ABSTRACT

WANG, YONG. Theory and algorithms for shape-preserving bivariate cubic L_1 splines. (Under the direction of Shu-Cherng Fang.)

A major objective of modelling geophysical features, biological objects, financial processes and many other irregular surfaces and functions is to develop "shape-preserving" methodologies for smoothly interpolating bivariate data with sudden changes in magnitude or spacing. Shape preservation usually means the elimination of extraneous non-physical oscillations. Classical splines do not preserve shape well in this sense.

Empirical experiments have shown that the recently proposed cubic L_1 splines are capable of providing C^1-smooth, shape-preserving, multi-scale interpolation of arbitrary data, including data with abrupt changes in spacing and magnitude, with no need for node adjustment or other user input. However, a theoretic treatment of the bivariate cubic L_1 splines is still in lack. The currently available approximation algorithms are not able to generate the exact coefficients of a bivariate cubic L_1 spline.

For theoretical treatment and the algorithm development, we propose to solve bivariate cubic L_1 spline problems in a generalized geometric programming framework. Our framework includes a primal problem, a geometric dual problem with a linear objective function and convex cubic constraints, and a linear system for dual-to-primal transformation. We show that bivariate cubic L_1 splines indeed preserve linearity under some mild conditions.

Since solving the dual geometric program involves heavy computation, to improve computational efficiency, we further develop three methods for generating bivariate cubic L_1 splines: a tensor-product approach that generates a good approximation for large scale bivariate cubic L_1 splines; a primal-dual interior point method that obtains discretized bivariate cubic L_1 splines robustly for small and medium size problems; and a compressed primal-dual method that efficiently and robustly generates discretized bivariate cubic L_1 splines of large size.

THEORY AND ALGORITHMS FOR SHAPE-PRESERVING BIVARIATE CUBIC L_1 SPLINES

by

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Dedicated to those I love, paticularly,

my parents Baozu Wang, Xiuhua Zhu

my brother Hao Wang

and

my wife Chuanhua Xing

Biography

WANG, YONG was born in Nanjing, China on June 5, 1976. He entered the Department of Applied Mathematics at Southeast University in 1994 and obtained his B.S. and M.S. in Applied Mathematics and Operations Research in 1998 and 2001, respectively. In August 2001, he started his Ph.D. study in the graduate program of Operations Research at North Carolina State University.

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Chapter 1

Introduction

The theory of spline functions with applications is a relatively recent development. As late as 1960, there were no more than a handful of papers explicitly mentioning spline functions by name. Today, there are well over 1000 research articles on the subject, and it remains as an active research area [3].

The rapid development of spline functions is due primarily to their great usefulness in applications. Classes of spline functions possess many nice structural properties as well as excellent approximation powers. Since they are easy to store, evaluate, and manipulate on a digital computer, a myriad of applications in the numerical solution of a variety of problems in applied mathematics and engineering have been found. These include computer aided geometric design (CAGD) [17], data fitting [7], function approximation [25], numerical quadrature, numerical solution of operator equations [9], integral equations, optimal control problems [32], etc.

The term of spline comes from drafting where splines were flexible strips guided by points on a paper, used to draw curves. Splines are a convenient method for drawing curves in two or three dimensional spaces.

According to the type of functions used, conventional splines ([1, 7, 23, 25, 32]) can be roughly classified as

1. Polynomial-based splines. There are a variety of polynomial-based splines, which are widely used in real life. Several classic polynomial-based splines are listed below:

   (a) Piecewise-linear splines. Such a spline is a piecewise-linear function passing through every given point. The coefficients of piecewise-linear splines can be easily calculated. No extraneous oscillations occur for this kind of splines. But piece-wise linear function is not differentiable in general [32].

   (b) Lagrangian splines. Lagrangian spline is a Lagrangian interpolating polynomial defined over a given set of data points [9]. The coefficients of Lagrangian interpolating polynomial can be easily determined. It has the continuous derivatives of every order. However, the degree of the polynomial depends on the number of knots. The change of one point in the give data causes the recalculation of all the coefficients of the Lagrangian interpolating polynomial.

   (c) Bézier curves. For a given array of vertices $\mathbf{P} = \{\mathbf{P}_i(x_i, y_i, z_i),\ i = 0, 1, \ldots, m\}$, a Bézier curve of degree m is defined by the vector equation $\mathbf{R}(t) = \sum_{i=0}^{m} B_i^m(t)\mathbf{P}_i$, where $B_i^m(t) = \frac{m!}{i!(m-i)!}t^i(1-t)^{m-i}$ [7]. Bézier curves are very popular approximating curves, since they can be parameterized and drawn. Bézier curve allows any number of control points. The shape of the spline can be easily controlled and adjusted. The change of a single point does not require total recomputation of the curve. The curve is constrained to be within a convex hull. But the degree of the equation that computes the Bézier curve depends on the number of vertices in **P**.

(d) B-splines. The B-spline $N_{i,k+1}$ of degree of k with knots $x_i, \ldots, x_{i+k+1}$ is defined as $N_{i,k+1}(t) = (x_{i+k+1} - x_i) \sum_{j=0}^{k+1} \frac{(x_{i+j}-t)_+^k}{\prod_{l=0,l\neq j}^{k+1}(x_{i+j}-x_{i+l})}$, where $(x_{i+j}-t)_+^k = (x_{i+j}-t)^k$, if $x_{i+j} \geq t$, $(x_{i+j}-t)_+^k = 0$, otherwise [7]. B-spline is another popular approximating spline curve. B-spline lies in the convex hull generated by data points. It is easy to control the shape of the spline. Adjusting a single point does not require total recomputation of the curve. However the computation complexity of B-spline is high.

2 Nonpolynomial-based splines. There are many kinds of nonpolynomial-based splines, such as trigonometric splines, complex splines, etc [32].

Based on the number of variables in a spline, people also classify splines as univariate splines, bivariate splines and higher dimensional splines.

Bivariate splines are potentially useful in many real life applications, for example,

- Fast recognition of faces and objects. A lot of security applications require that faces or other complex objects be recognized quickly. In these areas, bivariate splines may provide an efficient way to realize it.

- Virtual space simulation. Dangerous environments or vulnerable objects need to be virtually simulated so that people can be safely trained with low cost. Bivariate splines are powerful tools in these applications.

- Terrain description. Many military applications require that various terrains be quickly outlined. Due to the excellent capability of describing shapes, bivariate splines are widely used in such applications.

- Fast 3-D zooming. Using bivariate splines representing data instead of storing all of them can save computer memory and provide another efficient way to realize fast zoom-in and zoom-out in 3-D space.

- 3-D haptic devices design. In order to control machines or robots remotely and easily, many 3-D haptic devices which can simulate human senses are designed. Bivariate splines may be used in these applications.

In these applications, one important requirement for splines is that they should be "shape preserving", i.e., no "nonphysical" or "extraneous" oscillations are involved. From the geometric point of view, "shape-preserving" means the resulting curves retain geometric properties of the initial data, such as positivity, monotonicity, convexity, linear and planar sections [26]. From the computational perspective, conventional splines such as B-splines are ideal, since their coefficients can be calculated by using efficient, banded-matrix-based algorithms and their locally polynomial nature ensures efficient evaluation [33, 34]. However, experience has shown that conventional smooth splines often do not preserve shape well. In particular, none of them preserves shape well for arbitrary data with arbitrary changes in magnitude and in node spacing due to extensive extraneous oscillations.

Recently, Lavery [22] proposed a new kind of splines called bivariate cubic L_1 splines. Experiments have shown that bivariate cubic L_1 splines preserve shape well even for irregular data [21, 22]. Bivariate cubic L_1 splines are calculated by minimizing the L_1 norm of the second partial derivatives of a piecewise bivariate cubic polynomial under the C^1 smooth constraints i.e.

$$\arg\min_{z(x,y)} \left\{ \iint_{(x,y)\in D} \left[\left| \frac{\partial^2 z(x,y)}{\partial x^2} \right| + 2 \left| \frac{\partial^2 z(x,y)}{\partial x \partial y} \right| + \left| \frac{\partial^2 z(x,y)}{\partial y^2} \right| \right] dxdy \;\middle|\; z(x,y) \in C^1 \right\} \quad (1.1)$$

where $z(x, y)$ is piecewise bivariate cubic polynomial defined over a tensor-product grid with triangulations (see Definition 2.4.2) and D is the domain for $z(x, y)$.

Solving a bivariate cubic L_1 spline problem turns out to be equivalent to solving a nonsmooth convex optimization problem [19, 12, 16]. For a nonsmooth optimization problem, we can either (i) use smoothing techniques to make the objective function differentiable, (ii) apply subgradient based algorithms, or (iii) use derivative-free global optimization techniques, such as the Genetic Algorithms or simplex based direct search method, to tackle the problem . However, unless the special structure of the bivariate cubic L_1 splines is fully exploited, these algorithms will not be efficient enough for real applications.

Lavery proposed a primal affine scaling method to solve the bivariate cubic L_1 problem [22]. In his approach, the explicit expression of the integral in the objective function (1.1) is not computed. Instead, this integral is approximated by numerical integration.

In order to develop an algorithm for finding exact solutions and analyze the properties of bivariate cubic L_1 splines, it is necessary to establish a tractable analytical framework. In this dissertation, we transfer the nondifferentiable optimization problem to a convex programming problem by using the generalized geometric programming theory.

Geometric programming was originally developed for solving optimization problems in posynomial form [8]. E. L. Peterson [27, 28, 29] further generalized this approach for convex analysis. Generalized geometric programming theory provides strong existence, uniqueness, and characterization theorems, which are useful for parametric analysis and algorithm design.

We are able to formulate the bivariate cubic L_1 splines problem as a generalized geometric program. This framework includes a dual convex programming problem with a linear objective function and cubic constraints, plus a linear program for dual-to-primal transformation. It provides a platform for theoretical treatment of the bivariate cubic L_1 splines.

According to the numerical experiments, the bivariate cubic L_1 splines preserve linearity well. This means if the four corner points of a rectangle are on a common plane, then the corresponding bivariate cubic L_1 spline over this rectangular area must be linear, i.e., a plane geometrically. Based on the generalized geometric programming framework, we can show this shape-preserving property under some mild condition.

Since no efficient algorithm for the generalized geometric programming framework has been specifically designed, we have to solve it by using a general purpose nonlinear programming solver. As expected, current commercial solver cannot solve large size bivariate cubic L_1 splines problems. Therefore, a tensor-product approximation approach using the efficient active set algorithm for univariate cubic L_1 splines is proposed. This tensor-product approach can generate approximate bivariate cubic L_1 splines very efficiently for large scale problems. But in some cases, the difference between a tensor-product spline and the true bivariate cubic L_1 spline could be large.

Minimizing the L_1 spline functional is a non-smooth nonlinear program. Designing robust and efficient methods for this nonlinear program is not easy at all. Therefore, we restrict our attentions to the task of minimizing a discretized L_1 spline functional, which is a linear program. In this dissertation, two interior-point methods, the primal affine method and the primal-dual methods, are studied to solve this linear program to generate discretized bivariate cubic L_1 splines. Both methods can work well, but the primal-dual method also demonstrates its fast convergence and robustness for small and medium size problems. In order to handle large size problem, a "compressed" primal-dual method is also developed in this dissertation. It dramatically reduces the computational time as well as the storage.

This dissertation is organized as follows. In Chapter 2, we review the definitions of conventional bivariate splines. After that, the shape-preserving bivariate cubic L_1 splines are introduced.

In Chapter 3, we provide a generalized geometric programming framework for the bivariate cubic L_1 splines. The framework includes three parts, the primal problem, dual problem and a primal to dual transformation. At the end of this chapter, we report some computational results obtained by using a general purpose solver for true bivariate cubic L_1 splines.

In Chapter 4, we show that bivariate cubic L_1 splines preserve linearity under some mild conditions.

In Chapter 5, we design a tensor-product approximation approach for the bivariate cubic L_1 splines. Computational results are also reported.

In Chapter 6, we develop two interior-point methods, the primal affine scaling method and the primal-dual method, for generating discretized bivariate cubic L_1 splines.

In Chapter 7, a compressed primal-dual method is developed to efficiently generate large scale discretized bivariate cubic L_1 splines.

Finally, we conclude this dissertation with discussions and point out some directions for future research in Chapter 8.

Chapter 2

Bivariate Splines

This chapter provides introductory information of bivariate splines. After the presentation of conventional bivariate splines, we introduce a new shape-preserving bivariate cubic L_1 spline.

2.1 Polynomials

Polynomials have played a central role in approximation theory and numerical analysis for many years. Using the notation of Schumaker [32], we introduce the following definition of polynomial space.

Definition 2.1.1 (Polynomial Space) *[32] Given positive integers m and d, we define*

$$\mathbb{P}_m^{(d)} = \text{span}\left\{ \prod_{i=1}^{d} x_i^{\alpha_i} \,\middle|\, (x_1, \ldots, x_d)^T \in R^d,\ \sum_{i=1}^{d} \alpha_i < m, \quad \alpha_i \in Z_+,\ i = 1, \ldots, d \right\} \tag{2.1}$$

to be the d variate polynomial space of total order m, where Z_+ is the set of all the nonnegative integers and $\text{span}\left\{\prod_{i=1}^{d} x_i^{\alpha_i}\right\}$ *means the set of all linear combinations of* $\left\{\prod_{i=1}^{d} x_i^{\alpha_i}\right\}$.

Notice the following attractive features:

1. $\mathbb{P}_m^{(d)}$ is a finite dimensional linear space with a convenient basis;

2. Polynomials are smooth functions;

3. Polynomials are easy to store, manipulate, and evaluate on a digital computer;

4. The derivative and antiderivative of a polynomial are again polynomial whose coefficients can be found algebraically;

5. Given any continuous function on an area $[a, b] \times [\tilde{a}, \tilde{b}]$, there exists a bivariate polynomial that is uniformly close to it.

From this list, we can see polynomials possess many properties that are ideal for approximation. However, in reality, the curves generated by polynomial approximation may oscillate wildly in general.

2.2 Piecewise polynomials

As mentioned in the previous section, the main drawback of the space $\mathbb{P}_m^{(2)}$ of polynomials for approximation is that severe oscillations often appear, particularly when the length of interval $[a, b]$ or $[\tilde{a}, \tilde{b}]$ is large and m is bigger than 3 or 4 [32]. This observation suggests that in order to achieve flexibility, we should work with polynomials of relatively low degree, and should divide the rectangular area into smaller rectangles.

Given a *tensor-product grid* $\Delta = \{x_i, y_j\}$, for $i = 0, \cdots, I$ and $j = 0, \cdots, J$, it forms a strictly monotonic partition of the finite rectangle area $D \triangleq [a, b] \times [\tilde{a}, \tilde{b}]$, such that

$$\begin{aligned} a = x_0 < x_1 < \cdots < x_{I-1} < x_I = b \\ \tilde{a} = y_0 < y_1 < \cdots < y_{J-1} < y_J = \tilde{b}. \end{aligned} \tag{2.2}$$

The nodes, $x_i,\ i=0,\cdots,I$ and $y_j,\ j=0,\cdots,J$, need not to be uniformly spaced. The set Δ partitions the rectangle D into $I \times J$ subrectangles, $T_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, $i=0,\cdots,I-1,\ j=0,\cdots,J-1$.

Definition 2.2.1 (Piecewise Bivariate Polynomials Over Tensor-Product Grid) *Let Δ be a tensor-product grid described in (2.2) such that the set Δ partitions the rectangle $[a,b]\times[\tilde{a},\tilde{b}]$ into $I \times J$ subrectangles, $[x_i,x_{i+1}]\times[y_i,y_{j+1}]$, for $i=0,\dots,I-1$ and $j=0,\dots,J-1$. Given a positive integer m, let*

$$\mathcal{PP}_m^{(2)}(\Delta) = \Big\{ f(\mathbf{x}) | f(\mathbf{x}) = p_{ij}(\mathbf{x}) \text{ for } \mathbf{x} = (x,y)^T \in T_{ij} = [x_i, x_{i+1}] \times [y_i, y_{j+1}], \\ \text{where } p_{ij}(\mathbf{x}) \in \mathbb{P}_m^{(2)},\ i=0,\cdots,I-1; j=0,\cdots,J-1 \Big\} \tag{2.3}$$

be the space of piecewise bivariate polynomials of order m with knots $\{x_i, y_j\}$, $i=0,\dots,I$, $j=0,\dots,J$, over tensor-product grid.

While we have gained much flexibility, piecewise polynomial functions are not necessarily smooth.

2.3 Conventional bivariate cubic L_2 splines

In order to maintain the flexibility of piecewise polynomials and to achieve some degree of global smoothness, we define the following class of functions:

Definition 2.3.1 (Piecewise Bivariate Smooth Polynomial) *[32] Let Δ be a tensor-product grid over the rectangle $[a,b]\times[\tilde{a},\tilde{b}]$ as in Definition 2.2.1, and m be a positive integer. Define*

$$\mathcal{S}_m^{(2)}(\Delta) = \mathcal{PP}_m^{(2)}(\Delta) \cap C^{m-2}([a,b]\times[\tilde{a},\tilde{b}]), \tag{2.4}$$

where $C^r([a,b]\times[\tilde{a},\tilde{b}])$ *is the space of functions whose first* r^{th} *derivatives are continuous on* $[a,b]\times[\tilde{a},\tilde{b}]$*, for a given positive integer* r*. We call* $\mathcal{S}_m^{(2)}(\Delta)$ *the space of piecewise bivariate smooth polynomial of order* m *with knots* $\{x_i, y_j\}$*,* $i = 0,\ldots,I$*,* $j = 0,\ldots,J$*.*

The conventional bivariate cubic L_2 spline is introduced as follows.

Definition 2.3.2 (Conventional Bivariate Cubic L_2 Spline) *Let* $\Delta = \{(x_i, y_j)\}$*,* $i = 0,\cdots,I, j = 0,\cdots,J$*, be a partition of the rectangle* $D \triangleq [a,b]\times[\tilde{a},\tilde{b}]$*,* $T_{ij} = [x_i, x_{i+1}]\times[y_j, y_{j+1}]$ *be the rectangle partitioned by* Δ *over* D*, and* $\{(x_i, y_j, z_{ij})\}$*,* $i = 0,\cdots,I, j = 0,\cdots,J$*, be the given data set. A piecewise cubic polynomial* $\mathtt{z}(x,y)$ *is called a bivariate cubic* L_2 *spline, if*

$$\mathtt{z}(x,y) = \underset{z(x,y)}{\operatorname{argmin}}\left\{\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\iint_{(x,y)\in T_{ij}}\left[\left|\frac{\partial^2 z(x,y)}{\partial x^2}\right|^2 + 2\left|\frac{\partial^2 z(x,y)}{\partial x\partial y}\right|^2 + \left|\frac{\partial^2 z(x,y)}{\partial y^2}\right|^2\right]dxdy\right.$$

$$\left.\Bigg|\, z(x,y)\in\mathcal{S}_4^{(2)}(\Delta) \text{ and } z(x_i,y_j) = z_{ij},\ i = 0,\cdots,I, j = 0,\cdots,J\right\},$$

Although this kind of splines may reduce some degree of nonphysical oscillation, their performance in practice is not all satisfactory from the shape-preserving point of view.

2.4 Bivariate cubic L_1 splines

Conventional bivariate cubic L_2 splines are calculated by minimizing the L_2 norm of the second partial derivatives of the piecewise bivariate smooth polynomial. Computational experience has shown that they often exhibit excessive "nonphysical" oscillation and therefore do not preserve shape well. For this reason, variants of bivariate cubic splines that may preserve shape well are sought.

In order to generate shape preserving splines over an arbitrarily spacing grid, Han and Schumaker [16] interpolate data on rectangular grids by C^1 splines defined on a triangulation of each subrectangle T_{ij}, by dividing it into four subtriangles formed by drawing the two diagonals. The subtriangles are labelled 1,2,3,4, respectively, as shown in Figure 2.1 and denoted as $T_{ij}^k,\ k = 1, \cdots, 4$. This subdivision is called the *Sibson split*.

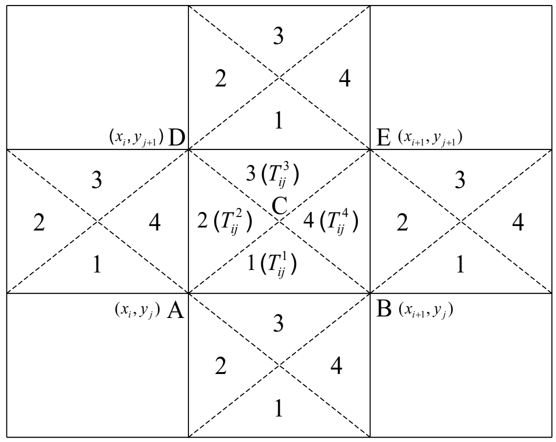


Figure 2.1: The Sibson split of rectangles.

Define the space of piecewise bivariate polynomials of total order m over Sibson split with knots $(x_i, y_j),\ i = 0, \cdots, I, j = 0, \cdots, J$, as follows,

$$\mathbb{PP}_m^{(2)}(\Delta) = \left\{ f(\mathbf{x}) = p_{ij}^k(\mathbf{x}) \,\middle|\, \mathbf{x} \in T_{ij}^k,\ p_{ij}^k(\mathbf{x}) \in \mathbb{P}_m^{(2)}, \right.$$
$$\left. i = 0, \cdots, I-1, j = 0, \cdots, J-1, k = 1, 2, 3, 4 \right\}.$$

By using piecewise bivariate cubic polynomials, Han and Schumaker [16] introduced a C^1 smooth spline function over the Sibson split, called Sibson element. This forms the base of the bivariate cubic L_p splines.

