\beta)>0$ and hence $\tau^{\overline{\alpha}^3}(\beta)>0$. Also, since it is assumed that $\lambda^{\overline{\alpha}^2}(\beta)/\alpha_2<\tau^{\overline{\alpha}^3}(\beta)$, we have $\lambda^{\overline{\alpha}^3}(\beta)>0$. Therefore none of the diagonal entries of this lower-triangular matrix can be zero except the first one, i.e. $\tau^{\overline{\alpha}^1}(\beta)$. Therefore either the matrix has a full rank or it has a rank of 3 with a zero first row. Both cases imply that the the points q_3^3, p_1^3, p_2^3 and p_3^3 are affinely independent. Thus (3.5) is a facet of $Q^{\overline{\alpha}^3,\beta}$. □

Based on Theorem 3.7, we define the 3-step MIR facet for $Q^{\overline{\alpha}^3,\beta}$:

Definition 3.8. *Let $\beta, \alpha_i \in \mathbb{R}$, $\alpha_i > 0$ for $i = 1,2,3$ and $\overline{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ such that $\beta/\alpha_1 < \tau^{\overline{\alpha}^1}(\beta)$, $\lambda^{\overline{\alpha}^1}(\beta)/\alpha_2 < \tau^{\overline{\alpha}^2}(\beta) \leq \alpha_1/\alpha_2$ and $\lambda^{\overline{\alpha}^2}(\beta)/\alpha_3 < \tau^{\overline{\alpha}^3}(\beta) \leq \alpha_2/\alpha_3$. Then facet* (3.5) *is called the **3-step MIR facet** for $Q^{\overline{\alpha}^3,\beta}$.* □

Example 3.9. Let $\overline{\alpha} = (\alpha_1, \alpha_2, \alpha_3) = (1, 0.3, 0.08)$ and $\beta = 2.8$. Then we have $Q^{\overline{\alpha}^3,\beta} = Q^{(1,0.3,0.08),2.8} = \{(y_1, y_2, y_3, v) \in \mathbb{Z} \times \mathbb{Z}_+^2 \times \mathbb{R}_+ : y_1 + 0.3y_2 + 0.08y_3 + v \geq 2.8\}$. Notice that $\beta/\alpha_1 = 2.8 < 3 = \tau^{\overline{\alpha}^1}(\beta)$. Since $\sigma^{\overline{\alpha}^2}(\beta) = 2 \geq 1$ and $\sigma^{\overline{\alpha}^3}(\beta) = 2 \geq 1$, by Lemma 3.5, inequality (3.2) will give a facet of $Q^{(1,0.3,0.08),2.8}$, which is

$$y_1 + 0.375y_2 + 0.1y_3 + 1.25v \geq 3.$$

We observe that the points $q_1^3 = (2,0,0,0.8)$, $q_2^3 = (2,2,0,0.2)$, $q_3^3 = (2,2,2,0.04)$ and $p_1^3 = (3,0,0,0)$ are four (affinely) independent points in $Q^{(1,0.3,0.08),2.8}$ which are on the plane of this facet.

Also notice that $\lambda^{\overline{\alpha}^1}(\beta)/\alpha_2 = 8/3 < 3 = \tau^{\overline{\alpha}^2}(\beta)$; therefore, since $\tau^{\overline{\alpha}^2}(\beta) = 3 \leq \frac{1}{0.3} = \frac{\alpha_1}{\alpha_2}$ and $\sigma^{\overline{\alpha}^3}(\beta) = 2 \geq 1$, by Lemma 3.6, inequality (3.4) will give a facet of $Q^{(1,0.3,0.08),2.8}$, which is

$$3y_1 + y_2 + 0.4y_3 + 5v \geq 9.$$

The points $q_2^3 = (2,2,0,0.2)$, $q_3^3 = (2,2,2,0.04)$, $p_1^3 = (3,0,0,0)$ and $p_2^3 = (2,3,0,0)$ are four (affinely) independent points in $Q^{(1,0.3,0.08),2.8}$ which are on the plane of this facet.

And finally, we have $\lambda^{\overline{\alpha}^2}(\beta)/\alpha_3 = 2.5 < 3 = \tau^{\overline{\alpha}^3}(\beta)$; therefore since $\tau^{\overline{\alpha}^2}(\beta) = 3 \leq \frac{1}{0.3} = \frac{\alpha_1}{\alpha_2}$ and $\tau^{\overline{\alpha}^3}(\beta) = 3 \leq \frac{0.3}{0.08} = \frac{\alpha_2}{\alpha_3}$, by Theorem 3.7, inequality (3.5) will give the 3-step MIR facet for $Q^{(1,0.3,0.08),2.8}$, which is

$$9y_1 + 3y_2 + y_3 + 25v \geq 27.$$

The points $q_3^3 = (2,2,2,0.04)$, $p_1^3 = (3,0,0,0)$, $p_2^3 = (2,3,0,0)$ and $p_3^3 = (2,2,3,0)$ are four (affinely) independent points in $Q^{(1,0.3,0.08),2.8}$, which are on the plane of this facet. □

3.3 3-step MIR Inequality for a General IP Constraint

In sections 2.2 and 2.4, we used the 1-step and 2-step MIR facets of $Q^{\alpha_1,\beta}$ and $Q^{(\alpha_1,\alpha_2),\beta}$ to generate valid inequalities for $Y_{\overline{a}}^{\alpha_1,b}$, the feasible set of a general IP constraint. In this section, we prove that this is also possible for the 3-step MIR facet of $Q^{\overline{\alpha}^3,\beta}$ by proposing the appropriate MIR procedure. In fact, the result of the MIR procedure is a new class of MIR functions, which we call the 3-step MIR functions. The 3-step MIR functions will be the tool for generating new valid inequalities for $Y_{\overline{a}}^{\alpha_1,b}$, which we refer to as the 3-step MIR inequalities.

We first define the 3-step MIR function and prove that it has the same properties of the 2-step MIR function which were stated in Theorem 2.18. Then, in Theorem 3.12, we present the MIR procedure leading to the 3-step MIR inequality.

Definition 3.10. *Let $\beta, \alpha_i \in \mathbb{R}$, $\alpha_i > 0$ for $i = 1,2,3$ and $\overline{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$. If $b/\alpha_1 < \tau^{\overline{\alpha}^1}(b)$, $\lambda^{\overline{\alpha}^1}(b)/\alpha_2 < \tau^{\overline{\alpha}^2}(b) \leq \alpha_1/\alpha_2$ and $\lambda^{\overline{\alpha}^2}(b)/\alpha_3 < \tau^{\overline{\alpha}^3}(b) \leq \alpha_2/\alpha_3$, the* ***positive 3-step MIR function*** *for the right hand side b and the parameter vector $\overline{\alpha}$, $g_+^{\overline{\alpha}^3,b}(u)$, is defined as*

$$g_+^{\overline{\alpha}^3,b}(u) = \frac{\alpha_1 \delta^{\overline{\alpha}^3,b}(u) - \lambda^{\overline{\alpha}^1}(u)\tau^{\overline{\alpha}^2}(b)\tau^{\overline{\alpha}^3}(b)}{[\alpha_1 - \lambda^{\overline{\alpha}^1}(b)]\tau^{\overline{\alpha}^2}(b)\tau^{\overline{\alpha}^3}(b)} \tag{3.13}$$

where

$$\delta^{\overline{\alpha}^3,b}(u) = \begin{cases} \tau^{\overline{\alpha}^3}(b)\tau^{\overline{\alpha}^2}(b) & \text{if } u \in I_0^{\overline{\alpha}^3,b} \\ \tau^{\overline{\alpha}^3}(b)\tau^{\overline{\alpha}^2}(u) & \text{if } u \in I_1^{\overline{\alpha}^3,b} \\ \tau^{\overline{\alpha}^3}(b)\sigma^{\overline{\alpha}^2}(u) + \tau^{\overline{\alpha}^3}(u) & \text{if } u \in I_2^{\overline{\alpha}^3,b} \\ \tau^{\overline{\alpha}^3}(b)\sigma^{\overline{\alpha}^2}(u) + \sigma^{\overline{\alpha}^3}(u) + \lambda^{\overline{\alpha}^3}(u)/\lambda^{\overline{\alpha}^3}(b) & \text{if } u \in I_3^{\overline{\alpha}^3,b} \end{cases}$$

and

$$\begin{aligned} I_0^{\overline{\alpha}^3,b} &= \left\{u \in \mathbb{R} : \lambda^{\overline{\alpha}^1}(u) \geq \lambda^{\overline{\alpha}^1}(b)\right\} \\ I_1^{\overline{\alpha}^3,b} &= \left\{u \in \mathbb{R} : \lambda^{\overline{\alpha}^1}(u) < \lambda^{\overline{\alpha}^1}(b), \lambda^{\overline{\alpha}^2}(u) \geq \lambda^{\overline{\alpha}^2}(b)\right\} \\ I_2^{\overline{\alpha}^3,b} &= \left\{u \in \mathbb{R} : \lambda^{\overline{\alpha}^1}(u) < \lambda^{\overline{\alpha}^1}(b), \lambda^{\overline{\alpha}^2}(u) < \lambda^{\overline{\alpha}^2}(b), \lambda^{\overline{\alpha}^3}(u) \geq \lambda^{\overline{\alpha}^3}(b)\right\} \\ I_3^{\overline{\alpha}^3,b} &= \left\{u \in \mathbb{R} : \lambda^{\overline{\alpha}^1}(u) < \lambda^{\overline{\alpha}^1}(b), \lambda^{\overline{\alpha}^2}(u) < \lambda^{\overline{\alpha}^2}(b), \lambda^{\overline{\alpha}^3}(u) < \lambda^{\overline{\alpha}^3}(b)\right\}. \end{aligned}$$

Also if $-b/\alpha_1 < \tau^{\overline{\alpha}^1}(-b)$, $\lambda^{\overline{\alpha}^1}(-b)/\alpha_2 < \tau^{\overline{\alpha}^2}(-b) \leq \alpha_1/\alpha_2$ and $\lambda^{\overline{\alpha}^2}(-b)/\alpha_3 < \tau^{\overline{\alpha}^3}(-b) \leq \alpha_2/\alpha_3$, the ***negative 3-step MIR function*** *for the right hand side b with the parameter vector $\overline{\alpha}$, $g_-^{\overline{\alpha}^3,b}(u)$, is defined as*

$$g_-^{\overline{\alpha}^3,b}(u) = g_+^{\overline{\alpha}^3,-b}(-u). \tag{3.14}$$

□

Theorem 3.11. *Let $\beta, \alpha_i \in \mathbb{R}$, $\alpha_i > 0$ for $i = 1,2,3$ and $\overline{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$. The following statements are true for any $k_b, k_u \in \mathbb{Z}$:*

(i). *if* $b/\alpha_1 < \tau^{\overline{\alpha}^1}(b)$, $\lambda^{\overline{\alpha}^1}(b)/\alpha_2 < \tau^{\overline{\alpha}^2}(b) \leq \alpha_1/\alpha_2$ *and* $\lambda^{\overline{\alpha}^2}(b)/\alpha_3 < \tau^{\overline{\alpha}^3}(b) \leq \alpha_2/\alpha_3$, *then* $g_+^{\overline{\alpha}^3,b}(u) = g_+^{\overline{\alpha}^3,k_b\alpha_1+b}(k_u\alpha_1+u)$,

(ii). *if* $-b/\alpha_1 < \tau^{\overline{\alpha}^1}(-b)$, $\lambda^{\overline{\alpha}^1}(-b)/\alpha_2 < \tau^{\overline{\alpha}^2}(-b) \leq \alpha_1/\alpha_2$ *and* $\lambda^{\overline{\alpha}^2}(-b)/\alpha_3 < \tau^{\overline{\alpha}^3}(-b) \leq \alpha_2/\alpha_3$, *then* $g_-^{\overline{\alpha}^3,b}(u) = g_-^{\overline{\alpha}^3,k_b\alpha_1+b}(k_u\alpha_1+u)$ *and* $g_-^{\overline{\alpha}^3,b}(u) = g_+^{\overline{\alpha}^3,k_b\alpha_1-b}(k_u\alpha_1-u)$.

Proof. From (3.13), it can be seen that $g_+^{\overline{\alpha}^3,b}(u)$ is a function of only $\lambda^{\overline{\alpha}^1}(u)$ and $\lambda^{\overline{\alpha}^1}(b)$. This is true because $\lambda^{\overline{\alpha}^2}(.)$, $\lambda^{\overline{\alpha}^3}(.)$, $\sigma^{\overline{\alpha}^2}(.)$, $\sigma^{\overline{\alpha}^3}(.)$, $\tau^{\overline{\alpha}^2}(.)$ and $\tau^{\overline{\alpha}^3}(.)$ are all functions of $\lambda^{\overline{\alpha}^1}(.)$. In the proof of Theorem 2.6, we saw that $\lambda^{\overline{\alpha}^1}(.)$ is periodic with period α_1; therefore $g_+^{\overline{\alpha}^3,b}(u)$, as a function of $\lambda^{\overline{\alpha}^1}(u)$ and $\lambda^{\overline{\alpha}^1}(b)$, is also periodic in u and b with period α_1 and *(i)* is true. The argument for *(ii)* is very similar to the argument for *(ii)* in the proof of Theorem 2.6. □

Theorem 3.12. *Let* $\beta, \alpha_i \in \mathbb{R}$, $\alpha_i > 0$ *for* $i = 1,2,3$ *and* $\overline{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$. *The* ***positive 3-step MIR inequality***

$$\sum_{j\in J} g_+^{\overline{\alpha}^3,b}(a_j)x_j \geq 1 \tag{3.15}$$

is valid for $Y_{\overline{a}}^{\alpha_1,b}$ *if* $b/\alpha_1 < \tau^{\overline{\alpha}^1}(b)$, $\lambda^{\overline{\alpha}^1}(b)/\alpha_2 < \tau^{\overline{\alpha}^2}(b) \leq \alpha_1/\alpha_2$, *and* $\lambda^{\overline{\alpha}^2}(b)/\alpha_3 < \tau^{\overline{\alpha}^3}(b) \leq \alpha_2/\alpha_3$, *and the* ***negative 3-step MIR inequality***

$$\sum_{j\in J} g_-^{\overline{\alpha}^3,b}(a_j)x_j \geq 1 \tag{3.16}$$

is valid for $Y_{\overline{a}}^{\alpha_1,b}$ *if* $-b/\alpha_1 < \tau^{\overline{\alpha}^1}(-b)$, $\lambda^{\overline{\alpha}^1}(-b)/\alpha_2 < \tau^{\overline{\alpha}^2}(-b) \leq \alpha_1/\alpha_2$, *and* $\lambda^{\overline{\alpha}^2}(-b)/\alpha_3 < \tau^{\overline{\alpha}^3}(-b) \leq \alpha_2/\alpha_3$.

Proof. First, we prove (3.15) assuming its required conditions hold true. We start from (2.9), i.e.

$$\sum_{j\in J} a_j x_j + \alpha_1 z = b.$$

Assume that J_0, J_1, J_2 and J_3 form a partition of J. We can relax (2.9) as

$$\begin{aligned}\alpha_1 z + \sum_{j\in J_0} \alpha_1 \tau^{\overline{\alpha}^1}(a_j)x_j + \sum_{j\in J_1} \left[\alpha_1\sigma^{\overline{\alpha}^1}(a_j) + \alpha_2\tau^{\overline{\alpha}^2}(a_j)\right] x_j \\ + \sum_{j\in J_2} \left[\alpha_1\sigma^{\overline{\alpha}^1}(a_j) + \alpha_2\sigma^{\overline{\alpha}^2}(a_j) + \alpha_3\tau^{\overline{\alpha}^3}(a_j)\right] x_j \\ + \sum_{j\in J_3} \left[\alpha_1\sigma^{\overline{\alpha}^1}(a_j) + \alpha_2\sigma^{\overline{\alpha}^2}(a_j) + \alpha_3\sigma^{\overline{\alpha}^3}(a_j) + \lambda^{\overline{\alpha}^3}(a_j)\right] x_j \geq b.\end{aligned}$$

To see that this is a relaxation of (2.9), notice that, for $j \in J_0$, $\alpha_1 \tau^{\overline{\alpha}^1}(a_j) \geq a_j$ is obvious. For $j \in J_1$, we have $\alpha_1 \sigma^{\overline{\alpha}^1}(a_j) + \alpha_2 \tau^{\overline{\alpha}^2}(a_j) \geq a_j$ because, based on identities (3.1), this can be written as $\alpha_2 \tau^{\overline{\alpha}^2}(a_j) \geq \lambda^{\overline{\alpha}^1}(a_j)$, which is trivial. Similarly, for $j \in J_2$, we have $\alpha_1 \sigma^{\overline{\alpha}^1}(a_j) + \alpha_2 \sigma^{\overline{\alpha}^2}(a_j) + \alpha_3 \tau^{\overline{\alpha}^3}(a_j) \geq a_j$ because, based on (3.1), this can be written as $\alpha_3 \tau^{\overline{\alpha}^3}(a_j) \geq \lambda^{\overline{\alpha}^2}(a_j)$, which is also trivial. Finally, for $j \in J_3$, no relaxation of the coefficients is done because, by (3.1), we have $\alpha_1 \sigma^{\overline{\alpha}^1}(a_j) + \alpha_2 \sigma^{\overline{\alpha}^2}(a_j) + \alpha_3 \sigma^{\overline{\alpha}^3}(a_j) + \lambda^{\overline{\alpha}^3}(a_j) = a_j$.

Now the inequality above can be written as

$$\begin{aligned}\alpha_1 \left[z + \sum_{j \in J_0} \tau^{\overline{\alpha}^1}(a_j) x_j + \sum_{j \in J_1} \sigma^{\overline{\alpha}^1}(a_j) x_j + \sum_{j \in J_2} \sigma^{\overline{\alpha}^1}(a_j) x_j + \sum_{j \in J_3} \sigma^{\overline{\alpha}^1}(a_j) x_j \right] \\ + \alpha_2 \left[\sum_{j \in J_1} \tau^{\overline{\alpha}^2}(a_j) x_j + \sum_{j \in J_2} \sigma^{\overline{\alpha}^2}(a_j) x_j + \sum_{j \in J_3} \sigma^{\overline{\alpha}^2}(a_j) x_j \right] \\ + \alpha_3 \left[\sum_{j \in J_2} \tau^{\overline{\alpha}^3}(a_j) x_j + \sum_{j \in J_3} \sigma^{\overline{\alpha}^3}(a_j) x_j \right] + \left[\sum_{j \in J_3} \lambda^{\overline{\alpha}^3}(a_j) x_j \right] \geq b \end{aligned} \tag{3.17}$$

The terms in the first to the fourth brackets in (3.17) play the roles of y_1, y_2, y_3 and v in $Q^{\overline{\alpha}^3,b}$, respectively. The sign and integrality conditions match. The first expression is an integer, the second and third ones are non-negative integers and the fourth one a non-negative real number. Since it is assumed $b/\alpha_1 < \tau^{\overline{\alpha}^1}(b)$, $\lambda^{\overline{\alpha}^1}(b)/\alpha_2 < \tau^{\overline{\alpha}^2}(b) \leq \alpha_1/\alpha_2$ and $\lambda^{\overline{\alpha}^2}(b)/\alpha_3 < \tau^{\overline{\alpha}^3}(b) \leq \alpha_2/\alpha_3$, if y_1, y_2, y_3 and v in the 3-step MIR facet (3.5) are replaced by the corresponding terms in (3.17), by Theorem 3.7, the inequality obtained will be valid for $Y_{\overline{a}}^{\alpha_1,b}$. Doing so, we get

$$\begin{aligned}&\tau^{\overline{\alpha}^2}(b)\tau^{\overline{\alpha}^3}(b) \left[z + \sum_{j \in J_0} \tau^{\overline{\alpha}^1}(a_j) x_j + \sum_{j \in J_1} \sigma^{\overline{\alpha}^1}(a_j) x_j + \sum_{j \in J_2} \sigma^{\overline{\alpha}^1}(a_j) x_j + \sum_{j \in J_3} \sigma^{\overline{\alpha}^1}(a_j) x_j \right] \\ &+ \tau^{\overline{\alpha}^3}(b) \left[\sum_{j \in J_1} \tau^{\overline{\alpha}^2}(a_j) x_j + \sum_{j \in J_2} \sigma^{\overline{\alpha}^2}(a_j) x_j + \sum_{j \in J_3} \sigma^{\overline{\alpha}^2}(a_j) x_j \right] \\ &+ \left[\sum_{j \in J_2} \tau^{\overline{\alpha}^3}(a_j) x_j + \sum_{j \in J_3} \sigma^{\overline{\alpha}^3}(a_j) x_j \right] + \frac{1}{\lambda^{\overline{\alpha}^3}(b)} \left[\sum_{j \in J_3} \lambda^{\overline{\alpha}^3}(a_j) x_j \right] \geq \tau^{\overline{\alpha}^1}(b)\tau^{\overline{\alpha}^2}(b)\tau^{\overline{\alpha}^3}(b)\end{aligned}$$

If we multiply both sides by α_1 and replace for $\alpha_1 z$ from (2.9), we get

$$\begin{aligned}&\tau^{\overline{\alpha}^2}(b)\tau^{\overline{\alpha}^3}(b) \left[b - \sum_{j \in J} a_j x_j \right] \\ &+ \alpha_1 \tau^{\overline{\alpha}^2}(b)\tau^{\overline{\alpha}^3}(b) \left[\sum_{j \in J_0} \tau^{\overline{\alpha}^1}(a_j) x_j + \sum_{j \in J_1} \sigma^{\overline{\alpha}^1}(a_j) x_j + \sum_{j \in J_2} \sigma^{\overline{\alpha}^1}(a_j) x_j + \sum_{j \in J_3} \sigma^{\overline{\alpha}^1}(a_j) x_j \right]\end{aligned}$$

$$+\alpha_1\tau^{\overline{\alpha}^3}(b)\left[\sum_{j\in J_1}\tau^{\overline{\alpha}^2}(a_j)x_j+\sum_{j\in J_2}\sigma^{\overline{\alpha}^2}(a_j)x_j+\sum_{j\in J_3}\sigma^{\overline{\alpha}^2}(a_j)x_j\right]$$

$$+\alpha_1\left[\sum_{j\in J_2}\tau^{\overline{\alpha}^3}(a_j)x_j+\sum_{j\in J_3}\sigma^{\overline{\alpha}^3}(a_j)x_j\right]+\frac{\alpha_1}{\lambda^{\overline{\alpha}^3}(b)}\left[\sum_{j\in J_3}\lambda^{\overline{\alpha}^3}(a_j)x_j\right]\geq\alpha_1\tau^{\overline{\alpha}^1}(b)\tau^{\overline{\alpha}^2}(b)\tau^{\overline{\alpha}^3}(b)$$

By rearranging some terms, we will have

$$\sum_{j\in J}\left(\alpha_1\delta^{\overline{\alpha}^3,b}(a_j)-\lambda^{\overline{\alpha}^1}(a_j)\tau^{\overline{\alpha}^2}(b)\tau^{\overline{\alpha}^3}(b)\right)x_j\geq[\alpha_1-\lambda^{\overline{\alpha}^1}(b)]\tau^{\overline{\alpha}^2}(b)\tau^{\overline{\alpha}^3}(b) \tag{3.18}$$

where

$$\delta^{\overline{\alpha}^3,b}(a_j)=\begin{cases}\tau^{\overline{\alpha}^3}(b)\tau^{\overline{\alpha}^2}(b) & \text{if } j\in J_0\\ \tau^{\overline{\alpha}^3}(b)\tau^{\overline{\alpha}^2}(a_j) & \text{if } j\in J_1\\ \tau^{\overline{\alpha}^3}(b)\sigma^{\overline{\alpha}^2}(a_j)+\tau^{\overline{\alpha}^3}(a_j) & \text{if } j\in J_2\\ \tau^{\overline{\alpha}^3}(b)\sigma^{\overline{\alpha}^2}(a_j)+\sigma^{\overline{\alpha}^3}(a_j)+\lambda^{\overline{\alpha}^3}(a_j)/\lambda^{\overline{\alpha}^3}(b) & \text{if } j\in J_3\end{cases}$$

To obtain the strongest inequality the coefficients of x_j's should be minimized. In other words, the partitioning of J into J_0, J_1, J_2 and J_3 should be determined such that $\delta^{\overline{\alpha}^3,b}(a_j)$ gets the minimum of the four values above. It is not difficult to verify that the partitioning should be as follows

$$\begin{aligned}J_0 &= \left\{j\in J:\lambda^{\overline{\alpha}^1}(a_j)\geq\lambda^{\overline{\alpha}^1}(b)\right\}\\ J_1 &= \left\{j\in J:\lambda^{\overline{\alpha}^1}(a_j)<\lambda^{\overline{\alpha}^1}(b),\lambda^{\overline{\alpha}^2}(a_j)\geq\lambda^{\overline{\alpha}^2}(b)\right\}\\ J_2 &= \left\{j\in J:\lambda^{\overline{\alpha}^1}(a_j)<\lambda^{\overline{\alpha}^1}(b),\lambda^{\overline{\alpha}^2}(a_j)<\lambda^{\overline{\alpha}^2}(b),\lambda^{\overline{\alpha}^3}(a_j)\geq\lambda^{\overline{\alpha}^3}(b)\right\}\\ J_3 &= \left\{j\in J:\lambda^{\overline{\alpha}^1}(a_j)<\lambda^{\overline{\alpha}^1}(b),\lambda^{\overline{\alpha}^2}(a_j)<\lambda^{\overline{\alpha}^2}(b),\lambda^{\overline{\alpha}^3}(a_j)<\lambda^{\overline{\alpha}^3}(b)\right\}.\end{aligned}$$

This partitioning along with (3.18) gives (3.15). Thus (3.15) is valid for $Y_{\overline{a}}^{\alpha_1,b}$.

Now to prove the validity of (3.16), assuming its required conditions hold true, the argument is similar to the proof of Theorem 2.19. The set $Y_{\overline{a}}^{\alpha_1,b}$ can also be expressed as in (2.29). Therefore the above argument can be repeated exactly starting with

$$\sum_{j\in J}-a_jx_j+\alpha_1z=-b$$

and with a_j and b replaced by $-a_j$ and $-b$. Since it is assumed $-b/\alpha_1<\tau^{\overline{\alpha}^1}(-b)$, $\lambda^{\overline{\alpha}^1}(-b)/\alpha_2<\tau^{\overline{\alpha}^2}(-b)\leq\alpha_1/\alpha_2$ and $\lambda^{\overline{\alpha}^2}(-b)/\alpha_3<\tau^{\overline{\alpha}^3}(-b)\leq\alpha_2/\alpha_3$, the argument leads to

$$\sum_{j\in J}g_+^{\overline{\alpha}^3,-b}(-a_j)x_j\geq1$$

which, by Definition 3.10, is the same as (3.16). □

As in the case of the 1-step and 2-step MIR inequalities, the special case of $Y_{\overline{a}}^{1,b}$ is of particular interest:

Corollary 3.13. *Let $t \in \mathbb{N}$, $\alpha_2, \alpha_3, b \in \mathbb{R}$, $\alpha_2, \alpha_3 > 0$. The 3-step MIR inequality*

$$\sum_{j \in J} g_+^{(1/t,\alpha_2,\alpha_3),b}(a_j)x_j \geq 1 \tag{3.19}$$

is valid for $Y_{\overline{a}}^{1,b}$ if $tb < \tau^{1/t}(b)$, $\lambda^{1/t}(b)/\alpha_2 < \tau^{(1/t,\alpha_2)}(b) \leq 1/(t\alpha_2)$ and $\lambda^{(1/t,\alpha_2)}(b)/\alpha_3 < \tau^{(1/t,\alpha_2,\alpha_3)}(b) \leq \alpha_2/\alpha_3$, the 3-step MIR inequality

$$\sum_{j \in J} g_-^{(1/t,\alpha_2,\alpha_3),b}(a_j)x_j \geq 1 \tag{3.20}$$

is valid for $Y_{\overline{a}}^{1,b}$ if $-tb < \tau^{1/t}(-b)$, $\lambda^{1/t}(-b)/\alpha_2 < \tau^{(1/t,\alpha_2)}(-b) \leq 1/(t\alpha_2)$ and $\lambda^{(1/t,\alpha_2)}(-b)/\alpha_3 < \tau^{(1/t,\alpha_2,\alpha_3)}(-b) \leq \alpha_2/\alpha_3$.