Definition 2.4.1 (Sibson element) *Given a rectangle* $T_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, *divide it into four triangles by drawing the two diagonals. The Sibson element,* $\mathsf{z}(x, y)$, *is a piecewise bivariate cubic polynomial such that it is* (i) C^1 *on the lines separating the four* *triangles,* (ii) C^1 *on the boundary of the rectangle that match with adjacent piecewise bivariate cubic polynomials,* (iii) *has derivative* $\frac{\partial z(x,y)}{\partial x}$ *being linear along the edges* $x = x_i$ *and* $x = x_{i+1}$ *of the rectangle, and* (iv) *has derivative* $\frac{\partial z(x,y)}{\partial y}$ *being linear along the edges* $y = y_i$ *and* $y = y_{i+1}$ *of the rectangle.*

Lavery proposed the so-called bivariate cubic L_p splines in [22] as follows:

Definition 2.4.2 (Bivariate cubic L_p splines) *Let* $\Delta = \{(x_i, y_j)\}$, $i = 0, \cdots, I$ *and* $j = 0, \cdots, J$, *be a partition of the rectangle* $D \triangleq [a,b] \times [\tilde{a}, \tilde{b}]$, $T_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ *be the triangle partitioned by* Δ *over* D, *and* $\{(x_i, y_j, z_{ij})\}$, $i = 0, \cdots, I, j = 0, \cdots, J$, *be the given data set. A piecewise cubic polynomial* $\mathcal{Z}(x,y)$ *is called a bivariate cubic* L_p *spline, if, for* $1 \leq p < \infty$,

$$\mathcal{Z}(x,y) = \underset{z(x,y)}{\operatorname{argmin}} \left\{ \sum_{i=0}^{I-1} \sum_{j=0}^{J-1} \iint_{(x,y)\in T_{ij}} \left[\left| \frac{\partial^2 z(x,y)}{\partial x^2} \right|^p + 2 \left| \frac{\partial^2 z(x,y)}{\partial x \partial y} \right|^p + \left| \frac{\partial^2 z(x,y)}{\partial y^2} \right|^p \right] dxdy \right.$$

$$\left| \; z(x,y) \in \mathbb{PP}_4^{(2)}(\Delta) \cap C^1 [a,b] \times [\tilde{a}, \tilde{b}] \text{ is a Sibson element on each } T_{ij}, \right.$$

$$\left. \text{and } z(x_i, y_j) = z_{ij}, \; i = 0, \cdots, I, j = 0, \cdots, J \right\},$$

and for $p = \infty$,

$$\mathcal{Z}(x,y) = \underset{z(x,y)}{\operatorname{argmin}} \left\{ \sum_{i=0}^{I-1} \sum_{j=0}^{J-1} \left[\max_{(x,y)\in T_{ij}} \left| \frac{\partial^2 z(x,y)}{\partial x^2} \right| + 2 \max_{(x,y)\in T_{ij}} \left| \frac{\partial^2 z(x,y)}{\partial x \partial y} \right| \right. \right.$$

$$\left. + \max_{(x,y)\in T_{ij}} \left| \frac{\partial^2 z(x,y)}{\partial y^2} \right| \right] dxdy \; \Big| \; z(x,y) \in \mathbb{PP}_4^{(2)}(\Delta) \cap C^1 [a,b] \times [\tilde{a}, \tilde{b}]$$

$$\text{is a Sibson element on each } T_{ij} \text{ and } z(x_i, y_j) = z_{ij},$$

$$\left. i = 0, \cdots, I, j = 0, \cdots, J \right\}.$$

Note that when $p = 1$, we have the *bivariate cubic* L_1 *splines.*

Lavery proved the existence of the bivariate cubic L_p splines for $1 \leq p \leq \infty$. The computational results in [21, 22] show that the properties of L_p splines depend strongly on p. The smaller the p is, the better the bivariate cubic L_p splines preserve the shape. Moreover, experiments show that bivariate cubic L_1 splines preserve shape better than conventional bivariate splines. We focus on bivariate cubic L_1 splines in this dissertation.

If $\mathsf{z}(x,y)$ is a Sibson element defined on $T_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ in Figure 2.1, then on each triangle T_{ij}^k of the rectangle T_{ij} , there exists a bivariate cubic polynomial $z^{ijk}(x,y)$ such that $\mathsf{z}(x,y) = z^{ijk}(x,y)$, $\forall x \in T_{ij}^k$. For $i = 0, \cdots, I-1, j = 0, \cdots, J-1$, $z^{ijk}(x,y)$ can be expressed as

$$
\begin{aligned}
z^{ij1}(x,y) = {} & c_{30}^{ij1}(x-x_i)^3 + c_{21}^{ij1}(x-x_i)^2(y-y_j) \\
& + c_{12}^{ij1}(x-x_i)(y-y_j)^2 + c_{03}^{ij1}(y-y_j)^3 \\
& + c_{20}^{ij1}(x-x_i)^2 + c_{11}^{ij1}(x-x_i)(y-y_j) + c_{02}^{ij1}(y-y_j)^2 \\
& + c_{10}^{ij1}(x-x_i) + c_{01}^{ij1}(y-y_j) + c_{00}^{ij1}, \\
z^{ij2}(x,y) = {} & c_{30}^{ij2}(x-x_i)^3 + c_{21}^{ij2}(x-x_i)^2(y-y_{j+1}) \\
& + c_{12}^{ij2}(x-x_i)(y-y_{j+1})^2 + c_{03}^{ij2}(y-y_{j+1})^3 \\
& + c_{20}^{ij2}(x-x_i)^2 + c_{11}^{ij2}(x-x_i)(y-y_{j+1}) + c_{02}^{ij2}(y-y_{j+1})^2 \\
& + c_{10}^{ij2}(x-x_i) + c_{01}^{ij2}(y-y_{j+1}) + c_{00}^{ij2}, \\
z^{ij3}(x,y) = {} & c_{30}^{ij3}(x-x_{i+1})^3 + c_{21}^{ij3}(x-x_{i+1})^2(y-y_{j+1}) \\
& + c_{12}^{ij3}(x-x_{i+1})(y-y_{j+1})^2 + c_{03}^{ij3}(y-y_{j+1})^3 \\
& + c_{20}^{ij3}(x-x_{i+1})^2 + c_{11}^{ij3}(x-x_{i+1})(y-y_{j+1}) + c_{02}^{ij3}(y-y_{j+1})^2 \\
& + c_{10}^{ij3}(x-x_{i+1}) + c_{01}^{ij3}(y-y_{j+1}) + c_{00}^{ij3},
\end{aligned}
$$

$$
\begin{aligned}
z^{ij4}(x,y) = {}& c_{30}^{ij4}(x-x_{i+1})^3 + c_{21}^{ij4}(x-x_{i+1})^2(y-y_j) \\
&+ c_{12}^{ij4}(x-x_{i+1})(y-y_j)^2 + c_{03}^{ij4}(y-y_j)^3 \\
&+ c_{20}^{ij4}(x-x_{i+1})^2 + c_{11}^{ij4}(x-x_{i+1})(y-y_j) + c_{02}^{ij4}(y-y_j)^2 \\
&+ c_{10}^{ij4}(x-x_{i+1}) + c_{01}^{ij4}(y-y_j) + c_{00}^{ij4},
\end{aligned}
$$

where the ten coefficients of each bivariate cubic interpolation function $z^{ijk}(x,y)$ are unknown variables.

Let us consider $z^{ij1}(x,y)$ first. The first order partial derivatives of $z^{ij1}(x,y)$ are

$$
\begin{aligned}
\frac{\partial z^{ij1}(x,y)}{\partial x} = {}& 3c_{30}^{ij1}(x-x_i)^2 + 2c_{21}^{ij1}(x-x_i)(y-y_j) + c_{12}^{ij1}(y-y_j)^2 \\
&+ 2c_{20}^{ij1}(x-x_i) + c_{11}^{ij1}(y-y_j) + c_{10}^{ij1},
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial z^{ij1}(x,y)}{\partial y} = {}& c_{21}^{ij1}(x-x_i)^2 + 2c_{12}^{ij1}(x-x_i)(y-y_j) + 3c_{03}^{ij1}(y-y_j)^2 \\
&+ c_{11}^{ij1}(x-x_i) + 2c_{02}^{ij1}(y-y_j) + c_{01}^{ij1}.
\end{aligned}
$$

By the construction of Sibson element, $\frac{\partial z^{ij1}(x,y)}{\partial y}$ is linear along the edge $y = y_j$, i.e.,

$$
\left.\frac{\partial z^{ij1}(x,y)}{\partial y}\right|_{y=y_j} = c_{21}^{ij1}(x-x_i)^2 + c_{11}^{ij1}(x-x_i) + c_{01}^{ij1}
$$

is a linear function. Hence,

$$
c_{21}^{ij1} = 0. \tag{2.5}
$$

It is obvious that

$$
c_{00}^{ij1} = z_{ij}. \tag{2.6}
$$

The second order partial derivative of $z^{ij1}(x,y)$ can be written as

$$\begin{aligned}
\frac{\partial^2 z^{ij1}(x,y)}{\partial x^2} &= 6c_{30}^{ij1}(x-x_i)+2c_{20}^{ij1}, \\
\frac{\partial^2 z^{ij1}(x,y)}{\partial y^2} &= 2c_{12}^{ij1}(x-x_i)+6c_{03}^{ij1}(y-y_j)+2c_{02}^{ij1}, \\
\frac{\partial^2 z^{ij1}(x,y)}{\partial x \partial y} &= 2c_{12}^{ij1}(y-y_j)+c_{11}^{ij1}.
\end{aligned}$$

Focus on $z^{ij2}(x,y)$. The first order partial derivatives of $z^{ij2}(x,y)$ are

$$\begin{aligned}
\frac{\partial z^{ij2}(x,y)}{\partial x} = {} & 3c_{30}^{ij2}(x-x_i)^2+2c_{21}^{ij2}(x-x_i)(y-y_{j+1})+c_{12}^{ij2}(y-y_{j+1})^2 \\
& +2c_{20}^{ij2}(x-x_i)+c_{11}^{ij2}(y-y_{j+1})+c_{10}^{ij2},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial z^{ij2}(x,y)}{\partial y} = {} & c_{21}^{ij2}(x-x_i)^2+2c_{12}^{ij2}(x-x_i)(y-y_{j+1})+3c_{03}^{ij2}(y-y_{j+1})^2 \\
& +c_{11}^{ij2}(x-x_i)+2c_{02}^{ij2}(y-y_{j+1})+c_{01}^{ij2}.
\end{aligned}$$

By the construction of Sibson element, $\frac{\partial z^{ij2}(x,y)}{\partial x}$ is linear along the edge $x = x_i$, i.e.,

$$\left.\frac{\partial z^{ij2}(x,y)}{\partial x}\right|_{x=x_i} = c_{12}^{ij2}(y-y_{j+1})^2+c_{11}^{ij2}(y-y_{j+1})+c_{10}^{ij2}$$

is a linear function. Hence,

$$c_{12}^{ij2} = 0. \tag{2.7}$$

And it is easy to see

$$c_{00}^{ij2} = z_{i,j+1}. \tag{2.8}$$

Therefore, the second partial derivative of $z^{ij2}(x,y)$ are

$$\begin{aligned}
\frac{\partial^2 z^{ij2}(x,y)}{\partial x^2} &= 6c_{30}^{ij2}(x-x_i)+2c_{21}^{ij2}(y-y_{j+1})+2c_{20}^{ij2}, \\
\frac{\partial^2 z^{ij2}(x,y)}{\partial y^2} &= 6c_{03}^{ij2}(y-y_{j+1})+2c_{02}^{ij2}, \\
\frac{\partial^2 z^{ij2}(x,y)}{\partial x \partial y} &= 2c_{21}^{ij2}(x-x_i)+c_{11}^{ij2}.
\end{aligned}$$

Similarly, the corresponding second order derivatives of $z^{ij3}(x,y)$ and $z^{ij4}(x,y)$ can be obtained as follows.

For $z^{ij3}(x,y)$:

$$\begin{aligned}
\frac{\partial^2 z^{ij3}(x,y)}{\partial x^2} &= 6c_{30}^{ij3}\,(x-x_{i+1})+2c_{20}^{ij3},\\
\frac{\partial^2 z^{ij3}(x,y)}{\partial y^2} &= 2c_{12}^{ij3}\,(x-x_{i+1})+6c_{03}^{ij3}\,(y-y_{j+1})+2c_{02}^{ij3},\\
\frac{\partial^2 z^{ij3}(x,y)}{\partial x\partial y} &= 2c_{12}^{ij3}\,(y-y_{j+1})+c_{11}^{ij3}.\\
c_{21}^{ij3} &= 0. \qquad (2.9)\\
c_{00}^{ij3} &= z_{i+1,j+1}. \qquad (2.10)
\end{aligned}$$

For $z^{ij4}(x,y)$:

$$\begin{aligned}
\frac{\partial^2 z^{ij4}(x,y)}{\partial x^2} &= 6c_{30}^{ij4}\,(x-x_{i+1})+2c_{21}^{ij4}(y-y_j)+2c_{20}^{ij4},\\
\frac{\partial^2 z^{ij4}(x,y)}{\partial y^2} &= 6c_{03}^{ij4}\,(y-y_j)+2c_{02}^{ij4},\\
\frac{\partial^2 z^{ij4}(x,y)}{\partial x\partial y} &= 2c_{21}^{ij4}\,(x-x_{i+1})+c_{11}^{ij4}.\\
c_{12}^{ij4} &= 0. \qquad (2.11)\\
c_{00}^{ij4} &= z_{i+1,j}. \qquad (2.12)
\end{aligned}$$

Denote the coefficients of these bivariate cubic polynomials (2.4) as

$$\mathbf{c}^{ijk} = \{c_{30}^{ijk}, c_{21}^{ijk}, c_{12}^{ijk}, c_{03}^{ijk}, c_{20}^{ijk}, c_{11}^{ijk}, c_{02}^{ijk}, c_{10}^{ijk}, c_{01}^{ijk}, c_{00}^{ijk}\}$$

with $i = 0, \cdots, I-1,\ j = 0, \cdots, J-1,\ k = 1, \cdots, 4.$

Then the objective function of bivariate cubic L_1 splines can be written as

$$
\begin{aligned}
&\min_{\mathbf{c}^{ijk}} \sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\sum_{k=1}^{4}\left(\left\|\frac{\partial^2 z^{ijk}}{\partial x^2}\right\|_1 + 2\left\|\frac{\partial^2 z^{ijk}}{\partial x\partial y}\right\|_1 + \left\|\frac{\partial^2 z^{ijk}}{\partial y^2}\right\|_1\right)\\
&= \min_{\mathbf{c}^{ijk}} \sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\Bigg[\iint_{(x,y)\in T_{ij}^1}\Bigg(\left|6c_{30}^{ij1}(x-x_i)+2c_{20}^{ij1}\right| + 2\left|2c_{12}^{ij1}(y-y_j)+c_{11}^{ij1}\right|\\
&\qquad + \left|2c_{12}^{ij1}(x-x_i)+6c_{03}^{ij1}(y-y_j)+2c_{02}^{ij1}\right|\Bigg)dxdy\\
&\quad + \iint_{(x,y)\in T_{ij}^2}\Bigg(\left|6c_{30}^{ij2}(x-x_i)+2c_{21}^{ij2}(y-y_{j+1})+2c_{20}^{ij2}\right|\\
&\qquad + 2\left|2c_{21}^{ij2}(x-x_i)+c_{11}^{ij2}\right| + \left|6c_{03}^{ij2}(y-y_{j+1})+2c_{02}^{ij2}\right|\Bigg)dxdy\\
&\quad + \iint_{(x,y)\in T_{ij}^3}\Bigg(\left|6c_{30}^{ij3}(x-x_{i+1})+2c_{20}^{ij3}\right| + 2\left|2c_{12}^{ij3}(y-y_{j+1})+c_{11}^{ij3}\right|\\
&\qquad + \left|2c_{12}^{ij3}(x-x_{i+1})+6c_{03}^{ij3}(y-y_{j+1})+2c_{02}^{ij3}\right|\Bigg)dxdy\\
&\quad + \iint_{(x,y)\in T_{ij}^4}\Bigg(\left|6c_{30}^{ij4}(x-x_{i+1})+2c_{21}^{ij4}(y-y_j)+2c_{20}^{ij4}\right|\\
&\qquad + \left|6c_{03}^{ij4}(y-y_j)+2c_{02}^{ij4}\right| + 2\left|2c_{21}^{ij4}(x-x_{i+1})+c_{11}^{ij4}\right|\Bigg)dxdy\Bigg].
\end{aligned}
\tag{2.13}
$$

Note that this function is not differentiable. In the next subsection, we will derive the C^1 smooth constraints for bivariate cubic L_1 splines.

2.5 C^1 smooth constraints

According to the definition of Sibson element, bivariate cubic L_1 splines are required to satisfy the C^1 smooth conditions on the boundary of each triangle. We check the rectangle $[x_i, x_{i+1}] \times [y_i, y_{i+1}]$ in Figure 2.1. Note that (A, E) and (B, D) are the two diagonals crossing at point C. Then the C^1 smooth condition requires that

$$
\begin{aligned}
z^{ij1}(x,y)\big|_{(A,C)} &= z^{ij2}(x,y)\big|_{(A,C)} \\
\left.\frac{\partial z^{ij1}(x,y)}{\partial x}\right|_{(A,C)} &= \left.\frac{\partial z^{ij2}(x,y)}{\partial x}\right|_{(A,C)} \\
\left.\frac{\partial z^{ij1}(x,y)}{\partial y}\right|_{(A,C)} &= \left.\frac{\partial z^{ij2}(x,y)}{\partial y}\right|_{(A,C)}
\end{aligned}
$$

$$
\begin{aligned}
z^{ij1}(x,y)\big|_{(B,C)} &= z^{ij4}(x,y)\big|_{(B,C)} \\
\left.\frac{\partial z^{ij1}(x,y)}{\partial x}\right|_{(B,C)} &= \left.\frac{\partial z^{ij4}(x,y)}{\partial x}\right|_{(B,C)} \\
\left.\frac{\partial z^{ij1}(x,y)}{\partial y}\right|_{(B,C)} &= \left.\frac{\partial z^{ij4}(x,y)}{\partial y}\right|_{(B,C)}
\end{aligned}
$$

$$
\begin{aligned}
z^{ij2}(x,y)\big|_{(C,D)} &= z^{ij3}(x,y)\big|_{(C,D)} \\
\left.\frac{\partial z^{ij2}(x,y)}{\partial x}\right|_{(C,D)} &= \left.\frac{\partial z^{ij3}(x,y)}{\partial x}\right|_{(C,D)} \\
\left.\frac{\partial z^{ij2}(x,y)}{\partial y}\right|_{(C,D)} &= \left.\frac{\partial z^{ij3}(x,y)}{\partial y}\right|_{(C,D)}
\end{aligned}
$$

$$
\begin{aligned}
z^{ij3}(x,y)\big|_{(C,E)} &= z^{ij4}(x,y)\big|_{(C,E)} \\
\left.\frac{\partial z^{ij3}(x,y)}{\partial x}\right|_{(C,E)} &= \left.\frac{\partial z^{ij4}(x,y)}{\partial x}\right|_{(C,E)} \\
\left.\frac{\partial z^{ij3}(x,y)}{\partial y}\right|_{(C,E)} &= \left.\frac{\partial z^{ij4}(x,y)}{\partial y}\right|_{(C,E)}
\end{aligned}
$$

$$
\begin{aligned}
z^{ij1}(x,y)\big|_{(A,B)} &= z^{i,j-1,3}(x,y)\big|_{(A,B)} \\
\left.\frac{\partial z^{ij1}(x,y)}{\partial x}\right|_{(A,B)} &= \left.\frac{\partial z^{i,j-1,3}(x,y)}{\partial x}\right|_{(A,B)} \\
\left.\frac{\partial z^{ij1}(x,y)}{\partial y}\right|_{(A,B)} &= \left.\frac{\partial z^{i,j-1,3}(x,y)}{\partial y}\right|_{(A,B)}
\end{aligned}
$$

$$
\begin{aligned}
z^{ij4}(x,y)\big|_{(B,E)} &= z^{i+1,j2}(x,y)\big|_{(B,E)} \\
\left.\frac{\partial z^{ij4}(x,y)}{\partial x}\right|_{(B,E)} &= \left.\frac{\partial z^{i+1,j2}(x,y)}{\partial x}\right|_{(B,E)} \\
\left.\frac{\partial z^{ij4}(x,y)}{\partial y}\right|_{(B,E)} &= \left.\frac{\partial z^{i+1,j2}(x,y)}{\partial y}\right|_{(B,E)}
\end{aligned}
$$

$$
\begin{aligned}
z^{ij3}(x,y)\big|_{(D,E)} &= z^{i,j+1,1}(x,y)\big|_{(D,E)} \\
\left.\frac{\partial z^{ij3}(x,y)}{\partial x}\right|_{(D,E)} &= \left.\frac{\partial z^{i,j+1,1}(x,y)}{\partial x}\right|_{(D,E)} \\
\left.\frac{\partial z^{ij3}(x,y)}{\partial y}\right|_{(D,E)} &= \left.\frac{\partial z^{i,j+1,1}(x,y)}{\partial y}\right|_{(D,E)}
\end{aligned}
$$

$$
\begin{aligned}
z^{ij2}(x,y)\big|_{(A,D)} &= z^{i-1,j4}(x,y)\big|_{(A,D)} \\
\left.\frac{\partial z^{ij2}(x,y)}{\partial x}\right|_{(A,D)} &= \left.\frac{\partial z^{i-1,j4}(x,y)}{\partial x}\right|_{(A,D)} \\
\left.\frac{\partial z^{ij2}(x,y)}{\partial y}\right|_{(A,D)} &= \left.\frac{\partial z^{i-1,j4}(x,y)}{\partial y}\right|_{(A,D)}
\end{aligned}
$$