Proof. As argued in the proof of Corollary 2.9, $Y_{\overline{a}}^{1/t,b}$, where $t \in \mathbb{N}$, is a relaxation of $Y_{\overline{a}}^{1,b}$. Therefore any valid inequality for $Y_{\overline{a}}^{1/t,b}$ is also valid for $Y_{\overline{a}}^{1,b}$. By Theorem 3.12, (3.19) and (3.20) are valid for $Y_{\overline{a}}^{1/t,b}$ if the respective conditions are satisfied. Hence they are valid for $Y_{\overline{a}}^{1,b}$ if the same conditions hold true. □

Example 3.14. Let $\overline{\alpha} = (\alpha_1, \alpha_2, \alpha_3) = (1, 0.3, 0.08)$ and $b = 0.8$. In Example 3.9, we observed that these values satisfy the conditions of Theorem 3.12 for the positive 3-step MIR inequality. Figures 3.1 and 3.2 show the graphs of $g_+^{(1,0.3,0.08),k_1+0.8}(u)$ and $g_-^{(1,0.3,0.08),k_2+0.2}(u)$ for $u \in [0,1]$. These functions are periodic with period $\alpha_1 = 1$. Also observe that $g_-^{(1,0.3,0.08),k_1+0.2}(u) = g_+^{(1,0.3,0.08),k_2+0.8}(1-u)$, as stated in Theorem 2.18. By Theorem 3.12, using $g_+^{(1,0.3,0.08),2.8}(u)$, a 3-step MIR valid inequality for the set

$$\left\{(x_1,x_2,x_3,x_4,x_5) \in \mathbb{Z}_+^5 : 0.1x_1 + 0.28x_2 + 0.45x_3 + 0.6x_4 + 0.95x_5 + z = 2.8; z \in \mathbb{Z}\right\}$$

is

$$g_+^{(1,0.3,0.08),2.8}(0.1)x_1 + g_+^{(1,0.3,0.08),2.8}(0.28)x_2 + g_+^{(1,0.3,0.08),2.8}(0.45)x_3 \\ + g_+^{(1,0.3,0.08),2.8}(0.6)x_4 + g_+^{(1,0.3,0.08),2.8}(0.95)x_5 \geq 1,$$

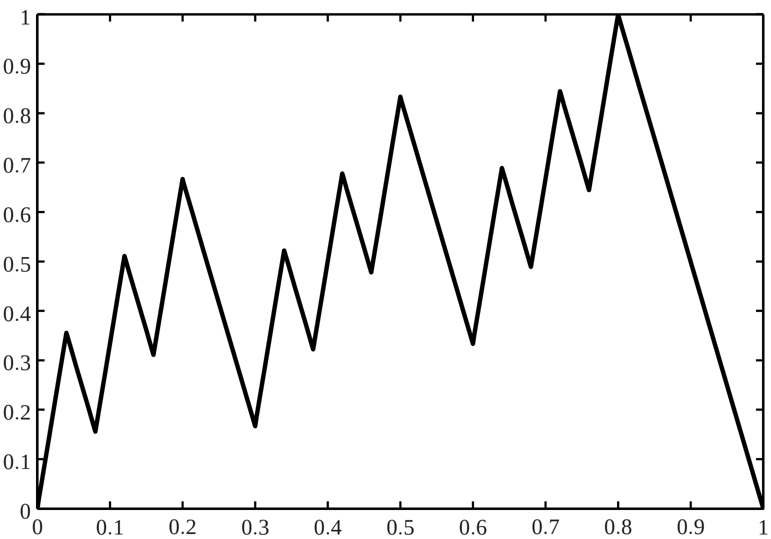


Figure 3.1: Graph of $g_{+}^{(1,0.3,0.08),k_1+0.8}(u)$ for $u \in [0,1]$

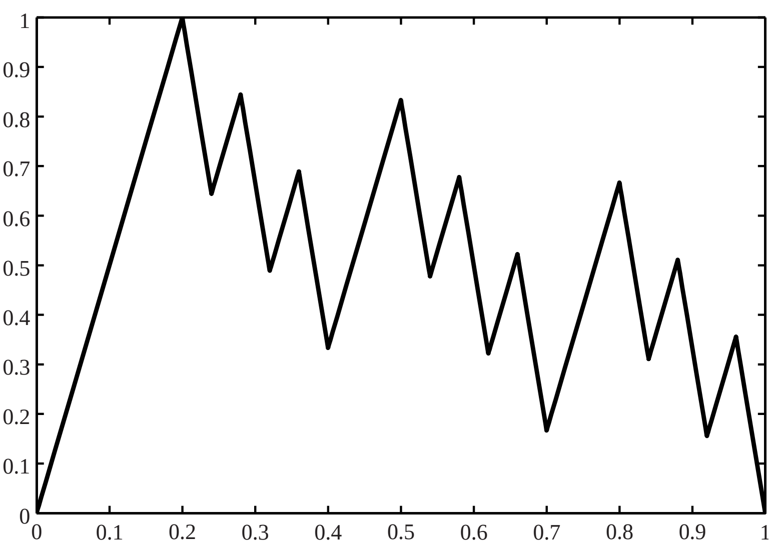


Figure 3.2: Graph of $g_{-}^{(1,0.3,0.08),k_2+0.2}(u)$ for $u \in [0,1]$

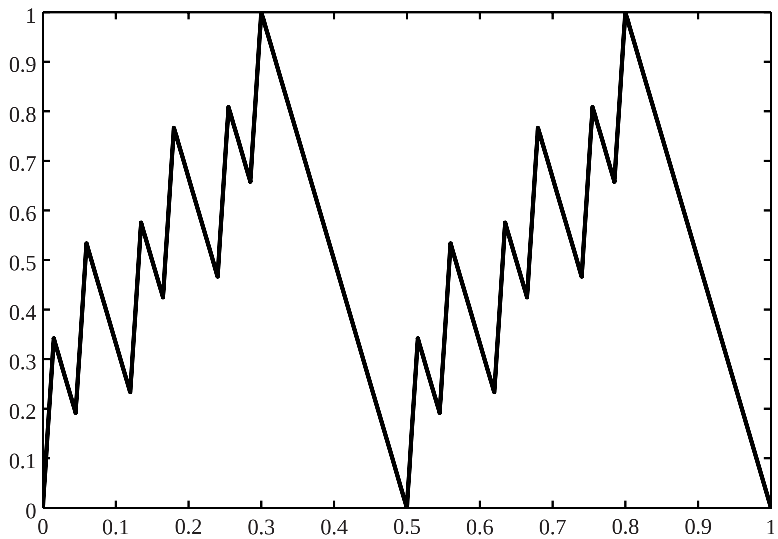


Figure 3.3: Graph of $g_+^{(0.5,0.12,0.045),k_1+0.8}(u)$ for $u \in [0,1]$

or

$$0.333x_1 + 0.267x_2 + 0.528x_3 + 0.333x_4 + 0.25x_5 \geq 1.$$

Observe that this inequality dominates the 2-step inequality developed for this set in Example 2.21 using $g_+^{(1,0.3),2.8}(u)$.

Another example for the right hand side $b = k_1 + 0.8$ could be obtained by setting $\alpha_1 = 0.5$ and $\alpha_2 = 0.12$. Figure 3.3 shows the graph of $g_+^{(0.5,0.12,0.045),k_1+0.8}(u)$ for $u \in [0,1]$. The valid inequality obtained for the set above using this 3-step MIR function is

$$g_+^{(0.5,0.12,0.045),2.8}(0.1)x_1 + g_+^{(0.5,0.12,0.045),2.8}(0.28)x_2 + g_+^{(0.5,0.12,0.045),2.8}(0.45)x_3 + \\ g_+^{((0.5,0.12,0.045),2.8}(0.6)x_4 + g_+^{(0.5,0.12,0.045),2.8}(0.95)x_5 \geq 1,$$

or

$$0.333x_1 + 0.683x_2 + 0.25x_3 + 0.333x_4 + 0.25x_5 \geq 1.$$

Again this inequality dominates the 2-step inequality developed in Example 2.21 using $g_+^{(0.5,0.12),2.8}(u)$. □

Chapter 4

Generalization to n-step MIR

In this chapter, we show that the generalization presented in chapter 3 is not limited to the 3-step MIR and can be extended to any higher dimension. The result is the introduction of the *n-step MIR* for any $n \in \mathbb{N}$. More specifically, in section 4.1 we prove that for any $n \in \mathbb{N}$, under certain simple conditions, an *n-step MIR facet* exists for an $(n+1)$-dimensional mixed integer set. Similar to the case of 3-step MIR, along the path of our proof, we will show that for each n, each of the previous $n-1$ MIR facets (i.e. the 1-step MIR to the $(n-1)$-step MIR facet) generates a facet for this $(n+1)$-dimensional set under similar simple conditions. All these facets will be used in derivation of the n-step MIR facet. Then in section 4.2, we show that for any $n \in \mathbb{N}$, the n-step MIR facet can be used to derive the *n-step MIR inequality* for the feasible set of a general IP constraint, i.e. $Y_{\overline{a}}^{\alpha_1,b}$. The result will be introduction of the *n-step MIR functions* as easy tools for generating these inequalities. In section 4.3, we will discuss that the development of n-step MIR inequalities can be easily extended to the feasible set of a general MIP constraint, that is a set which includes continuous variables in addition to integer variables. In section 4.4, we discuss the computer codes that generate values of the n-step MIR function at any given point, as well as the codes that plot graphs of these functions. The codes are brought in Appendix A.

4.1 n-step MIR Facet

In this section we develop the n-step MIR facet for our certain $(n+1)$-dimensional mixed integer set. The form of this set is the natural generalization of the sets $Q^{\overline{\alpha}^1,\beta}$, $Q^{\overline{\alpha}^2,\beta}$, and $Q^{\overline{\alpha}^3,\beta}$, which were discussed in chapters 2 and 3. We present this general form as a definition:

Definition 4.1. *For any $n \in \mathbb{N}$, $\beta \in \mathbb{R}$ and $\overline{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \mathbb{R}$, $\alpha_i > 0$, we define the set $Q^{\overline{\alpha}^n,\beta}$ as follows:*

$$Q^{\overline{\alpha}^n,\beta} = \Big\{(y_1, \ldots, y_n, v) \in \mathbb{Z} \times \mathbb{Z}_+^{n-1} \times \mathbb{R}_+ : \sum_{i=1}^{n} \alpha_i y_i + v \geq \beta\Big\}.$$

□

It can be seen that the sets considered in chapters 2 and 3 are the special cases for $n = 1, 2,$ and 3. In Definition 4.2, we define one inequality for each $n \in \mathbb{N}$, that depends on n and we refer to as MIR_n. MIR_n is the inequality which we intend to prove that is the n-step MIR facet for $Q^{\overline{\alpha}^n,\beta}$. However, to avoid confusion we will not refer to it as the n-step MIR facet until the proof is complete. After the completion of the proof at the end of this section, in Definition 4.9 we will define it as the n-step MIR facet. For notational convenience, we assume that if $a > b$, $\sum_{l=a}^{b}(.) = 0$ and $\prod_{l=a}^{b}(.) = 1$, throughout the whole dissertation.

Definition 4.2. *For $n \in \mathbb{N}$, $\beta \in \mathbb{R}$ and $\overline{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \mathbb{R}$, $\alpha_i > 0$ for $i = 1, \ldots, n$, such that $\beta/\alpha_1 < \tau^{\overline{\alpha}^1}(\beta)$ and $\lambda^{\overline{\alpha}^{i-1}}(\beta)/\alpha_i < \tau^{\overline{\alpha}^i}(\beta) \leq \alpha_{i-1}/\alpha_i$ for $i = 2, \ldots, n$, we define the inequality MIR_n (that depends on n) as follows:*

$$\sum_{i=1}^{n} \Big(\prod_{l=i+1}^{n} \tau^{\overline{\alpha}^l}(\beta) \Big) y_i + \frac{1}{\lambda^{\overline{\alpha}^n}(\beta)} v \geq \prod_{l=1}^{n} \tau^{\overline{\alpha}^l}(\beta) \qquad (MIR_n)$$

□

Therefore our goal is to prove the following main theorem:

Theorem 4.3. *For any $n \in \mathbb{N}$, let $\beta \in \mathbb{R}$ be any scalar and $\overline{\alpha} = (\alpha_1, \ldots, \alpha_n)$ be any n-vector such that $\alpha_i \in \mathbb{R}$, $\alpha_i > 0$ for $i = 1, \ldots, n$ and $\beta/\alpha_1 < \tau^{\overline{\alpha}^1}(\beta)$ and $\lambda^{\overline{\alpha}^{i-1}}(\beta)/\alpha_i < \tau^{\overline{\alpha}^i}(\beta) \leq \alpha_{i-1}/\alpha_i$ for $i = 2, \ldots, n$. Then the inequality MIR_n is valid and facet-defining for $Q^{\overline{\alpha}^n,\beta}$.*

Proof. The proof will be presented at the end of this section after several preliminary lemmas are proved. □

It can be observed that MIR_1, MIR_2 and MIR_3 are the 1-, 2- and 3-step MIR facets presented in inequalities (2.1), (2.18), and (3.5), respectively. As mentioned, we require to prove several lemmas which we will use in the proof of Theorem 4.3. The line of proof followed by these lemmas and Theorem 4.3 is the generalization of the line of proof in Theorems 2.14 and 3.7 for the 2- and 3-step MIR facets, respectively. This generalization will be based on induction.

We start with defining a general notation for a collection of points which we will use to prove the facet-defining property of MIR_n. To this end, in many of the lemmas and theorems that follow, we will use a corresponding affinely-independent subset of these points.

Definition 4.4. *Let $n \in \mathbb{N}$, $\beta \in \mathbb{R}$ and $\overline{\alpha} = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \in \mathbb{R}$, $\alpha_i > 0$ for $i = 1, \dots, n$. For $l \in \mathbb{N}$, $l \le n$, we define the point $p_l^n \in \mathbb{Z} \times \mathbb{Z}_+^{n-1} \times \mathbb{R}_+$ as $p_l^n = (y_1, \dots, y_n, v)$, such that*

$$y_i = \begin{cases} \sigma^{\overline{\alpha}^i}(\beta) & \text{if } i = 1, \dots, l-1 \\ \tau^{\overline{\alpha}^i}(\beta) & \text{if } i = l \\ 0 & \text{if } i = l+1, \dots, n \end{cases} \qquad \text{and } v = 0;$$

Also we define the point $q_l^n \in \mathbb{Z} \times \mathbb{Z}_+^{n-1} \times \mathbb{R}_+$ as $q_l^n = (y_1, \dots, y_n, v)$, such that

$$y_i = \begin{cases} \sigma^{\overline{\alpha}^i}(\beta) & \text{if } i = 1, \dots, l \\ 0 & \text{if } i = l+1, \dots, n \end{cases} \qquad \text{and } v = \lambda^{\overline{\alpha}^l}(\beta).$$

□

In order to employ these points for the aforementioned purpose, we need to prove that for each n, all of the points p_l^n and q_l^n, for $l = 1, \dots, n$, belong to $Q^{\overline{\alpha}^n, \beta}$. Lemma 4.5 proves this statement.

Lemma 4.5. *Let $n \in \mathbb{N}$, $\beta \in \mathbb{R}$ and $\overline{\alpha} = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \in \mathbb{R}$, $\alpha_i > 0$ for $i = 1, \dots, n$. The points p_l^n and q_l^n, $l = 1, \dots, n$, belong to $Q^{\overline{\alpha}^n, \beta}$.*

Proof. By definition it is clear that all the points p_l^n and q_l^n, for $l = 1, \dots, n$, belong to $\mathbb{Z} \times \mathbb{Z}_+^{n-1} \times \mathbb{R}_+$. Therefore it remains to verify that they satisfy the inequality $\sum_{i=1}^n \alpha_i y_i + v \ge \beta$. If we substitute the coordinates of p_l^n into the defining inequality of $Q^{\overline{\alpha}^n, \beta}$, we get $\sum_{i=1}^{l-1} \alpha_i \sigma^{\overline{\alpha}^i}(\beta) + \alpha_l \tau^{\overline{\alpha}^l}(\beta) \ge \beta$. Using identities (3.1), this is equivalent to $\alpha_l \tau^{\overline{\alpha}^l}(\beta) \ge \lambda^{\overline{\alpha}^{l-1}}(\beta)$, which is trivial. For q_l^n, doing the same, gives $\sum_{i=1}^l \alpha_i \sigma^{\overline{\alpha}^i}(\beta) + \lambda^{\overline{\alpha}^l}(\beta) \ge \beta$, which again by (3.1) is equivalent to $\lambda^{\overline{\alpha}^l}(\beta) \ge \lambda^{\overline{\alpha}^l}(\beta)$, which is also trivial. Thus $p_l^n, q_l^n \in Q^{\overline{\alpha}^n, \beta}$, for $l = 1, \dots, n$. □

In the proof of Theorem 3.7, we observed that inequalities (3.2) and (3.4) developed in Lemmas 3.5 and 3.6 were used to derive the 3-step MIR facet. The next two lemmas are in a sense the generalization of the results presented in Lemmas 3.5 and 3.6. First, we show that in general if the inequality MIR_{n_1} is valid for $Q^{\overline{\alpha}^{n_1}, \beta}$, $n_1 \in \mathbb{N}$, then it can be used to produce a valid inequality for $Q^{\overline{\alpha}^n, \beta}$ for any $n \in \mathbb{N}$ where $n > n_1$.

Lemma 4.6. *Let $n_1, n \in \mathbb{N}$, $n_1 < n$, $\beta \in \mathbb{R}$ and $\overline{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \mathbb{R}$, $\alpha_i > 0$ for $i = 1, \ldots, n$, such that $\beta/\alpha_1 < \tau^{\overline{\alpha}^1}(\beta)$ and $\lambda^{\overline{\alpha}^{i-1}}(\beta)/\alpha_i < \tau^{\overline{\alpha}^i}(\beta) \le \alpha_{i-1}/\alpha_i$ for $i = 2, \ldots, n_1$. If the inequality MIR_{n_1} is valid for $Q^{\overline{\alpha}^{n_1},\beta}$, then the following inequality is valid for $Q^{\overline{\alpha}^n,\beta}$:*

$$\sum_{i=1}^{n_1} \left(\prod_{l=i+1}^{n_1} \tau^{\overline{\alpha}^l}(\beta) \right) y_i + \sum_{i=n_1+1}^{n} \left(\frac{\alpha_i}{\lambda^{\overline{\alpha}^{n_1}}(\beta)} \right) y_i + \left(\frac{1}{\lambda^{\overline{\alpha}^{n_1}}(\beta)} \right) v \ge \prod_{l=1}^{n_1} \tau^{\overline{\alpha}^l}(\beta). \tag{4.1}$$

Proof. MIR_{n_1} is the following inequality:

$$\sum_{i=1}^{n_1} \left(\prod_{l=i+1}^{n_1} \tau^{\overline{\alpha}^l}(\beta) \right) y_i + \frac{1}{\lambda^{\overline{\alpha}^{n_1}}(\beta)} v \ge \prod_{l=1}^{n_1} \tau^{\overline{\alpha}^l}(\beta)$$

Similar to Lemmas 3.5 and 3.6, the proof is by making correspondence between the variables in $Q^{\overline{\alpha}^n,\beta}$ and the variables in $Q^{\overline{\alpha}^{n_1},\beta}$. In $Q^{\overline{\alpha}^n,\beta}$, we have $y_1 \in \mathbb{Z}$, $y_2, \ldots, y_{n_1} \in \mathbb{Z}_+$ and $\sum_{i=n_1+1}^{n} \alpha_i y_i + v \in \mathbb{R}_+$. Therefore if we treat y_1, $y_2, \ldots$, y_{n_1} and $\sum_{i=n_1+1}^{n} \alpha_i y_i + v$ in $Q^{\overline{\alpha}^n,\beta}$ as y_1, $y_2, \ldots$, y_{n_1} and v in $Q^{\overline{\alpha}^{n_1},\beta}$, respectively, and substitute into MIR_{n_1}, the result would be valid for $Q^{\overline{\alpha}^{n_1},\beta}$. If we do so, we get (4.1). □

In Lemma 3.5 we observed that if $\sigma^{\overline{\alpha}^2}(\beta) \ge 1$ and $\sigma^{\overline{\alpha}^3}(\beta) \ge 1$, then inequality (3.2) is a facet for $Q^{\overline{\alpha}^3,\beta}$. Also, in Lemma 3.6 we observed the same fact for inequality (3.4) if $\sigma^{\overline{\alpha}^3}(\beta) \ge 1$ is satisfied. Lemma 4.7 is a generalization of these results and gives the general condition for the inequality (4.1) to be a facet for $Q^{\overline{\alpha}^n,\beta}$ in addition to being valid. The facet-defining property of (4.1) will not be used in the proof of Theorem 4.3. It is, however, an interesting result which we prove here in Lemma 4.7, and later we will mention that based on this result, in addition to MIR_n, $n-1$ other facets of the form (4.1) also exist for $Q^{\overline{\alpha}^n,\beta}$, which are developed based on the facets MIR_1 to MIR_{n-1} for $Q^{\overline{\alpha}^1,\beta}$ to $Q^{\overline{\alpha}^{n-1},\beta}$, respectively.

Lemma 4.7. *Let $n_1, n \in \mathbb{N}$, $n_1 < n$, $\beta \in \mathbb{R}$ and $\overline{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \mathbb{R}$, $\alpha_i > 0$ for $i = 1, \ldots, n$, such that $\beta/\alpha_1 < \tau^{\overline{\alpha}^1}(\beta)$ and $\lambda^{\overline{\alpha}^{i-1}}(\beta)/\alpha_i < \tau^{\overline{\alpha}^i}(\beta) \le \alpha_{i-1}/\alpha_i$ for $i = 2, \ldots, n_1$. If the inequality MIR_{n_1} is valid for $Q^{\overline{\alpha}^{n_1},\beta}$ and the additional conditions $\sigma^{\overline{\alpha}^i}(\beta) \ge 1$ are satisfied for $i = n_1+1, ..., n$, then inequality* (4.1) *is facet-defining for $Q^{\overline{\alpha}^n,\beta}$.*

Proof. By Lemma 4.6, we know that (4.1) is valid for $Q^{\overline{\alpha}^n,\beta}$. It remains to find $n+1$ affinely independent points in $Q^{\overline{\alpha}^n,\beta}$ that lie on the hyperplane of inequality (4.1). By Lemma 4.5, we know that the points $q_{n_1}^n, \ldots, q_n^n, p_1^n, \ldots, p_{n_1}^n \in Q^{\overline{\alpha}^n,\beta}$ belong to $Q^{\overline{\alpha}^n,\beta}$. These points satisfy (4.1) at equality: for any $q_{n_2}^n$, $n_1 \le n_2 \le n$, if we replace the coordinates into the left-hand side of (4.1), we will have

$$\sum_{i=1}^{n_1} \left(\prod_{l=i+1}^{n_1} \tau^{\overline{\alpha}^l}(\beta) \right) \sigma^{\overline{\alpha}^i}(\beta) + \sum_{i=n_1+1}^{n_2} \left(\frac{\alpha_i \sigma^{\overline{\alpha}^i}(\beta)}{\lambda^{\overline{\alpha}^{n_1}}(\beta)} \right) + \frac{\lambda^{\overline{\alpha}^{n_2}}(\beta)}{\lambda^{\overline{\alpha}^{n_1}}(\beta)}$$

By identities (3.1), we have $\sum_{i=n_1+1}^{n_2} \alpha_i \sigma^{\overline{\alpha}^i}(\beta) + \lambda^{\overline{\alpha}^{n_2}}(\beta) = \lambda^{\overline{\alpha}^{n_1}}(\beta)$. Therefore the expression above simplifies to

$$
\begin{aligned}
\sum_{i=1}^{n_1}\left(\prod_{l=i+1}^{n_1}\tau^{\overline{\alpha}^l}(\beta)\right)\sigma^{\overline{\alpha}^i}(\beta)+1 &= \sum_{i=1}^{n_1-1}\left(\prod_{l=i+1}^{n_1}\tau^{\overline{\alpha}^l}(\beta)\right)\sigma^{\overline{\alpha}^i}(\beta)+\sigma^{\overline{\alpha}^{n_1}}(\beta)+1\\
&= \sum_{i=1}^{n_1-1}\left(\prod_{l=i+1}^{n_1}\tau^{\overline{\alpha}^l}(\beta)\right)\sigma^{\overline{\alpha}^i}(\beta)+\tau^{\overline{\alpha}^{n_1}}(\beta)\\
&= \left[\sum_{i=1}^{n_1-2}\left(\prod_{l=i+1}^{n_1-1}\tau^{\overline{\alpha}^l}(\beta)\right)\sigma^{\overline{\alpha}^i}(\beta)+\sigma^{\overline{\alpha}^{n_1-1}}(\beta)+1\right]\tau^{\overline{\alpha}^{n_1}}(\beta)\\
&= \left[\sum_{i=1}^{n_1-2}\left(\prod_{l=i+1}^{n_1-1}\tau^{\overline{\alpha}^l}(\beta)\right)\sigma^{\overline{\alpha}^i}(\beta)+\tau^{\overline{\alpha}^{n_1-1}}(\beta)\right]\tau^{\overline{\alpha}^{n_1}}(\beta)\\
&= \left[\sum_{i=1}^{n_1-3}\left(\prod_{l=i+1}^{n_1-2}\tau^{\overline{\alpha}^l}(\beta)\right)\sigma^{\overline{\alpha}^i}(\beta)+\sigma^{\overline{\alpha}^{n_1-2}}(\beta)+1\right]\tau^{\overline{\alpha}^{n_1-1}}(\beta)\tau^{\overline{\alpha}^{n_1}}(\beta)\\
&= \cdots = \prod_{l=1}^{n_1}\tau^{\overline{\alpha}^l}(\beta)
\end{aligned}
$$

As we can see, if we continue to take the terms out of the summation and factor out, the result will be $\prod_{l=1}^{n_1} \tau^{\overline{\alpha}^l}(\beta)$. Therefore inequality (4.1) is satisfied at equality.

For any $p^n_{n_2}$, $1 \le n_2 \le n_1$, if we replace the coordinates into the left-hand side of (4.1), we will have $\sum_{i=1}^{n_1-1}\left(\prod_{l=i+1}^{n_1}\tau^{\overline{\alpha}^l}(\beta)\right)\sigma^{\overline{\alpha}^i}(\beta)+\tau^{\overline{\alpha}^{n_1}}(\beta)$, which as we calculated above is equal to $\prod_{l=1}^{n_1} \tau^{\overline{\alpha}^l}(\beta)$. Therefore again inequality (4.1) holds at equality. To verify that these points are affinely independent in $\mathbb{R}^{n+1}$, we form an $(n+1)\times(n+1)$ matrix whose rows are $p^n_1,\dots,p^n_{n_1},q^n_{n_1},\dots,q^n_n$ and rearrange its column as $(y_1,\dots,y_{n_1},v,y_{n_1+1},\dots,y_n)$. If we do so, we will have a lower-triangular matrix as follows

$$
\begin{bmatrix}
\tau^{\overline{\alpha}^1}(\beta) & 0 & \cdots & & & & & \cdots & 0\\
\sigma^{\overline{\alpha}^1}(\beta) & \tau^{\overline{\alpha}^2}(\beta) & 0 & \cdots & & & & \cdots & 0\\
\vdots & & \ddots & & & & & & \vdots\\
\sigma^{\overline{\alpha}^1}(\beta) & \cdots & \sigma^{\overline{\alpha}^{n_1-1}}(\beta) & \tau^{\overline{\alpha}^{n_1}}(\beta) & 0 & \cdots & & \cdots & 0\\
\sigma^{\overline{\alpha}^1}(\beta) & \cdots & \cdots & \sigma^{\overline{\alpha}^{n_1}}(\beta) & \lambda^{\overline{\alpha}^{n_1}}(\beta) & 0 & \cdots & \cdots & 0\\
\sigma^{\overline{\alpha}^1}(\beta) & \cdots & \cdots & \sigma^{\overline{\alpha}^{n_1}}(\beta) & \lambda^{\overline{\alpha}^{n_1+1}}(\beta) & \sigma^{\overline{\alpha}^{n_1+1}}(\beta) & 0 & \cdots & 0\\
\vdots & & & \vdots & \vdots & \vdots & \ddots & & \vdots\\
\sigma^{\overline{\alpha}^1}(\beta) & \cdots & \cdots & \sigma^{\overline{\alpha}^{n_1}}(\beta) & \lambda^{\overline{\alpha}^{n-1}}(\beta) & \sigma^{\overline{\alpha}^{n_1+1}}(\beta) & \cdots & \sigma^{\overline{\alpha}^{n-1}}(\beta) & 0\\
\sigma^{\overline{\alpha}^1}(\beta) & \cdots & \cdots & \sigma^{\overline{\alpha}^{n_1}}(\beta) & \lambda^{\overline{\alpha}^{n}}(\beta) & \sigma^{\overline{\alpha}^{n_1+1}}(\beta) & \cdots & \sigma^{\overline{\alpha}^{n-1}}(\beta) & \sigma^{\overline{\alpha}^{n}}(\beta)
\end{bmatrix}.
$$

In the assumptions we have $\beta/\alpha_1 < \tau^{\overline{\alpha}^1}(\beta)$, therefore $\lambda^{\overline{\alpha}^1}(\beta) > 0$ and hence $\tau^{\overline{\alpha}^2}(\beta) > 0$. Again in the assumptions we have $\lambda^{\overline{\alpha}^1}(\beta)/\alpha_2 < \tau^{\overline{\alpha}^2}(\beta)$, therefore $\lambda^{\overline{\alpha}^2}(\beta) > 0$ and hence $\tau^{\overline{\alpha}^3}(\beta) > 0$. We can continue in this manner and use the assumption that $\lambda^{\overline{\alpha}^{i-1}}(\beta)/\alpha_i < \tau^{\overline{\alpha}^i}(\beta)$, for each $2 \leq i \leq n_1$, to show that $\lambda^{\overline{\alpha}^i}(\beta) > 0$ and hence $\tau^{\overline{\alpha}^{i+1}}(\beta) > 0$. Therefore the diagonal entries in rows 2 to n_1+1 of the above matrix are non-zero. Moreover, the additional conditions $\sigma^{\overline{\alpha}^i}(\beta) \geq 1$, for $i = n_1+1, ..., n$, make the diagonal entries in rows n_1+2 to n non-zero too. Therefore the lower-triangular square matrix we have built either has full rank or has a rank equal to n with its first row being entirely zero. Both cases imply that the rows are affinely independent. Thus (4.1) is facet-defining for $Q^{\overline{\alpha}^n,\beta}$. □

In derivation of the 3-step MIR facet in the proof of Theorem 3.7, we observed that in addition to inequalities (3.2) and (3.4) (which are inequality (4.1) for $n = 1$ and 2), the intermediate inequalities (3.7), (3.9), and (3.12) were also generated and used. The following lemma generalizes the derivation of the inequalities of this form.

Lemma 4.8. *Let $n \in \mathbb{N}$, $\beta \in \mathbb{R}$ and $\overline{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \mathbb{R}$, $\alpha_i > 0$ for $i = 1, \ldots, n$, such that $\beta/\alpha_1 < \tau^{\overline{\alpha}^1}(\beta)$ and $\lambda^{\overline{\alpha}^{i-1}}(\beta)/\alpha_i < \tau^{\overline{\alpha}^i}(\beta) \leq \alpha_{i-1}/\alpha_i$ for $i = 2, \ldots, n$. If all the inequalities MIR_1 to MIR_{n-1} are valid for $Q^{\overline{\alpha}^1,\beta}$ to $Q^{\overline{\alpha}^{n-1},\beta}$, respectively, then the following inequality is valid for $Q^{\overline{\alpha}^n,\beta}$ for all $k = 1, ..., n$:*

$$\alpha_k \left[\sum_{i=1}^{k} \left(\prod_{l=i+1}^{k} \tau^{\overline{\alpha}^l}(\beta) \right) y_i - \prod_{l=1}^{k} \tau^{\overline{\alpha}^l}(\beta) + 1 \right] + \sum_{i=k+1}^{n} \alpha_i y_i + v \geq \lambda^{\overline{\alpha}^k}(\beta). \tag{4.2}$$

Proof. The proof is by induction on k. For $k = 1$ inequality (4.2) reduces to $\sum_{i=1}^{n} \alpha_i y_i + v \geq \beta$, which is the defining inequality of $Q^{\overline{\alpha}^n,\beta}$ and hence valid. Now, as the induction hypothesis, we assume inequality (4.2) is valid for $Q^{\overline{\alpha}^n,\beta}$ for $k = n_1$ where $1 \leq n_1 < n$. We prove that (4.2) is valid for $Q^{\overline{\alpha}^n,\beta}$ for $k = n_1+1$ too. For $k = n_1$, $1 \leq n_1 < n$, inequality (4.2) can be written as follows if we multiply the expression in brackets by $1 = \alpha_{n_1+1}/\alpha_{n_1+1}$:

$$\frac{\alpha_{n_1}}{\alpha_{n_1+1}} \alpha_{n_1+1} \left[\sum_{i=1}^{n_1} \left(\prod_{l=i+1}^{n_1} \tau^{\overline{\alpha}^l}(\beta) \right) y_i - \prod_{l=1}^{n_1} \tau^{\overline{\alpha}^l}(\beta) + 1 \right] + \sum_{i=n_1+1}^{n} \alpha_i y_i + v \geq \lambda^{\overline{\alpha}^{n_1}}(\beta). \tag{4.3}$$

Since $1 \leq n_1 < n$, by the assumption, MIR_{n_1} is valid for $Q^{\overline{\alpha}^{n_1},\beta}$. Hence, by Lemma 4.6, inequality (4.1) is valid for $Q^{\overline{\alpha}^n,\beta}$:

$$\sum_{i=1}^{n_1} \left(\prod_{l=i+1}^{n_1} \tau^{\overline{\alpha}^l}(\beta) \right) y_i + \sum_{i=n_1+1}^{n} \left(\frac{\alpha_i}{\lambda^{\overline{\alpha}^{n_1}}(\beta)} \right) y_i + \left(\frac{1}{\lambda^{\overline{\alpha}^{n_1}}(\beta)} \right) v \geq \prod_{l=1}^{n_1} \tau^{\overline{\alpha}^l}(\beta)$$

If we add $-\prod_{l=1}^{n_1} \tau^{\overline{\alpha}^l}(\beta)+1$ to both sides of (4.1) and multiply its both sides by $\lambda^{\overline{\alpha}^{n_1}}(\beta)$, we get

$$\lambda^{\overline{\alpha}^{n_1}}(\beta)\left[\sum_{i=1}^{n_1}\left(\prod_{l=i+1}^{n_1}\tau^{\overline{\alpha}^l}(\beta)\right)y_i-\prod_{l=1}^{n_1}\tau^{\overline{\alpha}^l}(\beta)+1\right]+\sum_{i=n_1+1}^{n}\alpha_i y_i+v\geq\lambda^{\overline{\alpha}^{n_1}}(\beta).$$

Multiply the expression in brackets by $1=\alpha_{n_1+1}/\alpha_{n_1+1}$, we get

$$\frac{\lambda^{\overline{\alpha}^{n_1}}(\beta)}{\alpha_{n_1+1}}\alpha_{n_1+1}\left[\sum_{i=1}^{n_1}\left(\prod_{l=i+1}^{n_1}\tau^{\overline{\alpha}^l}(\beta)\right)y_i-\prod_{l=1}^{n_1}\tau^{\overline{\alpha}^l}(\beta)+1\right]+\sum_{i=n_1+1}^{n}\alpha_i y_i+v\geq\lambda^{\overline{\alpha}^{n_1}}(\beta). \tag{4.4}$$

Therefore,

$$\gamma\alpha_{n_1+1}\left[\sum_{i=1}^{n_1}\left(\prod_{l=i+1}^{n_1}\tau^{\overline{\alpha}^l}(\beta)\right)y_i-\prod_{l=1}^{n_1}\tau^{\overline{\alpha}^l}(\beta)+1\right]+\sum_{i=n_1+1}^{n}\alpha_i y_i+v\geq\lambda^{\overline{\alpha}^{n_1}}(\beta) \tag{4.5}$$

is valid for $Q^{\overline{\alpha}^n,\beta}$ for any $\lambda^{\overline{\alpha}^{n_1}}(\beta)/\alpha_{n_1+1}\leq\gamma\leq\alpha_{n_1}/\alpha_{n_1+1}$, as it can be expressed as a convex combination of (4.3) and (4.4). Since we have the assumption that $\lambda^{\overline{\alpha}^{n_1}}(\beta)/\alpha_{n_1+1}<\tau^{\overline{\alpha}^{n_1+1}}(\beta)\leq\alpha_{n_1}/\alpha_{n_1+1}$, we can set $\gamma=\tau^{\overline{\alpha}^{n_1+1}}(\beta)$. Substituting for γ in (4.5) yields

$$\alpha_{n_1+1}\left[\sum_{i=1}^{n_1}\left(\prod_{l=i+1}^{n_1+1}\tau^{\overline{\alpha}^l}(\beta)\right)y_i-\prod_{l=1}^{n_1+1}\tau^{\overline{\alpha}^l}(\beta)+\tau^{\overline{\alpha}^{n_1+1}}(\beta)\right]+\sum_{i=n_1+1}^{n}\alpha_i y_i+v\geq\lambda^{\overline{\alpha}^{n_1}}(\beta),$$

or

$$\alpha_{n_1+1}\left[\sum_{i=1}^{n_1+1}\left(\prod_{l=i+1}^{n_1+1}\tau^{\overline{\alpha}^l}(\beta)\right)y_i-\prod_{l=1}^{n_1+1}\tau^{\overline{\alpha}^l}(\beta)+\tau^{\overline{\alpha}^{n_1+1}}(\beta)\right]+\sum_{i=n_1+2}^{n}\alpha_i y_i+v\geq\lambda^{\overline{\alpha}^{n_1}}(\beta).$$

Subtracting $\alpha_{n_1+1}\sigma^{\overline{\alpha}^{n_1+1}}(\beta)$ from both sides gives

$$\alpha_{n_1+1}\left[\sum_{i=1}^{n_1+1}\left(\prod_{l=i+1}^{n_1+1}\tau^{\overline{\alpha}^l}(\beta)\right)y_i-\prod_{l=1}^{n_1+1}\tau^{\overline{\alpha}^l}(\beta)+1\right]+\sum_{i=n_1+2}^{n}\alpha_i y_i+v\geq\lambda^{\overline{\alpha}^{n_1+1}}(\beta),$$

which is (4.2) for $k=n_1+1$. Thus (4.2) for $k=n_1+1$ is also valid for $Q^{\overline{\alpha}^n,\beta}$, and the proof is complete. □

Now we are ready to prove Theorem 4.3.

Proof of Theorem 4.3. We prove the validity of MIR_n for $Q^{\overline{\alpha}^n,\beta}$ by induction on $n\in\mathbb{N}$. For $n=1$, MIR_1 was proved valid for $Q^{\overline{\alpha}^1,\beta}$ in Theorem 2.1. In fact, we also proved the theorem for $n=2$ and $n=3$ in Theorems 2.14 and 3.7, respectively.

Now, as the induction hypothesis, assume validity holds for $n=1,\ldots,n_1-1$, where $n_1>1$, meaning that MIR_1 to MIR_{n_1-1} are valid for $Q^{\overline{\alpha}^1,\beta}$ to $Q^{\overline{\alpha}^{n_1-1},\beta}$, respectively, if the respective conditions of the theorem are satisfied for each of them. We will prove validity for $n=n_1$. In other words, we prove if $\overline{\alpha} = (\alpha_1, \ldots, \alpha_{n_1})$ and β satisfy the conditions of the theorem, then MIR_{n_1} is valid for $Q^{\overline{\alpha}^{n_1},\beta}$. The conditions of the theorem for $n = n_1$ subsume the conditions of the theorem for $n = 1, \ldots, n_1 - 1$; therefore, by the induction hypothesis, MIR_1 to MIR_{n_1-1} are valid for $Q^{\overline{\alpha}^1,\beta}$ to $Q^{\overline{\alpha}^{n_1-1},\beta}$, respectively. Thus, by Lemma 4.8, inequality (4.2) is valid for $Q^{\overline{\alpha}^{n_1},\beta}$ for $n = n_1$ and $k = 1, \ldots, n_1$. In particular, for $n = k = n_1$, (4.2) is

$$\alpha_{n_1}\left[\sum_{i=1}^{n_1}\left(\prod_{l=i+1}^{n_1}\tau^{\overline{\alpha}^l}(\beta)\right)y_i - \prod_{l=1}^{n_1}\tau^{\overline{\alpha}^l}(\beta) + 1\right] + v \geq \lambda^{\overline{\alpha}^{n_1}}(\beta) \tag{4.6}$$

and is valid for $Q^{\overline{\alpha}^{n_1},\beta}$. Now the expression in brackets belongs to $\mathbb{Z}$ and $v \in \mathbb{R}_+$. Therefore, we can treat them as y_1 and v in $Q^{\alpha_{n_1},\lambda^{\overline{\alpha}^{n_1}}(\beta)}$, respectively. By the assumption, we have $\lambda^{\overline{\alpha}^{n_1-1}}(\beta)/\alpha_{n_1} < \tau^{\overline{\alpha}^{n_1}}(\beta)$. This means that $\lambda^{\overline{\alpha}^{n_1}}(\beta) > 0$. Therefore, $0 < \lambda^{\overline{\alpha}^{n_1}}(\beta)/\alpha_{n_1} < 1$. This implies that $\tau^{\alpha_{n_1}}(\lambda^{\overline{\alpha}^{n_1}}(\beta)) = 1$, and hence $\lambda^{\overline{\alpha}^{n_1}}(\beta)/\alpha_{n_1} < \tau^{\alpha_{n_1}}(\lambda^{\overline{\alpha}^{n_1}}(\beta))$. Thus the condition of Theorem 2.1 is satisfied for $Q^{\alpha_{n_1},\lambda^{\overline{\alpha}^{n_1}}(\beta)}$, and using this theorem, the 1-step MIR facet for this set, when y_1 is replaced with its substitute, leads to

$$\sum_{i=1}^{n_1}\left(\prod_{l=i+1}^{n_1}\tau^{\overline{\alpha}^l}(\beta)\right)y_i - \prod_{l=1}^{n_1}\tau^{\overline{\alpha}^l}(\beta) + 1 + \frac{1}{\lambda^{\alpha_{n_1}}(\lambda^{\overline{\alpha}^{n_1}}(\beta))}v \geq \tau^{\alpha_{n_1}}(\lambda^{\overline{\alpha}^{n_1}}(\beta))$$

But we have $\lambda^{\alpha_{n_1}}(\lambda^{\overline{\alpha}^{n_1}}(\beta)) = \lambda^{\overline{\alpha}^{n_1}}(\beta)$ and $\tau^{\alpha_{n_1}}(\lambda^{\overline{\alpha}^{n_1}}(\beta)) = 1$, therefore we have

$$\sum_{i=1}^{n_1}\left(\prod_{l=i+1}^{n_1}\tau^{\overline{\alpha}^l}(\beta)\right)y_i - \prod_{l=1}^{n_1}\tau^{\overline{\alpha}^l}(\beta) + 1 + \frac{1}{\lambda^{\overline{\alpha}^{n_1}}(\beta)}v \geq 1,$$

or

$$\sum_{i=1}^{n_1}\left(\prod_{l=i+1}^{n_1}\tau^{\overline{\alpha}^l}(\beta)\right)y_i + \frac{1}{\lambda^{\overline{\alpha}^{n_1}}(\beta)}v \geq \prod_{l=1}^{n_1}\tau^{\overline{\alpha}^l}(\beta),$$

which is MIR_{n_1}. Thus the validity holds for $n = n_1$. This proves the validity part of the theorem.

Next, we show that MIR_n is also a facet for $Q^{\overline{\alpha}^n,\beta}$. For this, note that the points $p_1^n, \ldots, p_n^n, q_n^n$ are $n+1$ points which, by Lemma 4.5, belong to $Q^{\overline{\alpha}^n,\beta}$. They also lie on the hyperplane of MIR_n. The argument for this is exactly the same as the one in the proof of Lemma 4.7 if we set $n = n_1$. These points are also affinely independent. The argument is again similar to the one in the proof of Lemma 4.7. If we form an $(n+1)\times(n+1)$ matrix whose rows are $p_1^n, \ldots, p_n^n, q_n^n$ without any change to the order of columns, the result will be the following lower-triangular matrix:

$$\begin{bmatrix} \tau^{\overline{\alpha}^1}(\beta) & 0 & \cdots & \cdots & 0 \\ \sigma^{\overline{\alpha}^1}(\beta) & \tau^{\overline{\alpha}^2}(\beta) & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \sigma^{\overline{\alpha}^1}(\beta) & \cdots & \sigma^{\overline{\alpha}^{n-1}}(\beta) & \tau^{\overline{\alpha}^n}(\beta) & 0 \\ \sigma^{\overline{\alpha}^1}(\beta) & \cdots & \cdots & \sigma^{\overline{\alpha}^n}(\beta) & \lambda^{\overline{\alpha}^n}(\beta) \end{bmatrix}.$$

With the same argument as in the proof of Lemma 4.7, the diagonal entries of this matrix are all non-zero except for possibly the first row. Therefore this lower-triangular matrix either has a full rank or has a rank of n with its first row being entirely zero. Both cases imply that the points $p_1^n, \ldots, p_n^n, q_n^n$ are affinely independent. Therefore MIR_n is a facet of $Q^{\overline{\alpha}^n,\beta}$. □

Now that we have proved MIR_n is a facet for $Q^{\overline{\alpha}^n,\beta}$ for any $n \in \mathbb{N}$, we define it as the *n-step MIR facet*.

Definition 4.9. *Let $n \in \mathbb{N}$, $\beta \in \mathbb{R}$ and $\overline{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \mathbb{R}$, $\alpha_i > 0$ for $i = 1, \ldots, n$, such that $\beta/\alpha_1 < \tau^{\overline{\alpha}^1}(\beta)$ and $\lambda^{\overline{\alpha}^{i-1}}(\beta)/\alpha_i < \tau^{\overline{\alpha}^i}(\beta) \leq \alpha_{i-1}/\alpha_i$ for $i = 2, \ldots, n$. We refer to the facet MIR_n as the **n-step MIR facet** for $Q^{\overline{\alpha}^n,\beta}$.* □

Note that if in addition to the conditions of Theorem 4.3, we have $\sigma^{\overline{\alpha}^i}(\beta) \geq 1$ for $i = 2, \ldots, n$, based on Lemma 4.7, in addition to the n-step MIR facet, we have also $n-1$ other facets for $Q^{\overline{\alpha}^n,\beta}$ from inequality (4.1). Therefore overall, for each $n \in \mathbb{N}$ we have generated n facets of the set $Q^{\overline{\alpha}^n,\beta}$.

4.2 n-step MIR Inequality for a General IP Constraint

In this section, we show that, for each $n \in \mathbb{N}$, the n-step MIR facet of $Q^{\overline{\alpha}^n,\beta}$ developed in section 4.1 can be used to generate a family of valid inequalities for $Y_{\overline{a}}^{\alpha_1,b}$, called the *n-step MIR inequalities*, whose members are distinguished by different values of the parameter vector $\overline{\alpha}$. As a result, we introduce an infinite number of new families of valid inequalities (one for each n) for the feasible set of a general IP constraint. The interesting fact is that the n-step MIR inequalities can be simply produced using some closed-form functions that we refer to as the *n-step MIR functions*. We first define this function and state its properties and then present the n-step MIR inequality in Theorem 4.13. The proof includes the generalized MIR procedure for this purpose.

First, we generalize the definition of the sets I that we used in Definitions 2.5, 2.17, and 3.10. These sets will be used in definition of n-step MIR functions.

Definition 4.10. *For any $n \in \mathbb{N}$, $b \in \mathbb{R}$ and $\overline{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, $\alpha_i > 0$ for $i = 1, \ldots, n$, we define the set $I_m^{\overline{\alpha}^n, b}$ for $m = 0, \ldots, n$ as follows:*

– *for* $m = 0, \ldots, n-1$: $I_m^{\overline{\alpha}^n, b} = \{u \in \mathbb{R} : \lambda^{\overline{\alpha}^i}(u) < \lambda^{\overline{\alpha}^i}(b),\ \text{for } i = 1, \ldots, m,\ \lambda^{\overline{\alpha}^{m+1}}(u) \geq \lambda^{\overline{\alpha}^{m+1}}(b)\}$;

– *for* $m = n$: $I_n^{\overline{\alpha}^n, b} = \{u \in \mathbb{R} : \lambda^{\overline{\alpha}^i}(u) < \lambda^{\overline{\alpha}^i}(b), \text{for } i = 1, \ldots, n\}$.

□

Now we define the n-step MIR function:

Definition 4.11. *Let $n \in \mathbb{N}$, $b \in \mathbb{R}$ and $\overline{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \mathbb{R}$, $\alpha_i > 0$ for $i = 1, \ldots, n$. If $b/\alpha_1 < \tau^{\overline{\alpha}^1}(b)$ and $\lambda^{\overline{\alpha}^{i-1}}(b)/\alpha_i < \tau^{\overline{\alpha}^i}(b) \leq \alpha_{i-1}/\alpha_i$ for $i = 2, \ldots, n$, then the **positive n-step MIR function** for the right hand side b and the parameter vector $\overline{\alpha}^n$, denoted by $g_+^{\overline{\alpha}^n, b}(u)$, is defined as*

$$g_+^{\overline{\alpha}^n, b}(u) = \frac{\alpha_1 \delta^{\overline{\alpha}^n, b}(u) - \lambda^{\overline{\alpha}^1}(u) \prod_{l=2}^n \tau^{\overline{\alpha}^l}(b)}{\left[\alpha_1 - \lambda^{\overline{\alpha}^1}(b)\right] \prod_{l=2}^n \tau^{\overline{\alpha}^l}(b)} \tag{4.7}$$

where

$$\delta^{\overline{\alpha}^n, b}(u) = \begin{cases} \prod_{l=2}^n \tau^{\overline{\alpha}^l}(b) & \text{if } u \in I_0^{\overline{\alpha}^n, b} \\ \sum_{i=2}^m \left[\prod_{l=i+1}^n \tau^{\overline{\alpha}^l}(b)\right] \sigma^{\overline{\alpha}^i}(u) + \left[\prod_{l=m+2}^n \tau^{\overline{\alpha}^l}(b)\right] \tau^{\overline{\alpha}^{m+1}}(u) & \text{if } u \in I_m^{\overline{\alpha}^n, b},\ m = 1, \ldots, n-1 \\ \sum_{i=2}^n \left[\prod_{l=i+1}^n \tau^{\overline{\alpha}^l}(b)\right] \sigma^{\overline{\alpha}^i}(u) + \lambda^{\overline{\alpha}^n}(u)/\lambda^{\overline{\alpha}^n}(b) & \text{if } u \in I_n^{\overline{\alpha}^n, b} \end{cases}$$

*Also if $-b/\alpha_1 < \tau^{\overline{\alpha}^1}(-b)$ and $\lambda^{\overline{\alpha}^{i-1}}(-b)/\alpha_i < \tau^{\overline{\alpha}^i}(-b) \leq \alpha_{i-1}/\alpha_i$ for $i = 2, \ldots, n$, then the **negative n-step MIR function** for the right hand side b and the parameter vector $\overline{\alpha}^n$, denoted by $g_-^{\overline{\alpha}^n, b}(u)$, is defined as*

$$g_-^{\overline{\alpha}^n, b}(u) = g_+^{\overline{\alpha}^n, -b}(-u). \tag{4.8}$$

□

The general n-step MIR function has the same periodic properties proved for $n = 1, 2$ and 3, in Theorems 2.6, 2.18, and 3.11, respectively:

Theorem 4.12. *Let $n \in \mathbb{N}$, $b \in \mathbb{R}$ and $\overline{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \mathbb{R}$, $\alpha_i > 0$ for $i = 1, \ldots, n$. The following statements are true for any $k_b, k_u \in \mathbb{Z}$:*

(i). if $b/\alpha_1 < \tau^{\overline{\alpha}^1}(b)$ *and* $\lambda^{\overline{\alpha}^{i-1}}(b)/\alpha_i < \tau^{\overline{\alpha}^i}(b) \leq \alpha_{i-1}/\alpha_i$ *for* $i = 2, \ldots, n,$ *then* $g_+^{\overline{\alpha}^n,b}(u) = g_+^{\overline{\alpha}^n,k_b\alpha_1+b}(k_u\alpha_1 + u),$

(ii). if $-b/\alpha_1 < \tau^{\overline{\alpha}^1}(-b)$ *and* $\lambda^{\overline{\alpha}^{i-1}}(-b)/\alpha_i < \tau^{\overline{\alpha}^i}(-b) \leq \alpha_{i-1}/\alpha_i$ *for* $i = 2, \ldots, n,$ *then* $g_-^{\overline{\alpha}^n,b}(u) = g_-^{\overline{\alpha}^n,k_b\alpha_1+b}(k_u\alpha_1 + u)$ *and* $g_-^{\overline{\alpha}^n,b}(u) = g_+^{\overline{\alpha}^n,k_b\alpha_1-b}(k_u\alpha_1 - u).$

Proof. From (4.7), it can be seen that $g_+^{\overline{\alpha}^n,b}(u)$ is a function of only $\lambda^{\overline{\alpha}^i}(u)$, $\sigma^{\overline{\alpha}^i}(u)$, and $\tau^{\overline{\alpha}^i}(u)$, as well as $\lambda^{\overline{\alpha}^i}(b)$, and $\tau^{\overline{\alpha}^i}(b)$, for $i = 1, \ldots, n$. By Definitions 3.3 and 3.4, these are all functions of $\lambda^{\overline{\alpha}^1}(u)$ and $\lambda^{\overline{\alpha}^1}(b)$. In the proof of Theorem 2.6, we saw that $\lambda^{\overline{\alpha}^1}(.)$ is periodic with period α_1; therefore $g_+^{\overline{\alpha}^n,b}(u)$ is also periodic in u and b with period α_1 and *(i)* is true. The argument for *(ii)* is very similar to the argument for *(ii)* in the proof of Theorem 2.6. □

Theorem 4.13 presents the n-step MIR inequalities for the set $Y_{\overline{a}}^{\alpha_1,b}$. The proof is the generalization of the MIR procedure presented in the proofs of Theorems 2.8, 2.19, and 3.12.