From these C^1 smooth constraints, we get the following linear homogeneous constraints for the coefficients of the cubic L_1 splines. For $i = 0, \ldots, I-1$ and $j = 0, \ldots, J-1$,

$$
\begin{aligned}
& c_{30}^{ij1} h_i^{x3} + c_{21}^{ij1} h_i^{x2} h_j^y + c_{12}^{ij1} h_i^x h_j^{y2} + c_{03}^{ij1} h_j^{y3} \\
& \qquad - c_{30}^{ij2} h_i^{x3} - c_{21}^{ij2} h_i^{x2} h_j^y - c_{12}^{ij2} h_i^x h_j^{y2} - c_{03}^{ij2} h_j^{y3} = 0 \\
& c_{20}^{ij1} h_i^{x2} + c_{11}^{ij1} h_i^x h_j^y + c_{02}^{ij1} h_j^{y2} + c_{21}^{ij2} h_i^{x2} h_j^y + 2c_{12}^{ij2} h_i^x h_j^{y2} \\
& \qquad + 3c_{03}^{ij2} h_j^{y3} - c_{20}^{ij2} h_i^{x2} - c_{11}^{ij2} h_i^x h_j^y - c_{02}^{ij2} h_j^{y2} = 0 \\
& c_{10}^{ij1} h_i^x + c_{01}^{ij1} h_j^y - c_{12}^{ij2} h_i^x h_j^{y2} - 3c_{03}^{ij2} h_j^{y3} \\
& \qquad + c_{11}^{ij2} h_i^x h_j^y + 2c_{02}^{ij2} h_j^{y2} - c_{10}^{ij2} h_i^x - c_{01}^{ij2} h_j^y = 0 \\
& c_{00}^{ij1} + c_{03}^{ij2} h_j^{y3} - c_{02}^{ij2} h_j^{y2} + c_{01}^{ij2} h_j^y - c_{00}^{ij2} = 0 \qquad (2.14) \\
& 3c_{30}^{ij1} h_i^{x2} + 2c_{21}^{ij1} h_i^x h_j^y + c_{12}^{ij1} h_j^{y2} - 3c_{30}^{ij2} h_i^{x2} - 2c_{21}^{ij2} h_i^x h_j^y - c_{12}^{ij2} h_j^{y2} = 0 \\
& 2c_{20}^{ij1} h_i^x + c_{11}^{ij1} h_j^y + 2c_{21}^{ij2} h_i^x h_j^y + 2c_{12}^{ij2} h_j^{y2} - 2c_{20}^{ij2} h_i^x - c_{11}^{ij2} h_j^y = 0 \\
& c_{10}^{ij1} - c_{12}^{ij2} h_j^{y2} + c_{11}^{ij2} h_j^y - c_{10}^{ij2} = 0 \\
& c_{21}^{ij1} h_i^{x2} + 2c_{12}^{ij1} h_i^x h_j^y + 3c_{03}^{ij1} h_j^{y2} - c_{21}^{ij2} h_i^{x2} - 2c_{12}^{ij2} h_i^x h_j^y - 3c_{03}^{ij2} h_j^{y2} = 0 \\
& c_{11}^{ij1} h_i^x + 2c_{02}^{ij1} h_j^y + 2c_{12}^{ij2} h_i^x h_j^y + 6c_{03}^{ij2} h_j^{y2} - c_{11}^{ij2} h_i^x - 2c_{02}^{ij2} h_j^y = 0 \\
& c_{01}^{ij1} - 3c_{03}^{ij2} h_j^{y2} + 2c_{02}^{ij2} h_j^y - c_{01}^{ij2} = 0
\end{aligned}
$$

$$
\begin{aligned}
c_{30}^{ij1} h_i^{x3} - c_{21}^{ij1} h_i^{x2} h_j^{y} + c_{12}^{ij1} h_i^{x} h_j^{y2} - c_{03}^{ij1} h_j^{y3}& \\
- c_{30}^{ij4} h_i^{x3} + c_{21}^{ij4} h_i^{x2} h_j^{y} - c_{12}^{ij4} h_i^{x} h_j^{y2} + c_{03}^{ij4} h_j^{y3} &= 0 \\
3c_{30}^{ij1} h_i^{x3} - 2c_{21}^{ij1} h_i^{x2} h_j^{y} + c_{12}^{ij1} h_i^{x} h_j^{y2} + c_{20}^{ij1} h_i^{x2} - c_{11}^{ij1} h_i^{x} h_j^{y} + c_{02}^{ij1} h_j^{y2}& \\
- c_{20}^{ij4} h_i^{x2} + c_{11}^{ij4} h_i^{x} h_j^{y} - c_{02}^{ij4} h_j^{y2} &= 0 \\
3c_{30}^{ij1} h_i^{x3} - c_{21}^{ij1} h_i^{x2} h_j^{y} + 2c_{20}^{ij1} h_i^{x2} - c_{11}^{ij1} h_i^{x} h_j^{y}& \\
+ c_{10}^{ij1} h_i^{x} - c_{01}^{ij1} h_j^{y} - c_{10}^{ij4} h_i^{x} + c_{01}^{ij4} h_j^{y} &= 0 \\
c_{30}^{ij1} h_i^{x3} + c_{20}^{ij1} h_i^{x2} + c_{10}^{ij1} h_i^{x} + c_{00}^{ij1} - c_{00}^{ij4} &= 0 \qquad (2.15) \\
3c_{30}^{ij1} h_i^{x2} - 2c_{21}^{ij1} h_i^{x} h_j^{y} + c_{12}^{ij1} h_j^{y2} - 3c_{30}^{ij4} h_i^{x2} + 2c_{21}^{ij4} h_i^{x} h_j^{y} - c_{12}^{ij4} h_j^{y2} &= 0 \\
6c_{30}^{ij1} h_i^{x2} - 2c_{21}^{ij1} h_i^{x} h_j^{y} + 2c_{20}^{ij1} h_i^{x} - c_{11}^{ij1} h_j^{y} - 2c_{20}^{ij4} h_i^{x} + c_{11}^{ij4} h_j^{y} &= 0 \\
3c_{30}^{ij1} h_i^{x2} + 2c_{20}^{ij1} h_i^{x} + c_{10}^{ij1} - c_{10}^{ij4} &= 0 \\
c_{21}^{ij1} h_i^{x2} - 2c_{12}^{ij1} h_i^{x} h_j^{y} + 3c_{03}^{ij1} h_j^{y2} - c_{21}^{ij4} h_i^{x2} + 2c_{12}^{ij4} h_i^{x} h_j^{y} - 3c_{03}^{ij4} h_j^{y2} &= 0 \\
2c_{21}^{ij1} h_i^{x2} - 2c_{12}^{ij1} h_i^{x} h_j^{y} + c_{11}^{ij1} h_i^{x} - 2c_{02}^{ij1} h_j^{y} - c_{11}^{ij4} h_i^{x} + 2c_{02}^{ij4} h_j^{y} &= 0 \\
c_{21}^{ij1} h_i^{x2} + c_{11}^{ij1} h_i^{x} + c_{01}^{ij1} - c_{01}^{ij4} &= 0
\end{aligned}
$$

$$
\begin{aligned}
&c_{30}^{ij3} h_i^{x3} - c_{21}^{ij3} h_i^{x2} h_j^{y} + c_{12}^{ij3} h_i^{x} h_j^{y2} - c_{03}^{ij3} h_j^{y3} \\
&\qquad - c_{30}^{ij2} h_i^{x3} + c_{21}^{ij2} h_i^{x2} h_j^{y} - c_{12}^{ij2} h_i^{x} h_j^{y2} + c_{03}^{ij2} h_j^{y3} = 0 \\
&-3c_{30}^{ij3} h_i^{x3} + 2c_{21}^{ij3} h_i^{x2} h_j^{y} - c_{12}^{ij3} h_i^{x} h_j^{y2} + c_{20}^{ij3} h_i^{x2} - c_{11}^{ij3} h_i^{x} h_j^{y} + c_{02}^{ij3} h_j^{y2} \\
&\qquad - c_{20}^{ij2} h_i^{x2} + c_{11}^{ij2} h_i^{x} h_j^{y} - c_{02}^{ij2} h_j^{y2} = 0 \\
&3c_{30}^{ij3} h_i^{x3} - c_{21}^{ij3} h_i^{x2} h_j^{y} - 2c_{20}^{ij3} h_i^{x2} + c_{11}^{ij3} h_i^{x} h_j^{y} + c_{10}^{ij3} h_i^{x} - c_{01}^{ij3} h_j^{y} \\
&\qquad - c_{10}^{ij2} h_i^{x} + c_{01}^{ij2} h_j^{y} = 0 \\
&-c_{30}^{ij3} h_i^{x3} + c_{20}^{ij3} h_i^{x2} - c_{10}^{ij3} h_i^{x} + c_{00}^{ij3} - c_{00}^{ij2} = 0 \\
&3c_{30}^{ij3} h_i^{x2} - 2c_{21}^{ij3} h_i^{x} h_j^{y} + c_{12}^{ij3} h_j^{y2} - 3c_{30}^{ij2} h_i^{x2} + 2c_{21}^{ij2} h_i^{x} h_j^{y} - c_{12}^{ij2} h_j^{y2} = 0 \\
&-6c_{30}^{ij3} h_i^{x2} + 2c_{21}^{ij3} h_i^{x} h_j^{y} + 2c_{20}^{ij3} h_i^{x} - c_{11}^{ij3} h_j^{y} - 2c_{20}^{ij2} h_i^{x} + c_{11}^{ij2} h_j^{y} = 0 \\
&3c_{30}^{ij3} h_i^{x2} - 2c_{20}^{ij3} h_i^{x} + c_{10}^{ij3} - c_{10}^{ij2} = 0 \\
&c_{21}^{ij3} h_i^{x2} - 2c_{12}^{ij3} h_i^{x} h_j^{y} + 3c_{03}^{ij3} h_j^{y2} - c_{21}^{ij2} h_i^{x2} + 2c_{12}^{ij2} h_i^{x} h_j^{y} - 3c_{03}^{ij2} h_j^{y2} = 0 \\
&-2c_{21}^{ij3} h_i^{x2} + 2c_{12}^{ij3} h_i^{x} h_j^{y} + c_{11}^{ij3} h_i^{x} - 2c_{02}^{ij3} h_j^{y} - c_{11}^{ij2} h_i^{x} + 2c_{02}^{ij2} h_j^{y} = 0 \\
&c_{21}^{ij3} h_i^{x2} - c_{11}^{ij3} h_i^{x} + c_{01}^{ij3} - c_{01}^{ij2} = 0
\end{aligned}
\tag{2.16}
$$

$$
\begin{aligned}
c_{30}^{ij3}h_i^{x3} + c_{21}^{ij3}h_i^{x2}h_j^{y} + c_{12}^{ij3}h_i^{x}h_j^{y2} + c_{03}^{ij3}h_j^{y3}& \\
-c_{30}^{ij4}h_i^{x3} - c_{21}^{ij4}h_i^{x2}h_j^{y} - c_{12}^{ij4}h_i^{x}h_j^{y2} - c_{03}^{ij4}h_j^{y3} &= 0 \\
c_{20}^{ij3}h_i^{x2} + c_{11}^{ij3}h_i^{x}h_j^{y} + c_{02}^{ij3}h_j^{y2}& \\
-c_{21}^{ij4}h_i^{x2}h_j^{y} - 2c_{12}^{ij4}h_i^{x}h_j^{y2} - 3c_{03}^{ij4}h_j^{y3} - c_{20}^{ij4}h_i^{x2} - c_{11}^{ij4}h_i^{x}h_j^{y} - c_{02}^{ij4}h_j^{y2} &= 0 \\
c_{10}^{ij3}h_i^{x} + c_{01}^{ij3}h_j^{y}& \\
-c_{12}^{ij4}h_i^{x}h_j^{y2} - 3c_{03}^{ij4}h_j^{y3} - c_{11}^{ij4}h_i^{x}h_j^{y} - 2c_{02}^{ij4}h_j^{y2} - c_{10}^{ij4}h_i^{x} - c_{01}^{ij4}h_j^{y} &= 0 \\
c_{00}^{ij3} - c_{03}^{ij4}h_j^{y3} - c_{02}^{ij4}h_j^{y2} - c_{01}^{ij4}h_j^{y} - c_{00}^{ij4} &= 0 \\
3c_{30}^{ij3}h_i^{x2} + 2c_{21}^{ij3}h_i^{x}h_j^{y} + c_{12}^{ij3}h_j^{y2} - 3c_{30}^{ij4}h_i^{x2} - 2c_{21}^{ij4}h_i^{x}h_j^{y} - c_{12}^{ij4}h_j^{y2} &= 0 \\
2c_{20}^{ij3}h_i^{x} + c_{11}^{ij3}h_j^{y} - 2c_{21}^{ij4}h_i^{x}h_j^{y} - 2c_{12}^{ij4}h_j^{y2} - 2c_{20}^{ij4}h_i^{x} - c_{11}^{ij4}h_j^{y} &= 0 \\
c_{10}^{ij3} - c_{12}^{ij4}h_j^{y2} - c_{11}^{ij4}h_j^{y} - c_{10}^{ij4} &= 0 \\
c_{21}^{ij3}h_i^{x2} + 2c_{12}^{ij3}h_i^{x}h_j^{y} + 3c_{03}^{ij3}h_j^{y2} - c_{21}^{ij4}h_i^{x2} - 2c_{12}^{ij4}h_i^{x}h_j^{y} - 3c_{03}^{ij4}h_j^{y2} &= 0 \\
c_{11}^{ij3}h_i^{x} + 2c_{02}^{ij3}h_j^{y} - 2c_{12}^{ij4}h_i^{x}h_j^{y} - 6c_{03}^{ij4}h_j^{y2} - c_{11}^{ij4}h_i^{x} - 2c_{02}^{ij4}h_j^{y} &= 0 \\
c_{01}^{ij3} - 3c_{03}^{ij4}h_j^{y2} - 2c_{02}^{ij4}h_j^{y} - c_{01}^{ij4} &= 0
\end{aligned}
\tag{2.17}
$$

$$
\begin{aligned}
c_{30}^{i,j-1,3} - c_{30}^{ij1} &= 0 \\
c_{20}^{i,j-1,3} - 3c_{30}^{ij1}h_i^{x} - c_{20}^{ij1} &= 0 \\
c_{10}^{i,j-1,3} - 3c_{30}^{ij1}h_i^{x2} - 2c_{20}^{ij1}h_i^{x} - c_{10}^{ij1} &= 0 \\
c_{00}^{i,j-1,3} - c_{30}^{ij1}h_i^{x3} - c_{20}^{ij1}h_i^{x2} - c_{10}^{ij1}h_i^{x} - c_{00}^{ij1} &= 0 \\
c_{21}^{i,j-1,3} - c_{21}^{ij1} &= 0 \\
c_{11}^{i,j-1,3} - 2c_{21}^{ij1}h_i^{x} - c_{11}^{ij1} &= 0 \\
c_{01}^{i,j-1,3} - c_{21}^{ij1}h_i^{x2} - c_{11}^{ij1}h_i^{x} - c_{01}^{ij1} &= 0
\end{aligned}
\tag{2.18}
$$

$$
\begin{aligned}
c_{03}^{i+1,j,2} - c_{03}^{ij4} &= 0 \\
c_{02}^{i+1,j,2} - 3c_{03}^{ij4}h_j^y - c_{02}^{ij4} &= 0 \\
c_{01}^{i+1,j,2} - 3c_{03}^{ij4}h_j^{y2} - 2c_{02}^{ij4}h_j^y - c_{01}^{ij4} &= 0 \\
c_{00}^{i+1,j,2} - c_{03}^{ij4}h_j^{y3} - c_{02}^{ij4}h_j^{y2} - c_{01}^{ij4}h_j^y - c_{00}^{ij4} &= 0 \\
c_{12}^{i+1,j,2} - c_{12}^{ij4} &= 0 \\
c_{11}^{i+1,j,2} - 2c_{12}^{ij4}h_j^y - c_{11}^{ij4} &= 0 \\
c_{10}^{i+1,j,2} - c_{12}^{ij4}h_j^{y2} - c_{11}^{ij4}h_j^y - c_{10}^{ij4} &= 0
\end{aligned}
\tag{2.19}
$$

$$
\begin{aligned}
c_{30}^{ij3} - c_{30}^{i,j+1,1} &= 0 \\
c_{20}^{ij3} - 3c_{30}^{i,j+1,1}h_i^x - c_{20}^{i,j+1,1} &= 0 \\
c_{10}^{ij3} - 3c_{30}^{i,j+1,1}h_i^{x2} - 2c_{20}^{i,j+1,1}h_i^x - c_{10}^{i,j+1,1} &= 0 \\
c_{00}^{ij3} - c_{30}^{i,j+1,1}h_i^{x3} - c_{20}^{i,j+1,1}h_i^{x2} - c_{10}^{i,j+1,1}h_i^x - c_{00}^{i,j+1,1} &= 0 \\
c_{21}^{ij3} - c_{21}^{i,j+1,1} &= 0 \\
c_{11}^{ij3} - 2c_{21}^{i,j+1,1}h_i^x - c_{11}^{i,j+1,1} &= 0 \\
c_{01}^{ij3} - c_{21}^{i,j+1,1}h_i^{x2} - c_{11}^{i,j+1,1}h_i^x - c_{01}^{i,j+1,1} &= 0
\end{aligned}
\tag{2.20}
$$

$$
\begin{aligned}
c_{03}^{ij2} - c_{03}^{i-1,j4} &= 0 \\
c_{02}^{ij2} - 3c_{03}^{i-1,j4}h_j^y - c_{02}^{i-1,j4} &= 0 \\
c_{01}^{ij2} - 3c_{03}^{i-1,j4}h_j^{y2} - 2c_{02}^{i-1,j4}h_j^y - c_{01}^{i-1,j4} &= 0 \\
c_{00}^{ij2} - c_{03}^{i-1,j4}h_j^{y3} - c_{02}^{i-1,j4}h_j^{y2} - c_{01}^{i-1,j4}h_j^y - c_{00}^{i-1,j4} &= 0 \\
c_{12}^{ij2} - c_{12}^{i-1,j4} &= 0 \\
c_{11}^{ij2} - 2c_{12}^{i-1,j4}h_j^y - c_{11}^{i-1,j4} &= 0 \\
c_{10}^{ij2} - c_{12}^{i-1,j4}h_j^{y2} - c_{11}^{i-1,j4}h_j^y - c_{10}^{i-1,j4} &= 0
\end{aligned}
\tag{2.21}
$$

The above C^1 smooth constraints (2.14)-(2.21) together with the objective function (2.13) give us an explicit mathematical model of the bivariate cubic L_1 splines.

Chapter 3

Generalized Geometric Programming Approach for Bivariate Cubic L_1 Splines

In this chapter, we introduce the basic theory of generalized geometric programming. After that, a generalized geometric programming framework for bivariate cubic L_1 spline is proposed. The computational results obtained by using a general nonlinear programming solver are also reported.

3.1 Generalized geometric programming

Geometric programming [27, 28, 29] is an optimization theory with a wide range of applications. In this section, we briefly introduce the the basic theory of generalized geometric programming.

3.1.1 Primal program

In generalized geometric programming, the *primal problem* is to find the minimizer of a real-valued convex function $\mathfrak{g}(x)$ over a given subset $\mathfrak{F}$, which is the intersection of the function domain $\mathfrak{C} \subseteq R^n$ and a cone $\mathfrak{X} \subseteq R^n$, i.e.

$$(\textbf{Primal}) \qquad \begin{cases} \min \mathfrak{g}(\mathbf{x}) \\ \mathbf{x} \in \mathfrak{C} \cap \mathfrak{X} \end{cases} \tag{3.1}$$

3.1.2 Conjugate transform

Definition 3.1.1 (Conjugate Transform) *Given a function $w(\mathbf{z})$ with domain $W \subseteq R^n$, the conjugate transform of $w(\mathbf{z})$ is a function $\omega(\zeta)$ with domain $\Omega \subseteq R^n$, where*

$$\Omega = \left\{ \zeta \in R^n \,\middle|\, \sup_{\mathbf{z} \in W} [\langle \zeta, \mathbf{z} \rangle - w(\mathbf{z})] < +\infty \right\}$$

and

$$\omega(\zeta) = \sup_{\mathbf{z} \in W} [\langle \zeta, \mathbf{z} \rangle - w(\mathbf{z})], \quad \forall \zeta \in \Omega$$

For a given function w, if the domain of its conjugate transform is empty, we say that its conjugate transform *does not exist*. It is know that the conjugate transform of a convex function always exists.

Theorem 3.1.1 [27, 28] *Given a function $w(\mathbf{z})$ with domain $W \subseteq R^n$. If $w(\mathbf{z})$ is a convex function and W is a nonempty convex set, then there exists a conjugate transform of $w(\mathbf{z})$.*

The above theorem and the definition of the conjugate transform give us the following important inequality:

Theorem 3.1.2 (Conjugate Inequality) [28] *For each $\mathbf{z} \in W$ and $\zeta \in \Omega$,*

$$\langle \zeta, \mathbf{z} \rangle \leq w(\mathbf{z}) + \omega(\zeta) \tag{3.2}$$

with equality holding if and only if $\zeta \in \partial w(\mathbf{z})$.

3.1.3 Dual program

Given a convex function $\mathfrak{g}(x)$ over domain $\mathfrak{C}$, denoted by $\mathfrak{g} : \mathfrak{C}$, the primal problem is given by (3.1). The conjugate transform of $\mathfrak{g} : \mathfrak{C}$ is $\mathfrak{h}$ with domain $\mathfrak{D}$, denoted by $\mathfrak{h} : \mathfrak{D}$, where

$$\mathfrak{D} = \left\{ \mathbf{y} \in R^n \,\middle|\, \sup_{\mathbf{x} \in \mathfrak{C}} [\langle \mathbf{y}, \mathbf{x} \rangle - \mathfrak{g}(\mathbf{x})] < +\infty \right\}$$

and

$$\mathfrak{h}(\mathbf{y}) = \sup_{\mathbf{x} \in \mathfrak{C}} [\langle \mathbf{y}, \mathbf{x} \rangle - \mathfrak{g}(\mathbf{x})], \quad \forall \mathbf{y} \in \mathfrak{D}$$

The feasible region of the primal problem is the intersection of domain $\mathfrak{C}$ with some cone $\mathfrak{X} \subseteq R^n$. Let $\mathfrak{Y}$ be the dual cone of $\mathfrak{X}$, which is defined by

$$\mathfrak{Y} = \{ \mathbf{y} \in R^n \,| \langle \mathbf{y}, \mathbf{x} \rangle \geq 0, \quad \forall x \in \mathfrak{X} \}$$

Then the dual problem becomes

$$(\mathbf{Dual}) \qquad \begin{cases} \min \mathfrak{h}(\mathbf{y}) \\ \mathbf{y} \in \mathfrak{D} \cap \mathfrak{Y} \end{cases} \tag{3.3}$$

3.1.4 Optimality conditions

Theorem 3.1.3 (Optimality Conditions) [*28*] $\mathbf{x}^*$ *and* $\mathbf{y}^*$ *are optimal solutions of the primal problem (3.1) and the dual problem (3.3), respectively, if and only if*

(I) $\mathbf{x}^* \in \mathfrak{C} \cap \mathfrak{X}, \mathbf{y}^* \in \mathfrak{D} \cap \mathfrak{Y}$

(II) $\langle \mathbf{x}^*, \mathbf{y}^* \rangle = 0$

(III) $\mathbf{y}^* \in \partial \mathfrak{g}(\mathbf{x}^*) \stackrel{\triangle}{=} \{ \mathbf{y} \in R^n \,| \mathfrak{g}(\mathbf{x}^*) + \langle \mathbf{y}, \mathbf{x} - \mathbf{x}^* \rangle \leq \mathfrak{g}(\mathbf{x}), \quad \forall \mathbf{x} \in \mathfrak{C} \}$

Optimality condition (I) indicates primal and dual feasibility. Optimality condition (II) is called the "orthogonality condition". If the primal cone $\mathfrak{X}$ is actually a vector space, then its dual cone $\mathfrak{Y} = \mathfrak{X}^{\perp}$. Hence, the orthogonality condition is automatically satisfied and can be omitted. Optimality condition (III) is called the "subgradient condition". When both function $\mathfrak{g} : \mathfrak{C}$ and cone $\mathfrak{X}$ are convex and closed, the primal problem (3.1) and the dual problem (3.3) are symmetric and the optimality condition (III) can be restated as

$$\text{(IIIa)} \quad \mathbf{x}^* \in \partial\mathfrak{h}(\mathbf{y}^*) \text{ and } \mathbf{y}^* \in \partial\mathfrak{g}(\mathbf{x}^*)$$

Theorem 3.1.4 *[28] If* $\mathbf{x}$ *and* $\mathbf{y}$ *are feasible solutions of the primal problem (3.1) and the dual problem (3.3), respectively, then*

$$0 \leq \mathfrak{g}(\mathbf{x}) + \mathfrak{h}(\mathbf{y}),$$

with equality holding if and only if the optimality conditions (II) *and* (III) *are satisfied. In this case,* $\mathbf{x}$ *and* $\mathbf{y}$ *are optimal solutions of the primal problem (3.1)and dual problem (3.3) respectively.*

Let us denote the *relative interior* of convex set $\mathfrak{D}$ by $ri(\mathfrak{D}) \equiv \{x \in \text{aff}(\mathfrak{D}) \,|\, \exists \epsilon > 0, (x + \epsilon \text{B}) \cap \text{aff}(\mathfrak{D}) \subset \mathfrak{D}\}$, where $\text{aff}(\mathfrak{D})$ is the affine hull of $\mathfrak{D}$ and B is the Euclidean unit ball in R^n, i.e., $\text{B} = \{x \,|\, \|x\|_2 \leq 1,\ x \in R^n\}$ [30].