Theorem 4.13. *Let* $n \in \mathbb{N}$*,* $b \in \mathbb{R}$ *and* $\overline{\alpha} = (\alpha_1, \ldots, \alpha_n)$*,* $\alpha_i \in \mathbb{R}$*,* $\alpha_i > 0$ *for* $i = 1, \ldots, n$*. Then*

$$\sum_{j \in J} g_+^{\overline{\alpha}^n,b}(a_j)x_j \geq 1, \tag{4.9}$$

called the ***positive*** ***n-step MIR inequality****, is valid for* $Y_{\overline{a}}^{\alpha_1,b}$ *if* $b/\alpha_1 < \tau^{\overline{\alpha}^1}(b)$ *and* $\lambda^{\overline{\alpha}^{i-1}}(b)/\alpha_i < \tau^{\overline{\alpha}^i}(b) \leq \alpha_{i-1}/\alpha_i$ *for* $i = 2, \ldots, n$*, and*

$$\sum_{j \in J} g_-^{\overline{\alpha}^n,b}(a_j)x_j \geq 1, \tag{4.10}$$

called the ***negative*** ***n-step MIR inequality****, is valid for* $Y_{\overline{a}}^{\alpha_1,b}$ *if* $-b/\alpha_1 < \tau^{\overline{\alpha}^1}(-b)$ *and* $\lambda^{\overline{\alpha}^{i-1}}(-b)/\alpha_i < \tau^{\overline{\alpha}^i}(-b) \leq \alpha_{i-1}/\alpha_i$ *for* $i = 2, \ldots, n$*.*

Proof. First, we prove the validity of (4.9) assuming the required conditions hold true. We start from equation (2.9), i.e.

$$\sum_{j \in J} a_j x_j + \alpha_1 z = b.$$

We partition J into $J_0, J_1, \ldots, J_n$. Then we can relax (2.9) as

$$\alpha_1 z + \sum_{m=0}^{n-1} \sum_{j \in J_m} \left[\sum_{i=1}^{m} \alpha_i \sigma^{\overline{\alpha}^i}(a_j) + \alpha_{m+1} \tau^{\overline{\alpha}^{m+1}}(a_j) \right] x_j + \sum_{j \in J_n} \left[\sum_{i=1}^{n} \alpha_i \sigma^{\overline{\alpha}^i}(a_j) + \lambda^{\overline{\alpha}^n}(a_j) \right] x_j \geq b \tag{4.11}$$

The reason why inequality (4.11) is a relaxation of (2.9) is as follows. By identities (3.1), for any $1 \le m \le n$, a_j can be written as

$$a_j = \sum_{i=1}^{m} \alpha_i \sigma^{\overline{\alpha}^i}(a_j) + \lambda^{\overline{\alpha}^m}(a_j).$$

For $j \in J_n$, a_j has been simply substituted from this identity for $m = n$ with no relaxation. It is trivial that $a_j \le \alpha_1 \tau^{\overline{\alpha}^1}(a_j)$, and $\lambda^{\overline{\alpha}^m}(a_j) \le \alpha_{m+1} \tau^{\overline{\alpha}^{m+1}}(a_j)$ for $m = 1, \dots n-1$. Therefore, the identity above implies that

$$a_j \le \sum_{i=1}^{m} \alpha_i \sigma^{\overline{\alpha}^i}(a_j) + \alpha_{m+1} \tau^{\overline{\alpha}^{m+1}}(a_j),$$

for $0 \le m < n$. Thus if, as in (4.11), a_j is replaced with the right-hand side of this last inequality for $j \in J_m$, $m = 0, \dots, n-1$, the result will be a relaxation of (2.9).

Now, by rearrangement of terms, (4.11) can be written as

$$\alpha_1 \left[z + \sum_{j \in J_0} \tau^{\overline{\alpha}^1}(a_j) x_j + \sum_{m=1}^{n} \sum_{j \in J_m} \sigma^{\overline{\alpha}^1}(a_j) x_j \right] + \sum_{i=2}^{n} \alpha_i \left[\sum_{j \in J_{i-1}} \tau^{\overline{\alpha}^i}(a_j) x_j + \sum_{m=i}^{n} \sum_{j \in J_m} \sigma^{\overline{\alpha}^i}(a_j) x_j \right] + \left[\sum_{j \in J_n} \lambda^{\overline{\alpha}^n}(a_j) x_j \right] \ge b. \quad (4.12)$$

The expressions in the first to the last brackets in (4.12) play the roles of $y_1, y_2, \dots, y_n$ and v in $Q^{\overline{\alpha}^n, b}$, respectively. The sign and integrality conditions match. Since it is assumed $b/\alpha_1 < \tau^{\overline{\alpha}^1}(b)$ and $\lambda^{\overline{\alpha}^{i-1}}(b)/\alpha_i < \tau^{\overline{\alpha}^i}(b) \le \alpha_{i-1}/\alpha_i$ for $i = 2, \dots, n$, if $y_1, \dots, y_n$ and v in the n-step MIR facet MIR_n are replaced with the corresponding terms in (4.12), by Theorem 4.3, the inequality obtained will be valid for $Y_{\overline{a}}^{\alpha_1, b}$. Doing so and substituting for z from (2.9) gives

$$\sum_{j \in J} \left(\alpha_1 \delta^{\overline{\alpha}^n, b}(u) - \lambda^{\overline{\alpha}^1}(u) \prod_{l=2}^{n} \tau^{\overline{\alpha}^l}(b) \right) x_j \ge \left[\alpha_1 - \lambda^{\overline{\alpha}^1}(b) \right] \prod_{l=2}^{n} \tau^{\overline{\alpha}^l}(b) \quad (4.13)$$

where

$$\delta^{\overline{\alpha}^n, b}(a_j) = \begin{cases} \prod_{l=2}^{n} \tau^{\overline{\alpha}^l}(b) & \text{if } j \in J_0 \\ \sum_{i=2}^{m} \left[\prod_{l=i+1}^{n} \tau^{\overline{\alpha}^l}(b) \right] \sigma^{\overline{\alpha}^i}(a_j) + \left[\prod_{l=m+2}^{n} \tau^{\overline{\alpha}^l}(b) \right] \tau^{\overline{\alpha}^{m+1}}(a_j) & \text{if } j \in J_m,\ m = 1, \dots, n-1 \\ \sum_{i=2}^{n} \left[\prod_{l=i+1}^{n} \tau^{\overline{\alpha}^l}(b) \right] \sigma^{\overline{\alpha}^i}(a_j) + \lambda^{\overline{\alpha}^n}(a_j) / \lambda^{\overline{\alpha}^n}(b) & \text{if } j \in J_n \end{cases}$$

To obtain the strongest inequality the coefficients of x_j's should be minimized. In other words, the partitioning of J into $J_0, \dots, J_n$ should be determined such that $\delta^{\overline{\alpha}^n, b}(a_j)$ gets the minimum of the

$n+1$ values above. It is not difficult to verify that the partitioning should be as follows

$$J_m = \{ j \in J : \lambda^{\overline{\alpha}^i}(a_j) < \lambda^{\overline{\alpha}^i}(b), \text{ for } i = 1,\dots,m,\ \lambda^{\overline{\alpha}^{m+1}}(a_j) \geq \lambda^{\overline{\alpha}^{m+1}}(b) \};\ \text{for } m = 0,\dots,n-1,$$
$$J_n = \{ j \in J : \lambda^{\overline{\alpha}^i}(a_j) < \lambda^{\overline{\alpha}^i}(b), \text{ for } i = 1,\dots,n \}.$$

This partitioning along with (4.13), and Definitions 4.10 and 4.11 results in inequality (4.9). Therefore the positive n-step MIR inequality is valid for $Y_{\overline{a}}^{\alpha_1,b}$.

Now to prove the validity of (4.10), assuming its required conditions hold true, the argument is similar to the argument in the proof of Theorem 2.19. The set $Y_{\overline{a}}^{\alpha_1,b}$ can also be expressed as in (2.29). Therefore the above argument can be repeated exactly starting with

$$\sum_{j\in J} -a_j x_j + \alpha_1 z = -b$$

and a_j and b replaced by $-a_j$ and $-b$. Since it is assumed $-b/\alpha_1 < \tau^{\overline{\alpha}^1}(-b)$ and $\lambda^{\overline{\alpha}^{i-1}}(-b)/\alpha_i < \tau^{\overline{\alpha}^i}(-b) \leq \alpha_{i-1}/\alpha_i$ for $i = 2,\dots,n$, the argument leads to (4.10). □

The special case of $Y_{\overline{a}}^{1,b}$ is of importance because a typical row of the optimal simplex tableaux and also the group problems have the form of this set:

Corollary 4.14. *Let $b \in \mathbb{R}$, $n,t \in \mathbb{N}$, $\overline{\alpha} = (\alpha_1,\alpha_2,\dots,\alpha_n)$ such that $\alpha_1 = 1/t$, $\alpha_i \in \mathbb{R}$, $\alpha_i > 0$ for $i = 2,\dots,n$. The positive n-step MIR inequality*

$$\sum_{j\in J} g_+^{\overline{\alpha}^n,b}(a_j)x_j \geq 1 \tag{4.14}$$

is valid for $Y_{\overline{a}}^{1,b}$ if $b/\alpha_1 < \tau^{\overline{\alpha}^1}(b)$ and $\lambda^{\overline{\alpha}^{i-1}}(b)/\alpha_i < \tau^{\overline{\alpha}^i}(b) \leq \alpha_{i-1}/\alpha_i$ for $i = 2,\dots,n$, and the negative n-step MIR inequality

$$\sum_{j\in J} g_-^{\overline{\alpha}^n,b}(a_j)x_j \geq 1 \tag{4.15}$$

is valid for $Y_{\overline{a}}^{1,b}$ if $-b/\alpha_1 < \tau^{\overline{\alpha}^1}(-b)$ and $\lambda^{\overline{\alpha}^{i-1}}(-b)/\alpha_i < \tau^{\overline{\alpha}^i}(-b) \leq \alpha_{i-1}/\alpha_i$ for $i = 2,\dots,n$.

Proof. As argued in the proof of Corollary 2.9, $Y_{\overline{a}}^{1/t,b}$, where $t \in \mathbb{N}$, is a relaxation of $Y_{\overline{a}}^{1,b}$. Therefore any valid inequality for $Y_{\overline{a}}^{1/t,b}$ is also valid for $Y_{\overline{a}}^{1,b}$. By Theorem 4.13, (4.14) and (4.15) are valid for $Y_{\overline{a}}^{1/t,b}$ if the respective conditions are satisfied. Hence they are valid for $Y_{\overline{a}}^{1,b}$ given the same conditions hold true. □

Example 4.15. Examples 2.21 and 3.14 were examples of the n-step MIR functions and inequalities for $n = 2,3$, respectively. Here, we consider two examples of higher dimensions. For the same right-hand side $b = k_1 + 0.8$, consider the parameter vector $\overline{\alpha} = (1, 0.45, 0.2, 0.0558, 0.011)$. It can be easily verified that the parameters satisfy the conditions of Theorem 4.13. The graph of the 5-step MIR function $g_+^{(1,0.45,0.2,0.0558,0.011),k_1+0.8}(u)$ is shown in Figure 4.1. The period is $\alpha_1 = 1$. By Theorem 4.13, using this function, a 5-step MIR valid inequality for the set of Examples 2.21 and 3.14, i.e.

$$\left\{(x_1,x_2,x_3,x_4,x_5) \in \mathbb{Z}_+^5 : 0.1x_1 + 0.28x_2 + 0.45x_3 + 0.6x_4 + 0.95x_5 + z = 2.8; z \in \mathbb{Z}\right\}$$

is

$$0.334x_1 + 0.518x_2 + 0.25x_3 + 0.75x_4 + 0.25x_5 \geq 1.$$

Like before, based on Theorem 4.12, the negative 5-step MIR function with the same $\overline{\alpha}$ and the right-hand side $k_2 + 0.2$ is the mirror image and can be seen in Figure 4.2.

Another example for the right-hand side $b = k_3 + 0.8$ could be the 6-step MIR inequality with $\overline{\alpha} = (1, 0.48, 0.19, 0.08, 0.032, 0.012)$. Figure 4.3 shows the graph of $g_+^{(1,0.48,0.19,0.08,0.032,0.012),k_3+0.8}(u)$ whose period is $\alpha_1 = 1$. By Theorem 4.13, the valid inequality obtained for the same set using this 6-step MIR function is

$$0.438x_1 + 0.632x_2 + 0.25x_3 + 0.594x_4 + 0.25x_5 \geq 1.$$

Figure 4.4 is the mirror image negative function with the same $\overline{\alpha}$ and the right-hand side $k_4 + 0.2$. □

An interesting observation is that none of the n-step MIR functions with the same period (α_1) dominates the other on the whole period. This means that in general they all can be potentially strong valid inequalities. Of course, domination can certainly happen on the coefficients (a_j's) in a particular instance of the set $Y_{\overline{a}}^{\alpha_1,b}$ and it depends on the coefficients and the parameters of the functions.

4.3 Extension of n-step MIR Inequalities to a General MIP Constraint

The development of n-step MIR inequalities we have presented so far has been for a general pure IP constraint. In other words, all variables in the set $Y_{\overline{a}}^{\alpha_1,b}$ are integer. The n-step MIR inequal-

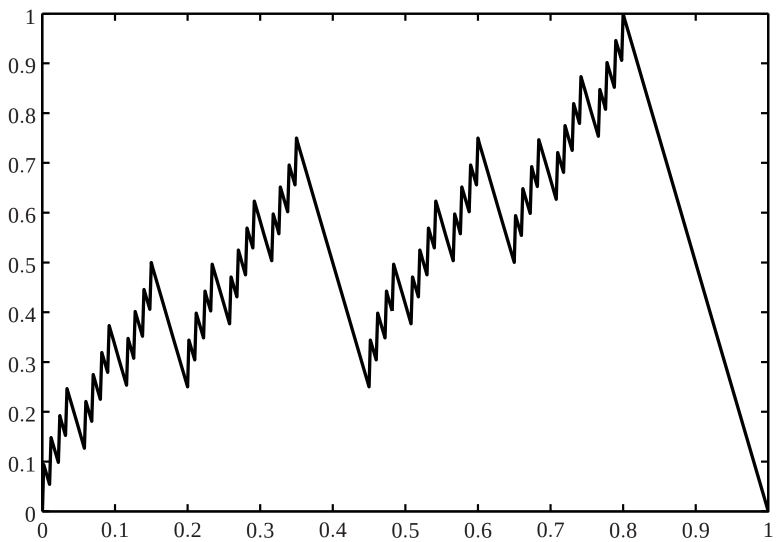


Figure 4.1: Graph of $g_{+}^{(1,0.45,0.2,0.0558,0.011),k_1+0.8}(u)$ for $u \in [0,1]$

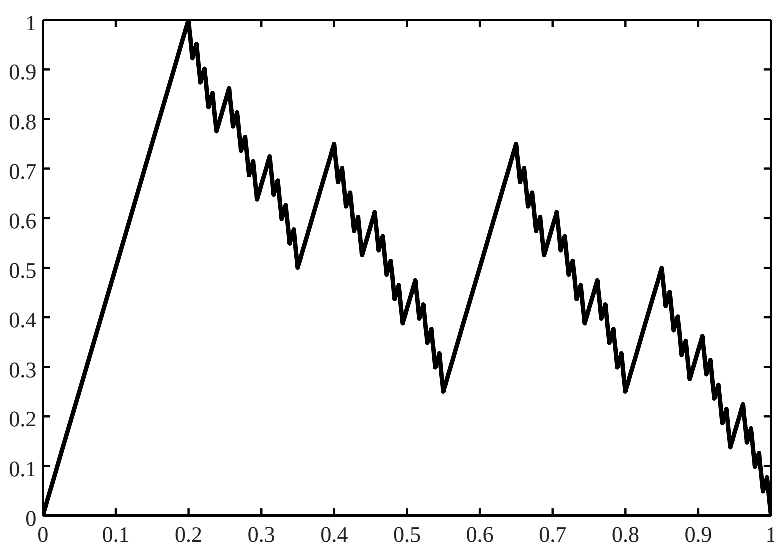


Figure 4.2: Graph of $g_{-}^{(1,0.45,0.2,0.0558,0.011),k_2+0.2}(u)$ for $u \in [0,1]$

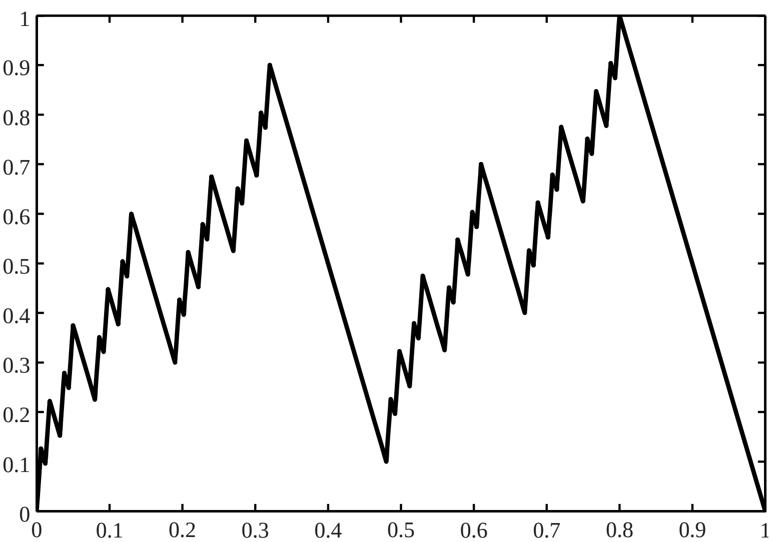


Figure 4.3: Graph of $g_{+}^{(1,0.48,0.19,0.08,0.032,0.012),k_3+0.8}(u)$ for $u \in [0,1]$

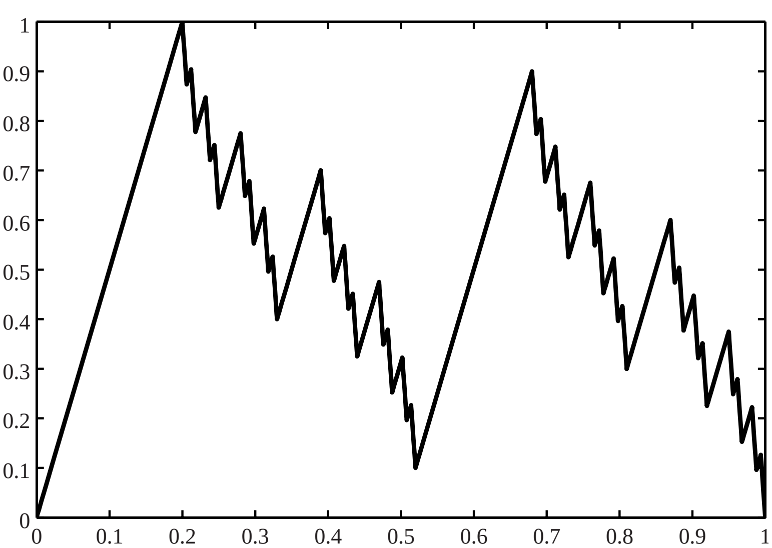


Figure 4.4: Graph of $g_{-}^{(1,0.48,0.19,0.08,0.032,0.012),k_4+0.2}(u)$ for $u \in [0,1]$

ity can be easily extended to the case where this set includes continuous variables, i.e. the feasible set of a general mixed integer constraint. We define this set as follows:

Definition 4.16. *Let $N_H, N_J \in \mathbb{N}$, $H = \{1,\dots,N_H\}$ and $J = \{1,\dots,N_J\}$. For any $\alpha_1, b \in \mathbb{R}$, $\overline{c}_d = (c_1,\dots,c_{N_H}) \in \mathbb{R}^{N_H}$ and $\overline{a} = (a_1,\dots,a_{N_J}) \in \mathbb{R}^{N_J}$, we define the set $Y^{\alpha_1,b}_{\overline{c},\overline{a}}$ as*

$$Y^{\alpha_1,b}_{\overline{c},\overline{a}} = \{(u_1,\dots,u_{N_H},x_1,\dots,x_{N_J}) \in \mathbb{R}^{N_H}_{+} \times \mathbb{Z}^{N_J}_{+} : \sum_{h\in H} c_h u_h + \sum_{j\in J} a_j x_j + \alpha_1 z = b, z \in \mathbb{Z}\}.$$

□

In the n-step MIR inequality for the set $Y^{\alpha_1,b}_{\overline{c},\overline{a}}$, the slopes of the n-step MIR function appear in the inequality in addition to the value of the function. The following lemma presents the slopes of the n-step MIR function:

Lemma 4.17. *If $\overline{\alpha}^n$ and b satisfy the conditions stated for the positive n-step MIR function in Definition 4.11, then this function is a piecewise-linear continuous function with two slopes, and the absolute values of its positive and negative slopes, which we denote by $s^+(g^{\overline{\alpha}^n,b}_{+})$ and $s^-(g^{\overline{\alpha}^n,b}_{+})$, respectively, are*

$$s^+(g^{\overline{\alpha}^n,b}_{+}) = \frac{\alpha_1 - \lambda^{\overline{\alpha}^n}(b)\prod_{l=2}^{n}\tau^{\overline{\alpha}^l}(b)}{\lambda^{\overline{\alpha}^n}(b)\left[\alpha_1 - \lambda^{\overline{\alpha}^1}(b)\right]\prod_{l=2}^{n}\tau^{\overline{\alpha}^l}(b)}\ ,\ \text{ and }\ s^-(g^{\overline{\alpha}^n,b}_{+}) = \frac{1}{\alpha_1 - \lambda^{\overline{\alpha}^1}(b)}.$$

Also, if $\overline{\alpha}^n$ and b satisfy the conditions stated for the negative n-step MIR function in Definition 4.11, then this function is a piecewise-linear continuous function with two slopes, and the absolute values of its positive and negative slopes, which we denote by $s^+(g^{\overline{\alpha}^n,b}_{-})$ and $s^-(g^{\overline{\alpha}^n,b}_{-})$, respectively, are

$$s^+(g^{\overline{\alpha}^n,b}_{-}) = s^-(g^{\overline{\alpha}^n,-b}_{+})\ ,\ \text{ and }\ s^-(g^{\overline{\alpha}^n,b}_{-}) = s^+(g^{\overline{\alpha}^n,-b}_{+}). \tag{4.16}$$

Proof. From Definition 4.11, it can easily be seen that the n-step MIR function $g^{\overline{\alpha}^n,b}_{+}(u)$ is linear. It is also continuous. Continuity needs to be verified only at the points on the boundary of the sets $I^{\overline{\alpha}^n,b}_{m}$, $m = 0,1,\dots,n$. From Definition 4.10, it can be seen that these are the set of all points where $\lambda^{\overline{\alpha}^i}(u) = 0$ or $\lambda^{\overline{\alpha}^i}(u) = \lambda^{\overline{\alpha}^i}(b)$ for some $1 \le i \le n$. It is easy to verify that the respective formulas from (4.7) give the same values at the left and right limits of these points and the points themselves, and hence the function is continuous. It is also easy to see from (4.7) that on $I^{\overline{\alpha}^n,b}_{n}$ the function has a positive slope equal to $s^+(g^{\overline{\alpha}^n,b}_{+})$, and on all $I^{\overline{\alpha}^n,b}_{m}$, $m = 0,\dots,n-1$, the function has a negative slope whose absolute value is $s^-(g^{\overline{\alpha}^n,b}_{+})$. The results for the negative n-step MIR function are the direct consequence of the results for the positive function and identity (4.8). □

Theorem 4.18 presents the extension of Theorem 4.13 to the set $Y^{\alpha_1,b}_{\overline{c},\overline{a}}$:

Theorem 4.18. *Let $n \in \mathbb{N}$, $b \in \mathbb{R}$ and $\overline{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \mathbb{R}$, $\alpha_i > 0$ for $i = 1, \ldots, n$. The* ***positive n-step MIR inequality***

$$\sum_{h \in \{H: c_h \geq 0\}} s^+(g_+^{\overline{\alpha}^n,b}) c_h u_h \; - \sum_{h \in \{H: c_h < 0\}} s^-(g_+^{\overline{\alpha}^n,b}) c_h u_h + \sum_{j \in J} g_+^{\overline{\alpha}^n,b}(a_j) x_j \geq 1 \tag{4.17}$$

is valid for $Y^{\alpha_1,b}_{\overline{c},\overline{a}}$ if $b/\alpha_1 < \tau^{\overline{\alpha}^1}(b)$ and $\lambda^{\overline{\alpha}^{i-1}}(b)/\alpha_i < \tau^{\overline{\alpha}^i}(b) \leq \alpha_{i-1}/\alpha_i$ for $i = 2, \ldots, n$, and the ***negative n-step MIR inequality***

$$\sum_{h \in \{H: c_h \geq 0\}} s^+(g_-^{\overline{\alpha}^n,b}) c_h u_h \; - \sum_{h \in \{H: c_h < 0\}} s^-(g_-^{\overline{\alpha}^n,b}) c_h u_h + \sum_{j \in J} g_-^{\overline{\alpha}^n,b}(a_j) x_j \geq 1 \tag{4.18}$$

is valid for $Y^{\alpha_1,b}_{\overline{c},\overline{a}}$ if $-b/\alpha_1 < \tau^{\overline{\alpha}^1}(-b)$ and $\lambda^{\overline{\alpha}^{i-1}}(-b)/\alpha_i < \tau^{\overline{\alpha}^i}(-b) \leq \alpha_{i-1}/\alpha_i$ for $i = 2, \ldots, n$.