Theorem 3.1.5 *If the dual problem (3.3) has a feasible solution* $\mathbf{y}^* \in ri(\mathfrak{D})$ *and* $\inf_{y \in \mathfrak{D} \cap \mathfrak{Y}} \mathfrak{h}(\mathbf{y}) < +\infty$*, then the primal problem (3.1) has a nonempty solution set and*

$$0 = \inf_{x \in \mathfrak{C} \cap \mathfrak{X}} \mathfrak{g}(\mathbf{x}) + \inf_{y \in \mathfrak{D} \cap \mathfrak{Y}} \mathfrak{h}(\mathbf{y}).$$

3.2 Geometric programming approach for bivariate cubic L_1 splines

3.2.1 Primal problem

Note that the objective function (2.13) is not separable, i.e. some variables appear in more than one terms in the objective function. In order to make the objective function $\mathfrak{g}(\mathbf{c})$ separable so that we can calculate the conjugate transform of the objective function more easily, we introduce four variables $\tilde{c}_{12}^{ij1}$, $\tilde{c}_{21}^{ij2}$, $\tilde{c}_{12}^{ij3}$ and $\tilde{c}_{21}^{ij4}$ on each rectangle T_{ij} such that

$$\begin{aligned}
\tilde{c}_{12}^{ij1} &= c_{12}^{ij1} \\
\tilde{c}_{21}^{ij2} &= c_{21}^{ij2} \\
\tilde{c}_{12}^{ij3} &= c_{12}^{ij3} \\
\tilde{c}_{21}^{ij4} &= c_{21}^{ij4},
\end{aligned}$$

or

$$c_{12}^{ij1} - \tilde{c}_{12}^{ij1} = 0 \tag{3.4}$$

$$c_{21}^{ij2} - \tilde{c}_{21}^{ij2} = 0 \tag{3.5}$$

$$c_{12}^{ij3} - \tilde{c}_{12}^{ij3} = 0 \tag{3.6}$$

$$c_{21}^{ij4} - \tilde{c}_{21}^{ij4} = 0, \tag{3.7}$$

Denote the modified coefficients of the bivariate cubic function $z^{ijk}(x, y)$ as

$$\mathbf{c}^{ijk} = \{c_{00}^{ijk}, c_{21}^{ijk}, c_{10}^{ijk}, c_{01}^{ijk}, c_{30}^{ijk}, c_{20}^{ijk}, c_{12}^{ijk}, c_{03}^{ijk}, c_{02}^{ijk}, \tilde{c}_{12}^{ijk}, c_{11}^{ijk}\}$$

$$i = 0, \cdots, I-1, \; j = 0, \cdots, J-1,$$

for $k = 1, 3$, and

$$\mathbf{c}^{ijk} = \{c_{00}^{ijk}, c_{12}^{ijk}, c_{10}^{ijk}, c_{01}^{ijk}, c_{30}^{ijk}, c_{21}^{ijk}, c_{20}^{ijk}, c_{03}^{ijk}, c_{02}^{ijk}, \tilde{c}_{21}^{ijk}, c_{11}^{ijk}\}$$

$$i = 0, \cdots, I-1,\; j = 0, \cdots, J-1,$$

for $k = 2, 4$.

Moreover, denote the coefficients of the bivariate cubic L_1 spline as

$$\begin{aligned} \mathbf{c} &= (\mathbf{c}^{001}, \mathbf{c}^{002}, \mathbf{c}^{003}, \mathbf{c}^{004}; \mathbf{c}^{011}, \mathbf{c}^{012}, \mathbf{c}^{013}, \mathbf{c}^{014}; \\ &\quad \cdots; \mathbf{c}^{I-1,J-1,1}, \mathbf{c}^{I-1,J-1,2}, \mathbf{c}^{I-1,J-1,3}, \mathbf{c}^{I-1,J-1,4})^T \end{aligned}$$

For $i = 0, \ldots, I-1$ and $j = 0, \ldots, J-1$, define $\mathbf{c}^{ij1}$, $\mathbf{c}^{ij2}$, $\mathbf{c}^{ij3}$ and $\mathbf{c}^{ij4}$ on the subsets

$$\mathfrak{C}^{ij1} = \{z_{ij}\} \times \{0\} \times R^9 \subset R^{11}$$

$$\mathfrak{C}^{ij2} = \{z_{i,j+1}\} \times \{0\} \times R^9 \subset R^{11}$$

$$\mathfrak{C}^{ij3} = \{z_{i+1,j+1}\} \times \{0\} \times R^9 \subset R^{11}$$

$$\mathfrak{C}^{ij4} = \{z_{i+1,j}\} \times \{0\} \times R^9 \subset R^{11}$$

Then, $\mathbf{c}$ is defined on the Cartesian product of $\mathfrak{C}^{ijk}$, i.e.,

$$\mathfrak{C} = \prod_{i=0}^{I-1} \prod_{j=0}^{J-1} \prod_{k=1}^{4} \mathfrak{C}^{ijk} \subset R^{44I \times J}$$

In this case, the constraints of (2.6) (2.8) (2.10) (2.12) and (2.5) (2.7) (2.9) (2.11) are automatically satisfied for any $\mathbf{c} \in \mathfrak{C}$. Moreover, the C^1 smooth constraints (2.14)-(2.21) and (3.4)-(3.7) can be treated as a cone constraint, since they can be expressed as

$$\mathcal{A}\mathbf{c} = 0 \tag{3.8}$$

where $\mathcal{A}$ is the coefficient matrix of (2.14)-(2.21) and (3.4)-(3.7). We define the cone $\mathfrak{X}$ to be the null space of matrix $\mathcal{A}$ for the primal problem, i.e.

$$\mathfrak{X} = \{\mathbf{c} \,|\, \mathcal{A}\mathbf{c} = 0\}$$

The primal objective function now becomes

$$\begin{aligned}\mathfrak{g}(\mathbf{c}) &= \sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\sum_{k=1}^{4}\mathfrak{g}^{ijk}(\mathbf{c}^{ijk}) \\ &= \sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\sum_{k=1}^{4}\Big(\mathfrak{g}^{ijk}_{xx}(\mathbf{c}^{ijk})+\mathfrak{g}^{ijk}_{yy}(\mathbf{c}^{ijk})+\mathfrak{g}^{ijk}_{xy}(\mathbf{c}^{ijk})\Big) \qquad (3.9)\end{aligned}$$

where

$$\mathfrak{g}^{ij1}_{xx}(\mathbf{c}^{ij1}) = \iint\limits_{(x,y)\in T^1_{ij}} \left|6c^{ij1}_{30}\,(x-x_i)+2c^{ij1}_{20}\right| dxdy$$

$$\mathfrak{g}^{ij1}_{yy}(\mathbf{c}^{ij1}) = \iint\limits_{(x,y)\in T^1_{ij}} \left|2c^{ij1}_{12}\,(x-x_i)+6c^{ij1}_{03}\,(y-y_j)+2c^{ij1}_{02}\right| dxdy$$

$$\mathfrak{g}^{ij1}_{xy}(\mathbf{c}^{ij1}) = 2\iint\limits_{(x,y)\in T^1_{ij}} \left|2\tilde{c}^{ij1}_{12}\,(y-y_j)+c^{ij1}_{11}\right| dxdy$$

$$\mathfrak{g}^{ij2}_{xx}(\mathbf{c}^{ij2}) = \iint\limits_{(x,y)\in T^2_{ij}} \left|6c^{ij2}_{30}\,(x-x_i)+2c^{ij2}_{21}(y-y_{j+1})+2c^{ij2}_{20}\right| dxdy$$

$$\mathfrak{g}^{ij2}_{yy}(\mathbf{c}^{ij2}) = \iint\limits_{(x,y)\in T^2_{ij}} \left|6c^{ij2}_{03}\,(y-y_{j+1})+2c^{ij2}_{02}\right| dxdy$$

$$\mathfrak{g}^{ij2}_{xy}(\mathbf{c}^{ij2}) = 2\iint\limits_{(x,y)\in T^2_{ij}} \left|2\tilde{c}^{ij2}_{21}\,(x-x_i)+c^{ij2}_{11}\right| dxdy$$

$$\mathfrak{g}^{ij3}_{xx}(\mathbf{c}^{ij3}) = \iint\limits_{(x,y)\in T^3_{ij}} \left|6c^{ij3}_{30}\,(x-x_{i+1})+2c^{ij3}_{20}\right| dxdy$$

$$\mathfrak{g}^{ij3}_{yy}(\mathbf{c}^{ij3}) = \iint\limits_{(x,y)\in T^3_{ij}} \left|2c^{ij3}_{12}\,(x-x_{i+1})+6c^{ij3}_{03}\,(y-y_{j+1})+2c^{ij3}_{02}\right| dxdy$$

$$\mathfrak{g}^{ij3}_{xy}(\mathbf{c}^{ij3}) = 2\iint\limits_{(x,y)\in T^3_{ij}} \left|2\tilde{c}^{ij3}_{12}\,(y-y_{j+1})+c^{ij3}_{11}\right| dxdy$$

$$\mathfrak{g}_{xx}^{ij4}(\mathbf{c}^{ij4}) = \iint\limits_{(x,y)\in T_{ij}^4} \left| 6c_{30}^{ij4}\,(x - x_{i+1}) + 2c_{21}^{ij4}(y - y_j) + 2c_{20}^{ij4} \right| dxdy$$

$$\mathfrak{g}_{yy}^{ij4}(\mathbf{c}^{ij4}) = \iint\limits_{(x,y)\in T_{ij}^4} \left| 6c_{03}^{ij4}\,(y - y_j) + 2c_{02}^{ij4} \right| dxdy$$

$$\mathfrak{g}_{xy}^{ij4}(\mathbf{c}^{ij4}) = 2\iint\limits_{(x,y)\in T_{ij}^4} \left| 2\tilde{c}_{21}^{ij4}\,(x - x_{i+1}) + c_{11}^{ij4} \right| dxdy$$

Consequently, the modified objective function (3.9) becomes separable and the primal problem becomes

$$(\textbf{Primal}) \qquad \begin{cases} \min \mathfrak{g}(\mathbf{c}) \\ \mathbf{c} \in \mathfrak{C} \cap \mathfrak{X} \end{cases} \tag{3.10}$$

3.2.2 Dual problem

For simplicity, we introduce more notations. Let $h_i^x = x_{i+1} - x_i$ and $h_j^y = y_{j+1} - y_j$ be the width and height of T_{ij}. Denote $K = \frac{1}{12} h_i^x h_j^y$.

For $k = 1, 3$, denote the dual vector corresponding to $\mathbf{c}^{ijk}$ as

$$\mathbf{d}^{ijk} = \{d_{00}^{ijk}, d_{21}^{ijk}, d_{10}^{ijk}, d_{01}^{ijk}, d_{30}^{ijk}, d_{20}^{ijk}, d_{12}^{ijk}, d_{03}^{ijk}, d_{02}^{ijk}, \tilde{d}_{12}^{ijk}, d_{11}^{ijk}\},$$

for $k = 2, 4$, denote the dual vector corresponding to $\mathbf{c}^{ijk}$ as

$$\mathbf{d}^{ijk} = \{d_{00}^{ijk}, d_{12}^{ijk}, d_{10}^{ijk}, d_{01}^{ijk}, d_{30}^{ijk}, d_{21}^{ijk}, d_{20}^{ijk}, d_{03}^{ijk}, d_{02}^{ijk}, \tilde{d}_{21}^{ijk}, d_{11}^{ijk}\},$$

where $i = 0, \cdots, I-1$ and $0 = 1, \cdots, J-1$.

Thus dual variables can be expressed as a vector

$$\begin{aligned} \mathbf{d} \;\; = \;\; & (\mathbf{d}^{001}, \mathbf{d}^{002}, \mathbf{d}^{003}, \mathbf{d}^{004}; \mathbf{d}^{011}, \mathbf{d}^{012}, \mathbf{d}^{013}, \mathbf{d}^{014}; \\ & \cdots; \mathbf{d}^{I-1,J-1,1}, \mathbf{d}^{I-1,J-1,2}, \mathbf{d}^{I-1,J-1,3}, \mathbf{d}^{I-1,J-1,4})^T. \end{aligned}$$

Let us denote the conjugate transform of $\mathfrak{g}(\mathbf{c}) : \mathfrak{C}$ as $\mathfrak{h}(\mathbf{d}) : \mathfrak{D}$ and $\mathfrak{g}^{ijk}(\mathbf{c}^{ijk}) : \mathfrak{C}^{ijk}$ as

$\mathfrak{h}^{ijk}(\mathbf{d}^{ijk}) : \mathfrak{D}^{ijk}$. By the definition of conjugate transform, we have

$$\mathfrak{h}(\mathbf{d}) = \sum_{i=0}^{I-1} \sum_{j=0}^{J-1} \sum_{k=1}^{4} \mathfrak{h}^{ijk}(\mathbf{d}^{ijk}) \tag{3.11}$$

where $\mathfrak{h}^{ijk}(\mathbf{d}^{ijk})$ needs to be further defined.

Now, for $k = 1$,

$$\begin{aligned}
\mathfrak{h}^{ij1}(\mathbf{d}^{ij1}) &= \sup_{\mathbf{c}^{ij1} \in \mathfrak{C}^{ij1}} \left[\langle \mathbf{d}^{ij1}, \mathbf{c}^{ij1} \rangle - \mathfrak{g}^{ij1}(\bar{\mathbf{c}}^{ij1}) \right] \\
&= \sup_{c_{00}^{ij1} = z_{ij}} \left[d_{00}^{ij1} c_{00}^{ij1} \right] + \sup_{c_{21}^{ij1} = 0} \left[d_{21}^{ij1} c_{21}^{ij1} \right] + \sup_{c_{10}^{ij1} \in R} \left[d_{10}^{ij1} c_{10}^{ij1} \right] \\
&+ \sup_{c_{01}^{ij1} \in R} \left[d_{01}^{ij1} c_{01}^{ij1} \right] \\
&+ \sup_{c_{30}^{ij1}, c_{20}^{ij1} \in R} \left[d_{30}^{ij1} c_{30}^{ij1} + d_{20}^{ij1} c_{20}^{ij1} - \mathfrak{g}_{xx}^{ij1}(\mathbf{d}^{ij1}) \right] \\
&+ \sup_{c_{12}^{ij1}, c_{03}^{ij1}, c_{02}^{ij1} \in R} \left[d_{12}^{ij1} c_{12}^{ij1} + d_{03}^{ij1} c_{03}^{ij1} + d_{02}^{ij1} c_{02}^{ij1} - \mathfrak{g}_{yy}^{ij1}(\mathbf{d}^{ij1}) \right] \\
&+ \sup_{\tilde{c}_{12}^{ij1}, c_{11}^{ij1} \in R} \left[\tilde{d}_{12}^{ij1} \tilde{c}_{12}^{ij1} + d_{11}^{ij1} c_{11}^{ij1} - \mathfrak{g}_{xy}^{ij1}(\mathbf{d}^{ij1}) \right]
\end{aligned} \tag{3.12}$$

In order to make $\mathfrak{h}^{ij1}(\mathbf{d}^{ij1}) < +\infty$, each of the seven terms in (3.12) should be finite. Since $c_{00}^{ij1} = z_{ij}$ is a fixed real number, the first term is always finite and the supremum is given by $d_{00}^{ij1} z_{ij}$. For the second term, since $c_{21}^{ij1} = 0$ and d_{21}^{ij1} can be any real number, this term becomes 0, and $\sup_{c_{21}^{ij1} \in R} \left[d_{21}^{ij1} c_{21}^{ij1} \right] = 0$. In the third term, since c_{10}^{ij1} can be any real number, the only value of d_{10}^{ij1} which makes this term finite is 0, correspondingly $\sup_{c_{10}^{ij1} \in R} \left[d_{10}^{ij1} c_{10}^{ij1} \right] = 0$. Similarly, in the fourth term, $d_{01}^{ij1} = 0$ and $\sup_{c_{01}^{ij1} \in R} \left[d_{01}^{ij1} c_{01}^{ij1} \right] = 0$. Consequently,

$$
\begin{aligned}
\mathfrak{h}^{ij1}(\mathbf{d}^{ij1}) &= d_{00}^{ij1} z_{ij} + \sup_{c_{30}^{ij1}, c_{20}^{ij1} \in R} \left[d_{30}^{ij1} c_{30}^{ij1} + d_{20}^{ij1} c_{20}^{ij1} - \mathfrak{g}_{xx}^{ij1}(\mathbf{d}^{ij1}) \right] \\
&+ \sup_{c_{12}^{ij1}, c_{03}^{ij1}, c_{02}^{ij1} \in R} \left[d_{12}^{ij1} c_{12}^{ij1} + d_{03}^{ij1} c_{03}^{ij1} + d_{02}^{ij1} c_{02}^{ij1} - \mathfrak{g}_{yy}^{ij1}(\mathbf{d}^{ij1}) \right] \\
&+ \sup_{\tilde{c}_{12}^{ij1}, c_{11}^{ij1} \in R} \left[\tilde{d}_{12}^{ij1} \tilde{c}_{12}^{ij1} + d_{11}^{ij1} c_{11}^{ij1} - \mathfrak{g}_{xy}^{ij1}(\mathbf{d}^{ij1}) \right] \\
&= d_{00}^{ij1} z_{ij} + \mathfrak{h}_{xx}^{ij1}(\mathbf{d}^{ij1}) + \mathfrak{h}_{yy}^{ij1}(\mathbf{d}^{ij1}) + 2\mathfrak{h}_{xy}^{ij1}(\mathbf{d}^{ij1})
\end{aligned}
$$

From the Appendix, we see $\mathfrak{h}_{xx}^{ij1}(\mathbf{d}^{ij1}) = 0$, $\mathfrak{h}_{yy}^{ij1}(\mathbf{d}^{ij1}) = 0$, $\mathfrak{h}_{xy}^{ij1}(\mathbf{d}^{ij1}) = 0$. Thus.

$$
\mathfrak{h}^{ij1}(\mathbf{d}^{ij1}) = d_{00}^{ij1} z_{ij} \tag{3.13}
$$

Similar to the case of $\mathfrak{h}^{ij1}(\mathbf{d}^{ij1})$, we see that

$$
\mathfrak{h}^{ij2}(\mathbf{d}^{ij2}) = d_{00}^{ij2} z_{i,j+1} \tag{3.14}
$$

$$
\mathfrak{h}^{ij3}(\mathbf{d}^{ij3}) = d_{00}^{ij3} z_{i+1,j+1} \tag{3.15}
$$

$$
\mathfrak{h}^{ij4}(\mathbf{d}^{ij4}) = d_{00}^{ij4} z_{i+1,j} \tag{3.16}
$$

By substituting (3.13), (3.14), (3.15) and (3.16) into (3.11) we get the dual objective function

$$
\begin{aligned}
\mathfrak{h}(\mathbf{d}) &= \sum_{i=0}^{I-1} \sum_{j=0}^{J-1} \sum_{k=1}^{4} \mathfrak{h}^{ijk}(\mathbf{d}^{ijk}) \\
&= \sum_{i=0}^{I-1} \sum_{j=0}^{J-1} \left(z_{ij} d_{00}^{ij1} + z_{i,j+1} d_{00}^{ij2} + z_{i+1,j+1} d_{00}^{ij3} + z_{i+1,j} d_{00}^{ij4} \right) \qquad (3.17)
\end{aligned}
$$

whose domain is

$$
\mathfrak{D} = \prod_{i=0}^{I-1} \prod_{j=0}^{J-1} \prod_{k=1}^{4} \mathfrak{D}^{ijk} \subset R^{44I \times J},
$$

where

$$
\mathfrak{D}^{ijk} = R \times R \times \{0\} \times \{0\} \times \Omega_{xx}^{ijk} \times \Omega_{yy}^{ijk} \times \Omega_{xy}^{ijk},
$$

for $i=0,\ldots,I-1,\ j=0,\ldots,J-1,\ k=1,\ldots,4,$

and

$$\Omega_{xx}^{ij1}=\Bigg\{\left(d_{30}^{ij1},d_{20}^{ij1}\right)\in R^2\Bigg|\left(1-\frac{2d_{20}^{ij1}}{h_i^x h_j^y}\right)^3\le\left(\frac{3}{2}-\frac{2d_{30}^{ij1}}{h_i^{x2}h_j^y}\right)^2,$$

$$\left(1+\frac{2d_{20}^{ij1}}{h_i^x h_j^y}\right)^3\le\left(\frac{2d_{30}^{ij1}}{h_i^{x2}h_j^y}-\frac{6d_{20}^{ij1}}{h_i^x h_j^y}-\frac{3}{2}\right)^2,$$

$$\left(1+\frac{2d_{20}^{ij1}}{h_i^x h_j^y}\right)^3\le\left(\frac{3}{2}+\frac{2d_{30}^{ij1}}{h_i^{x2}h_j^y}\right)^2,$$

$$\left(1-\frac{2d_{20}^{ij1}}{h_i^x h_j^y}\right)^3\le\left(\frac{2d_{30}^{ij1}}{h_i^{x2}h_j^y}-\frac{6d_{20}^{ij1}}{h_i^x h_j^y}+\frac{3}{2}\right)^2\Bigg\},$$

$$\Omega_{yy}^{ij1}=\Bigg\{\left(d_{12}^{ij1},d_{03}^{ij1},d_{02}^{ij1}\right)\in R^3\Bigg|$$

$$x\le -y+z-3K+3\sqrt[3]{(-2K)\,(y+K)\,(z-K)},$$

$$x\ge -y+z+3K-3\sqrt[3]{(-2K)\,(y-K)\,(z+K)},$$

$$y\le -x+z-3K+3\sqrt[3]{(-2K)\,(x+K)\,(z-K)},$$

$$y\ge -x+z+3K-3\sqrt[3]{(-2K)\,(x-K)\,(z+K)},$$

$$z\le x+y-3K+3\sqrt[3]{2K\,(x-K)\,(y-K)},$$

$$z\ge x+y+3K-3\sqrt[3]{2K\,(x+K)\,(y+K)}\Bigg\},$$

$$\text{where}\begin{cases}x=\frac{d_{03}^{ij1}}{3h_j^y},\\ y=\frac{d_{12}^{ij1}}{2h_i^x}-\frac{d_{03}^{ij1}}{6h_j^y},\\ z=\frac{d_{12}^{ij1}}{2h_i^x}+\frac{d_{03}^{ij1}}{6h_j^y}-\frac{d_{02}^{ij1}}{2},\end{cases}$$

$$\Omega_{xy}^{ij1}=\Bigg\{\left(\tilde{d}_{12}^{ij1},d_{11}^{ij1}\right)\in R^3\Bigg|\left(\frac{1}{2}+\frac{4d_{11}^{ij1}}{h_i^x h_j^y}\right)^3\le\left(-\frac{6\tilde{d}_{12}^{ij1}}{h_i^x h_j^{y2}}+\frac{6d_{11}^{ij1}}{h_i^x h_j^y}+\frac{1}{2}\right)^2,$$

$$\left(\frac{1}{2}-\frac{4d_{11}^{ij1}}{h_i^x h_j^y}\right)^3\le\left(\frac{6\tilde{d}_{12}^{ij1}}{h_i^x h_j^{y2}}-\frac{6d_{11}^{ij1}}{h_i^x h_j^y}+\frac{1}{2}\right)^2\Bigg\},$$

$$\Omega_{xx}^{ij2} = \Bigg\{ \left(d_{21}^{ij2}, d_{30}^{ij2}, d_{20}^{ij2}\right) \in R^3 \Bigg|$$

$$x \le -y + z - 3K + 3\sqrt[3]{(-2K)(y+K)(z-K)},$$

$$x \ge -y + z + 3K - 3\sqrt[3]{(-2K)(y-K)(z+K)},$$

$$y \le -x + z - 3K + 3\sqrt[3]{(-2K)(x+K)(z-K)},$$

$$y \ge -x + z + 3K - 3\sqrt[3]{(-2K)(x-K)(z+K)},$$

$$z \le x + y - 3K + 3\sqrt[3]{2K(x-K)(y-K)},$$

$$z \ge x + y + 3K - 3\sqrt[3]{2K(x+K)(y+K)}\Bigg\},$$

$$\text{where } \begin{cases} x = \frac{d_{30}^{ij2}}{3h_i^x}, \\ y = \frac{d_{21}^{ij2}}{2h_j^y} - \frac{d_{30}^{ij2}}{6h_i^x} + \frac{d_{20}^{ij2}}{2}, \\ z = \frac{d_{21}^{ij2}}{2h_j^y} + \frac{d_{30}^{ij2}}{6h_i^x}, \end{cases}$$

$$\Omega_{yy}^{ij2} = \Bigg\{ \left(d_{03}^{ij2}, d_{02}^{ij2}\right) \in R^2 \Bigg| \left(1 - \frac{2d_{02}^{ij2}}{h_i^x h_j^y}\right)^3 \le \left(\frac{3}{2} + \frac{2d_{03}^{ij2}}{h_i^x h_j^{y2}}\right)^2,$$

$$\left(1 + \frac{2d_{02}^{ij2}}{h_i^x h_j^y}\right)^3 \le \left(-\frac{2d_{03}^{ij2}}{h_i^x h_j^{y2}} - \frac{6d_{02}^{ij2}}{h_i^x h_j^y} - \frac{3}{2}\right)^2,$$

$$\left(1 + \frac{2d_{02}^{ij2}}{h_i^x h_j^y}\right)^3 \le \left(\frac{3}{2} - \frac{2d_{03}^{ij2}}{h_i^x h_j^{y2}}\right)^2,$$

$$\left(1 - \frac{2d_{02}^{ij2}}{h_i^x h_j^y}\right)^3 \le \left(-\frac{2d_{03}^{ij2}}{h_i^x h_j^{y2}} - \frac{6d_{02}^{ij2}}{h_i^x h_j^y} + \frac{3}{2}\right)^2\Bigg\},$$

$$\Omega_{xy}^{ij2} = \Bigg\{ \left(\tilde{d}_{21}^{ij2}, d_{11}^{ij2}\right) \in R^3 \Bigg| \left(\frac{1}{2} + \frac{4d_{11}^{ij2}}{h_i^x h_j^y}\right)^3 \le \left(-\frac{6\tilde{d}_{21}^{ij2}}{h_i^{x2} h_j^y} + \frac{6d_{11}^{ij2}}{h_i^x h_j^y} + \frac{1}{2}\right)^2,$$

$$\left(\frac{1}{2} - \frac{4d_{11}^{ij2}}{h_i^x h_j^y}\right)^3 \le \left(\frac{6\tilde{d}_{21}^{ij2}}{h_i^{x2} h_j^y} - \frac{6d_{11}^{ij2}}{h_i^x h_j^y} + \frac{1}{2}\right)^2\Bigg\},$$

$$\Omega_{xx}^{ij3} = \left\{ \left(d_{30}^{ij3}, d_{20}^{ij3}\right) \in R^2 \;\middle|\; \left(1 - \frac{2d_{20}^{ij3}}{h_i^x h_j^y}\right)^3 \le \left(\frac{3}{2} + \frac{2d_{30}^{ij3}}{h_i^{x2} h_j^y}\right)^2, \right.$$

$$\left(1 + \frac{2d_{20}^{ij3}}{h_i^x h_j^y}\right)^3 \le \left(-\frac{2d_{30}^{ij3}}{h_i^{x2} h_j^y} - \frac{6d_{20}^{ij3}}{h_i^x h_j^y} - \frac{3}{2}\right)^2,$$

$$\left(1 + \frac{2d_{20}^{ij3}}{h_i^x h_j^y}\right)^3 \le \left(\frac{3}{2} - \frac{2d_{30}^{ij3}}{h_i^{x2} h_j^y}\right)^2,$$

$$\left.\left(1 - \frac{2d_{20}^{ij3}}{h_i^x h_j^y}\right)^3 \le \left(-\frac{2d_{30}^{ij3}}{h_i^{x2} h_j^y} - \frac{6d_{20}^{ij3}}{h_i^x h_j^y} + \frac{3}{2}\right)^2 \right\},$$

$$\Omega_{yy}^{ij3} = \left\{ \left(d_{12}^{ij3}, d_{03}^{ij3}, d_{02}^{ij3}\right) \in R^3 \;\middle|\; \right.$$

$$x \le -y + z - 3K + 3\sqrt[3]{(-2K)(y+K)(z-K)},$$

$$x \ge -y + z + 3K - 3\sqrt[3]{(-2K)(y-K)(z+K)},$$

$$y \le -x + z - 3K + 3\sqrt[3]{(-2K)(x+K)(z-K)},$$

$$y \ge -x + z + 3K - 3\sqrt[3]{(-2K)(x-K)(z+K)},$$

$$z \le x + y - 3K + 3\sqrt[3]{2K(x-K)(y-K)},$$

$$\left. z \ge x + y + 3K - 3\sqrt[3]{2K(x+K)(y+K)} \right\},$$

$$\text{where } \begin{cases} x = -\frac{d_{03}^{ij3}}{3h_j^y}, \\ y = -\frac{d_{12}^{ij3}}{2h_i^x} + \frac{d_{03}^{ij3}}{6h_j^y}, \\ z = -\frac{d_{12}^{ij3}}{2h_i^x} - \frac{d_{03}^{ij3}}{6h_j^y} - \frac{d_{02}^{ij3}}{2}, \end{cases}$$

$$\Omega_{xy}^{ij3} = \left\{ \left(\tilde{d}_{12}^{ij3}, d_{11}^{ij3}\right) \in R^3 \;\middle|\; \left(\frac{1}{2} + \frac{4d_{11}^{ij3}}{h_i^x h_j^y}\right)^3 \le \left(\frac{6\tilde{d}_{12}^{ij3}}{h_i^x h_j^{y2}} + \frac{6d_{11}^{ij3}}{h_i^x h_j^y} + \frac{1}{2}\right)^2, \right.$$

$$\left. \left(\frac{1}{2} - \frac{4d_{11}^{ij3}}{h_i^x h_j^y}\right)^3 \le \left(-\frac{\tilde{d}_{12}^{ij3}}{h_i^x h_j^{y2}} - \frac{6d_{11}^{ij3}}{h_i^x h_j^y} + \frac{1}{2}\right)^2 \right\},$$

$$\Omega_{xx}^{ij4} = \Bigg\{ \left(d_{21}^{ij4}, d_{30}^{ij4}, d_{20}^{ij4}\right) \in R^3 \Bigg|$$

$$x \le -y + z - 3K + 3\sqrt[3]{(-2K)(y+K)(z-K)},$$

$$x \ge -y + z + 3K - 3\sqrt[3]{(-2K)(y-K)(z+K)},$$

$$y \le -x + z - 3K + 3\sqrt[3]{(-2K)(x+K)(z-K)},$$

$$y \ge -x + z + 3K - 3\sqrt[3]{(-2K)(x-K)(z+K)},$$

$$z \le x + y - 3K + 3\sqrt[3]{2K(x-K)(y-K)},$$

$$z \ge x + y + 3K - 3\sqrt[3]{2K(x+K)(y+K)}\Bigg\},$$

$$\text{where} \begin{cases} x = \frac{d_{30}^{ij4}}{3h_i^x} + \frac{d_{20}^{ij4}}{2}, \\ y = \frac{d_{21}^{ij4}}{2h_j^y} - \frac{d_{30}^{ij4}}{6h_i^x} - \frac{d_{20}^{ij4}}{4}, \\ z = \frac{d_{21}^{ij4}}{2h_j^y} + \frac{d_{30}^{ij4}}{6h_i^x} - \frac{d_{20}^{ij4}}{4}, \end{cases}$$

$$\Omega_{yy}^{ij4} = \Bigg\{ \left(d_{03}^{ij4}, d_{02}^{ij4}\right) \in R^2 \Bigg| \left(1 - \frac{2d_{02}^{ij4}}{h_i^x h_j^y}\right)^3 \le \left(\frac{3}{2} - \frac{2d_{03}^{ij4}}{h_i^x h_j^{y2}}\right)^2,$$

$$\left(1 + \frac{2d_{02}^{ij4}}{h_i^x h_j^y}\right)^3 \le \left(\frac{2d_{03}^{ij4}}{h_i^x h_j^{y2}} - \frac{6d_{02}^{ij4}}{h_i^x h_j^y} - \frac{3}{2}\right)^2,$$

$$\left(1 + \frac{2d_{02}^{ij4}}{h_i^x h_j^y}\right)^3 \le \left(\frac{3}{2} + \frac{2d_{03}^{ij4}}{h_i^x h_j^{y2}}\right)^2,$$

$$\left(1 - \frac{2d_{02}^{ij4}}{h_i^x h_j^y}\right)^3 \le \left(\frac{2d_{03}^{ij4}}{h_i^x h_j^{y2}} - \frac{6d_{02}^{ij4}}{h_i^x h_j^y} + \frac{3}{2}\right)^2\Bigg\},$$

$$\Omega_{xy}^{ij4} = \Bigg\{ \left(\tilde{d}_{21}^{ij4}, d_{11}^{ij4}\right) \in R^3 \Bigg| \left(\frac{1}{2} + \frac{4d_{11}^{ij4}}{h_i^x h_j^y}\right)^3 \le \left(\frac{6\tilde{d}_{21}^{ij4}}{h_i^{x2} h_j^y} - \frac{1}{2}\right)^2,$$

$$\left(\frac{1}{2} - \frac{4d_{11}^{ij4}}{h_i^x h_j^y}\right)^3 \le \left(\frac{6\tilde{d}_{21}^{ij4}}{h_i^{x2} h_j^y} + \frac{1}{2}\right)^2\Bigg\}.$$

From the Appendix, we see the sets Ω_{xx}^{ijk}, Ω_{xy}^{ijk} and Ω_{yy}^{ijk} are all convex.

Therefore, the dual problem becomes

$$(\textbf{Dual}) \qquad \begin{cases} \min \mathfrak{h}(\mathbf{d}) \\ \mathbf{d} \in \mathfrak{D} \cap \mathfrak{Y} \end{cases} \tag{3.18}$$

where the dual cone $\mathfrak{Y}$ is the row space of the matrix $\mathcal{A}$ defined in (3.8). This dual problem is a convex programming problem with a linear objective function and convex cubic constraints.

3.2.3 Dual to primal transformation

If a dual optimal solution $\mathbf{d}^*$ is obtained by solving problem (3.3), according to Theorem 3.1.3, a primal optimal solution can be obtained by solving the following optimality conditions:

$$\begin{aligned} &\text{(I)} \quad \mathbf{c}^* \in \mathfrak{C} \cap \mathfrak{X}, \mathbf{d}^* \in \mathfrak{D} \cap \mathfrak{Y} \\ &\text{(II)} \quad \langle \mathbf{c}^*, \mathbf{d}^* \rangle = 0 \\ &\text{(III)} \quad \mathbf{c}^* \in \partial \mathfrak{h}(\mathbf{d}^*). \end{aligned}$$

This means that the primal optimal solution $\mathbf{c}^*$ is a vector such that

$$\mathbf{c}^* \in \mathfrak{C} \cap \mathfrak{X} \cap \partial \mathfrak{h}(\mathbf{d}^*) \tag{3.19}$$

Now we try to find $\partial \mathfrak{h}(\hat{\mathbf{d}})$ for any given dual vector $\hat{\mathbf{d}}$. For any $\hat{\mathbf{d}} \in \mathfrak{D}$ and $\boldsymbol{\gamma} \in \partial \mathfrak{h}(\hat{\mathbf{d}})$, let

$$\begin{aligned} \boldsymbol{\gamma} \;=\; &(\boldsymbol{\gamma}^{001}, \boldsymbol{\gamma}^{002}, \boldsymbol{\gamma}^{003}, \boldsymbol{\gamma}^{004}; \boldsymbol{\gamma}^{011}, \boldsymbol{\gamma}^{012}, \boldsymbol{\gamma}^{013}, \boldsymbol{\gamma}^{014}; \\ &\cdots; \boldsymbol{\gamma}^{I-1,J-1,1}, \boldsymbol{\gamma}^{I-1,J-1,2}, \boldsymbol{\gamma}^{I-1,J-1,3}, \boldsymbol{\gamma}^{I-1,J-1,4}) \end{aligned}$$

where

$$\boldsymbol{\gamma}^{ijk} = \{\gamma_{00}^{ijk}, \gamma_{21}^{ijk}, \gamma_{10}^{ijk}, \gamma_{01}^{ijk}, \gamma_{30}^{ijk}, \gamma_{20}^{ijk}, \gamma_{12}^{ijk}, \gamma_{03}^{ijk}, \gamma_{02}^{ijk}, \tilde{\gamma}_{12}^{ijk}, \gamma_{11}^{ijk}\},$$

for $k = 1, 3$, and

$$\boldsymbol{\gamma}^{ijk} = \{\gamma_{00}^{ijk}, \gamma_{12}^{ijk}, \gamma_{10}^{ijk}, \gamma_{01}^{ijk}, \gamma_{30}^{ijk}, \gamma_{21}^{ijk}, \gamma_{20}^{ijk}, \gamma_{03}^{ijk}, \gamma_{02}^{ijk}, \tilde{\gamma}_{21}^{ijk}, \gamma_{11}^{ijk}\},$$

for $k = 2, 4$, and $i = 0, \cdots, I-1,\ j = 0, \cdots, J-1$.

From the definition of subgradient, $\boldsymbol{\gamma}$ satisfies

$$\left\langle \boldsymbol{\gamma}, \mathbf{d} - \hat{\mathbf{d}} \right\rangle \leq \mathfrak{h}(\mathbf{d}) - \mathfrak{h}(\hat{\mathbf{d}}), \quad \forall \mathbf{d} \in \mathfrak{D}. \tag{3.20}$$

Since d_{10}^{ijk} and d_{01}^{ijk} can only be zero, γ_{10}^{ijk} and γ_{01}^{ijk} can be any real number. Assume $\mathbf{d}$ is a vector whose elements are the same as the given vector $\hat{\mathbf{d}}$, except for the component $d_{00}^{ij1} = \hat{d}_{00}^{ij1} + \delta$, where δ can be any real number. Then

$$\left\langle \boldsymbol{\gamma}, \mathbf{d} - \hat{\mathbf{d}} \right\rangle = \gamma_{00}^{ij1}\delta,$$

$$\mathfrak{h}(\mathbf{d}) - \mathfrak{h}(\hat{\mathbf{d}}) = z_{ij}\delta.$$

Since δ can be either a positive or a negative number, we have $\gamma_{00}^{ij1} = z_{ij}$.

Based on a similar argument, we have

$$\gamma_{00}^{ij2} = z_{i,j+1}$$

$$\gamma_{00}^{ij3} = z_{i+1,j+1}$$

$$\gamma_{00}^{ij4} = z_{i+1,j}$$

Assume $\mathbf{d}$ is a vector whose elements are the same as the given vector $\hat{\mathbf{d}}$, except for the component $d_{21}^{ij1} = \hat{d}_{21}^{ij1} + \delta$, where δ can be any real number. Then

$$\left\langle \boldsymbol{\gamma}, \mathbf{d} - \hat{\mathbf{d}} \right\rangle = \gamma_{21}^{ij1}\delta,$$

$$\mathfrak{h}(\mathbf{d}) - \mathfrak{h}(\hat{\mathbf{d}}) = 0.$$

Since δ can be either a positive or a negative number, from (3.20) we have $\gamma_{21}^{ij1} = 0$.

Similarly, we also have $\gamma_{12}^{ij2} = 0, \gamma_{21}^{ij3} = 0$ and $\gamma_{12}^{ij4} = 0$.

Assume $\mathbf{d}$ is a vector whose elements are the same as the given vector $\hat{\mathbf{d}}$, except for the components d_{30}^{ij1} and d_{20}^{ij1}. Then

$$\left\langle \boldsymbol{\gamma}, \mathbf{d} - \hat{\mathbf{d}} \right\rangle = \gamma_{30}^{ij1}\left(d_{30}^{ij1} - \hat{d}_{30}^{ij1}\right) + \gamma_{20}^{ij1}\left(d_{20}^{ij1} - \hat{d}_{20}^{ij1}\right),$$

$$\mathfrak{h}(\mathbf{d}) - \mathfrak{h}(\hat{\mathbf{d}}) = 0.$$

Formula (3.20) now becomes

$$\gamma_{30}^{ij1}\left(d_{30}^{ij1} - \hat{d}_{30}^{ij1}\right) + \gamma_{20}^{ij1}\left(d_{20}^{ij1} - \hat{d}_{20}^{ij1}\right) \le 0, \quad \forall \left(d_{30}^{ij1}, d_{20}^{ij1}\right) \in \Omega_{xx}^{ij1}. \tag{3.21}$$

If we consider the above condition in the two-dimensional $\left(d_{30}^{ij1}, d_{20}^{ij1}\right)$ space, then $\left(\gamma_{30}^{ij1}, \gamma_{20}^{ij1}\right)$ must lie in the normal cone of Ω_{xx}^{ij1} at point $\left(\hat{d}_{30}^{ij1}, \hat{d}_{20}^{ij1}\right)$.

Recall that $\left(d_{30}^{ij1}, d_{20}^{ij1}\right)$ is defined on the set

$$\begin{aligned}
\Omega_{xx}^{ij1} = \Bigg\{ \left(d_{30}^{ij1}, d_{20}^{ij1}\right) \in R^2 \,\Bigg|\, & \left(1 - \frac{2d_{20}^{ij1}}{h_i^x h_j^y}\right)^3 \le \left(\frac{3}{2} - \frac{2d_{30}^{ij1}}{h_i^{x2} h_j^y}\right)^2, \\
& \left(1 + \frac{2d_{20}^{ij1}}{h_i^x h_j^y}\right)^3 \le \left(\frac{2d_{30}^{ij1}}{h_i^{x2} h_j^y} - \frac{6d_{20}^{ij1}}{h_i^x h_j^y} - \frac{3}{2}\right)^2, \\
& \left(1 + \frac{2d_{20}^{ij1}}{h_i^x h_j^y}\right)^3 \le \left(\frac{3}{2} + \frac{2d_{30}^{ij1}}{h_i^{x2} h_j^y}\right)^2, \\
& \left(1 - \frac{2d_{20}^{ij1}}{h_i^x h_j^y}\right)^3 \le \left(\frac{2d_{30}^{ij1}}{h_i^{x2} h_j^y} - \frac{6d_{20}^{ij1}}{h_i^x h_j^y} + \frac{3}{2}\right)^2 \Bigg\},
\end{aligned} \tag{3.22}$$

which is a convex set bounded by four cubic curves:

$$\begin{aligned}
C_{xx1}^{ij1}: \quad & \left(1 - \frac{2d_{20}^{ij1}}{h_i^x h_j^y}\right)^3 = \left(\frac{3}{2} - \frac{2d_{30}^{ij1}}{h_i^{x2} h_j^y}\right)^2, \\
C_{xx2}^{ij1}: \quad & \left(1 + \frac{2d_{20}^{ij1}}{h_i^x h_j^y}\right)^3 = \left(\frac{2d_{30}^{ij1}}{h_i^{x2} h_j^y} - \frac{6d_{20}^{ij1}}{h_i^x h_j^y} - \frac{3}{2}\right)^2, \\
C_{xx3}^{ij1}: \quad & \left(1 + \frac{2d_{20}^{ij1}}{h_i^x h_j^y}\right)^3 = \left(\frac{3}{2} + \frac{2d_{30}^{ij1}}{h_i^{x2} h_j^y}\right)^2, \\
C_{xx4}^{ij1}: \quad & \left(1 - \frac{2d_{20}^{ij1}}{h_i^x h_j^y}\right)^3 = \left(\frac{2d_{30}^{ij1}}{h_i^{x2} h_j^y} - \frac{6d_{20}^{ij1}}{h_i^x h_j^y} + \frac{3}{2}\right)^2.
\end{aligned}$$

If $\left(\hat{d}_{30}^{ij1}, \hat{d}_{20}^{ij1}\right)$ is an interior point of Ω_{xx}^{ij1}, then obviously

$$\left(\gamma_{30}^{ij1}, \gamma_{20}^{ij1}\right) = (0,0).$$

If $\left(\hat{d}_{30}^{ij1}, \hat{d}_{20}^{ij1}\right)$ is a boundary point on the curve C_{xx1}^{ij1} but not on the curve C_{xx2}^{ij1}, C_{xx3}^{ij1} and C_{xx4}^{ij1}, then one normal vector of Ω_{xx}^{ij1} is

$$\boldsymbol{\eta}_{xx1}^{ij1} = \left(-\frac{4}{h_{ij}^{x2}h_{ij}^{y}}\left(\frac{3}{2} - \frac{2\hat{d}_{30}^{ij1}}{h_{i}^{x2}h_{j}^{y}}\right), \frac{6}{h_{ij}^{x}h_{ij}^{y}}\left(1 - \frac{2\hat{d}_{20}^{ij1}}{h_{i}^{x}h_{j}^{y}}\right)^{2}\right)^{T}.$$

Hence

$$\begin{pmatrix} \gamma_{30}^{ij1} \\ \\ \gamma_{20}^{ij1} \end{pmatrix} = \lambda_{xx}^{ij1}\boldsymbol{\eta}_{xx1}^{ij1} \quad \text{for some } \lambda_{xx}^{ij1} \geq 0.$$