Proof. The proof is very similar to the proof of Theorem 4.13. Here, we just highlight the differences. We start with the defining equation of $Y^{\alpha_1,b}_{\overline{c},\overline{a}}$, i.e.

$$\sum_{h \in H} c_h u_h + \sum_{j \in J} a_j x_j + \alpha_1 z = b, \tag{4.19}$$

and develop a relaxation of this equation. The relaxations for the integer variables x_j are the same as the relaxations in inequality (4.11). For the continuous variables u_h, we simply drop the ones with negative coefficients. Therefore, the resulting relaxation of (4.19) will be

$$\alpha_1 z + \sum_{m=0}^{n-1} \sum_{j \in J_m} \left[\sum_{i=1}^{m} \alpha_i \sigma^{\overline{\alpha}^i}(a_j) + \alpha_{m+1} \tau^{\overline{\alpha}^{m+1}}(a_j) \right] x_j \\ + \sum_{j \in J_n} \left[\sum_{i=1}^{n} \alpha_i \sigma^{\overline{\alpha}^i}(a_j) + \lambda^{\overline{\alpha}^n}(a_j) \right] x_j + \sum_{h \in \{H: c_h \geq 0\}} c_h u_h \geq b.$$

Thus after regrouping the terms, an additional summation appears in the last pair of brackets in inequality (4.12), and the result will be

$$\alpha_1 \left[z + \sum_{j \in J_0} \tau^{\overline{\alpha}^1}(a_j) x_j + \sum_{m=1}^{n} \sum_{j \in J_m} \sigma^{\overline{\alpha}^1}(a_j) x_j \right] \\ + \sum_{i=2}^{n} \alpha_i \left[\sum_{j \in J_{i-1}} \tau^{\overline{\alpha}^i}(a_j) x_j + \sum_{m=i}^{n} \sum_{j \in J_m} \sigma^{\overline{\alpha}^i}(a_j) x_j \right] + \left[\sum_{j \in J_n} \lambda^{\overline{\alpha}^n}(a_j) x_j + \sum_{h \in \{H: c_h \geq 0\}} c_h u_h \right] \geq b$$

The expression in the last pair of brackets is still a non-negative real value and can represent v in $Q^{\overline{\alpha}^n,b}$. As a result, the rest of the proof can be followed exactly similar to the proof of Theorem 4.13.

z will be replaced from (4.19) and the final result will be inequality (4.17). The validity of (4.18) can also be proved exactly like Theorem 4.13 using the fact that the defining inequality of $Y_{\overline{c},\overline{a}}^{\alpha_1,b}$ can also be written as $-\sum_{h\in H} c_h u_h - \sum_{j\in J} a_j x_j + \alpha_1 z = -b$, since z is an integer unrestricted in sign. □

4.4 Computer Codes for Calculating *n*-step MIR Functions

Having the formulas (4.7) and (4.8), generating the value of positive or negative n-step MIR functions with given parameters at any point is straightforward. Appendix A contains MATLAB functions that generate values and plot the graph of n-step MIR functions. The details are as follows:

- `lambda`**,** `tau`**, and** `sigma` **functions (section A.1)**: These are auxiliary functions that are used inside the functions described below. They receive a parameter vector $\overline{\alpha}^n$, a right-hand side b as input, and return the vectors $(\lambda^{\overline{\alpha}^1}(b),\dots,\lambda^{\overline{\alpha}^n}(b))$, $(\tau^{\overline{\alpha}^1}(b),\dots,\tau^{\overline{\alpha}^n}(b))$, and $(\sigma^{\overline{\alpha}^1}(b),\dots,\sigma^{\overline{\alpha}^n}(b))$, respectively.

- `mirp` **and** `mirpplot` **functions (section A.2)**: The function `mirp` receives a parameter vector $\overline{\alpha}^n$, a right-hand side b, and a vector $\overline{u} = (u_1,\dots,u_k)$ as input, and returns a vector whose elements are the values of the positive n-step MIR function $g_+^{\overline{\alpha}^n,b}(.)$ at $u_1,\dots,u_k$, i.e. $(g_+^{\overline{\alpha}^n,b}(u_1),\dots,g_+^{\overline{\alpha}^n,b}(u_k))$. If $\overline{\alpha}^n$ and b do not satisfy the conditions of Definition 4.11 for the positive n-step MIR function, the returned value will be -1.

  The function `mirpplot` receives a parameter vector $\overline{\alpha}^n$ and a right-hand side b and generates the graph of the positive n-step MIR function $g_+^{\overline{\alpha}^n,b}(u)$ for $u \in [0,1]$. If $\overline{\alpha}^n$ and b do not satisfy the conditions of Definition 4.11 for the positive n-step MIR function, the graph will be empty.

- `mirn` **and** `mirnplot` **functions (section A.3)**: The function `mirn` receives a parameter vector $\overline{\alpha}^n$, a right-hand side b, and a vector $\overline{u} = (u_1,\dots,u_k)$ as input, and returns a vector whose elements are the values of the negative n-step MIR function $g_-^{\overline{\alpha}^n,b}(.)$ at $u_1,\dots,u_k$, i.e. $(g_-^{\overline{\alpha}^n,b}(u_1),\dots,g_-^{\overline{\alpha}^n,b}(u_k))$. If $\overline{\alpha}^n$ and b do not satisfy the conditions of Definition 4.11 for the negative n-step MIR function, the returned value will be -1.

  The function `mirnplot` receives a parameter vector $\overline{\alpha}^n$ and a right-hand side b and generates the graph of the negative n-step MIR function $g_-^{\overline{\alpha}^n,b}(u)$ for $u \in [0,1]$. If $\overline{\alpha}^n$ and b do not satisfy the conditions of Definition 4.11 for the negative n-step MIR function, the graph will be empty.

Chapter 5

n-step MIR Functions: Facets for Finite and Infinite Group Problems

Gomory first introduced the notion of integer programming over cones in [22] and then in [23], he studied *corner polyhedra* and their relation with *group problems* and discussed that facets of polyhedra associated with group problems are sources for generating cutting planes for general (mixed) integer programming problems. Therefore development of facets for these polyhedra has been of particular importance in the area of cutting planes. Gomory and Johnson [24, 25] continued study of group problems and derived numerous results on development of facets for finite and infinite group problems. Development of these facets has also received special attention in some recent works [26, 27, 1, 11, 12, 15, 36].

In this chapter, we show that the n-step MIR functions developed in chapter 4, generate facets for group polyhedra. This is a significant property because of at least three reasons: first, this property proves that n-step MIR functions are potentially strong valid inequalities since they belong to the family of facets (strongest valid inequalities) of group polyhedra; second, new facets of group polyhedra generated by n-step MIR functions are sources for producing a wide range of new valid inequalities for general MIP problems; and third, these new facets provide more insight into the structure of corner polyhedra as building blocks of integer programming problems.

In section 5.1, we review some of the basic concepts form group theory. In section 5.2, we introduce corner polyhedra as well as finite and infinite group polyhedra, and discuss their relationship. Then, in section 5.3, we show that the n-step MIR function, for any $n \in \mathbb{N}$, defines a two-slope facet for the infinite group polyhedron. In section 5.4, we prove that the n-step MIR functions also generate facets for the finite master cyclic group polyhedron if appropriate conditions on their parameters are satisfied. Finally, in section 5.5, we discuss that the facet-defining properties of sections 5.3 and 5.4 extend to the finite and infinite group problems with continuous variables.

5.1 Basic Concepts from Group Theory

In this section, we define some basic concepts form group theory which will be used later in this chapter. The definitions are taken from [2].

Definition 5.1. *A **group** is a set G together with an operation '+', which has the following properties:*

- *the operation '+' is associative, that is* $(a+b)+c = a+(b+c)$ *for all* $a,b,c \in G$,
- *there is a zero element* $0 \in G$ *such that* $0+a = a+0 = a$ *for all* $a \in G$,
- *For every* $a \in G$, *there exists an inverse element* $-a \in G$ *such that* $a+(-a) = (-a)+a = 0$. □

Definition 5.2. *An **abelian group** is a group G such that the operation '+' is commutative, that is* $a+b = b+a$ *for all* $a,b \in G$. □

Definition 5.3. *A subset H of group G is a **subgroup** of G if it has the following properties:*

- *if* $a \in H$ *and* $b \in H$, *then* $a+b \in H$,
- $0 \in H$,
- *if* $a \in H$, *then* $-a \in H$. □

Definition 5.4. *Let G be a group. For any* $a \in G$ *and* $N \in \mathbb{N}$, *we define*

$$Na = \underbrace{a+\cdots+a}_{N \text{ times}}.$$

□

Definition 5.5. *A **cyclic group of order** N is an abelian group with the form*

$$\{0, a, 2a, \ldots, (N-1)a\},$$

where $Na = 0$. □

According to Definition 5.5, a cyclic group is a group which can be generated by only one element like a. The group

$$C_N = \left\{0, \frac{1}{N}, \ldots, \frac{N-1}{N}\right\},$$

where the operation '+' is defined as $a+b = (a+b) \mod 1 = \lambda^1(a+b)$ for any $a, b \in C_N$, is a cyclic group of order N. This is the cyclic group which we are interested in in this chapter.

Definition 5.6. *Let G and G' be two groups. A* ***homomorphism*** *ϕ from G to G' is a mapping $\phi : G \longrightarrow G'$ satisfying the rule*

$$\phi(a+b) = \phi(a) + \phi(b)$$

for all $a, b \in G$. The set of elements of G that are mapped to 0 of G' is called the ***kernel*** *of the homomorphism.* □

Definition 5.7. *An* ***isomorphism*** *ϕ from G to G' is a bijective homomorphism from G to G'. If there exists an isomorphism from G to G', the two groups are called* ***isomorphic****.* □

Definition 5.8. *An* ***automorphism*** *$\phi : G \longrightarrow G$ is an isomorphism from G to itself.* □

If N' divides N, then the mapping $\phi : C_N \longrightarrow C_{N'}$ defined by

$$\phi(\frac{j}{N}) = \frac{(j \mod N')}{N'}$$

is a homomorphism with the kernel $\left\{\frac{j}{N} \in C_N : j = kN', k = 0, \ldots, \frac{N}{N'} - 1\right\}$. Also, if m and N are relatively prime, then the mapping $\phi : C_N \longrightarrow C_N$ defined by

$$\phi(\frac{j}{N}) = \frac{(mj \mod N)}{N}$$

is an automorphism. If we combine these two statements we can say the mapping $\phi : C_N \longrightarrow C_{N'}$ defined by

$$\phi(\frac{j}{N}) = \frac{(mj \mod N')}{N'}$$

is a homomorphism with the same kernel if N' divides N, and m and N' are relatively prime.

5.2 Corner Polyhedra and Group Polyhedra

In this section, we review the concepts of corner polyhedra for integer programming problems, group problems, and their connection. Corner polyhedra and group problems are studied in many references, particularly [23, 17, 1, 26, 27]. Our presentation is based on these resources. Consider the (IP) problem where the constraints are written as equalities:

$$\begin{aligned} \min \bar{c}\bar{x} & \\ \mathbf{A}\bar{x} &= \bar{b} \\ \bar{x} &\geq 0 \\ \bar{x} & \text{ integer,} \end{aligned} \tag{5.1}$$

and its linear programming relaxation

$$\begin{aligned} \min \bar{c}\bar{x} & \\ \mathbf{A}\bar{x} &= \bar{b} \\ \bar{x} &\geq 0. \end{aligned} \tag{5.2}$$

Let $(\mathbf{B},\mathbf{N})$ represent the columns of $\mathbf{A}$ associated with basic and non-basic variables for an optimal basic solution to (5.2). Then (5.1) can be written as

$$\begin{aligned} \min \bar{c}_{\mathbf{B}}\bar{x}_{\mathbf{B}} + \bar{c}_{\mathbf{N}}\bar{x}_{\mathbf{N}} & \\ \mathbf{B}\bar{x}_{\mathbf{B}} + \mathbf{N}\bar{x}_{\mathbf{N}} &= \bar{b} \\ \bar{x}_{\mathbf{B}}, \bar{x}_{\mathbf{N}} &\geq 0 \\ \bar{x}_{\mathbf{B}}, \bar{x}_{\mathbf{N}} & \text{ integer,} \end{aligned} \tag{5.3}$$

Now if we remove the non-negativity constraints on basic variables $\bar{x}_{\mathbf{B}}$ in (5.3), we get the following problem

$$\begin{aligned} \min \bar{c}_{\mathbf{B}}\bar{x}_{\mathbf{B}} + \bar{c}_{\mathbf{N}}\bar{x}_{\mathbf{N}} & \\ \mathbf{B}\bar{x}_{\mathbf{B}} + \mathbf{N}\bar{x}_{\mathbf{N}} &= \bar{b} \\ \bar{x}_{\mathbf{N}} &\geq 0 \\ \bar{x}_{\mathbf{B}}, \bar{x}_{\mathbf{N}} & \text{ integer,} \end{aligned} \tag{5.4}$$

The convex hull of the feasible region of this problem is the corner polyhedron associated with the basis B.

Definition 5.9. *The convex hull of the feasible region of problem* (5.4)*, i.e.*

$$conv\left(\left\{(\bar{x}_{\mathbf{B}}, \bar{x}_{\mathbf{N}}) : \mathbf{B}\bar{x}_{\mathbf{B}} + \mathbf{N}\bar{x}_{\mathbf{N}} = \bar{b}, \bar{x}_{\mathbf{N}} \geq 0, \bar{x}_{\mathbf{B}}, \bar{x}_{\mathbf{N}} \; integer\right\}\right)$$

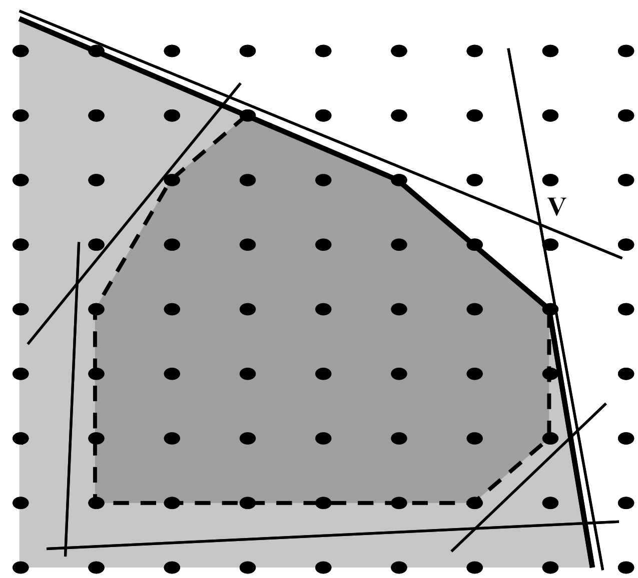


Figure 5.1: Integer programming polyhedron and corner polyhedron (figure taken from [27])

is called the ***corner polyhedron*** *of integer programming problem* (5.1) *associated with the basis* **B**. □

Figure 5.1, taken from [27], shows the convex hull polyhedron of integer programming problem and the corner polyhedron in a two-dimensional case. The dark gray area is the convex hull polyhedron of an integer programming problem whose constraints are shown as lines. The letter *V* marks the vertex of the current basic feasible solution, where the non-basic constraints meet. The light gray area, which continues off the figure, is obtained when the non-negativity restrictions on basic variables are relaxed. The dark and light gray areas together make up the *corner polyhedron* which is the convex hull of the integer points in the area defined only by the non-negativity of non-basic variables.

As explained in [27], corner polyhedra are connected to the original integer programming problem in two ways: first, the facets of the corner polyhedra are cutting planes for problem (5.2), which is the aspect of corner polyhedra that we are interested in; and second, the problem is identical with the integer programming problem if the right-hand sides are large enough. This gives rise to the theory of asymptotic integer programming, which is discussed in [23]. The corner polyhedra are the simplest integer programming problems in the sense that they are present in every integer programming problem, therefore study of them is useful.

Now we establish the relationship between corner polyhedra and the group problem. The argument is taken primarily from [1, 27]. Assume problem (5.1) has integer data meaning $\mathbf{A}$ and $\bar{b}$ have integer elements. Let m be the number of basic variables and NBV be the set of indices of non-basic variables in (5.4). Multiplying both sides of the constraints $\mathbf{B}\bar{x}_{\mathbf{B}} + \mathbf{N}\bar{x}_{\mathbf{N}} = \bar{b}$ by $\mathbf{B}^{-1}$ gives

$$\mathbf{I}\bar{x}_{\mathbf{B}} + \mathbf{B}^{-1}\mathbf{N}\bar{x}_{\mathbf{N}} = \mathbf{B}^{-1}\bar{b}. \tag{5.5}$$

The set of all integer combinations of the columns of $\mathbf{B}$ gives a lattice L in m-dimensional space. In an integer solution to the corner polyhedron problem (5.4), the non-basic variables times their (integer) columns must add up to the right-hand side m-vector $\bar{b}$, modulo the lattice L. An equivalent statement from equation (5.5) is that the non-basic columns, transformed (pre-multiplied) by $\mathbf{B}^{-1}$, must add up to the right-hand side, transformed by $\mathbf{B}^{-1}$, modulo 1. We write

$$\mathbf{B}^{-1}\mathbf{N}\bar{x}_{\mathbf{N}} \equiv \mathbf{B}^{-1}\bar{b} \mod 1. \tag{5.6}$$

In equation (5.6), equivalence of two vectors mod 1 means the corresponding elements of the vectors are congruent mod 1. The quotient group of all integer vectors in m-dimensional space taken mod $\mathbf{B}$ forms a finite group G. G is also obtained from reducing mod 1 the (usually non-integer) vectors in m-dimensional space that are obtained by pre-multiplying integer vectors by $\mathbf{B}^{-1}$. Either way, it is shown in [17] that G has det($\mathbf{B}$) elements. The group element corresponding to $\mathbf{B}^{-1}\mathbf{N}_i$, the transformed ith column of $\mathbf{N}$, is $\mathbf{B}^{-1}\mathbf{N}_i \mod 1$. Consequently, if we use g_j for the group element corresponding to columns of $\mathbf{B}^{-1}\mathbf{N}$ and g_r for the group element corresponding to the right-hand side $\mathbf{B}^{-1}\bar{b}$, instead of (5.6), we can write the following *group problem*:

$$\sum_j g_j t_j = g_r$$
$$t_j \geq 0, \text{ integer},$$

where the variables t_j are non-negative integers and the sum is the group sum. The variables t_j are just the non-basic variables $\bar{x}_{\mathbf{N}}$ adjusted for possible duplication because more than one column of $\mathbf{B}^{-1}\mathbf{N}$ may map into same group element g_j.

For a general integer programming problem, group G can have a very complex structure. But we can take only a single row of (5.6) as a relaxation:

$$\bar{a}'\bar{x}_{\mathbf{N}} \equiv b' \mod 1,$$

where $\overline{a}' = (a'_1, \ldots, a'_{|NBV|})$ and b' are elements in the selected row of $(\mathbf{B}^{-1}\mathbf{N}|\mathbf{B}^{-1}\overline{b})$. Since we are taking mod 1, we only need to take the fractional part of the coefficients, i.e.

$$\sum_{i \in NBV} \lambda^1(a'_i)x_i \equiv \lambda^1(b') \mod 1, \tag{5.7}$$

Then the group resulting from the general process explained above will be a cyclic group with elements between 0 and 1. The maximum number of elements will be again det($\mathbf{B}$) because it is a common denominator for all elements of $\mathbf{B}^{-1}\mathbf{N}$ and $\mathbf{B}^{-1}\overline{b}$. Therefore the resulting *cyclic group problem* will have the generic form

$$\sum_{j \in S} g_j t_j \equiv g_r \mod 1 \tag{5.8}$$

$$t_j \geq 0, \text{ integer, for all } j \in S,$$

where S is the set of variable indices and $g_j \in C_N = \left\{0, \frac{1}{N}, \ldots, \frac{N-1}{N}\right\}$. The order of the group for the cyclic group problem associated with (5.7) is det($\mathbf{B}$). Typically only a subset of elements of C_N are present in the cyclic group problem derived as above.

The solution for variables t_j in (5.8) can be translated back to the non-basic variables, possibly in more than one way. The non-basic variables then determine the basic variable associated with the original row, which will necessarily turn out to be integer, although not necessarily non-negative. The facets of the convex hull of the solutions t_j to (5.8), when translated back, leave all the corner polyhedra on one side. These solutions include all the integer solutions to the original integer programming problem because cyclic group problem is a relaxation of the original problem. Therefore the facets of (5.8), are always cutting planes for the integer programming problem. This is the major reason for their importance.

The complete version of the cyclic group problem, which has each group element present in it exactly once, is the *finite master cyclic group problem*. This problem and its associated polyhedron are formally defined in Definition 5.10. To be consistent in using x as the notation for integer variables, from now on we use x_j's, instead of t_j's, to denote the variables of a generic group problem. The relationship between integer variables of the group problems and integer variables of the integer programming problems, which was explained above, should be kept in mind.

Definition 5.10. *Any point $(x_1, \ldots, x_{N-1}) \in \mathbb{Z}_+^{N-1}$ that satisfies*

$$\sum_{j=1}^{N-1} \frac{j}{N} x_j \equiv \frac{r}{N} \mod 1 \tag{5.9}$$

is called a solution to the ***finite master cyclic group problem*** *over the cyclic group* C_N*, with the right-hand side* $\frac{r}{N}$*. The* ***finite master cyclic group polyhedron****, denoted by* $P(C_N, \frac{r}{N})$*, is the polyhedron of the convex hull of all these solutions, i.e.*

$$P(C_N, \frac{r}{N}) = conv\Big\{(x_1, \dots, x_{N-1}) \in \mathbb{Z}_+^{N-1} : \sum_{j=1}^{N-1} \frac{j}{N} x_j \equiv \frac{r}{N} \mod 1\Big\}.$$

□

If in the set $Y_{\overline{a}}^{\alpha_1, b}$, defined in Definition 2.3, we set $\alpha_1 = 1$, $b = \frac{r}{N}$ and $\overline{a}_{C_N} = \big(\frac{1}{N}, \dots, \frac{N-1}{N}\big)$, we get the set

$$Y_{\overline{a}_{C_N}}^{1, \frac{r}{N}} = \Big\{(x_1, \dots, x_{N-1}) \in \mathbb{Z}_+^{N-1} : \sum_{j=1}^{N-1} \frac{j}{N} x_j + z = \frac{r}{N}, z \in \mathbb{Z}\Big\}.$$

or

$$Y_{\overline{a}_{C_N}}^{1, \frac{r}{N}} = \Big\{(x_1, \dots, x_{N-1}) \in \mathbb{Z}_+^{N-1} : \sum_{j=1}^{N-1} \frac{j}{N} x_j \equiv \frac{r}{N} \mod 1\Big\}.$$

Therefore $Y_{\overline{a}_{C_N}}^{1, \frac{r}{N}}$ is exactly the feasible set to the master cyclic group problem (5.9) and

$$P(C_N, \frac{r}{N}) = conv(Y_{\overline{a}_{C_N}}^{1, \frac{r}{N}}).$$

The concepts of valid inequality and facet for $P(C_N, \frac{r}{N})$ can be formally defined as follows:

Definition 5.11. *The vector* $\overline{\pi} = (\pi_1, \dots, \pi_{N-1}) \in \mathbb{R}_+^{N-1}$ *is a* ***valid inequality*** *for* $P(C_N, \frac{r}{N})$ *if* $\sum_{j=1}^{N-1} \pi_j x_j \geq 1$ *for every* $(x_1, \dots, x_{N-1}) \in Y_{\overline{a}_{C_N}}^{1, \frac{r}{N}}$*. A valid inequality* $\overline{\pi}$ *for* $P(C_N, \frac{r}{N})$ *is a* ***facet*** *if* $\overline{\pi}$ *cannot be written as a convex combination of two distinct valid inequalities for* $P(C_N, \frac{r}{N})$*.* □

Gomory [23] showed that the facets of the convex hull of solutions to the cyclic group problem (5.8) may be obtained from a subset of facets $P(C_N, \frac{r}{N})$ by simply deleting the facet coefficients corresponding to elements that are not present in (5.8). The remaining facets for the master problem give valid inequalities for the cyclic group problem. Therefore, facets of master cyclic group polyhedron can provide cutting planes for any integer programming problem. Based on this fact, generation of facets for the master cyclic group problem and study of its facial structure are always of particular importance.

For exactly the same reason, the so-called *infinite group problem* is also of particular significance. The group in this problem is the group of all real numbers between 0 and 1 with addition modulo 1. The infinite group problem, introduced in [25], is defined as follows:

Definition 5.12. *Let $U=[0,1)$. We define $X(U,u_0)$ for a right-hand side $u_0 \in U-\{0\}$ as the set of all integer-valued functions $x(u)$ on U such that*

$$\sum_{u\in U} ux(u) \equiv u_0 \mod 1,$$

*where $x(u)\geq 0$ for all $u\in U$ and $x(u)$ has a finite support, i.e. $x(u)>0$ only for a finite subset of U. An $x(u)\in X(U,u_0)$ is a solution to the **infinite group problem** with the right hand side u_0. The **infinite group polyhedron**, denoted by $P(U,u_0)$, is the polyhedron of the convex hull of all these solutions, i.e.*

$$P(U,u_0) = conv\Big\{x(u)\in X(U,u_0)\Big\}.$$

□

The valid inequality and facet for $P(U,u_0)$ can be defined as follows:

Definition 5.13. *A **valid inequality** for $P(U,u_0)$ is a real-valued function π defined for all $u\in U$ such that $\pi(0)=0, \pi(u)\geq 0, u\in U$ and $\sum_{u\in U}\pi(u)x(u)\geq 1$ for any $x(u)\in X(U,u_0)$. A valid inequality π for $P(U,u_0)$ is a **facet (extreme valid inequality)** if π cannot be written as a convex combination of two distinct valid inequalities for $P(U,u_0)$.* □

In the next sections, we prove that the n-step MIR functions are facet-defining for $P(U,u_0)$ and $P(C_N,\frac{r}{N})$. This fact not only shows that the n-step MIR inequalities are potentially strong valid inequalities, but also results in many families of new valid inequalities for general MIP problems.

5.3 n-step MIR Functions: Facets for Infinite Group Polyhedra

In this section, we show that the n-step MIR functions are facets for the infinite group polyhedron $P(U,u_0)$. Throughout the discussion we assume all additions in U are mod 1 and refrain from writing "mod 1" explicitly each time. We start with the definitions of subadditivity and minimality concepts.

Definition 5.14. *A valid inequality π for $P(U,u_0)$ is **subadditive** if $\pi(u_1)+\pi(u_2)\geq \pi(u_1+u_2)$ for all $u_1,u_2\in U$.* □

Definition 5.15. *A valid inequality π for $P(U,u_0)$ is **minimal** if there is no other valid inequality $\rho\neq\pi$ for $P(U,u_0)$ satisfying $\rho(u)\leq\pi(u)$ for all $u\in U$.* □

We use some key results from [25] and [26] to prove the facet-defining property of the n-step MIR functions for $P(U,u_0)$. We replicate these results here without proof in a format which is most convenient for our arguments.

Theorem 5.16. *The valid inequality π for $P(U,u_0)$ is minimal if and only if $\pi(u)+\pi(u_0-u)=1$ for all $u\in U$.* $\square$

Theorem 5.17. *Any minimal valid inequality for $P(U,u_0)$ is subadditive.* $\square$

Theorem 5.18. *If π is a piecewise linear continuous function on U with only two slopes and it is subadditive and minimal, then it is a facet for $P(U,u_0)$.* $\square$

The way we will use these results is as follows: first in Theorem 5.20, we prove that the n-step MIR function is a valid inequality for $P(U,u_0)$. Then, in Lemmas 5.21 and 5.22 and Theorem 5.23, we will prove it is a minimal valid inequality based on Theorem 5.16. Theorem 5.17 implies it is also subadditive. Therefore, Theorem 5.18 can be used to prove that it is a facet, and this will be done in Theorem 5.24.

Before starting the above line of proof, we present an alternative formulation for the n-step MIR function, which is slightly different from formulation in Definition 4.11. We will use this alternative formulation in this chapter. This formulation makes our future arguments in proving the minimality more straightforward. The main difference is in the definition of the sets $I_m^{\overline{\alpha}^n,b}$. We define the sets $\mathscr{I}_m^{\overline{\alpha}^n,b}$, and use them instead of $I_m^{\overline{\alpha}^n,b}$. The difference between these two definitions is on the points where $\lambda^{\overline{\alpha}^i}(u)=0$. More exactly, for any $m=0,\ldots,n-1$ we have moved the points where $\lambda^{\overline{\alpha}^{m+1}}(u)=0$ from $I_n^{\overline{\alpha}^n,b}$ to $I_m^{\overline{\alpha}^n,b}$.

Definition 5.19. *For any $n\in\mathbb{N}$, $b\in\mathbb{R}$ and $\overline{\alpha}=(\alpha_1,\ldots,\alpha_n)\in\mathbb{R}^n$, $\alpha_i>0$ for $i=1,\ldots,n$, we define the set $\mathscr{I}_m^{\overline{\alpha}^n,b}$ for $m=0,\ldots,n$ as follows:*

– *for* $m=0,\ldots,n-1$:

$$\mathscr{I}_m^{\overline{\alpha}^n,b}=\Big\{u\in\mathbb{R}:\lambda^{\overline{\alpha}^i}(u)\in(0,\lambda^{\overline{\alpha}^i}(b)),\ \text{for } i=1,\ldots,m,\ \lambda^{\overline{\alpha}^{m+1}}(u)\in\{0\}\cup[\lambda^{\overline{\alpha}^{m+1}}(b),\alpha_{m+1})\Big\};$$

– *for* $m=n$: $\mathscr{I}_n^{\overline{\alpha}^n,b}=\{u\in\mathbb{R}:\lambda^{\overline{\alpha}^i}(u)\in(0,\lambda^{\overline{\alpha}^i}(b)),\ \text{for } i=1,\ldots,n\}$. $\square$

With Definition 5.19, the new formulation for the n-step MIR function, instead of formulation 4.7, will be

$$g_+^{\overline{\alpha}^n,b}(u)=\frac{\alpha_1\delta^{\overline{\alpha}^n,b}(u)-\lambda^{\overline{\alpha}^1}(u)\prod_{l=2}^n\tau^{\overline{\alpha}^l}(b)}{\left[\alpha_1-\lambda^{\overline{\alpha}^1}(b)\right]\prod_{l=2}^n\tau^{\overline{\alpha}^l}(b)} \tag{5.10}$$

where

$$\delta^{\overline{\alpha}^n,b}(u)=\begin{cases}\left[\prod_{l=2}^n \tau^{\overline{\alpha}^l}(b)\right]\tau^{\overline{\alpha}^1}(\lambda^{\overline{\alpha}^1}(u)) & \text{if } u\in\mathscr{I}_0^{\overline{\alpha}^n,b}\\ \sum_{i=2}^m\left[\prod_{l=i+1}^n \tau^{\overline{\alpha}^l}(b)\right]\sigma^{\overline{\alpha}^i}(u)+\left[\prod_{l=m+2}^n \tau^{\overline{\alpha}^l}(b)\right]\tau^{\overline{\alpha}^{m+1}}(u) & \text{if } u\in\mathscr{I}_m^{\overline{\alpha}^n,b},\ m=1,\dots,n-1\\ \sum_{i=2}^n\left[\prod_{l=i+1}^n \tau^{\overline{\alpha}^l}(b)\right]\sigma^{\overline{\alpha}^i}(u)+\lambda^{\overline{\alpha}^n}(u)/\lambda^{\overline{\alpha}^n}(b) & \text{if } u\in\mathscr{I}_n^{\overline{\alpha}^n,b}\end{cases}$$

Other than the change from I sets to $\mathscr{I}$ sets, everything is the same as before except for the additional coefficient $\tau^{\overline{\alpha}^1}(\lambda^{\overline{\alpha}^1}(u))$ in $\delta^{\overline{\alpha}^n,b}(u)$ in the case of $u\in\mathscr{I}_0^{\overline{\alpha}^n,b}$, which makes sure the value of the function is zero where $u=0$.

Now back to the line of our proof, we first show that the n-step MIR function gives a valid inequality for $P(U,u_0)$.

Theorem 5.20. *Let $u_0\in U$, $n,t\in\mathbb{N}$, $\overline{\alpha}=(\alpha_1,\alpha_2,\dots,\alpha_n)$ such that $\alpha_1=1/t$, $\alpha_i\in\mathbb{R}$, $\alpha_i>0$ for $i=2,\dots,n$. The real-valued function $g_+^{\overline{\alpha}^n,u_0}(u)$ is a valid inequality for $P(U,u_0)$ if $u_0/\alpha_1<\tau^{\overline{\alpha}^1}(u_0)$ and $\lambda^{\overline{\alpha}^{i-1}}(u_0)/\alpha_i<\tau^{\overline{\alpha}^i}(u_0)\le\alpha_{i-1}/\alpha_i$ for $i=2,\dots,n$. The same is true for $g_-^{\overline{\alpha}^n,u_0}(u)$ if $-u_0/\alpha_1<\tau^{\overline{\alpha}^1}(-u_0)$ and $\lambda^{\overline{\alpha}^{i-1}}(-u_0)/\alpha_i<\tau^{\overline{\alpha}^i}(-u_0)\le\alpha_{i-1}/\alpha_i$ for $i=2,\dots,n$.*

Proof. Based on formula (5.10), it is easy to verify that $g_+^{\overline{\alpha}^n,u_0}(0)=0$. It is also not difficult to verify that if the stated conditions on the parameters hold, then $g_+^{\overline{\alpha}^n,u_0}(u)\ge 0$ for all $u\in U$. It remains to prove that for any fixed $x(u)\in X(U,u_0)$, we have $\sum_{u\in U} g_+^{\overline{\alpha}^n,u_0}(u)x(u)\ge 1$. By definition, for an $x(u)\in X(U,u_0)$ there are a finite number, say N, of points $u\in U$ such that $x(u)>0$. If we call these points u_1 to u_N and define $\overline{u}=(u_1,\dots,u_N)$, then by Corollary 4.14, the inequality $\sum_{j=1}^N g_+^{\overline{\alpha}^n,u_0}(u_j)x_j\ge 1$ is valid for the set $Y_{\overline{u}}^{1,u_0}$. Clearly $(x(u_1),\dots,x(u_N))\in Y_{\overline{u}}^{1,u_0}$, therefore $\sum_{j=1}^N g_+^{\overline{\alpha}^n,u_0}(u_j)x(u_j)\ge 1$ or $\sum_{u\in U} g_+^{\overline{\alpha}^n,u_0}(u)x(u)\ge 1$. A similar argument proves the result for $g_-^{\overline{\alpha}^n,u_0}(u)$. □

Next we will show that the valid inequality for $P(U,u_0)$ obtained from the n-step MIR function is in fact a minimal valid inequality. For this we will first prove the following lemmas:

Lemma 5.21. *Let $m,n\in\mathbb{N}$, $m\le n$, $b\in\mathbb{R}$ and $\overline{\alpha}=(\alpha_1,\dots,\alpha_n)\in\mathbb{R}^n$, $\alpha_i>0$ for $i=1,\dots,n$ such that $b/\alpha_1<\tau^{\overline{\alpha}^1}(b)$ and $\lambda^{\overline{\alpha}^{i-1}}(b)/\alpha_i<\tau^{\overline{\alpha}^i}(b)$ for $i=2,\dots,n$. If $\lambda^{\overline{\alpha}^i}(u)\in(0,\lambda^{\overline{\alpha}^i}(b))$ for $i=1,\dots,m$, then*

$$\lambda^{\overline{\alpha}^i}(b-u)=\lambda^{\overline{\alpha}^i}(b)-\lambda^{\overline{\alpha}^i}(u),$$

and

$$\sigma^{\overline{\alpha}^i}(b-u)=\sigma^{\overline{\alpha}^i}(b)-\sigma^{\overline{\alpha}^i}(u)$$

for $i = 1,\dots,m$.

Proof. The proof is by induction on m. For $m = 1$, $\lambda^{\overline{\alpha}^1}(u) \in (0, \lambda^{\overline{\alpha}^1}(b))$ is the assumption. By identities (3.1), we can write $u = \alpha_1\sigma^{\overline{\alpha}^1}(u) + \lambda^{\overline{\alpha}^1}(u)$. Similarly, $b = \alpha_1\sigma^{\overline{\alpha}^1}(b) + \lambda^{\overline{\alpha}^1}(b)$. Subtracting the first identity from the second one, we get $b-u = \alpha_1[\sigma^{\overline{\alpha}^1}(b) - \sigma^{\overline{\alpha}^1}(u)] + \lambda^{\overline{\alpha}^1}(b) - \lambda^{\overline{\alpha}^1}(u)$. This identity along with the fact that $\lambda^{\overline{\alpha}^1}(u) \in (0, \lambda^{\overline{\alpha}^1}(b))$ implies that $\lambda^{\overline{\alpha}^1}(b-u) = \lambda^{\overline{\alpha}^1}(b) - \lambda^{\overline{\alpha}^1}(u)$ and $\sigma^{\overline{\alpha}^1}(b-u) = \sigma^{\overline{\alpha}^1}(b) - \sigma^{\overline{\alpha}^1}(u)$ because $\lambda^{\overline{\alpha}^1}(b-u)$ is a unique non-negative number smaller than α_1.

Now, as the induction hypothesis, we assume the lemma is true for $m = k$. We then prove that it is also true for $m = k+1$. In other words we prove that if $\lambda^{\overline{\alpha}^i}(u) \in (0, \lambda^{\overline{\alpha}^i}(b))$ for $i = 1,\dots,k+1$, then $\lambda^{\overline{\alpha}^i}(b-u) = \lambda^{\overline{\alpha}^i}(b) - \lambda^{\overline{\alpha}^i}(u)$ and $\sigma^{\overline{\alpha}^i}(b-u) = \sigma^{\overline{\alpha}^i}(b) - \sigma^{\overline{\alpha}^i}(u)$ for $i = 1,\dots,k+1$. Since $\lambda^{\overline{\alpha}^i}(u) \in (0, \lambda^{\overline{\alpha}^i}(b))$ and $\sigma^{\overline{\alpha}^i}(b-u) = \sigma^{\overline{\alpha}^i}(b) - \sigma^{\overline{\alpha}^i}(u)$ for $i = 1,\dots,k$, by the induction hypothesis $\lambda^{\overline{\alpha}^i}(b-u) = \lambda^{\overline{\alpha}^i}(b) - \lambda^{\overline{\alpha}^i}(u)$ for $i = 1,\dots,k$. Now, again from identities (3.1), we can write $\lambda^{\overline{\alpha}^k}(u) = \alpha_{k+1}\sigma^{\overline{\alpha}^{k+1}}(u) + \lambda^{\overline{\alpha}^{k+1}}(u)$. Similarly, $\lambda^{\overline{\alpha}^k}(b) = \alpha_{k+1}\sigma^{\overline{\alpha}^{k+1}}(b) + \lambda^{\overline{\alpha}^{k+1}}(b)$. Since $\lambda^{\overline{\alpha}^k}(b-u) = \lambda^{\overline{\alpha}^k}(b) - \lambda^{\overline{\alpha}^k}(u)$, if we subtract the first identity from the second one, we get $\lambda^{\overline{\alpha}^k}(b-u) = \alpha_{k+1}[\sigma^{\overline{\alpha}^{k+1}}(b) - \sigma^{\overline{\alpha}^{k+1}}(u)] + \lambda^{\overline{\alpha}^{k+1}}(b) - \lambda^{\overline{\alpha}^{k+1}}(u)$. This identity along with the fact that $\lambda^{\overline{\alpha}^{k+1}}(u) \in (0, \lambda^{\overline{\alpha}^{k+1}}(b))$ implies that $\lambda^{\overline{\alpha}^{k+1}}(b-u) = \lambda^{\overline{\alpha}^{k+1}}(b) - \lambda^{\overline{\alpha}^{k+1}}(u)$ because $\lambda^{\overline{\alpha}^{k+1}}(b-u)$ is a unique non-negative number smaller than α_{k+1}. Therefore, it also implies $\sigma^{\overline{\alpha}^{k+1}}(b-u) = \sigma^{\overline{\alpha}^{k+1}}(b) - \sigma^{\overline{\alpha}^{k+1}}(u)$. This concludes the proof. □

Lemma 5.22. *Let $n \in \mathbb{N}$, $b \in \mathbb{R}$ and $\overline{\alpha} = (\alpha_1,\dots,\alpha_n) \in \mathbb{R}^n$, $\alpha_i > 0$ for $i = 1,\dots,n$ such that $b/\alpha_1 < \tau^{\overline{\alpha}^1}(b)$ and $\lambda^{\overline{\alpha}^{i-1}}(b)/\alpha_i < \tau^{\overline{\alpha}^i}(b)$ for $i = 2,\dots,n$. Then the following statements are true:*

(i). For any $1 \le m \le n$, $u \in \mathscr{I}_m^{\overline{\alpha}^n,b}$ if and only if $b-u \in \mathscr{I}_m^{\overline{\alpha}^n,b}$,

(ii). If $u, b-u \in \mathscr{I}_m^{\overline{\alpha}^n,b}$, then $\lambda^{\overline{\alpha}^i}(b-u) = \lambda^{\overline{\alpha}^i}(b) - \lambda^{\overline{\alpha}^i}(u)$ and $\sigma^{\overline{\alpha}^i}(b-u) = \sigma^{\overline{\alpha}^i}(b) - \sigma^{\overline{\alpha}^i}(u)$ for $i = 1,\dots,m$,

(iii). If $1 \le m < n$ and $u, b-u \in \mathscr{I}_m^{\overline{\alpha}^n,b}$, then $\tau^{\overline{\alpha}^{m+1}}(b-u) = \tau^{\overline{\alpha}^{m+1}}(b) - \tau^{\overline{\alpha}^{m+1}}(u)$.

Proof. (i). Because of symmetry we only need to prove the forward direction. Assume $u \in \mathscr{I}_m^{\overline{\alpha}^n,b}$. This means $\lambda^{\overline{\alpha}^i}(u) \in (0, \lambda^{\overline{\alpha}^i}(b))$ for $i = 1,\dots,m$ and $\lambda^{\overline{\alpha}^{m+1}}(u) \in \{0\} \cup [\lambda^{\overline{\alpha}^{m+1}}(b), \alpha_{m+1})$. Since $\lambda^{\overline{\alpha}^i}(u) \in (0, \lambda^{\overline{\alpha}^i}(b))$ for $i = 1,\dots,m$, by Lemma 5.21, we have $\lambda^{\overline{\alpha}^i}(b-u) = \lambda^{\overline{\alpha}^i}(b) - \lambda^{\overline{\alpha}^i}(u)$ for $i = 1,\dots,m$. For each $i = 1,\dots,m$, $\lambda^{\overline{\alpha}^i}(b-u) = \lambda^{\overline{\alpha}^i}(b) - \lambda^{\overline{\alpha}^i}(u)$ and $\lambda^{\overline{\alpha}^i}(u) \in (0, \lambda^{\overline{\alpha}^i}(b))$ imply that

$$\lambda^{\overline{\alpha}^i}(b-u) \in (0, \lambda^{\overline{\alpha}^i}(b)), \quad \text{for } i = 1,\dots,m. \tag{5.11}$$

On the other hand, $\lambda^{\overline{\alpha}^{m+1}}(b-u) \notin (0, \lambda^{\overline{\alpha}^{m+1}}(b))$. This can be proved by contradiction. Assume $\lambda^{\overline{\alpha}^{m+1}}(b-u) \in (0, \lambda^{\overline{\alpha}^{m+1}}(b))$. This along with (5.11), based on Lemma 5.21, implies that $\lambda^{\overline{\alpha}^{i}}(u) = \lambda^{\overline{\alpha}^{i}}(b) - \lambda^{\overline{\alpha}^{i}}(b-u)$ for $i = 1, \ldots, m+1$. $\lambda^{\overline{\alpha}^{m+1}}(u) = \lambda^{\overline{\alpha}^{m+1}}(b) - \lambda^{\overline{\alpha}^{m+1}}(b-u)$ and $\lambda^{\overline{\alpha}^{m+1}}(b-u) \in (0, \lambda^{\overline{\alpha}^{m+1}}(b))$ imply that $\lambda^{\overline{\alpha}^{m+1}}(u) \in (0, \lambda^{\overline{\alpha}^{m+1}}(b))$ which contradicts the assumption. Therefore $\lambda^{\overline{\alpha}^{m+1}}(b-u) \notin (0, \lambda^{\overline{\alpha}^{m+1}}(b))$ or

$$\lambda^{\overline{\alpha}^{m+1}}(b-u) \in \{0\} \cup [\lambda^{\overline{\alpha}^{m+1}}(b), \alpha_{m+1}). \tag{5.12}$$

The proof is complete as (5.11) and (5.12) imply $b-u \in \mathscr{I}_m^{\overline{\alpha}^n, b}$.

(ii). The identities are true by Lemma 5.21 since $\lambda^{\overline{\alpha}^{i}}(u) \in (0, \lambda^{\overline{\alpha}^{i}}(b))$ for $i = 1, \ldots, m$.

(iii). By the argument in the proof of Lemma 5.21, since $\lambda^{\overline{\alpha}^{i}}(u) \in (0, \lambda^{\overline{\alpha}^{i}}(b))$ for $i = 1, \ldots, m$, the identity

$$\lambda^{\overline{\alpha}^{m}}(b-u) = \alpha_{m+1}[\sigma^{\overline{\alpha}^{m+1}}(b) - \sigma^{\overline{\alpha}^{m+1}}(u)] + \lambda^{\overline{\alpha}^{m+1}}(b) - \lambda^{\overline{\alpha}^{m+1}}(u) \tag{5.13}$$

is true. Since $\lambda^{\overline{\alpha}^{m+1}}(u) \in \{0\} \cup [\lambda^{\overline{\alpha}^{m+1}}(b), \alpha_{m+1})$, three cases are possible:

(a). $\lambda^{\overline{\alpha}^{m+1}}(u) = 0$; in this case clearly $\sigma^{\overline{\alpha}^{m+1}}(u) = \tau^{\overline{\alpha}^{m+1}}(u)$. Also, (5.13) implies $\lambda^{\overline{\alpha}^{m+1}}(b-u) = \lambda^{\overline{\alpha}^{m+1}}(b)$, and hence $\sigma^{\overline{\alpha}^{m+1}}(b-u) = \sigma^{\overline{\alpha}^{m+1}}(b) - \sigma^{\overline{\alpha}^{m+1}}(u)$. On the other hand, since by assumption $\lambda^{\overline{\alpha}^{m}}(b)/\alpha_{m+1} < \tau^{\overline{\alpha}^{m+1}}(b)$, we have $\lambda^{\overline{\alpha}^{m+1}}(b) > 0$. Therefore $\lambda^{\overline{\alpha}^{m+1}}(b-u) = \lambda^{\overline{\alpha}^{m+1}}(b) > 0$, and hence $\tau^{\overline{\alpha}^{m+1}}(b) = \sigma^{\overline{\alpha}^{m+1}}(b) + 1$ and $\tau^{\overline{\alpha}^{m+1}}(b-u) = \sigma^{\overline{\alpha}^{m+1}}(b-u) + 1$. Thus $\tau^{\overline{\alpha}^{m+1}}(b-u) = \tau^{\overline{\alpha}^{m+1}}(b) - \tau^{\overline{\alpha}^{m+1}}(u)$.

(b). $\lambda^{\overline{\alpha}^{m+1}}(u) = \lambda^{\overline{\alpha}^{m+1}}(b)$; by symmetry, u and $b-u$ can be interchanged. Then this case will be exactly the same as the case (a), and therefore again $\tau^{\overline{\alpha}^{m+1}}(b-u) = \tau^{\overline{\alpha}^{m+1}}(b) - \tau^{\overline{\alpha}^{m+1}}(u)$.

(c). $\lambda^{\overline{\alpha}^{m+1}}(u) \in (\lambda^{\overline{\alpha}^{m+1}}(b), \alpha_{m+1})$; in this case (5.13) implies $\lambda^{\overline{\alpha}^{m+1}}(b-u) = \alpha_{m+1} + \lambda^{\overline{\alpha}^{m+1}}(b) - \lambda^{\overline{\alpha}^{m+1}}(u)$, and hence $\sigma^{\overline{\alpha}^{m+1}}(b-u) = \sigma^{\overline{\alpha}^{m+1}}(b) - \sigma^{\overline{\alpha}^{m+1}}(u) - 1$. Also, $\lambda^{\overline{\alpha}^{m+1}}(b), \lambda^{\overline{\alpha}^{m+1}}(u), \lambda^{\overline{\alpha}^{m+1}}(b-u) > 0$. Therefore $\tau^{\overline{\alpha}^{m+1}}(b) = \sigma^{\overline{\alpha}^{m+1}}(b) + 1$, $\tau^{\overline{\alpha}^{m+1}}(u) = \sigma^{\overline{\alpha}^{m+1}}(u) + 1$ and $\tau^{\overline{\alpha}^{m+1}}(b-u) = \sigma^{\overline{\alpha}^{m+1}}(b-u) + 1$. Thus again $\tau^{\overline{\alpha}^{m+1}}(b-u) = \tau^{\overline{\alpha}^{m+1}}(b) - \tau^{\overline{\alpha}^{m+1}}(u)$.

Therefore, the result holds true in all three cases. □

Now we are ready to prove the minimality of the n-step MIR function:

Theorem 5.23. *Given all the conditions stated in Theorem 5.20, the valid inequality $g_+^{\overline{\alpha}^n,u_0}(u)$ is minimal for $P(U,u_0)$. The same is true for $g_-^{\overline{\alpha}^n,u_0}(u)$ if its respective conditions hold.*

Proof. By Theorem 5.20 $g_+^{\overline{\alpha}^n,u_0}(u)$ is a valid inequality for $P(U,u_0)$. Therefore, by Theorem 5.16, we need to prove that for any $u \in U$, $g_+^{\overline{\alpha}^n,u_0}(u) + g_+^{\overline{\alpha}^n,u_0}(u_0-u) = 1$. Although $u_0 - u$ is an addition modulo 1, we can use regular addition since, by Theorem 4.12, the function $g_+^{\overline{\alpha}^n,u_0}(u)$ is periodic with period $\alpha_1 = 1/t$, and hence also repeats itself on intervals of length 1. By Lemma 5.22, $u \in \mathscr{I}_m^{\overline{\alpha}^n,u_0}$ if and only if $u_0 - u \in \mathscr{I}_m^{\overline{\alpha}^n,u_0}$. Therefore, based on formulation (5.10), the minimality identity above must be shown true for $n+1$ cases which consist of $u \in \mathscr{I}_m^{\overline{\alpha}^n,u_0}$ for $m = 0,\dots,n$:

– *For $m = 0$:* from the case where $u \in \mathscr{I}_0^{\overline{\alpha}^n,u_0}$ in formula (5.10), we have

$$\begin{aligned}
g_+^{\overline{\alpha}^n,u_0}(u) + g_+^{\overline{\alpha}^n,u_0}(u_0-u) &= \frac{\alpha_1 \tau^{\overline{\alpha}^1}(\lambda^{\overline{\alpha}^1}(u)) + \alpha_1 \tau^{\overline{\alpha}^1}(\lambda^{\overline{\alpha}^1}(u_0-u)) - \lambda^{\overline{\alpha}^1}(u) - \lambda^{\overline{\alpha}^1}(u_0-u)}{\alpha_1 - \lambda^{\overline{\alpha}^1}(u_0)} \\
&= \frac{\alpha_1 - \lambda^{\overline{\alpha}^1}(u_0)}{\alpha_1 - \lambda^{\overline{\alpha}^1}(u_0)} = 1
\end{aligned}$$

The reason why the second identity is true is as follows. Three possible cases are those in the proof of Lemma 5.22, part *(iii)*: (a) If $\lambda^{\overline{\alpha}^1}(u) = 0$, then, as argued in Lemma 5.22*(iii)*, $\lambda^{\overline{\alpha}^1}(u_0-u) = \lambda^{\overline{\alpha}^1}(u_0)$. Also we will have $\tau^{\overline{\alpha}^1}(\lambda^{\overline{\alpha}^1}(u)) = 0$ and $\tau^{\overline{\alpha}^1}(\lambda^{\overline{\alpha}^1}(u_0-u)) = 1$. Hence, the identity holds. (b) If $\lambda^{\overline{\alpha}^1}(u_0-u) = 0$, the argument will be symmetric to the one in (a). (c) If $\lambda^{\overline{\alpha}^1}(u) \in (\lambda^{\overline{\alpha}^1}(u_0), \alpha_1)$, then, again as argued in Lemma 5.22*(iii)*, $\lambda^{\overline{\alpha}^1}(u) + \lambda^{\overline{\alpha}^1}(u-u_0) = \lambda^{\overline{\alpha}^1}(u_0) + \alpha_1$. Also $\tau^{\overline{\alpha}^1}(\lambda^{\overline{\alpha}^1}(u)) = 1$ and $\tau^{\overline{\alpha}^1}(\lambda^{\overline{\alpha}^1}(u_0-u)) = 1$. Therefore the identity is true in this case too.