If $\left(\hat{d}_{30}^{ij1}, \hat{d}_{20}^{ij1}\right)$ is a boundary point on the curve C_{xx2}^{ij1} but not on the curve C_{xx1}^{ij1}, C_{xx3}^{ij1} and C_{xx4}^{ij1}, then one normal vector of Ω_{xx}^{ij1} is

$$\begin{aligned} \boldsymbol{\eta}_{xx2}^{ij1} = \Bigg(& -\frac{4}{h_{ij}^{x2}h_{ij}^{y}}\left(\frac{2\hat{d}_{30}}{h_{ij}^{x2}h_{ij}^{y}} - \frac{6\hat{d}_{20}}{h_{ij}^{x}h_{ij}^{y}} - \frac{3}{2}\right), \\ & \frac{12}{h_{ij}^{x}h_{ij}^{y}}\left(\frac{2\hat{d}_{30}}{h_{ij}^{x2}h_{ij}^{y}} - \frac{6\hat{d}_{20}}{h_{ij}^{x}h_{ij}^{y}} - \frac{3}{2}\right) + \frac{6}{h_{ij}^{x}h_{ij}^{y}}\left(1 + \frac{2\hat{d}_{20}^{ij1}}{h_{i}^{x}h_{j}^{y}}\right)^{2}\Bigg)^{T}. \end{aligned}$$

Hence

$$\begin{pmatrix} \gamma_{30}^{ij1} \\ \\ \gamma_{20}^{ij1} \end{pmatrix} = \mu_{xx}^{ij1}\boldsymbol{\eta}_{xx2}^{ij1} \quad \text{for some } \mu_{xx}^{ij1} \geq 0.$$

If $\left(\hat{d}_{30}^{ij1}, \hat{d}_{20}^{ij1}\right)$ is a boundary point on the curve C_{xx3}^{ij1} but not on the curve C_{xx1}^{ij1}, C_{xx2}^{ij1} and C_{xx4}^{ij1}, then one normal vector of Ω_{xx}^{ij1} is

$$\boldsymbol{\eta}_{xx3}^{ij1} = \left(-\frac{4}{h_{ij}^{x2}h_{ij}^{y}}\left(\frac{3}{2} + \frac{2\hat{d}_{30}^{ij1}}{h_{i}^{x2}h_{j}^{y}}\right), \frac{6}{h_{ij}^{x}h_{ij}^{y}}\left(1 + \frac{2\hat{d}_{20}^{ij1}}{h_{i}^{x}h_{j}^{y}}\right)^{2}\right)^{T}.$$

Hence

$$\begin{pmatrix} \gamma_{30}^{ij1} \\ \\ \gamma_{20}^{ij1} \end{pmatrix} = \nu_{xx}^{ij1}\boldsymbol{\eta}_{xx3}^{ij1} \quad \text{for some } \nu_{xx}^{ij1} \geq 0.$$

If $\left(\hat{d}_{30}^{ij1}, \hat{d}_{20}^{ij1}\right)$ is a boundary point on the curve C_{xx4}^{ij1} but not on the curve C_{xx1}^{ij1}, C_{xx2}^{ij1} and C_{xx3}^{ij1}, then one normal vector of Ω_{xx}^{ij1} is

$$\boldsymbol{\eta}_{xx4}^{ij1} = \left(-\frac{4}{h_{ij}^{x2} h_{ij}^{y}} \left(\frac{2\hat{d}_{30}}{h_{ij}^{x2} h_{ij}^{y}} - \frac{6\hat{d}_{20}}{h_{ij}^{x} h_{ij}^{y}} + \frac{3}{2} \right), \right.$$
$$\left. \frac{12}{h_{ij}^{x} h_{ij}^{y}} \left(\frac{2\hat{d}_{30}}{h_{ij}^{x2} h_{ij}^{y}} - \frac{6\hat{d}_{20}}{h_{ij}^{x} h_{ij}^{y}} + \frac{3}{2} \right) - \frac{6}{h_{ij}^{x} h_{ij}^{y}} \left(1 - \frac{2\hat{d}_{20}^{ij1}}{h_{i}^{x} h_{j}^{y}} \right)^2 \right)^T.$$

Hence

$$\begin{pmatrix} \gamma_{30}^{ij1} \\ \gamma_{20}^{ij1} \end{pmatrix} = \tau_{xx}^{ij1} \boldsymbol{\eta}_{xx4}^{ij1} \quad \text{for some } \tau_{xx}^{ij1} \geq 0.$$

In summary, $\gamma_{30}^{ij1}, \gamma_{20}^{ij1}$ should be in the form of

$$\begin{pmatrix} \gamma_{30}^{ij1} \\ \gamma_{20}^{ij1} \end{pmatrix} = \lambda_{xx}^{ij1} \boldsymbol{n}_{xx1}^{ij1} + \mu_{xx}^{ij1} \boldsymbol{n}_{xx2}^{ij1} + \nu_{xx}^{ij1} \boldsymbol{n}_{xx3}^{ij1} + \tau_{xx}^{ij1} \boldsymbol{n}_{xx4}^{ij1}$$

$$\text{for some } \lambda_{xx}^{ij1}, \mu_{xx}^{ij1}, \nu_{xx}^{ij1}, \tau_{xx}^{ij1} \geq 0,$$

where $\boldsymbol{n}_{xx1}^{ij1} = \boldsymbol{\eta}_{xx1}^{ij1}$, if $\left(\hat{d}_{30}^{ij1}, \hat{d}_{20}^{ij1}\right)$ is on the curve C_{xx1}^{ij1}, otherwise $\boldsymbol{n}_{xx1}^{ij1} = (0,0)^T$; $\boldsymbol{n}_{xx2}^{ij1} = \boldsymbol{\eta}_{xx2}^{ij1}$, if $\left(\hat{d}_{30}^{ij1}, \hat{d}_{20}^{ij1}\right)$ is on the curve C_{xx2}^{ij1}, otherwise $\boldsymbol{n}_{xx2}^{ij1} = (0,0)^T$; $\boldsymbol{n}_{xx3}^{ij1} = \boldsymbol{\eta}_{xx3}^{ij1}$, if $\left(\hat{d}_{30}^{ij1}, \hat{d}_{20}^{ij1}\right)$ is on the curve C_{xx3}^{ij1}, otherwise $\boldsymbol{n}_{xx3}^{ij1} = (0,0)^T$; and $\boldsymbol{n}_{xx4}^{ij1} = \boldsymbol{\eta}_{xx4}^{ij1}$, if $\left(\hat{d}_{30}^{ij1}, \hat{d}_{20}^{ij1}\right)$ is on the curve C_{xx4}^{ij1}, otherwise $\boldsymbol{n}_{xx4}^{ij1} = (0,0)^T$.

Now let us consider γ_{12}^{ij1}, γ_{03}^{ij1} and γ_{02}^{ij1}. Assume $\mathbf{d}$ is a vector whose elements are the same as the given vector $\hat{\mathbf{d}}$, except for the components d_{12}^{ij1}, d_{03}^{ij1} and d_{02}^{ij1}. Then

$$\left\langle \boldsymbol{\gamma}, \mathbf{d} - \hat{\mathbf{d}} \right\rangle = \gamma_{12}^{ij1}\left(d_{12}^{ij1} - \hat{d}_{12}^{ij1}\right) + \gamma_{03}^{ij1}\left(d_{03}^{ij1} - \hat{d}_{03}^{ij1}\right) + \gamma_{02}^{ij1}\left(d_{02}^{ij1} - \hat{d}_{02}^{ij1}\right),$$
$$\mathfrak{h}(\mathbf{d}) - \mathfrak{h}(\hat{\mathbf{d}}) = 0.$$

Formula (3.20) now becomes

$$\gamma_{12}^{ij1}\left(d_{12}^{ij1} - \hat{d}_{12}^{ij1}\right) + \gamma_{03}^{ij1}\left(d_{03}^{ij1} - \hat{d}_{03}^{ij1}\right) + \gamma_{02}^{ij1}\left(d_{02}^{ij1} - \hat{d}_{02}^{ij1}\right) \le 0, \quad \forall \left(d_{12}^{ij1}, d_{03}^{ij1}, d_{02}^{ij1}\right) \in \Omega_{yy}^{ij1}. \tag{3.23}$$

If we consider the above condition in the three-dimensional $\left(d_{12}^{ij1}, d_{03}^{ij1}, d_{02}^{ij1}\right)$ space, then $\left(\gamma_{12}^{ij1}, \gamma_{03}^{ij1}, \gamma_{02}^{ij1}\right)$ must lie in the normal cone of Ω_{yy}^{ij1} at point $\left(\hat{d}_{12}^{ij1}, \hat{d}_{03}^{ij1}, \hat{d}_{02}^{ij1}\right)$.

Recall that $\left(d_{12}^{ij1}, d_{03}^{ij1}, d_{02}^{ij1}\right)$ is defined on the set

$$\Omega_{yy}^{ij1} = \Bigg\{ \left(d_{12}^{ij1}, d_{03}^{ij1}, d_{02}^{ij1}\right) \in R^3 \Bigg|$$
$$x \le -y + z - 3K + 3\sqrt[3]{(-2K)(y+K)(z-K)},$$
$$x \ge -y + z + 3K - 3\sqrt[3]{(-2K)(y-K)(z+K)},$$
$$y \le -x + z - 3K + 3\sqrt[3]{(-2K)(x+K)(z-K)},$$
$$y \ge -x + z + 3K - 3\sqrt[3]{(-2K)(x-K)(z+K)},$$
$$z \le x + y - 3K + 3\sqrt[3]{2K(x-K)(y-K)},$$
$$z \ge x + y + 3K - 3\sqrt[3]{2K(x+K)(y+K)}\Bigg\},$$

$$\text{where } \begin{cases} x = \frac{d_{03}^{ij1}}{3h_j^y}, \\ y = \frac{d_{12}^{ij1}}{2h_i^x} - \frac{d_{03}^{ij1}}{6h_j^y}, \\ z = \frac{d_{12}^{ij1}}{2h_i^x} + \frac{d_{03}^{ij1}}{6h_j^y} - \frac{d_{02}^{ij1}}{2}, \end{cases}$$

which is a convex set bounded by six cubic curves:

$$C_{yy1}^{ij1}: \qquad x = -y + z - 3K + 3\sqrt[3]{(-2K)(y+K)(z-K)},$$

$$C_{yy2}^{ij1}: \qquad x = -y + z + 3K - 3\sqrt[3]{(-2K)(y-K)(z+K)},$$

$$C_{yy3}^{ij1}: \qquad y = -x + z - 3K + 3\sqrt[3]{(-2K)(x+K)(z-K)},$$

$$C_{yy4}^{ij1}: \qquad y = -x + z + 3K - 3\sqrt[3]{(-2K)(x-K)(z+K)},$$

$$C_{yy5}^{ij1}: \qquad z = x + y - 3K + 3\sqrt[3]{2K(x-K)(y-K)},$$

$$C_{yy6}^{ij1}: \qquad z = x + y + 3K - 3\sqrt[3]{2K(x+K)(y+K)}.$$

If $\left(\hat{d}_{12}^{ij1}, \hat{d}_{03}^{ij1}, \hat{d}_{02}^{ij1}\right)$ is a boundary point on the curve C_{yy1}^{ij1} but not on the curves C_{yy2}^{ij1}, C_{yy3}^{ij1}, C_{yy4}^{ij1}, C_{yy5}^{ij1} and C_{yy6}^{ij1}, then one normal vector of Ω_{yy}^{ij1} is

$$\boldsymbol{\eta}_{yy1}^{ij1} = \begin{pmatrix} \left(\boldsymbol{\eta}_{yy1}^{ij1}\right)_1 \\ \left(\boldsymbol{\eta}_{yy1}^{ij1}\right)_2 \\ \left(\boldsymbol{\eta}_{yy1}^{ij1}\right)_3 \end{pmatrix} = \begin{array}{c} \Big(-\frac{1}{2h_{ij}^x}\left[27(-2K)(z-K) - 3(x+y-z+3K)^2\right] \\ -\frac{1}{2h_{ij}^x}\left[27(-2K)(y+K) + 3(x+y-z+3K)^2\right], \\ \frac{1}{3h_{ij}^y}(x+y-z+3K)^2 \\ +\frac{1}{6h_{ij}^y}\left[27(-2K)(z-K) - 3(x+y-z+3K)^2\right] \\ -\frac{1}{6h_{ij}^y}\left[27(-2K)(y+K) + 3(x+y-z+3K)^2\right], \\ \frac{1}{2}\left[27(-2K)(y+K) + 3(x+y-z+3K)^2\right]\Big)^T. \end{array}$$

Hence

$$\begin{pmatrix} \gamma_{12}^{ij1} \\ \gamma_{03}^{ij1} \\ \gamma_{02}^{ij1} \end{pmatrix} = \lambda_{yy}^{ij1}\boldsymbol{\eta}_{yy1}^{ij1} \quad \text{for some } \lambda_{yy}^{ij1} \geq 0.$$

If $\left(\hat{d}_{12}^{ij1}, \hat{d}_{03}^{ij1}, \hat{d}_{02}^{ij1}\right)$ is a boundary point on the curve C_{yy2}^{ij1} but not on the curves C_{yy1}^{ij1},

C^{ij1}_{yy3}, C^{ij1}_{yy4}, C^{ij1}_{yy5} and C^{ij1}_{yy6}, then one normal vector of Ω^{ij1}_{yy} is

$$\boldsymbol{\eta}^{ij1}_{yy2} = \begin{pmatrix} \left(\boldsymbol{\eta}^{ij1}_{yy2}\right)_1 \\ \left(\boldsymbol{\eta}^{ij1}_{yy2}\right)_2 \\ \left(\boldsymbol{\eta}^{ij1}_{yy2}\right)_3 \end{pmatrix} = \begin{matrix} \Big(-\frac{1}{2h^x_{ij}}\left[27(-2K)(z+K)+3(x+y-z-3K)^2\right] \\ -\frac{1}{2h^x_{ij}}\left[27(-2K)(y-K)-3(x+y-z-3K)^2\right], \\ -\frac{1}{3h^y_{ij}}(x+y-z-3K)^2 \\ +\frac{1}{6h^y_{ij}}\left[27(-2K)(z+K)+3(x+y-z-3K)^2\right] \\ -\frac{1}{6h^y_{ij}}\left[27(-2K)(y-K)-3(x+y-z-3K)^2\right], \\ \frac{1}{2}\left[27(-2K)(y-K)-3(x+y-z-3K)^2\right]\Big)^T. \end{matrix}$$

Hence

$$\begin{pmatrix} \gamma^{ij1}_{12} \\ \gamma^{ij1}_{03} \\ \gamma^{ij1}_{02} \end{pmatrix} = \mu^{ij1}_{yy}\boldsymbol{\eta}^{ij1}_{yy2} \text{ for some } \mu^{ij1}_{yy} \geq 0.$$

If $\left(\hat{d}^{ij1}_{12}, \hat{d}^{ij1}_{03}, \hat{d}^{ij1}_{02}\right)$ is a boundary point on the curve C^{ij1}_{yy3} but not on the curves C^{ij1}_{yy1}, C^{ij1}_{yy2}, C^{ij1}_{yy4}, C^{ij1}_{yy5} and C^{ij1}_{yy6}, then one normal vector of Ω^{ij1}_{yy} is

$$\boldsymbol{\eta}^{ij1}_{yy3} = \begin{pmatrix} \left(\boldsymbol{\eta}^{ij1}_{yy3}\right)_1 \\ \left(\boldsymbol{\eta}^{ij1}_{yy3}\right)_2 \\ \left(\boldsymbol{\eta}^{ij1}_{yy3}\right)_3 \end{pmatrix} = \begin{matrix} \Big(\frac{3}{2h^x_{ij}}(y+x-z+3K)^2 \\ -\frac{1}{2h^x_{ij}}\left[27(-2K)(x+K)+3(y+x-z+3K)^2\right], \\ -\frac{1}{3h^y_{ij}}\left[27(-2K)(z-K)-3(y+x-z+3K)^2\right] \\ -\frac{3}{6h^y_{ij}}(x+y-z+3K)^2 \\ -\frac{1}{6h^y_{ij}}\left[27(-2K)(x+K)+3(y+x-z+3K)^2\right], \\ \frac{1}{2}\left[27(-2K)(x+K)+3(y+x-z+3K)^2\right]\Big)^T. \end{matrix}$$

Hence

$$\begin{pmatrix} \gamma^{ij1}_{12} \\ \gamma^{ij1}_{03} \\ \gamma^{ij1}_{02} \end{pmatrix} = \nu^{ij1}_{yy}\boldsymbol{\eta}^{ij1}_{yy3} \text{ for some } \nu^{ij1}_{yy} \geq 0.$$

If $\left(\hat{d}_{12}^{ij1}, \hat{d}_{03}^{ij1}, \hat{d}_{02}^{ij1}\right)$ is a boundary point on the curve C_{yy4}^{ij1} but not on the curves C_{yy1}^{ij1}, C_{yy2}^{ij1}, C_{yy3}^{ij1}, C_{yy5}^{ij1} and C_{yy6}^{ij1}, then one normal vector of Ω_{yy}^{ij1} is

$$\boldsymbol{\eta}_{yy4}^{ij1} = \begin{pmatrix} \left(\boldsymbol{\eta}_{yy4}^{ij1}\right)_1 \\ \left(\boldsymbol{\eta}_{yy4}^{ij1}\right)_2 \\ \left(\boldsymbol{\eta}_{yy4}^{ij1}\right)_3 \end{pmatrix} = \begin{array}{c} \Big(-\frac{3}{2h_{ij}^x}(y+x-z-3K)^2 \\ -\frac{1}{2h_{ij}^x}\left[27(-2K)(x-K)+3(y+x-z-3K)^2\right], \\ -\frac{1}{3h_{ij}^y}\left[27(-2K)(z+K)-3(y+x-z-3K)^2\right] \\ +\frac{3}{6h_{ij}^y}(x+y-z-3K)^2 \\ -\frac{1}{6h_{ij}^y}\left[27(-2K)(x-K)-3(y+x-z-3K)^2\right], \\ \frac{1}{2}\left[27(-2K)(x-K)-3(y+x-z-3K)^2\right]\Big)^T. \end{array}$$

Hence

$$\begin{pmatrix} \gamma_{12}^{ij1} \\ \gamma_{03}^{ij1} \\ \gamma_{02}^{ij1} \end{pmatrix} = \tau_{yy}^{ij1}\boldsymbol{\eta}_{yy4}^{ij1} \quad \text{for some } \tau_{yy}^{ij1} \geq 0.$$

If $\left(\hat{d}_{12}^{ij1}, \hat{d}_{03}^{ij1}, \hat{d}_{02}^{ij1}\right)$ is a boundary point on the curve C_{yy5}^{ij1} but not on the curves C_{yy1}^{ij1}, C_{yy2}^{ij1}, C_{yy3}^{ij1}, C_{yy4}^{ij1} and C_{yy6}^{ij1}, then one normal vector of Ω_{yy}^{ij1} is

$$\boldsymbol{\eta}_{yy5}^{ij1} = \begin{pmatrix} \left(\boldsymbol{\eta}_{yy5}^{ij1}\right)_1 \\ \left(\boldsymbol{\eta}_{yy5}^{ij1}\right)_2 \\ \left(\boldsymbol{\eta}_{yy5}^{ij1}\right)_3 \end{pmatrix} = \begin{array}{c} \Big(-\frac{1}{2h_{ij}^x}\left[27(2K)(x-K)+3(z-x-y+3K)^2\right] \\ +\frac{3}{2h_{ij}^x}(z-x-y+3K)^2, \\ -\frac{1}{3h_{ij}^y}\left[27(2K)(y-K)+3(z-x-y+3K)^2\right] \\ +\frac{1}{6h_{ij}^y}\left[27(2K)(x-K)+3(z-x-y+3K)^2\right] \\ +\frac{3}{6h_{ij}^y}(z-x-y+3K)^2, \\ -\frac{3}{2}(z-x-y+3K)^2\Big)^T. \end{array}$$

Hence

$$\begin{pmatrix} \gamma_{12}^{ij1} \\ \gamma_{03}^{ij1} \\ \gamma_{02}^{ij1} \end{pmatrix} = \omega_{yy}^{ij1}\boldsymbol{\eta}_{yy5}^{ij1} \quad \text{for some } \omega_{yy}^{ij1} \geq 0.$$

If $\left(\hat{d}_{12}^{ij1}, \hat{d}_{03}^{ij1}, \hat{d}_{02}^{ij1}\right)$ is a boundary point on the curve C_{yy6}^{ij1} but not on the curves C_{yy1}^{ij1}, C_{yy2}^{ij1}, C_{yy3}^{ij1}, C_{yy4}^{ij1} and C_{yy5}^{ij1}, then one normal vector of Ω_{yy}^{ij1} is

$$\boldsymbol{\eta}_{yy6}^{ij1} = \begin{pmatrix} \left(\boldsymbol{\eta}_{yy6}^{ij1}\right)_1 \\ \left(\boldsymbol{\eta}_{yy6}^{ij1}\right)_2 \\ \left(\boldsymbol{\eta}_{yy6}^{ij1}\right)_3 \end{pmatrix} = \begin{matrix} \Big(-\frac{1}{2h_{ij}^x}\left[27(2K)(x+K) - 3(z-x-y-3K)^2\right] \\ -\frac{3}{2h_{ij}^x}(z-x-y-3K)^2, \\ -\frac{1}{3h_{ij}^y}\left[27(2K)(y+K) - 3(z-x-y-3K)^2\right] \\ +\frac{1}{6h_{ij}^y}\left[27(2K)(x+K) - 3(z-x-y-3K)^2\right] \\ -\frac{3}{6h_{ij}^y}(z-x-y-3K)^2, \\ \frac{3}{2}(z-x-y-3K)^2\Big)^T. \end{matrix}$$

Hence

$$\begin{pmatrix} \gamma_{12}^{ij1} \\ \gamma_{03}^{ij1} \\ \gamma_{02}^{ij1} \end{pmatrix} = \psi_{yy}^{ij1}\boldsymbol{\eta}_{yy6}^{ij1} \quad \text{for some } \psi_{yy}^{ij1} \geq 0.$$

In summary, γ_{12}^{ij1}, γ_{03}^{ij1} and γ_{02}^{ij1} should be in the form of

$$\begin{pmatrix} \gamma_{12}^{ij1} \\ \gamma_{03}^{ij1} \\ \gamma_{02}^{ij1} \end{pmatrix} = \lambda_{yy}^{ij1}\boldsymbol{n}_{yy1}^{ij1} + \mu_{yy}^{ij1}\boldsymbol{n}_{yy2}^{ij1} + \nu_{yy}^{ij1}\boldsymbol{n}_{yy3}^{ij1} + \tau_{yy}^{ij1}\boldsymbol{n}_{yy4}^{ij1} + \omega_{yy}^{ij1}\boldsymbol{n}_{yy5}^{ij1} + \psi_{yy}^{ij1}\boldsymbol{n}_{yy6}^{ij1}$$

$$\text{for some } \lambda_{yy}^{ij1}, \mu_{yy}^{ij1}, \nu_{yy}^{ij1}, \tau_{yy}^{ij1}, \omega_{yy}^{ij1}, \psi_{yy}^{ij1} \geq 0,$$

where $\boldsymbol{n}_{yy1}^{ij1} = \boldsymbol{\eta}_{yy1}^{ij1}$, if $\left(\hat{d}_{12}^{ij1}, \hat{d}_{03}^{ij1}, \hat{d}_{02}^{ij1}\right)$ is on the curve C_{yy1}^{ij1}, otherwise $\boldsymbol{n}_{yy1}^{ij1} = (0,0)^T$; $\boldsymbol{n}_{yy2}^{ij1} = \boldsymbol{\eta}_{yy2}^{ij1}$, if $\left(\hat{d}_{12}^{ij1}, \hat{d}_{03}^{ij1}, \hat{d}_{02}^{ij1}\right)$ is on the curve C_{yy2}^{ij1}, otherwise $\boldsymbol{n}_{yy2}^{ij1} = (0,0)^T$; $\boldsymbol{n}_{yy3}^{ij1} = \boldsymbol{\eta}_{yy3}^{ij1}$, if $\left(\hat{d}_{12}^{ij1}, \hat{d}_{03}^{ij1}, \hat{d}_{02}^{ij1}\right)$ is on the curve C_{yy3}^{ij1}, otherwise $\boldsymbol{n}_{yy3}^{ij1} = (0,0)^T$; $\boldsymbol{n}_{yy4}^{ij1} = \boldsymbol{\eta}_{yy4}^{ij1}$, if $\left(\hat{d}_{12}^{ij1}, \hat{d}_{03}^{ij1}, \hat{d}_{02}^{ij1}\right)$ is on the curve C_{yy4}^{ij1}, otherwise $\boldsymbol{n}_{yy4}^{ij1} = (0,0)^T$; $\boldsymbol{n}_{yy4}^{ij1} = \boldsymbol{\eta}_{yy5}^{ij1}$, if $\left(\hat{d}_{12}^{ij1}, \hat{d}_{03}^{ij1}, \hat{d}_{02}^{ij1}\right)$ is on the curve C_{yy5}^{ij1}, otherwise $\boldsymbol{n}_{yy5}^{ij1} = (0,0)^T$; and $\boldsymbol{n}_{yy4}^{ij1} = \boldsymbol{\eta}_{yy6}^{ij1}$, if $\left(\hat{d}_{12}^{ij1}, \hat{d}_{03}^{ij1}, \hat{d}_{02}^{ij1}\right)$ is on the curve C_{yy6}^{ij1}, otherwise $\boldsymbol{n}_{yy6}^{ij1} = (0,0)^T$.