– *For $m = 1,\dots,n-1$:* By Lemma 5.22 we have $\lambda^{\overline{\alpha}^i}(u_0-u) = \lambda^{\overline{\alpha}^i}(u_0) - \lambda^{\overline{\alpha}^i}(u)$ and $\sigma^{\overline{\alpha}^i}(u_0-u) = \sigma^{\overline{\alpha}^i}(u_0) - \sigma^{\overline{\alpha}^i}(u)$ for $i = 1,\dots,m$, and $\tau^{\overline{\alpha}^{m+1}}(u_0-u) = \tau^{\overline{\alpha}^{m+1}}(u_0) - \tau^{\overline{\alpha}^{m+1}}(u)$. Using these identities, for the cases where $u \in \mathscr{I}_m^{\overline{\alpha}^n,u_0}$, $m = 1,\dots,n-1$, in formula (5.10), we can write

$$\begin{aligned}
&g_+^{\overline{\alpha}^n,u_0}(u) + g_+^{\overline{\alpha}^n,u_0}(u_0-u) \\
&= \frac{1}{\left[\alpha_1 - \lambda^{\overline{\alpha}^1}(u_0)\right]\prod_{l=2}^n \tau^{\overline{\alpha}^l}(u_0)} \Bigg[\alpha_1 \Big[\sum_{i=2}^m \Big[\prod_{l=i+1}^n \tau^{\overline{\alpha}^l}(u_0)\Big](\sigma^{\overline{\alpha}^i}(u) + \sigma^{\overline{\alpha}^i}(u_0-u)) \\
&\qquad + \Big[\prod_{l=m+2}^n \tau^{\overline{\alpha}^l}(u_0)\Big](\tau^{\overline{\alpha}^{m+1}}(u) + \tau^{\overline{\alpha}^{m+1}}(u_0-u))\Big] - (\lambda^{\overline{\alpha}^1}(u) + \lambda^{\overline{\alpha}^1}(u_0-u))\prod_{l=2}^n \tau^{\overline{\alpha}^l}(u_0)\Bigg] \\
&= \frac{1}{\left[\alpha_1 - \lambda^{\overline{\alpha}^1}(u_0)\right]\prod_{l=2}^n \tau^{\overline{\alpha}^l}(u_0)} \Bigg[\alpha_1 \Big[\sum_{i=2}^m \Big[\prod_{l=i+1}^n \tau^{\overline{\alpha}^l}(u_0)\Big]\sigma^{\overline{\alpha}^i}(u_0) + \prod_{l=m+1}^n \tau^{\overline{\alpha}^l}(u_0)\Big] - \lambda^{\overline{\alpha}^1}(u_0)\prod_{l=2}^n \tau^{\overline{\alpha}^l}(u_0)\Bigg]
\end{aligned}$$

$$= \frac{1}{\left[\alpha_1 - \lambda^{\overline{\alpha}^1}(u_0)\right] \prod_{l=2}^{n} \tau^{\overline{\alpha}^l}(u_0)} \left[\alpha_1 \prod_{l=2}^{n} \tau^{\overline{\alpha}^l}(u_0) - \lambda^{\overline{\alpha}^1}(u_0) \prod_{l=2}^{n} \tau^{\overline{\alpha}^l}(u_0) \right] = 1.$$

In the next to the last identity, we have used the following equality which is true for $m = 1,\ldots,n$ and can be proved using the argument in the proof of Lemma 4.7:

$$\sum_{i=2}^{m} \Big[\prod_{l=i+1}^{n} \tau^{\overline{\alpha}^l}(u_0) \Big] \sigma^{\overline{\alpha}^i}(u_0) + \prod_{l=m+1}^{n} \tau^{\overline{\alpha}^l}(u_0) = \prod_{l=2}^{n} \tau^{\overline{\alpha}^l}(u_0).$$

– For $m = n$: By Lemma 5.22 we have $\lambda^{\overline{\alpha}^i}(u_0 - u) = \lambda^{\overline{\alpha}^i}(u_0) - \lambda^{\overline{\alpha}^i}(u)$ and $\sigma^{\overline{\alpha}^i}(u_0 - u) = \sigma^{\overline{\alpha}^i}(u_0) - \sigma^{\overline{\alpha}^i}(u)$ for $i = 1,\ldots,n$. Using these identities and from the case where $u \in \mathscr{I}_n^{\overline{\alpha}^n,u_0}$ in Definition 4.11, we can write

$$\begin{aligned}
& g_+^{\overline{\alpha}^n,u_0}(u) + g_+^{\overline{\alpha}^n,u_0}(u_0 - u) \\
&= \frac{1}{\left[\alpha_1 - \lambda^{\overline{\alpha}^1}(u_0)\right] \prod_{l=2}^{n} \tau^{\overline{\alpha}^l}(u_0)} \left[\alpha_1 \Big[\sum_{i=2}^{m} \Big[\prod_{l=i+1}^{n} \tau^{\overline{\alpha}^l}(u_0) \Big] (\sigma^{\overline{\alpha}^i}(u) + \sigma^{\overline{\alpha}^i}(u_0 - u)) \right. \\
&\qquad \left. + (\lambda^{\overline{\alpha}^n}(u) + \lambda^{\overline{\alpha}^n}(u_0 - u))/\lambda^{\overline{\alpha}^n}(u_0) \Big] - (\lambda^{\overline{\alpha}^1}(u) + \lambda^{\overline{\alpha}^1}(u_0 - u)) \prod_{l=2}^{n} \tau^{\overline{\alpha}^l}(u_0) \right] \\
&= \frac{1}{\left[\alpha_1 - \lambda^{\overline{\alpha}^1}(u_0)\right] \prod_{l=2}^{n} \tau^{\overline{\alpha}^l}(u_0)} \left[\alpha_1 \Big[\sum_{i=2}^{m} \Big[\prod_{l=i+1}^{n} \tau^{\overline{\alpha}^l}(u_0) \Big] \sigma^{\overline{\alpha}^i}(u_0) + 1 \Big] - \lambda^{\overline{\alpha}^1}(u_0) \prod_{l=2}^{n} \tau^{\overline{\alpha}^l}(u_0) \right] \\
&= \frac{1}{\left[\alpha_1 - \lambda^{\overline{\alpha}^1}(u_0)\right] \prod_{l=2}^{n} \tau^{\overline{\alpha}^l}(u_0)} \left[\alpha_1 \prod_{l=2}^{n} \tau^{\overline{\alpha}^l}(u_0) - \lambda^{\overline{\alpha}^1}(u_0) \prod_{l=2}^{n} \tau^{\overline{\alpha}^l}(u_0) \right] = 1.
\end{aligned}$$

This completes the proof of minimality for $g_+^{\overline{\alpha}^n,u_0}(u)$.

For the negative n-step MIR function, we observe that, by Theorem 5.20, $g_-^{\overline{\alpha}^n,u_0}(u)$ is a valid inequality for $P(U,u_0)$ given its respective conditions hold. On the other hand, based on (4.8), the validity of $g_-^{\overline{\alpha}^n,u_0}(u) + g_-^{\overline{\alpha}^n,u_0}(u_0 - u) = 1$ is equivalent to the validity of $g_+^{\overline{\alpha}^n,-u_0}(-u) + g_+^{\overline{\alpha}^n,-u_0}(-u_0 + u) = 1$, which was just proved. Hence $g_-^{\overline{\alpha}^n,u_0}(u)$ is a minimal valid inequality too. □

Finally, we can prove the facet-defining property of the n-step MIR functions for $P(U,u_0)$:

Theorem 5.24. *Let $u_0 \in U$, $n,t \in \mathbb{N}$, $\overline{\alpha} = (\alpha_1,\alpha_2,\ldots,\alpha_n)$ such that $\alpha_1 = 1/t$, $\alpha_i \in \mathbb{R}$, $\alpha_i > 0$ for $i = 2,\ldots,n$. If $u_0/\alpha_1 < \tau^{\overline{\alpha}^1}(u_0)$ and $\lambda^{\overline{\alpha}^{i-1}}(u_0)/\alpha_i < \tau^{\overline{\alpha}^i}(u_0) \leq \alpha_{i-1}/\alpha_i$ for $i = 2,\ldots,n$, then the function*

$g_+^{\overline{\alpha}^n,u_0}(u)$ *is a facet for* $P(U,u_0)$ *. Similarly, if* $-u_0/\alpha_1 < \tau^{\overline{\alpha}^1}(-u_0)$ *and* $\lambda^{\overline{\alpha}^{i-1}}(-u_0)/\alpha_i < \tau^{\overline{\alpha}^i}(-u_0) \leq \alpha_{i-1}/\alpha_i$ *for* $i = 2,\ldots,n$*, the function* $g_-^{\overline{\alpha}^n,u_0}(u)$ *is a facet for* $P(U,u_0)$*.*

Proof. From Lemma 4.17, the function $g_+^{\overline{\alpha}^n,u_0}(u)$ is a piecewise linear function with only two slopes. By Theorem 5.23, $g_+^{\overline{\alpha}^n,u_0}(u)$ is a minimal valid inequality for $P(U,u_0)$. Therefore, by Theorem 5.17, it is also subadditive. Thus all the conditions of Theorem 5.18 are satisfied, and hence $g_+^{\overline{\alpha}^n,u_0}(u)$ is a facet for $P(U,u_0)$. Exactly the same argument holds for $g_-^{\overline{\alpha}^n,u_0}(u)$. □

Example 5.25. Four examples of the positive n-step MIR functions for $n = 1,2,3,4$ are shown in Figures 5.2 to 5.5. These functions are the 1-step function $g_+^{1,0.8}(u)$, the 2-step function $g_+^{(1,0.45),0.8}(u)$, the 3-step function $g_+^{(1,0.45,0.2),0.8}(u)$ and the 4-step function $g_+^{(1,0.45,0.2,0.09),0.8}(u)$, respectively. By Theorem 5.24, all of them are facets for the infinite group polyhedron $P(U,0.8)$. It can be easily verified that the parameters of these functions satisfy the conditions of Theorem 5.24.
Figures 5.6 and 5.7 show two facets of $P(U,0.2)$ generated by two negative 3-step MIR functions based on Theorem 5.24. The facet $g_-^{(1,0.45,0.2),0.2}(u)$ of Figure 5.6 is the mirror image of the facet in Figure 5.4, and the facet $g_-^{(1,0.45,0.2,0.09),0.2}(u)$ of Figure 5.7 is the mirror image of the facet in Figure 5.5.

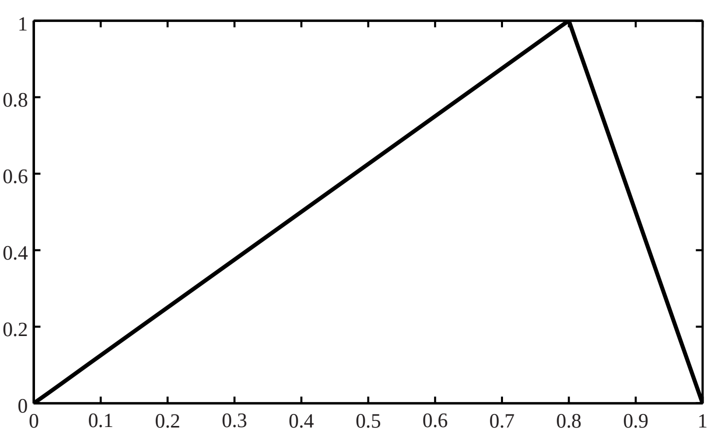


Figure 5.2: $g_+^{1,0.8}(u)$ facet for $P(U,0.8)$

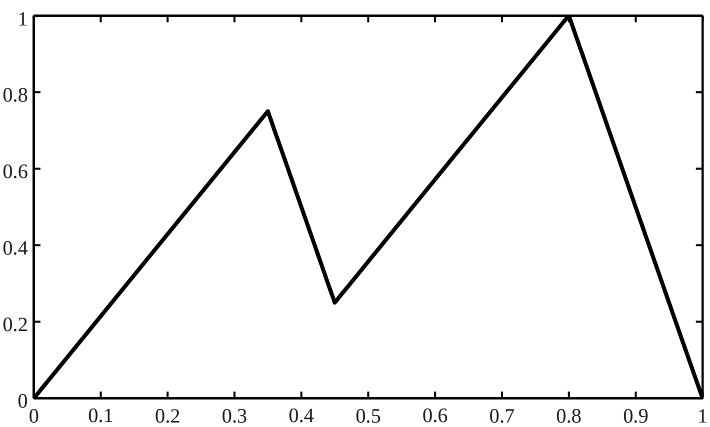


Figure 5.3: $g_{+}^{(1,0.45),0.8}(u)$ facet for $P(U,0.8)$

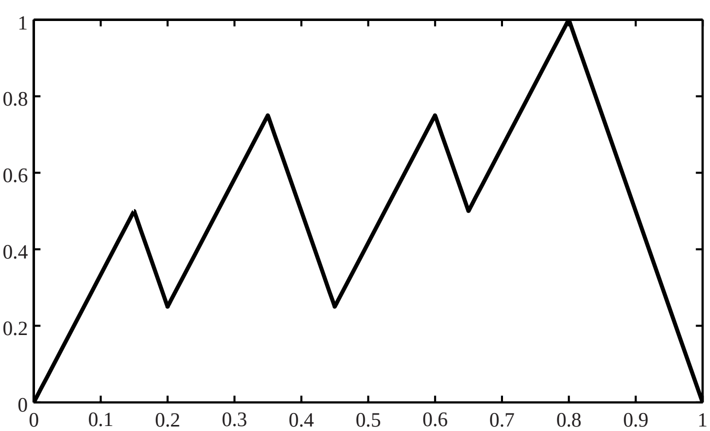


Figure 5.4: $g_{+}^{(1,0.45,0.2),0.8}(u)$ facet for $P(U,0.8)$

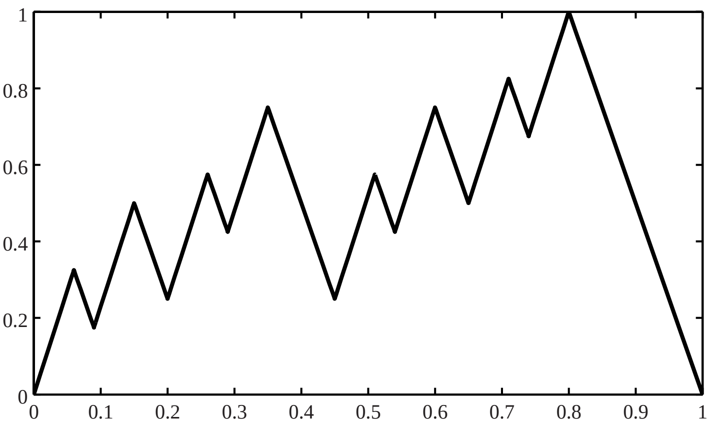


Figure 5.5: $g_{+}^{(1,0.45,0.2,0.09),0.8}(u)$ facet for $P(U,0.8)$

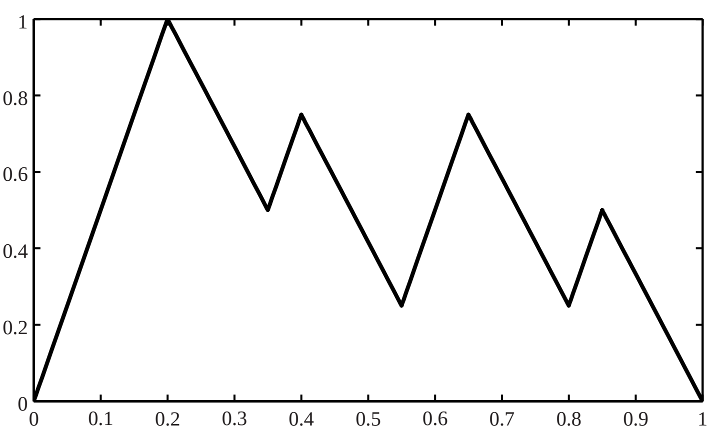


Figure 5.6: $g_{-}^{(1,0.45,0.2),0.2}(u)$ facet for $P(U,0.2)$

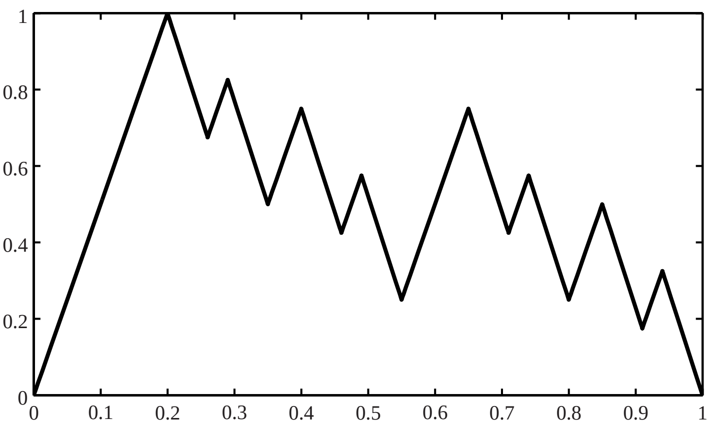


Figure 5.7: $g_{-}^{(1,0.45,0.2,0.09),0.2}(u)$ facet for $P(U,0.2)$

□

5.4 *n*-step MIR Functions: Facets for Finite Group Polyhedra

In this section, we show that the *n*-step MIR functions also define facets for the finite master group polyhedron $P(C_N, \frac{r}{N})$. This is of course contingent on the existence of the MIR function with appropriate parameters.

To prove the facet-defining property of the *n*-step MIR functions for the finite master cyclic group problem, we will use its facet-defining property for the infinite group polyhedra proved in section 5.3. We will also use the following theorems from [25] and [23]. They are stated in the format most appropriate for our arguments. Theorem 5.26 from [25] states a relationship between the facets of the finite and infinite group problems, and Theorem 5.27 from [23] is a fundamental result which presents a way for derivation of facets for a larger group using facets of a smaller group based on a homomorphism mapping of the larger one onto the smaller one.

Theorem 5.26. *If $\rho(u)$ is a facet (extreme valid inequality) for $P(U, \frac{r}{N})$ and consists of straight line segments connected at values $u = \frac{j}{N}$ for $j = 0, \ldots, N$, then $\overline{\pi} = (\pi_1, \ldots, \pi_{N-1})$, where $\pi_j = \rho(\frac{j}{N})$, is a facet for $P(C_N, \frac{r}{N})$.* □

Theorem 5.27. *Let $0 < r < N$ and the mapping $\phi : C_N \to C_{N'}$ be a homomorphism where $\frac{r'}{N'} = \phi(\frac{r}{N})$ and $r' \neq 0$ ($\frac{r}{N}$ is not in the kernel of ϕ). If $\overline{\pi}' = (\pi'_1, \ldots, \pi'_{N'-1})$ is a facet of $P(C_{N'}, \frac{r'}{N'})$, then $\overline{\pi} = (\pi_1, \ldots, \pi_{N-1})$, where $\pi_j = \pi'_{N'\phi(\frac{j}{N})}$ and π'_0 is defined to be zero, is a facet of $P(C_N, \frac{r}{N})$.* □

We present the facet-defining property of the n-step MIR functions for the finite master cyclic group problem in two theorems. In Theorem 5.28, we use Theorems 5.24 and 5.26 to prove this property for the n-step MIR functions where certain conditions on parameters hold. Then in Theorem 5.29, we use the results of Theorem 5.28 to extend this property to a larger collection of n-step MIR functions based on Theorem 5.27.

Theorem 5.28. *Let $n,t,N,r,d_i \in \mathbb{N}$, $i = 2,\ldots,n$, $r < N$ and assume $\frac{1}{t}$ is an integer multiple of $\frac{1}{N}$. Let $\overline{\alpha} = (\alpha_1,\ldots,\alpha_n)$ such that $\alpha_1 = \frac{1}{t}$, $\alpha_i = \frac{d_i}{N}$ for $i = 2,\ldots,n$. If $\frac{r}{N}/\alpha_1 < \tau^{\overline{\alpha}^1}(\frac{r}{N})$ and $\lambda^{\overline{\alpha}^{i-1}}(\frac{r}{N})/\alpha_i < \tau^{\overline{\alpha}^i}(\frac{r}{N}) \leq \alpha_{i-1}/\alpha_i$ for $i = 2,\ldots,n$, then $\overline{\pi} = (\pi_1,\ldots,\pi_{N-1})$, where $\pi_j = g_+^{\overline{\alpha}^n,\frac{r}{N}}(\frac{j}{N})$ for $j = 1,\ldots,N-1$, is a facet for $P(C_N,\frac{r}{N})$. Similarly, if $-\frac{r}{N} < \tau^{\overline{\alpha}^1}(-\frac{r}{N})$ and $\lambda^{\overline{\alpha}^{i-1}}(-\frac{r}{N})/\alpha_i < \tau^{\overline{\alpha}^i}(-\frac{r}{N}) \leq \alpha_{i-1}/\alpha_i$ for $i = 2,\ldots,n$, then $\overline{\pi} = (\pi_1,\ldots,\pi_{N-1})$, where $\pi_j = g_-^{\overline{\alpha}^n,\frac{r}{N}}(\frac{j}{N})$ for $j = 1,\ldots,N-1$, is a facet for $P(C_N,\frac{r}{N})$.*

Proof. Let $\overline{\pi} = (\pi_1,\ldots,\pi_{N-1})$, where $\pi_j = g_+^{\overline{\alpha}^n,\frac{r}{N}}(\frac{j}{N})$. Given the conditions on the parameters, by Theorem 5.24, the function $g_+^{\overline{\alpha}^n,\frac{r}{N}}(u)$ is a facet for $P(U,\frac{r}{N})$. Moreover, by Lemma 4.17, this function is a piece-wise linear continuous function with the two slopes. The break points of this function happen at the boundary points of the sets $\mathscr{I}_m^{\overline{\alpha}^n,\frac{r}{N}}$, $m = 0,1,\ldots,n$. These are the points at which either $\lambda^{\overline{\alpha}^i}(u) = 0$ or $\lambda^{\overline{\alpha}^i}(u) = \lambda^{\overline{\alpha}^i}(\frac{r}{N})$. Now since 0, $\frac{r}{N}$ and α_i, for $i = 1,\ldots,n$, are all integer multiples of $\frac{1}{N}$, all these boundary points and hence the break points of the function $g_+^{\overline{\alpha}^n,\frac{r}{N}}(u)$ occur on the elements of the group C_N. Therefore, $g_+^{\overline{\alpha}^n,\frac{r}{N}}(u)$ over U consists of straight line segments connected at values $u = \frac{j}{N}$ for $j = 0,\ldots,N$. Thus, by Theorem 5.26, $\overline{\pi}$ is a facet for $P(C_N,\frac{r}{N})$. A similar argument proves the result for the negative n-step MIR function. □

In the next theorem, we use Theorem 5.27 to extend the result of Theorem 5.28 to a larger group that can be appropriately mapped onto a smaller group in a homomorphism.

Theorem 5.29. *Let $n,t,s,N,r,d_i \in \mathbb{N}$, $i = 2,\ldots,n$, $r < N$ and $N = N'd$, where $d = gcd(s,N)$ is the greatest common divisor of s and N. Assume $\frac{1}{t}$ is an integer multiple of $\frac{1}{N}$ and $\frac{1}{t},\frac{d_2}{N},\ldots,\frac{d_n}{N}$ are all integer multiples of $\frac{1}{N'}$. Let $\overline{\alpha} = (\alpha_1,\ldots,\alpha_n)$ such that $\alpha_1 = \frac{1}{st}$, $\alpha_i = \frac{d_i}{sN}$ for $i = 2,\ldots,n$. If $\frac{r}{N}/\alpha_1 < \tau^{\overline{\alpha}^1}(\frac{r}{N})$ and $\lambda^{\overline{\alpha}^{i-1}}(\frac{r}{N})/\alpha_i < \tau^{\overline{\alpha}^i}(\frac{r}{N}) \leq \alpha_{i-1}/\alpha_i$ for $i = 2,\ldots,n$, then $\overline{\pi} = (\pi_1,\ldots,\pi_{N-1})$, where $\pi_j = g_+^{\overline{\alpha}^n,\frac{r}{N}}(\frac{j}{N})$ for $j = 1,\ldots,N-1$, is a facet for $P(C_N,\frac{r}{N})$. Similarly, if $-\frac{r}{N} < \tau^{\overline{\alpha}^1}(-\frac{r}{N})$ and $\lambda^{\overline{\alpha}^{i-1}}(-\frac{r}{N})/\alpha_i < \tau^{\overline{\alpha}^i}(-\frac{r}{N}) \leq \alpha_{i-1}/\alpha_i$ for $i = 2,\ldots,n$, then $\overline{\pi} = (\pi_1,\ldots,\pi_{N-1})$, where $\pi_j = g_-^{\overline{\alpha}^n,\frac{r}{N}}(\frac{j}{N})$ for $j = 1,\ldots,N-1$, is a facet for $P(C_N,\frac{r}{N})$.*

Proof. Let $\overline{\pi} = (\pi_1, \ldots, \pi_{N-1})$, where $\pi_j = g_+^{\overline{\alpha}^n, \frac{r}{N}}(\frac{j}{N})$ for $j = 1, \ldots, N-1$. We define $\overline{\omega} = (\omega_1, \ldots, \omega_n)$ $= s\overline{\alpha} = (\frac{1}{t}, \frac{d_2}{N}, \ldots, \frac{d_n}{N})$. Based on the Definitions 3.3 and 3.4, $\frac{r}{N}/\alpha_1 < \tau^{\overline{\alpha}^1}(\frac{r}{N})$ and $\lambda^{\overline{\alpha}^{i-1}}(\frac{r}{N})/\alpha_i <$ $\tau^{\overline{\alpha}^i}(\frac{r}{N}) \leq \alpha_{i-1}/\alpha_i$ for $i = 2, \ldots, n$ implies

$$\begin{aligned} \frac{sr}{N}/\omega_1 &< \tau^{\overline{\omega}^1}(\frac{sr}{N}) \\ \lambda^{\overline{\omega}^{i-1}}(\frac{sr}{N})/\omega_i &< \tau^{\overline{\omega}^i}(\frac{sr}{N}) \leq \omega_{i-1}/\omega_i \qquad \text{for } i = 2, \ldots, n. \end{aligned} \tag{5.14}$$

Therefore the conditions of Definition 4.11 for the function $g_+^{\overline{\omega}^n, \frac{sr}{N}}(u)$ are satisfied and this function is definable. Moreover, the following identities are true for $i = 1, \ldots, n$ and $j = 1, \ldots, N-1$:

$$\begin{aligned} \lambda^{\overline{\omega}^i}(\frac{sj}{N}) &= s\lambda^{\overline{\alpha}^i}(\frac{j}{N}) \\ \tau^{\overline{\omega}^i}(\frac{sj}{N}) &= \tau^{\overline{\alpha}^i}(\frac{j}{N}) \\ \sigma^{\overline{\omega}^i}(\frac{sj}{N}) &= \sigma^{\overline{\alpha}^i}(\frac{j}{N}) \end{aligned}$$

Based on these identities and Definition 4.11, it is easy to verify that

$$g_+^{\overline{\alpha}^n, \frac{r}{N}}(\frac{j}{N}) = g_+^{\overline{\omega}^n, \frac{sr}{N}}(\frac{sj}{N}) \qquad \text{for } j = 1, \ldots, N-1. \tag{5.15}$$

Now, as discussed in section 5.1, the mapping $\phi(\frac{j}{N}) = \frac{sj}{N}$ mod 1 is a homomorphism from C_N to $C_{N'}$. Moreover, if we let $\frac{r'}{N'} = \frac{sr}{N}$ mod 1, we have $\phi(\frac{r}{N}) = \frac{r'}{N'}$. Also by the first inequality in (5.14), $\frac{sr}{N}$ cannot be an integer, which means $\frac{r'}{N'} \neq 0$ (or $\frac{r}{N}$ does not belong to the kernel of ϕ).
On the other hand, by the periodical properties of Theorem 4.12, since 1 is a period for $g_+^{\overline{\omega}^n, \frac{sr}{N}}(u)$, we have $g_+^{\overline{\omega}^n, \frac{sr}{N}}(\frac{sj}{N}) = g_+^{\overline{\omega}^n, \frac{r'}{N'}}(\frac{sj}{N} \bmod 1) = g_+^{\overline{\omega}^n, \frac{r'}{N'}}(\phi(\frac{j}{N}))$. Thus by (5.15) we have

$$g_+^{\overline{\alpha}^n, \frac{r}{N}}(\frac{j}{N}) = g_+^{\overline{\omega}^n, \frac{r'}{N'}}(\phi(\frac{j}{N})) \qquad \text{for } j = 1, \ldots, N-1. \tag{5.16}$$

Now let $\overline{\pi}' = (\pi'_1, \ldots, \pi'_{N'-1})$ such that $\pi'_k = g_+^{\overline{\omega}^n, \frac{r'}{N'}}(\frac{k}{N'})$ for $k = 1, \ldots, N'-1$. By Theorem 5.28, $\overline{\pi}'$ is a facet for $P(C_{N'}, \frac{r'}{N'})$ because (5.14) holds and by assumption all elements of $\overline{\omega}$ are integer multiples of $\frac{1}{N'}$, and hence all conditions of that theorem are satisfied. Therefore $\overline{\pi}'$ and the homomorphism ϕ satisfy all the conditions of the Theorem 5.27 and hence $\overline{\pi} = (\pi_1, \ldots, \pi_{N-1})$, where $\pi_j = \pi'_{N'\phi(\frac{j}{N})} =$ $g_+^{\overline{\omega}^n, \frac{r'}{N'}}(\phi(\frac{j}{N}))$ for $j = 1, \ldots, N-1$, is a facet for $P(C_N, \frac{r}{N})$. By (5.16), $\pi_j = g_+^{\overline{\alpha}^n, \frac{r}{N}}(\frac{j}{N})$ for $j = 1, \ldots, N-$ 1, and hence the proof is complete. A similar argument proves the result for the negative n-step MIR function. $\square$

It should be noted that in Theorem 5.29, if $s = 1$ the homomorphism will be an isomorphism and the Theorem reduces to Theorem 5.28. Therefore Theorem 5.28 is in fact a special case of Theorem 5.29.

Example 5.30. Figures 5.8, 5.9 and 5.10 show the use of the same 1-, 2- and 3-step MIR functions of Example 5.25, respectively, to generate facets for the finite master group polyhedron $P(C_{20}, \frac{16}{20})$ based on Theorem 5.28. Let $\overline{\pi} = (\pi_1, \ldots, \pi_{19})$, $N = 20$, $t = 1$ and $r = 16$. If we have $n = 1$ in Theorem 5.28, the result is the facet $\pi_j = g_+^{1,0.8}(\frac{j}{20})$, $j = 1, \ldots, 19$, shown in Figure 5.8. By the same theorem, if $n = 2$ and $d_2 = 9$, $\pi_j = g_+^{(1,0.45),0.8}(\frac{j}{20})$, $j = 1, \ldots, 19$, gives the facet of Figure 5.9, and if $n = 3$, $d_2 = 9$ and $d_3 = 4$, the result is the facet $\pi_j = g_+^{(1,0.45,0.2),0.8}(\frac{j}{20})$, $j = 1, \ldots, 19$, shown in Figure 5.10.

The facet in Figure 5.11 is based on Theorem 5.29. Let $N = 20$, $r = 16$, $t = 1$, $s = 3$, $n = 3$, $d_2 = 5$ and $d_3 = 2$ in Theorem 5.29. We observe all conditions of this theorem are satisfied, and hence $\pi_j = g_+^{(1/3,0.25/3,0.1/3),0.8}(\frac{j}{20})$, $j = 1, \ldots, 19$, is a facet for $P(C_{20}, \frac{16}{20})$ too.
Figures 5.12 and 5.13 show two facets of $P(C_{20}, \frac{4}{20})$ generated by two negative 3-step MIR functions based on Theorems 5.28 and 5.29, respectively. The facet $\pi_j = g_-^{(1,0.45,0.2),0.2}(\frac{j}{20})$, $j = 1, \ldots, 19$, of Figure 5.12 is the mirror image of the facet in Figure 5.10, and the facet $\pi_j = g_-^{(1/3,0.25/3,0.1/3),0.2}(\frac{j}{20})$, $j = 1, \ldots, 19$, of Figure 5.13 is the mirror image of the facet in Figure 5.11. All parameters except $r = 4$ are the same as the parameters of the facets in Figures 5.10 and 5.11.

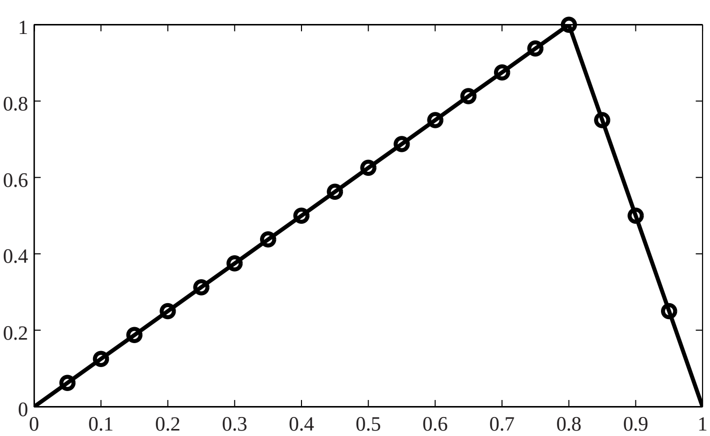


Figure 5.8: $\pi_j = g_+^{1,0.8}(\frac{j}{20})$ facet for $P(C_{20}, \frac{16}{20})$

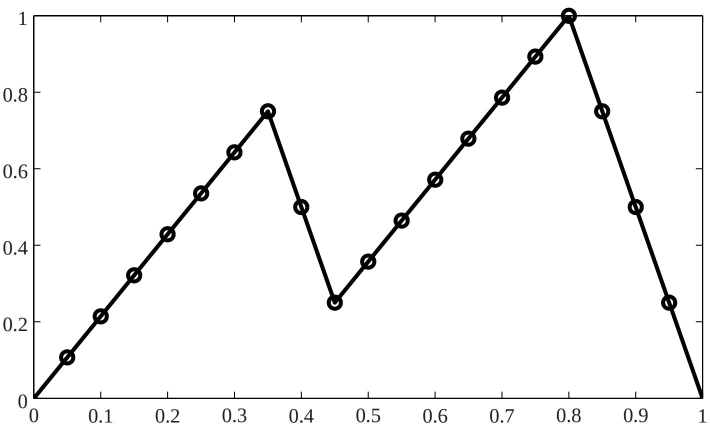


Figure 5.9: $\pi_j = g_+^{(1,0.45),0.8}(\frac{j}{20})$ facet for $P(C_{20}, \frac{16}{20})$

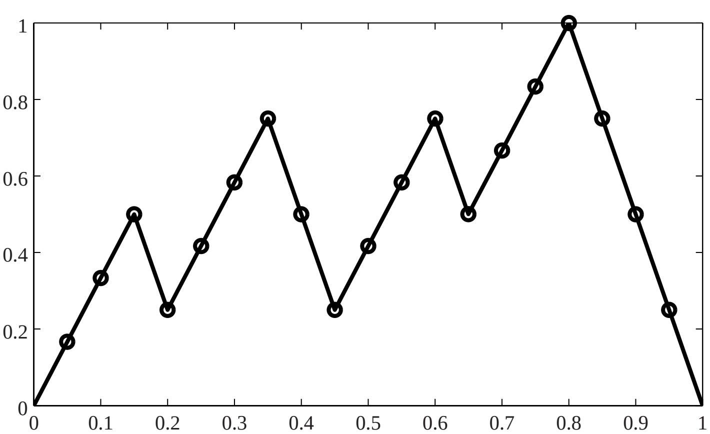


Figure 5.10: $\pi_j = g_+^{(1,0.45,0.2),0.8}(\frac{j}{20})$ facet for $P(C_{20}, \frac{16}{20})$

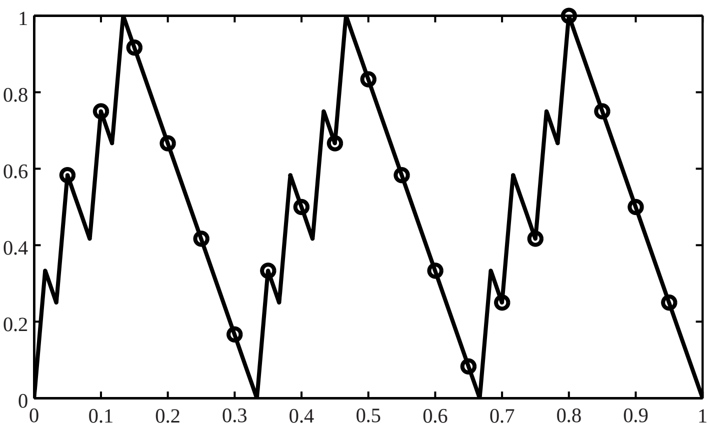


Figure 5.11: $\pi_j = g_+^{(1/3,0.25/3,0.1/3),0.8}(\frac{j}{20})$ facet for $P(C_{20}, \frac{16}{20})$

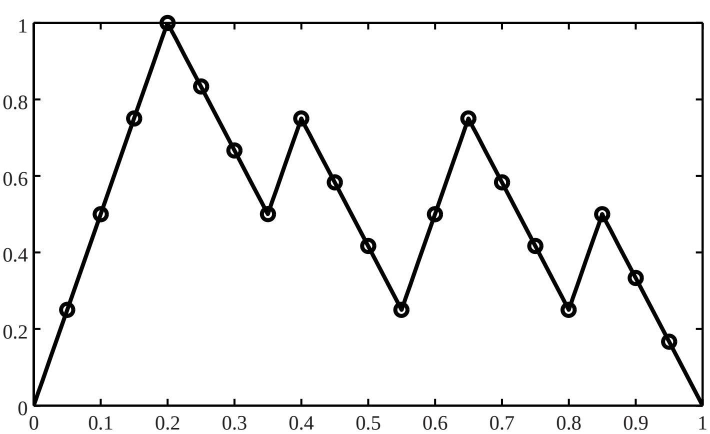


Figure 5.12: $\pi_j = g_-^{(1,0.45,0.2),0.2}(\frac{j}{20})$ facet for $P(C_{20}, \frac{4}{20})$

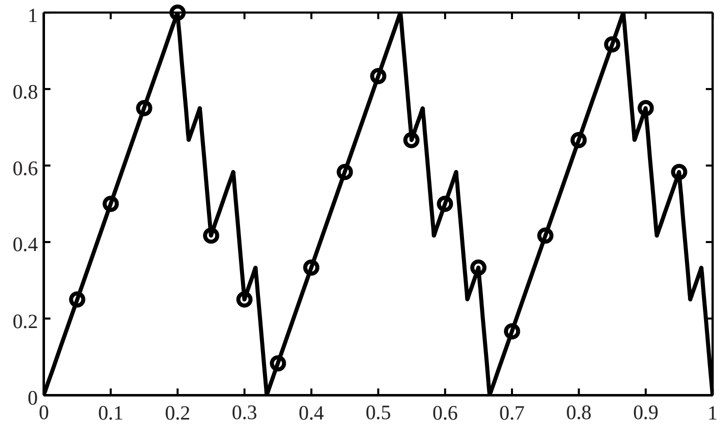


Figure 5.13: $\pi_j = g_{-}^{(1/3,0.25/3,0.1/3),0.2}(\frac{j}{20})$ facet for $P(C_{20}, \frac{4}{20})$

□

5.5 Facets for Group Polyhedra with Continuous Variables

As discussed in [24, 25], for a mixed integer constraint, facets of mixed integer group polyhedra, i.e. polyhedra of group problems with continuous variables, are sources for generating valid inequalities. In this section, we argue that the facet-defining properties of n-step MIR functions extend to the mixed integer group polyhedra. The infinite and finite mixed integer group polyhedra and the concepts of valid inequality and facet for them are formally defined as follows:

Definition 5.31. *Let $U = [0,1)$. We define $X_{-}^{+}(U,u_0)$ for a right-hand side $u_0 \in U - \{0\}$ as the set of triplets $(x(u),x^+,x^-)$ satisfying*

$$\sum_{u \in U} ux(u) + x^+ - x^- \equiv u_0 \mod 1,$$

*where $x^+, x^- \in \mathbb{R}_+$ and $x(u)$ is an integer-valued function on U such that $x(u) \geq 0$ for all $u \in U$ and it has a finite support., i.e. $x(u) > 0$ only for a finite subset of U. A triplet $(x(u),x^+,x^-) \in X_{-}^{+}(U,u_0)$ is a solution to the **infinite mixed integer group problem** with the right hand side u_0. The **infinite mixed integer group polyhedron**, denoted by $P_{-}^{+}(U,u_0)$, is the convex hull of all such solutions, i.e.*

$$P_{-}^{+}(U,u_0) = conv\Big\{(x(u),x^+,x^-) \in X_{-}^{+}(U,u_0)\Big\}.$$

□

Definition 5.32. *A* ***valid inequality*** *for $P_-^+(U,u_0)$ is a triplet $(\pi(u),\pi^+,\pi^-)$ such that $\sum_{u\in U}\pi(u)x(u)$ $+\pi^+x^+ + \pi^-x^- \geq 1$ for any $(x(u),x^+,x^-)\in X_-^+(U,u_0)$, where $\pi^+,\pi^-\in\mathbb{R}_+$, $\pi(u)$ is a real-valued function over $u\in U$, $\pi(0)=0$, and $\pi(u)\geq 0, u\in U$. A valid inequality $(\pi(u),\pi^+,\pi^-)$ for $P_-^+(U,u_0)$ is a* ***facet (extreme valid inequality)*** *if it cannot be written as a convex combination of two distinct valid inequalities for $P_-^+(U,u_0)$.* □

Definition 5.33. *Any point $(x_1,\ldots,x_{N-1},x^+,x^-)\in\mathbb{Z}_+^{N-1}\times\mathbb{R}_+^2$ that satisfies*

$$\sum_{j=1}^{N-1}\frac{j}{N}x_j + x^+ - x^- \equiv \frac{r}{N} \mod 1$$

is called a solution to the ***finite mixed integer group problem*** *over the cyclic group C_N, with the right-hand side $\frac{r}{N}$. The* ***finite mixed integer group polyhedron****, denoted by $P_-^+(C_N,\frac{r}{N})$, is the polyhedron of the convex hull of all these solutions, i.e.*

$$P_-^+(C_N,\frac{r}{N}) = conv\Big\{(x_1,\ldots,x_{N-1},x^+,x^-)\in\mathbb{Z}_+^{N-1}\times\mathbb{R}_+^2 : \sum_{j=1}^{N-1}\frac{j}{N}x_j + x^+ - x^- \equiv \frac{r}{N}\ mod\ 1\Big\}.$$

□

Definition 5.34. *The vector $\overline{\pi}_-^+ = (\pi_1,\ldots,\pi_{N-1},\pi^+,\pi^-)\in\mathbb{R}_+^{N+1}$ is a* ***valid*** *inequality for $P_-^+(C_N,\frac{r}{N})$ if $\sum_{j=1}^{N-1}\pi_j x_j + \pi^+x^+ + \pi^-x^- \geq 1$ for every $(x_1,\ldots,x_{N-1},x^+,x^-)\in\mathbb{Z}_+^{N-1}\times\mathbb{R}_+^2$ such that $\sum_{j=1}^{N-1}\frac{j}{N}x_j + x^+ - x^- \equiv \frac{r}{N}$ mod 1. A valid inequality $\overline{\pi}_-^+$ for $P_-^+(C_N,\frac{r}{N})$ is a* ***facet*** *if it cannot be written as a convex combination of two distinct valid inequalities for $P_-^+(C_N,\frac{r}{N})$.* □

Based on the results in [24, 25], there is a one-to-one correspondence between facets of $P_-^+(U,u_0)$ and $P(U,u_0)$ and also between facets of $P_-^+(C_N,\frac{r}{N})$ and $P(C_N,\frac{r}{N})$. These results from [24, 25] are stated in the following theorems:

Theorem 5.35. *If $\pi(u)$ is a facet for $P(U,u_0)$, and $\pi^+ = \lim_{u\downarrow 0}\frac{\pi(u)}{u}$, $\pi^- = \lim_{u\uparrow 1}\frac{\pi(u)}{1-u}$, then $(\pi(u),\pi^+,\pi^-)$ is a facet for $P_-^+(U,u_0)$. Conversely, if $(\pi(u),\pi^+,\pi^-)$ is a facet for $P_-^+(U,u_0)$, then $\pi(u)$ is a facet for $P(U,u_0)$.* □

Theorem 5.36. *If $(\pi_1,\ldots,\pi_{N-1})$ is a facet for $P(C_N,\frac{r}{N})$, and $\pi^+ = N\pi_1$, $\pi^- = N\pi_{N-1}$, then $(\pi_1,\ldots,\pi_{N-1},\pi^+,\pi^-)$ is a facet for $P_-^+(C_N,\frac{r}{N})$. Conversely, if $(\pi_1,\ldots,\pi_{N-1},\pi^+,\pi^-)$ is a facet for $P_-^+(C_N,\frac{r}{N})$, then $(\pi_1,\ldots,\pi_{N-1})$ is a facet for $P(C_N,\frac{r}{N})$.* □

Based on Theorem 5.35, the facet-defining properties for $P(U,u_0)$, stated in Theorem 5.24, can be easily extended to $P_-^+(U,u_0)$ as follows:

Theorem 5.37. *Let $u_0 \in U$, $n,t \in \mathbb{N}$, $\overline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ such that $\alpha_1 = 1/t$, $\alpha_i \in \mathbb{R}$, $\alpha_i > 0$ for $i = 2,\ldots,n$. If $u_0/\alpha_1 < \tau^{\overline{\alpha}^1}(u_0)$ and $\lambda^{\overline{\alpha}^{i-1}}(u_0)/\alpha_i < \tau^{\overline{\alpha}^i}(u_0) \leq \alpha_{i-1}/\alpha_i$ for $i = 2,\ldots,n$, then $(g_+^{\overline{\alpha}^n,u_0}(u), s^+(g_+^{\overline{\alpha}^n,u_0}), s^-(g_+^{\overline{\alpha}^n,u_0}))$ is a facet for $P_-^+(U,u_0)$. Similarly, if $-u_0/\alpha_1 < \tau^{\overline{\alpha}^1}(-u_0)$ and $\lambda^{\overline{\alpha}^{i-1}}(-u_0)/\alpha_i < \tau^{\overline{\alpha}^i}(-u_0) \leq \alpha_{i-1}/\alpha_i$ for $i = 2,\ldots,n$, then $(g_-^{\overline{\alpha}^n,u_0}(u), s^+(g_-^{\overline{\alpha}^n,u_0}), s^-(g_-^{\overline{\alpha}^n,u_0}))$ is a facet for $P_-^+(U,u_0)$.*

Proof. By Theorem 5.24, $g_+^{\overline{\alpha}^n,u_0}(u)$ gives a facet for $P(U,u_0)$. Now the limits in Theorem 5.35 are simply the two slopes of the n-step MIR function presented in Lemma 4.17, i.e. $\lim_{u\downarrow 0} \frac{g_+^{\overline{\alpha}^n,u_0}(u)}{u} = s^+(g_+^{\overline{\alpha}^n,u_0})$ and $\lim_{u\uparrow 1} \frac{g_+^{\overline{\alpha}^n,u_0}(u)}{1-u} = s^-(g_+^{\overline{\alpha}^n,u_0})$. Therefore by Theorem 5.35, $(g_+^{\overline{\alpha}^n,u_0}(u), s^+(g_+^{\overline{\alpha}^n,u_0}), s^-(g_+^{\overline{\alpha}^n,u_0}))$ is a facet for $P_-^+(U,u_0)$. The argument for the negative n-step MIR function is similar. □

Also based on Theorem 5.36, the facet-defining properties for $P(C_N, \frac{r}{N})$, stated in Theorems 5.28 and 5.29, can be easily extended to $P_-^+(C_N, \frac{r}{N})$. As Theorem 5.28 is a special case of Theorem 5.29, we only bring the extension of the latter:

Theorem 5.38. *Let $n,t,s,N,r,d_i \in \mathbb{N}$, $i = 2,\ldots,n$, $r < N$ and $N = N'd$, where $d = gcd(s,N)$ is the greatest common divisor of s and N. Assume $\frac{1}{t}$ is an integer multiple of $\frac{1}{N}$ and $\frac{1}{t}, \frac{d_2}{N}, \ldots, \frac{d_n}{N}$ are all integer multiples of $\frac{1}{N'}$. Let $\overline{\alpha} = (\alpha_1,\ldots,\alpha_n)$ such that $\alpha_1 = \frac{1}{st}$, $\alpha_i = \frac{d_i}{sN}$, for $i = 2,\ldots,n$. If $\frac{r}{N}/\alpha_1 < \tau^{\overline{\alpha}^1}(\frac{r}{N})$ and $\lambda^{\overline{\alpha}^{i-1}}(\frac{r}{N})/\alpha_i < \tau^{\overline{\alpha}^i}(\frac{r}{N}) \leq \alpha_{i-1}/\alpha_i$, for $i = 2,\ldots,n$, then $\overline{\pi} = (\pi_1,\ldots,\pi_{N-1}, N\pi_1, N\pi_{N-1})$, where $\pi_j = g_+^{\overline{\alpha}^n,\frac{r}{N}}(\frac{j}{N})$, for $j = 1,\ldots,N-1$, is a facet for $P_-^+(C_N, \frac{r}{N})$. Similarly, if $-\frac{r}{N} < \tau^{\overline{\alpha}^1}(-\frac{r}{N})$ and $\lambda^{\overline{\alpha}^{i-1}}(-\frac{r}{N})/\alpha_i < \tau^{\overline{\alpha}^i}(-\frac{r}{N}) \leq \alpha_{i-1}/\alpha_i$, for $i = 2,\ldots,n$, then $\overline{\pi} = (\pi_1,\ldots,\pi_{N-1}, N\pi_1, N\pi_{N-1})$, where $\pi_j = g_-^{\overline{\alpha}^n,\frac{r}{N}}(\frac{j}{N})$, for $j = 1,\ldots,N-1$, is a facet for $P_-^+(C_N, \frac{r}{N})$.*

Proof. Theorem 5.36, using Theorem 5.29, directly leads to the desired result. □

Chapter 6

Summary and Future Research

We presented a generalization of MIR. For any positive integer n, we developed n facets of a certain $(n+1)$-dimensional mixed integer set. We defined the last of these facets as the *n-step MIR facet*. As a result, we generated an infinite number of facets (one for each n), which we then used to produce valid inequalities for the general MIP constraints.

We introduced a generalized MIR procedure which, for any n, uses the n-step MIR facet to generate a family of valid inequalities for the feasible set of a general MIP constraint. We called these valid inequalities the *n-step MIR inequalities*. The Gomory Mixed Integer Cut and the 2-step MIR inequality are simply the first two families corresponding to $n = 1$ and 2, respectively. The n-step MIR inequalities are easily produced using closed-form periodic functions. which we defined as the *n-step MIR functions*. None of these functions dominates the other on its whole period.

We established an important connection between the n-step MIR functions and facets of group polyhedra. We proved that for any n, the n-step MIR function defines new families of facets for the finite and infinite group polyhedra. This fact shows that the n-step MIR inequalities are potentially strong. Many of the facets for the finite and infinite group polyhedra that we generate using n-step MIR functions are new facets that have not been introduced in the literature before.

Several future research paths can be followed based on the results of this dissertation:

- On the first path, further development of MIR based valid inequalities for general MIP problems can be investigated. In this dissertation, we have used simple sets of a special structure to derive MIR facets. We have also developed particular MIR procedures for derivation of MIR inequalities. The use of simple sets of other structures and new MIR procedures for this purpose is an interesting research subject. Along this path, development of valid inequalities with more than two slopes can be of particular interest. This research path would also provide a better understanding of the MIR process and would open the door to new problems in this regard.

- On the second path, valid inequalities based on group problems can be studied further. This area has been very active in recent years. A very interesting problem is whether a general relationship can be found between the MIR concept and facets of group polyhedra. This dissertation reveals a small part of the picture.

- On the third path, we can study the use of the new generalized MIR inequalities and valid inequalities based on group problems in development of valid inequalities for MIP problems with special structure. It has been shown that several strong valid inequalities for special MIP problems can be derived from variations of the 1-step MIR.

- On the fourth path, the very important issue of using these valid inequalities in practice can be investigated. With MIR inequalities and valid inequalities from group problems, we have an infinite number of different valid inequalities and what we need are intelligent methods to evaluate them and use them most effectively in general algorithms for solving MIP like branch-and-cut. The research in this area is in its elementary stages and can be done in both theoretical and experimental areas.

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Appendix

Appendix A

MATLAB Codes for Generating *n*-step MIR Functions

For details about the codes in this appendix refer to section 4.4.

A.1 `lambda`, `tau`, and `sigma` Functions

```
function l=lambda(alpha,beta)
l=beta;
for i=1:length(alpha)
    l=[l mod(l(i),alpha(i))];
end;
l(1)=[];
%%% end of the function lambda %%%


function t=tau(alpha,beta)
i=length(alpha);
t=ceil([beta/alpha(1) lambda(alpha(1:i-1),beta)./alpha(2:i)]-0.00000000001);
%%% end of the function tau %%%


function s=sigma(alpha,beta)
i=length(alpha);
s=floor([beta/alpha(1) lambda(alpha(1:i-1),beta)./alpha(2:i)]+0.00000000001);
%%% end of the function sigma %%%
```

A.2 mirp and mirpplot Functions

```
function g=mirp(alpha,b,xv)
g=[];
for i=1:length(xv)
    x=xv(i);
    n=length(alpha);
    if (sum([b/alpha(1) lambda(alpha(1:n-1),b)./alpha(2:n)]<tau(alpha,b))~=n) ...
     | (sum(tau(alpha,b)<=[tau(alpha(1),b) alpha(1:n-1)./alpha(2:n)])~=n)
        g=-1;
        return
    end
    mytaub=tau(alpha,b);
    mytaubcp=[fliplr(cumprod(fliplr(mytaub))) 1];
    mytaux=tau(alpha,x);
    mysigx=sigma(alpha,x);
    mylamb=lambda(alpha,b);
    mylamx=lambda(alpha,x);
    for m=0:n
        if m==n
            break
        end
        if mylamx(m+1)>=mylamb(m+1)
            break
        end
    end
    if m<n
        if m==0
            delta=mytaubcp(m+2)*mytaux(m+1);
        else
            delta=sum(mytaubcp(2:m+1).*mysigx(1:m))+mytaubcp(m+2)*mytaux(m+1);
        end
    end
    if m==n
        delta=sum(mytaubcp(2:n+1).*mysigx(1:n))+mylamx(n)/mylamb(n);
    end
    g=[g (alpha(1)*delta-mytaubcp(2)*x)/((alpha(1)*mytaub(1)-b)*mytaubcp(2))];
end
%%% end of the function mirp %%%


function mirpplot(alpha,b)
f=inline(['mir(' mat2str(alpha) ',' mat2str(b) ',x)']);
figure;
fplot(f,[0 1 0 1],0.00001);
h=findobj(gca,'color','b');
set(h,'color','k');
set(h,'linewidth',2);
%%% end of the function mirpplot %%%
```

A.3 mirn and mirnplot Functions

```
function g=mirn(alpha,b,xv)
g=mir(alpha,-b,-xv);
%%% end of the function mirn %%%

function mirnplot(alpha,b)
f=inline(['mirn(' mat2str(alpha) ',' mat2str(b) ',x)']);
figure;
fplot(f,[0 1 0 1],0.0001);
h=findobj(gca,'color','b');
set(h,'color','k');
set(h,'linewidth',2);
set(gca,'linewidth',1);
%%% end of the function mirnplot %%%
```