Let us consider $\tilde{\gamma}_{12}^{ij1}$ and γ_{11}^{ij1}. Assume $\mathbf{d}$ is a vector whose elements are the same as the given vector $\hat{\mathbf{d}}$, except for the components $\tilde{d}_{12}^{ij1}$ and d_{11}^{ij1}. Then

$$\begin{aligned}\left\langle \boldsymbol{\gamma}, \mathbf{d}-\hat{\mathbf{d}} \right\rangle &= \tilde{\gamma}_{12}^{ij1}\left(\tilde{d}_{12}^{ij1}-\hat{\tilde{d}}_{12}^{ij1}\right)+\gamma_{11}^{ij1}\left(d_{11}^{ij1}-\hat{d}_{11}^{ij1}\right),\\ \mathfrak{h}(\mathbf{d})-\mathfrak{h}(\hat{\mathbf{d}}) &= 0.\end{aligned}$$

Formula (3.20) now becomes

$$\tilde{\gamma}_{12}^{ij1}\left(\tilde{d}_{12}^{ij1}-\hat{\tilde{d}}_{12}^{ij1}\right)+\gamma_{11}^{ij1}\left(d_{11}^{ij1}-\hat{d}_{11}^{ij1}\right)\le 0,\quad \forall\left(\tilde{d}_{12}^{ij1}, d_{11}^{ij1}\right)\in\Omega_{xy}^{ij1}. \tag{3.24}$$

If we consider the above condition in the two-dimensional $\left(\tilde{d}_{12}^{ij1}, d_{11}^{ij1}\right)$ space, then $\left(\tilde{\gamma}_{12}^{ij1}, \gamma_{11}^{ij1}\right)$ must lie in the normal cone of Ω_{xx}^{ij1} at point $\left(\hat{\tilde{d}}_{12}^{ij1}, \hat{d}_{11}^{ij1}\right)$.

Recall that $\left(\tilde{d}_{12}^{ij1}, d_{11}^{ij1}\right)$ is defined on the set

$$\begin{aligned}\Omega_{xy}^{ij1} = \Bigg\{ \left(\tilde{d}_{12}^{ij1}, d_{11}^{ij1}\right)\in R^3 \Bigg| &\left(\frac{1}{2}+\frac{4d_{11}^{ij1}}{h_i^x h_j^y}\right)^3 \le \left(-\frac{6\tilde{d}_{12}^{ij1}}{h_i^x h_j^{y2}}+\frac{6d_{11}^{ij1}}{h_i^x h_j^y}+\frac{1}{2}\right)^2,\\ &\left(\frac{1}{2}-\frac{4d_{11}^{ij1}}{h_i^x h_j^y}\right)^3 \le \left(\frac{6\tilde{d}_{12}^{ij1}}{h_i^x h_j^{y2}}-\frac{6d_{11}^{ij1}}{h_i^x h_j^y}+\frac{1}{2}\right)^2\Bigg\},\end{aligned}$$

which is a convex set bounded by two cubic curvers:

$$\begin{aligned} C_{xy1}^{ij1}: &\quad \left(\frac{1}{2}+\frac{4d_{11}^{ij1}}{h_i^x h_j^y}\right)^3 = \left(-\frac{6\tilde{d}_{12}^{ij1}}{h_i^x h_j^{y2}}+\frac{6d_{11}^{ij1}}{h_i^x h_j^y}+\frac{1}{2}\right)^2,\\ C_{xy2}^{ij1}: &\quad \left(\frac{1}{2}-\frac{4d_{11}^{ij1}}{h_i^x h_j^y}\right)^3 = \left(\frac{6\tilde{d}_{12}^{ij1}}{h_i^x h_j^{y2}}-\frac{6d_{11}^{ij1}}{h_i^x h_j^y}+\frac{1}{2}\right)^2.\end{aligned}$$

If $\left(\hat{\tilde{d}}_{12}^{ij1}, \hat{d}_{11}^{ij1}\right)$ is an interior point of Ω_{xy}^{ij1}, then obviously

$$\left(\tilde{\gamma}_{12}^{ij1}, \gamma_{11}^{ij1}\right) = (0,0).$$

If $\left(\hat{\tilde{d}}_{12}^{ij1}, \hat{d}_{11}^{ij1}\right)$ is a boundary point on the curve C_{xy1}^{ij1} but not on the curve C_{xy2}^{ij1}, then one normal vector of Ω_{xy}^{ij1} is

$$\begin{aligned}\boldsymbol{\eta}_{xy1}^{ij1} = \Bigg(&\frac{12}{h_{ij}^x h_{ij}^{y2}}\left(-\frac{6\hat{\tilde{d}}_{12}^{ij1}}{h_i^x h_j^{y2}}+\frac{6\hat{d}_{11}^{ij1}}{h_i^x h_j^y}+\frac{1}{2}\right),\\ &-\frac{12}{h_{ij}^x h_{ij}^y}\left(-\frac{6\hat{\tilde{d}}_{12}^{ij1}}{h_i^x h_j^{y2}}+\frac{6\hat{d}_{11}^{ij1}}{h_i^x h_j^y}+\frac{1}{2}\right)+\frac{12}{h_{ij}^x h_{ij}^y}\left(\frac{1}{2}+\frac{4\hat{d}_{11}^{ij1}}{h_i^x h_j^y}\right)^2\Bigg)^T.\end{aligned}$$

Hence

$$\begin{pmatrix} \tilde{\gamma}_{12}^{ij1} \\ \\ \gamma_{11}^{ij1} \end{pmatrix} = \boldsymbol{\eta}_{xy1}^{ij1} \;\; \text{for some } \lambda_{xy}^{ij1} \geq 0.$$

If $\left(\hat{\tilde{d}}_{12}^{ij1}, \hat{d}_{11}^{ij1}\right)$ is a boundary point on the curve C_{xy2}^{ij1} but not on the curve C_{xy1}^{ij1}, then one normal vector of Ω_{xy}^{ij1} is

$$\boldsymbol{\eta}_{xy2}^{ij1} = \Bigg(-\frac{12}{h_{ij}^{x} h_{ij}^{y2}} \left(\frac{6\hat{\tilde{d}}_{12}^{ij1}}{h_i^x h_j^{y2}} - \frac{6\hat{d}_{11}^{ij1}}{h_i^x h_j^y} + \frac{1}{2} \right),$$
$$\frac{12}{h_{ij}^{x} h_{ij}^{y}} \left(\frac{6\hat{\tilde{d}}_{12}^{ij1}}{h_i^x h_j^{y2}} - \frac{6\hat{d}_{11}^{ij1}}{h_i^x h_j^y} + \frac{1}{2} \right) - \frac{12}{h_{ij}^{x} h_{ij}^{y}} \left(\frac{1}{2} - \frac{4\hat{d}_{11}^{ij1}}{h_i^x h_j^y} \right)^2 \Bigg)^T.$$

Hence

$$\begin{pmatrix} \tilde{\gamma}_{12}^{ij1} \\ \\ \gamma_{11}^{ij1} \end{pmatrix} = \boldsymbol{\eta}_{xy2}^{ij1} \;\; \text{for some } \mu_{xy}^{ij1} \geq 0.$$

In summary, $\tilde{\gamma}_{12}^{ij1}, \gamma_{11}^{ij1}$ should be in the form of

$$\begin{pmatrix} \tilde{\gamma}_{12}^{ij1} \\ \\ \gamma_{11}^{ij1} \end{pmatrix} = \lambda_{xy}^{ij1} \boldsymbol{n}_{xy1}^{ij1} + \mu_{xy}^{ij1} \boldsymbol{n}_{xy2}^{ij1} \;\; \text{for some } \lambda_{xy}^{ij1}, \mu_{xy}^{ij1} \geq 0,$$

where $\boldsymbol{n}_{xy1}^{ij1} = \boldsymbol{\eta}_{xy1}^{ij1}$, if $\left(\hat{\tilde{d}}_{12}^{ij1}, \hat{d}_{11}^{ij1}\right)$ is on the curve C_{xy1}^{ij1}, otherwise $\boldsymbol{n}_{xy1}^{ij1} = (0,0)^T$; and $\boldsymbol{n}_{xy2}^{ij1} = \boldsymbol{\eta}_{xy2}^{ij1}$, if $\left(\hat{\tilde{d}}_{12}^{ij1}, \hat{d}_{11}^{ij1}\right)$ is on the curve C_{xy2}^{ij1}, otherwise $\boldsymbol{n}_{xy2}^{ij1} = (0,0)^T$.

Based on the similar argument, let

$$\boldsymbol{\eta}_{xx1}^{ij2} = \begin{pmatrix} \left(\boldsymbol{\eta}_{xx1}^{ij2}\right)_1 \\ \left(\boldsymbol{\eta}_{xx1}^{ij2}\right)_2 \\ \left(\boldsymbol{\eta}_{xx1}^{ij2}\right)_3 \end{pmatrix} = \begin{matrix} \Big(-\frac{1}{2h_{ij}^y}\left[27(-2K)(z-K) - 3(x+y-z+3K)^2\right] \\ -\frac{1}{2h_{ij}^y}\left[27(-2K)(y+K) + 3(x+y-z+3K)^2\right], \\ \frac{1}{3h_{ij}^x}(x+y-z+3K)^2 \\ +\frac{1}{6h_{ij}^x}\left[27(-2K)(z-K) - 3(x+y-z+3K)^2\right] \\ -\frac{1}{6h_{ij}^x}\left[27(-2K)(y+K) + 3(x+y-z+3K)^2\right], \\ -\frac{1}{2}\left[27(-2K)(z-K) - 3(x+y-z+3K)^2\right]\Big)^T, \end{matrix}$$

$$\boldsymbol{\eta}_{xx2}^{ij2} = \begin{pmatrix} \left(\boldsymbol{\eta}_{xx2}^{ij2}\right)_1 \\ \left(\boldsymbol{\eta}_{xx2}^{ij2}\right)_2 \\ \left(\boldsymbol{\eta}_{xx2}^{ij2}\right)_3 \end{pmatrix} = \begin{matrix} \Big(-\frac{1}{2h_{ij}^y}\left[27(-2K)(z+K) + 3(x+y-z-3K)^2\right] \\ -\frac{1}{2h_{ij}^y}\left[27(-2K)(y-K) - 3(x+y-z-3K)^2\right], \\ -\frac{1}{3h_{ij}^x}(x+y-z-3K)^2 \\ +\frac{1}{6h_{ij}^x}\left[27(-2K)(z+K) + 3(x+y-z-3K)^2\right] \\ -\frac{1}{6h_{ij}^x}\left[27(-2K)(y-K) - 3(x+y-z-3K)^2\right], \\ -\frac{1}{2}\left[27(-2K)(z+K) + 3(x+y-z-3K)^2\right]\Big)^T, \end{matrix}$$

$$\boldsymbol{\eta}_{xx3}^{ij2} = \begin{pmatrix} \left(\boldsymbol{\eta}_{xx3}^{ij2}\right)_1 \\ \left(\boldsymbol{\eta}_{xx3}^{ij2}\right)_2 \\ \left(\boldsymbol{\eta}_{xx3}^{ij2}\right)_3 \end{pmatrix} = \begin{matrix} \Big(\frac{3}{2h_{ij}^y}(y+x-z+3K)^2 \\ -\frac{1}{2h_{ij}^y}\left[27(-2K)(x+K) + 3(y+x-z+3K)^2\right], \\ -\frac{1}{3h_{ij}^x}\left[27(-2K)(z-K) - 3(y+x-z+3K)^2\right] \\ -\frac{3}{6h_{ij}^x}(x+y-z+3K)^2 \\ -\frac{1}{6h_{ij}^x}\left[27(-2K)(x+K) + 3(y+x-z+3K)^2\right], \\ \frac{3}{2}(y+x-z+3K)^2\Big)^T, \end{matrix}$$

$$\boldsymbol{\eta}_{xx4}^{ij2} = \begin{pmatrix} \left(\boldsymbol{\eta}_{xx4}^{ij2}\right)_1 \\ \left(\boldsymbol{\eta}_{xx4}^{ij2}\right)_2 \\ \left(\boldsymbol{\eta}_{xx4}^{ij2}\right)_3 \end{pmatrix} = \begin{matrix} \Big(-\frac{3}{2h_{ij}^y}(y+x-z-3K)^2 \\ -\frac{1}{2h_{ij}^y}\left[27(-2K)(x-K)+3(y+x-z-3K)^2\right], \\ -\frac{1}{3h_{ij}^x}\left[27(-2K)(z+K)-3(y+x-z-3K)^2\right] \\ +\frac{3}{6h_{ij}^x}(x+y-z-3K)^2 \\ -\frac{1}{6h_{ij}^x}\left[27(-2K)(x-K)-3(y+x-z-3K)^2\right], \\ -\frac{3}{2}(y+x-z-3K)^2\Big)^T, \end{matrix}$$

$$\boldsymbol{\eta}_{xx5}^{ij2} = \begin{pmatrix} \left(\boldsymbol{\eta}_{xx5}^{ij2}\right)_1 \\ \left(\boldsymbol{\eta}_{xx5}^{ij2}\right)_2 \\ \left(\boldsymbol{\eta}_{xx5}^{ij2}\right)_3 \end{pmatrix} = \begin{matrix} \Big(-\frac{1}{2h_{ij}^y}\left[27(2K)(x-K)+3(z-x-y+3K)^2\right] \\ +\frac{3}{2h_{ij}^y}(z-x-y+3K)^2, \\ -\frac{1}{3h_{ij}^x}\left[27(2K)(y-K)+3(z-x-y+3K)^2\right] \\ +\frac{1}{6h_{ij}^x}\left[27(2K)(x-K)+3(z-x-y+3K)^2\right] \\ +\frac{3}{6h_{ij}^x}(z-x-y+3K)^2, \\ -\frac{1}{2}\left[27(2K)(x-K)+3(z-x-y+3K)^2\right]\Big)^T, \end{matrix}$$

$$\boldsymbol{\eta}_{xx6}^{ij2} = \begin{pmatrix} \left(\boldsymbol{\eta}_{xx6}^{ij2}\right)_1 \\ \left(\boldsymbol{\eta}_{xx6}^{ij2}\right)_2 \\ \left(\boldsymbol{\eta}_{xx6}^{ij2}\right)_3 \end{pmatrix} = \begin{matrix} \Big(-\frac{1}{2h_{ij}^y}\left[27(2K)(x+K)-3(z-x-y-3K)^2\right] \\ -\frac{3}{2h_{ij}^y}(z-x-y-3K)^2, \\ -\frac{1}{3h_{ij}^x}\left[27(2K)(y+K)-3(z-x-y-3K)^2\right] \\ +\frac{1}{6h_{ij}^x}\left[27(2K)(x+K)-3(z-x-y-3K)^2\right] \\ -\frac{3}{6h_{ij}^x}(z-x-y-3K)^2, \\ -\frac{1}{2}\left[27(2K)(x+K)-3(z-x-y-3K)^2\right]\Big)^T. \end{matrix}$$

Then, $\gamma_{21}^{ij2}\gamma_{30}^{ij2}\gamma_{20}^{ij2}$ should be in the form of

$$\begin{pmatrix} \gamma_{21}^{ij2} \\ \gamma_{30}^{ij2} \\ \gamma_{20}^{ij2} \end{pmatrix} = \lambda_{xx}^{ij2}\boldsymbol{n}_{xx1}^{ij2} + \mu_{xx}^{ij2}\boldsymbol{n}_{xx2}^{ij2} + \nu_{xx}^{ij2}\boldsymbol{n}_{xx3}^{ij2} + \tau_{xx}^{ij2}\boldsymbol{n}_{xx4}^{ij2} + \omega_{xx}^{ij2}\boldsymbol{n}_{xx5}^{ij2} + \psi_{xx}^{ij2}\boldsymbol{n}_{xx6}^{ij2}$$

$$\text{for some } \lambda_{xx}^{ij2}, \mu_{xx}^{ij2}, \nu_{xx}^{ij2}, \tau_{xx}^{ij2}, \omega_{xx}^{ij2}, \psi_{xx}^{ij2} \geq 0,$$

where $\boldsymbol{n}_{xx1}^{ij2} = \boldsymbol{\eta}_{xx1}^{ij2}$, if $\left(\hat{d}_{21}^{ij2}, \hat{d}_{30}^{ij2}, \hat{d}_{20}^{ij2}\right)$ is on the curve C_{xx1}^{ij2}, otherwise $\boldsymbol{n}_{xx1}^{ij2} = (0,0)^T$; $\boldsymbol{n}_{xx2}^{ij2} = \boldsymbol{\eta}_{xx2}^{ij2}$, if $\left(\hat{d}_{21}^{ij2}, \hat{d}_{30}^{ij2}, \hat{d}_{20}^{ij2}\right)$ is on the curve C_{xx2}^{ij2}, otherwise $\boldsymbol{n}_{xx2}^{ij2} = (0,0)^T$; $\boldsymbol{n}_{xx3}^{ij2} = \boldsymbol{\eta}_{xx3}^{ij2}$, if $\left(\hat{d}_{21}^{ij2}, \hat{d}_{30}^{ij2}, \hat{d}_{20}^{ij2}\right)$ is on the curve C_{xx3}^{ij2}, otherwise $\boldsymbol{n}_{xx3}^{ij2} = (0,0)^T$; $\boldsymbol{n}_{xx4}^{ij2} = \boldsymbol{\eta}_{xx4}^{ij2}$, if $\left(\hat{d}_{21}^{ij2}, \hat{d}_{30}^{ij2}, \hat{d}_{20}^{ij2}\right)$ is on the curve C_{xx4}^{ij2}, otherwise $\boldsymbol{n}_{xx4}^{ij2} = (0,0)^T$; $\boldsymbol{n}_{xx4}^{ij2} = \boldsymbol{\eta}_{xx5}^{ij2}$, if $\left(\hat{d}_{21}^{ij2}, \hat{d}_{30}^{ij2}, \hat{d}_{20}^{ij2}\right)$ is on the curve C_{xx5}^{ij2}, otherwise $\boldsymbol{n}_{xx5}^{ij2} = (0,0)^T$; and $\boldsymbol{n}_{xx4}^{ij2} = \boldsymbol{\eta}_{xx6}^{ij2}$, if $\left(\hat{d}_{21}^{ij2}, \hat{d}_{30}^{ij2}, \hat{d}_{20}^{ij2}\right)$ is on the curve C_{xx6}^{ij2}, otherwise $\boldsymbol{n}_{xx6}^{ij2} = (0,0)^T$.

Let

$$\boldsymbol{\eta}_{yy1}^{ij2} = \left(-\frac{4}{h_{ij}^x h_{ij}^{y2}}\left(\frac{3}{2} + \frac{2\hat{d}_{03}^{ij2}}{h_i^x h_j^{y2}}\right), -\frac{6}{h_{ij}^x h_{ij}^y}\left(1 - \frac{2\hat{d}_{02}^{ij2}}{h_i^x h_j^y}\right)^2\right)^T,$$

$$\begin{aligned}\boldsymbol{\eta}_{yy2}^{ij2} = \Bigg(&\frac{4}{h_{ij}^x h_{ij}^{y2}}\left(-\frac{2\hat{d}_{03}}{h_{ij}^x h_{ij}^{y2}} - \frac{6\hat{d}_{02}}{h_{ij}^x h_{ij}^y} - \frac{3}{2}\right),\\ &\frac{12}{h_{ij}^x h_{ij}^y}\left(-\frac{2\hat{d}_{03}}{h_{ij}^x h_{ij}^{y2}} - \frac{6\hat{d}_{02}}{h_{ij}^x h_{ij}^y} - \frac{3}{2}\right) + \frac{6}{h_{ij}^x h_{ij}^y}\left(1 + \frac{2\hat{d}_{02}^{ij2}}{h_i^x h_j^y}\right)^2\Bigg)^T,\end{aligned}$$

$$\boldsymbol{\eta}_{yy3}^{ij2} = \left(\frac{4}{h_{ij}^x h_{ij}^{y2}}\left(\frac{3}{2} - \frac{2\hat{d}_{03}^{ij2}}{h_i^x h_j^{y2}}\right), \frac{6}{h_{ij}^x h_{ij}^y}\left(1 + \frac{2\hat{d}_{02}^{ij2}}{h_i^x h_j^y}\right)^2\right)^T,$$

$$\begin{aligned}\boldsymbol{\eta}_{yy4}^{ij2} = \Bigg(&\frac{4}{h_{ij}^x h_{ij}^{y2}}\left(-\frac{2\hat{d}_{03}}{h_{ij}^x h_{ij}^{y2}} - \frac{6\hat{d}_{02}}{h_{ij}^x h_{ij}^y} + \frac{3}{2}\right),\\ &\frac{12}{h_{ij}^x h_{ij}^y}\left(-\frac{2\hat{d}_{03}}{h_{ij}^x h_{ij}^{y2}} - \frac{6\hat{d}_{02}}{h_{ij}^x h_{ij}^y} + \frac{3}{2}\right) - \frac{6}{h_{ij}^x h_{ij}^y}\left(1 - \frac{2\hat{d}_{02}^{ij2}}{h_i^x h_j^y}\right)^2\Bigg)^T.\end{aligned}$$

Then, $\gamma_{03}^{ij2}, \gamma_{02}^{ij2}$ should be in the form of

$$\begin{pmatrix}\gamma_{03}^{ij2}\\ \\ \gamma_{02}^{ij2}\end{pmatrix} = \lambda_{yy}^{ij2}\boldsymbol{n}_{yy1}^{ij2} + \mu_{yy}^{ij2}\boldsymbol{n}_{yy2}^{ij2} + \nu_{yy}^{ij2}\boldsymbol{n}_{yy3}^{ij2} + \tau_{yy}^{ij2}\boldsymbol{n}_{yy4}^{ij2} \quad \text{for some } \lambda_{yy}^{ij2}, \mu_{yy}^{ij2}, \nu_{yy}^{ij2}, \tau_{yy}^{ij2} \geq 0,$$

where $\boldsymbol{n}_{yy1}^{ij2} = \boldsymbol{\eta}_{yy1}^{ij2}$, if $\left(\hat{d}_{03}^{ij2}, \hat{d}_{02}^{ij2}\right)$ is on the curve C_{yy1}^{ij2}, otherwise $\boldsymbol{n}_{yy1}^{ij2} = (0,0)^T$; $\boldsymbol{n}_{yy2}^{ij2} = \boldsymbol{\eta}_{yy2}^{ij2}$, if $\left(\hat{d}_{03}^{ij2}, \hat{d}_{02}^{ij2}\right)$ is on the curve C_{yy2}^{ij2}, otherwise $\boldsymbol{n}_{yy2}^{ij2} = (0,0)^T$; $\boldsymbol{n}_{yy3}^{ij2} = \boldsymbol{\eta}_{yy3}^{ij2}$, if $\left(\hat{d}_{03}^{ij2}, \hat{d}_{02}^{ij2}\right)$ is on the curve C_{yy3}^{ij2}, otherwise $\boldsymbol{n}_{yy3}^{ij2} = (0,0)^T$ and $\boldsymbol{n}_{yy4}^{ij2} = \boldsymbol{\eta}_{yy4}^{ij2}$; if $\left(\hat{d}_{03}^{ij2}, \hat{d}_{02}^{ij2}\right)$ is on the curve C_{yy4}^{ij2}, otherwise $\boldsymbol{n}_{yy4}^{ij2} = (0,0)^T$.

Let

$$\boldsymbol{\eta}_{xy1}^{ij2} = \Bigg(\frac{12}{h_{ij}^{x2} h_{ij}^{y}} \left(-\frac{6\hat{\tilde{d}}_{21}^{ij2}}{h_i^{x2} h_j^{y}} + \frac{6\hat{d}_{11}^{ij2}}{h_i^{x} h_j^{y}} + \frac{1}{2} \right), \\ -\frac{12}{h_{ij}^{x} h_{ij}^{y}} \left(-\frac{6\hat{\tilde{d}}_{21}^{ij2}}{h_i^{x2} h_j^{y}} + \frac{6\hat{d}_{11}^{ij2}}{h_i^{x} h_j^{y}} + \frac{1}{2} \right) + \frac{12}{h_{ij}^{x} h_{ij}^{y}} \left(\frac{1}{2} + \frac{4\hat{d}_{11}^{ij2}}{h_i^{x} h_j^{y}} \right)^2 \Bigg)^T,$$

and

$$\boldsymbol{\eta}_{xy2}^{ij2} = \Bigg(-\frac{12}{h_{ij}^{x2} h_{ij}^{y}} \left(\frac{6\hat{\tilde{d}}_{21}^{ij2}}{h_i^{x2} h_j^{y}} - \frac{6\hat{d}_{11}^{ij2}}{h_i^{x} h_j^{y}} + \frac{1}{2} \right), \\ \frac{12}{h_{ij}^{x} h_{ij}^{y}} \left(\frac{6\hat{\tilde{d}}_{21}^{ij2}}{h_i^{x2} h_j^{y}} - \frac{6\hat{d}_{11}^{ij2}}{h_i^{x} h_j^{y}} + \frac{1}{2} \right) - \frac{12}{h_{ij}^{x} h_{ij}^{y}} \left(\frac{1}{2} - \frac{4\hat{d}_{11}^{ij2}}{h_i^{x} h_j^{y}} \right)^2 \Bigg)^T.$$

Then, $\tilde{\gamma}_{21}^{ij2}, \gamma_{11}^{ij2}$ should be in the form of

$$\begin{pmatrix} \tilde{\gamma}_{21}^{ij2} \\ \\ \gamma_{11}^{ij2} \end{pmatrix} = \lambda_{xy}^{ij2} \boldsymbol{n}_{xy1}^{ij2} + \mu_{xy}^{ij2} \boldsymbol{n}_{xy2}^{ij2} \quad \text{for some } \lambda_{xy}^{ij2} \mu_{xy}^{ij2} \geq 0,$$

where $\boldsymbol{n}_{xy1}^{ij2} = \boldsymbol{\eta}_{xy1}^{ij2}$, if $\left(\hat{\tilde{d}}_{21}^{ij2}, \hat{d}_{11}^{ij2}\right)$ is on the curve C_{xy1}^{ij2}, otherwise $\boldsymbol{n}_{xy1}^{ij2} = (0,0)^T$; and $\boldsymbol{n}_{xy2}^{ij2} = \boldsymbol{\eta}_{xy2}^{ij2}$, if $\left(\hat{\tilde{d}}_{21}^{ij2}, \hat{d}_{11}^{ij2}\right)$ is on the curve C_{xy2}^{ij2}, otherwise $\boldsymbol{n}_{xy2}^{ij2} = (0,0)^T$.

Let

$$\boldsymbol{\eta}_{xx1}^{ij3} = \left(-\frac{4}{h_{ij}^{x2} h_{ij}^{y}} \left(\frac{3}{2} + \frac{2\hat{d}_{30}^{ij3}}{h_i^{x2} h_j^{y}} \right), -\frac{6}{h_{ij}^{x} h_{ij}^{y}} \left(1 - \frac{2\hat{d}_{20}^{ij3}}{h_i^{x} h_j^{y}} \right)^2 \right)^T,$$

$$\boldsymbol{\eta}_{xx2}^{ij3} = \Bigg(\frac{4}{h_{ij}^{x2} h_{ij}^{y}} \left(-\frac{2\hat{d}_{30}}{h_{ij}^{x2} h_{ij}^{y}} - \frac{6\hat{d}_{20}}{h_{ij}^{x} h_{ij}^{y}} - \frac{3}{2} \right), \\ \frac{12}{h_{ij}^{x} h_{ij}^{y}} \left(-\frac{2\hat{d}_{30}}{h_{ij}^{x2} h_{ij}^{y}} - \frac{6\hat{d}_{20}}{h_{ij}^{x} h_{ij}^{y}} - \frac{3}{2} \right) + \frac{6}{h_{ij}^{x} h_{ij}^{y}} \left(1 + \frac{2\hat{d}_{20}^{ij3}}{h_i^{x} h_j^{y}} \right)^2 \Bigg)^T,$$

$$\boldsymbol{\eta}_{xx3}^{ij3} = \left(\frac{4}{h_{ij}^{x2} h_{ij}^{y}} \left(\frac{3}{2} - \frac{2\hat{d}_{30}^{ij3}}{h_{i}^{x2} h_{j}^{y}} \right), \frac{6}{h_{ij}^{x} h_{ij}^{y}} \left(1 + \frac{2\hat{d}_{20}^{ij3}}{h_{i}^{x} h_{j}^{y}} \right)^2 \right)^T,$$

$$\boldsymbol{\eta}_{xx4}^{ij3} = \left(\frac{4}{h_{ij}^{x2} h_{ij}^{y}} \left(-\frac{2\hat{d}_{30}}{h_{ij}^{x2} h_{ij}^{y}} - \frac{6\hat{d}_{20}}{h_{ij}^{x} h_{ij}^{y}} + \frac{3}{2} \right), \right.$$
$$\left. \frac{12}{h_{ij}^{x} h_{ij}^{y}} \left(-\frac{2\hat{d}_{30}}{h_{ij}^{x2} h_{ij}^{y}} - \frac{6\hat{d}_{20}}{h_{ij}^{x} h_{ij}^{y}} + \frac{3}{2} \right) - \frac{6}{h_{ij}^{x} h_{ij}^{y}} \left(1 - \frac{2\hat{d}_{20}^{ij3}}{h_{i}^{x} h_{j}^{y}} \right)^2 \right)^T.$$

Then, $\gamma_{30}^{ij3}, \gamma_{20}^{ij3}$ should be in the form of

$$\begin{pmatrix} \gamma_{30}^{ij3} \\ \gamma_{20}^{ij3} \end{pmatrix} = \lambda_{xx}^{ij3} \boldsymbol{n}_{xx1}^{ij3} + \mu_{xx}^{ij3} \boldsymbol{n}_{xx2}^{ij3} + \nu_{xx}^{ij3} \boldsymbol{n}_{xx3}^{ij3} + \tau_{xx}^{ij3} \boldsymbol{n}_{xx4}^{ij3} \quad \text{for some } \lambda_{xx}^{ij3}, \mu_{xx}^{ij3}, \nu_{xx}^{ij3}, \tau_{xx}^{ij3} \geq 0,$$

where $\boldsymbol{n}_{xx1}^{ij3} = \boldsymbol{\eta}_{xx1}^{ij3}$, if $\left(\hat{d}_{30}^{ij3}, \hat{d}_{20}^{ij3}\right)$ is on the curve C_{xx1}^{ij3}, otherwise $\boldsymbol{n}_{xx1}^{ij3} = (0,0)^T$; $\boldsymbol{n}_{xx2}^{ij3} = \boldsymbol{\eta}_{xx2}^{ij3}$, if $\left(\hat{d}_{30}^{ij3}, \hat{d}_{20}^{ij3}\right)$ is on the curve C_{xx2}^{ij3}, otherwise $\boldsymbol{n}_{xx2}^{ij3} = (0,0)^T$; $\boldsymbol{n}_{xx3}^{ij3} = \boldsymbol{\eta}_{xx3}^{ij3}$, if $\left(\hat{d}_{30}^{ij3}, \hat{d}_{20}^{ij3}\right)$ is on the curve C_{xx3}^{ij3}, otherwise $\boldsymbol{n}_{xx3}^{ij3} = (0,0)^T$; and $\boldsymbol{n}_{xx4}^{ij3} = \boldsymbol{\eta}_{xx4}^{ij3}$, if $\left(\hat{d}_{30}^{ij3}, \hat{d}_{20}^{ij3}\right)$ is on the curve C_{xx4}^{ij3}, otherwise $\boldsymbol{n}_{xx4}^{ij3} = (0,0)^T$.

Let

$$\boldsymbol{\eta}_{yy1}^{ij3} = \begin{pmatrix} \left(\boldsymbol{\eta}_{yy1}^{ij3}\right)_1 \\ \left(\boldsymbol{\eta}_{yy1}^{ij3}\right)_2 \\ \left(\boldsymbol{\eta}_{yy1}^{ij3}\right)_3 \end{pmatrix} = \begin{matrix} \Big(\frac{1}{2h_{ij}^{x}} \left[27(-2K)(z-K) - 3(x+y-z+3K)^2 \right] \\ + \frac{1}{2h_{ij}^{x}} \left[27(-2K)(y+K) + 3(x+y-z+3K)^2 \right], \\ -\frac{1}{3h_{ij}^{y}} (x+y-z+3K)^2 \\ -\frac{1}{6h_{ij}^{y}} \left[27(-2K)(z-K) - 3(x+y-z+3K)^2 \right] \\ + \frac{1}{6h_{ij}^{y}} \left[27(-2K)(y+K) + 3(x+y-z+3K)^2 \right], \\ \frac{1}{2} \left[27(-2K)(y+K) + 3(x+y-z+3K)^2 \right] \Big)^T, \end{matrix}$$

$$\boldsymbol{\eta}_{yy2}^{ij3} = \begin{pmatrix} \left(\boldsymbol{\eta}_{yy2}^{ij3}\right)_1 \\ \left(\boldsymbol{\eta}_{yy2}^{ij3}\right)_2 \\ \left(\boldsymbol{\eta}_{yy2}^{ij3}\right)_3 \end{pmatrix} = \begin{matrix} \Big(\frac{1}{2h_{ij}^x}\left[27(-2K)(z+K)+3(x+y-z-3K)^2\right] \\ +\frac{1}{2h_{ij}^x}\left[27(-2K)(y-K)-3(x+y-z-3K)^2\right], \\ \frac{1}{3h_{ij}^y}(x+y-z-3K)^2 \\ -\frac{1}{6h_{ij}^y}\left[27(-2K)(z+K)+3(x+y-z-3K)^2\right] \\ +\frac{1}{6h_{ij}^y}\left[27(-2K)(y-K)-3(x+y-z-3K)^2\right], \\ \frac{1}{2}\left[27(-2K)(y-K)-3(x+y-z-3K)^2\right]\Big)^T, \end{matrix}$$

$$\boldsymbol{\eta}_{yy3}^{ij3} = \begin{pmatrix} \left(\boldsymbol{\eta}_{yy3}^{ij3}\right)_1 \\ \left(\boldsymbol{\eta}_{yy3}^{ij3}\right)_2 \\ \left(\boldsymbol{\eta}_{yy3}^{ij3}\right)_3 \end{pmatrix} = \begin{matrix} \Big(-\frac{3}{2h_{ij}^x}(y+x-z+3K)^2 \\ +\frac{1}{2h_{ij}^x}\left[27(-2K)(x+K)+3(y+x-z+3K)^2\right], \\ \frac{1}{3h_{ij}^y}\left[27(-2K)(z-K)-3(y+x-z+3K)^2\right] \\ +\frac{3}{6h_{ij}^y}(x+y-z+3K)^2 \\ +\frac{1}{6h_{ij}^y}\left[27(-2K)(x+K)+3(y+x-z+3K)^2\right], \\ \frac{1}{2}\left[27(-2K)(x+K)+3(y+x-z+3K)^2\right]\Big)^T, \end{matrix}$$

$$\boldsymbol{\eta}_{yy4}^{ij3} = \begin{pmatrix} \left(\boldsymbol{\eta}_{yy4}^{ij3}\right)_1 \\ \left(\boldsymbol{\eta}_{yy4}^{ij3}\right)_2 \\ \left(\boldsymbol{\eta}_{yy4}^{ij3}\right)_3 \end{pmatrix} = \begin{matrix} \Big(\frac{3}{2h_{ij}^x}(y+x-z-3K)^2 \\ +\frac{1}{2h_{ij}^x}\left[27(-2K)(x-K)+3(y+x-z-3K)^2\right], \\ \frac{1}{3h_{ij}^y}\left[27(-2K)(z+K)-3(y+x-z-3K)^2\right] \\ -\frac{3}{6h_{ij}^y}(x+y-z-3K)^2 \\ +\frac{1}{6h_{ij}^y}\left[27(-2K)(x-K)-3(y+x-z-3K)^2\right], \\ \frac{1}{2}\left[27(-2K)(x-K)-3(y+x-z-3K)^2\right]\Big)^T, \end{matrix}$$

$$\boldsymbol{\eta}_{yy5}^{ij3} = \begin{pmatrix} \left(\boldsymbol{\eta}_{yy5}^{ij3}\right)_1 \\ \left(\boldsymbol{\eta}_{yy5}^{ij3}\right)_2 \\ \left(\boldsymbol{\eta}_{yy5}^{ij3}\right)_3 \end{pmatrix} = \begin{array}{c} \Big(\frac{1}{2h_{ij}^x}\left[27(2K)(x-K)+3(z-x-y+3K)^2\right] \\ -\frac{3}{2h_{ij}^x}(z-x-y+3K)^2, \\ \frac{1}{3h_{ij}^y}\left[27(2K)(y-K)+3(z-x-y+3K)^2\right] \\ -\frac{1}{6h_{ij}^y}\left[27(2K)(x-K)+3(z-x-y+3K)^2\right] \\ -\frac{3}{6h_{ij}^y}(z-x-y+3K)^2, \\ -\frac{3}{2}(z-x-y+3K)^2\Big)^T, \end{array}$$

$$\boldsymbol{\eta}_{yy6}^{ij3} = \begin{pmatrix} \left(\boldsymbol{\eta}_{yy6}^{ij3}\right)_1 \\ \left(\boldsymbol{\eta}_{yy6}^{ij3}\right)_2 \\ \left(\boldsymbol{\eta}_{yy6}^{ij3}\right)_3 \end{pmatrix} = \begin{array}{c} \Big(\frac{1}{2h_{ij}^x}\left[27(2K)(x+K)-3(z-x-y-3K)^2\right] \\ +\frac{3}{2h_{ij}^x}(z-x-y-3K)^2, \\ \frac{1}{3h_{ij}^y}\left[27(2K)(y+K)-3(z-x-y-3K)^2\right] \\ -\frac{1}{6h_{ij}^y}\left[27(2K)(x+K)-3(z-x-y-3K)^2\right] \\ +\frac{3}{6h_{ij}^y}(z-x-y-3K)^2, \\ \frac{3}{2}(z-x-y-3K)^2\Big)^T. \end{array}$$

Then, $\gamma_{12}^{ij3}\gamma_{03}^{ij3}\gamma_{02}^{ij3}$ should be in the form of

$$\begin{pmatrix} \gamma_{12}^{ij3} \\ \gamma_{03}^{ij3} \\ \gamma_{02}^{ij3} \end{pmatrix} = \lambda_{yy}^{ij3}\boldsymbol{n}_{yy1}^{ij3} + \mu_{yy}^{ij3}\boldsymbol{n}_{yy2}^{ij3} + \nu_{yy}^{ij3}\boldsymbol{n}_{yy3}^{ij3} + \tau_{yy}^{ij3}\boldsymbol{n}_{yy4}^{ij3} + \omega_{yy}^{ij3}\boldsymbol{n}_{yy5}^{ij3} + \psi_{yy}^{ij3}\boldsymbol{n}_{yy6}^{ij3}$$

$$\text{for some } \lambda_{yy}^{ij3}, \mu_{yy}^{ij3}, \nu_{yy}^{ij3}, \tau_{yy}^{ij3}, \omega_{yy}^{ij3}, \psi_{yy}^{ij3} \geq 0,$$

where $\boldsymbol{n}_{yy1}^{ij3} = \boldsymbol{\eta}_{yy1}^{ij3}$, if $\left(\hat{d}_{12}^{ij3}, \hat{d}_{03}^{ij3}, \hat{d}_{02}^{ij3}\right)$ is on the curve C_{yy1}^{ij3}, otherwise $\boldsymbol{n}_{yy1}^{ij3} = (0,0)^T$; $\boldsymbol{n}_{yy2}^{ij3} = \boldsymbol{\eta}_{yy2}^{ij3}$, if $\left(\hat{d}_{12}^{ij3}, \hat{d}_{03}^{ij3}, \hat{d}_{02}^{ij3}\right)$ is on the curve C_{yy2}^{ij3}, otherwise $\boldsymbol{n}_{yy2}^{ij3} = (0,0)^T$; $\boldsymbol{n}_{yy3}^{ij3} = \boldsymbol{\eta}_{yy3}^{ij3}$, if $\left(\hat{d}_{12}^{ij3}, \hat{d}_{03}^{ij3}, \hat{d}_{02}^{ij3}\right)$ is on the curve C_{yy3}^{ij3}, otherwise $\boldsymbol{n}_{yy3}^{ij3} = (0,0)^T$; $\boldsymbol{n}_{yy4}^{ij3} = \boldsymbol{\eta}_{yy4}^{ij3}$, if $\left(\hat{d}_{12}^{ij3}, \hat{d}_{03}^{ij3}, \hat{d}_{02}^{ij3}\right)$ is on the curve C_{yy4}^{ij3}, otherwise $\boldsymbol{n}_{yy4}^{ij3} = (0,0)^T$; $\boldsymbol{n}_{yy4}^{ij3} = \boldsymbol{\eta}_{yy5}^{ij3}$, if $\left(\hat{d}_{12}^{ij3}, \hat{d}_{03}^{ij3}, \hat{d}_{02}^{ij3}\right)$ is on the curve C_{yy5}^{ij3}, otherwise $\boldsymbol{n}_{yy5}^{ij3} = (0,0)^T$; and $\boldsymbol{n}_{yy4}^{ij3} = \boldsymbol{\eta}_{yy6}^{ij3}$, if $\left(\hat{d}_{12}^{ij3}, \hat{d}_{03}^{ij3}, \hat{d}_{02}^{ij3}\right)$ is on the curve C_{yy6}^{ij3}, otherwise $\boldsymbol{n}_{yy6}^{ij3} = (0,0)^T$.

Let

$$\boldsymbol{\eta}_{xy1}^{ij3} = \Bigg(-\frac{12}{h_{ij}^{x} h_{ij}^{y2}} \left(\frac{6\hat{\tilde{d}}_{12}^{ij3}}{h_i^x h_j^{y2}} + \frac{6\hat{d}_{11}^{ij3}}{h_i^x h_j^y} + \frac{1}{2} \right),$$

$$- \frac{12}{h_{ij}^{x} h_{ij}^{y}} \left(\frac{6\hat{\tilde{d}}_{12}^{ij3}}{h_i^x h_j^{y2}} + \frac{6\hat{d}_{11}^{ij3}}{h_i^x h_j^y} + \frac{1}{2} \right) + \frac{12}{h_{ij}^{x} h_{ij}^{y}} \left(\frac{1}{2} + \frac{4\hat{d}_{11}^{ij3}}{h_i^x h_j^y} \right)^2 \Bigg)^T,$$

and

$$\boldsymbol{\eta}_{xy2}^{ij3} = \Bigg(\frac{12}{h_{ij}^{x} h_{ij}^{y2}} \left(-\frac{6\hat{\tilde{d}}_{12}^{ij3}}{h_i^x h_j^{y2}} - \frac{6\hat{d}_{11}^{ij3}}{h_i^x h_j^y} + \frac{1}{2} \right),$$

$$\frac{12}{h_{ij}^{x} h_{ij}^{y}} \left(-\frac{6\hat{\tilde{d}}_{12}^{ij3}}{h_i^x h_j^{y2}} - \frac{6\hat{d}_{11}^{ij3}}{h_i^x h_j^y} + \frac{1}{2} \right) - \frac{12}{h_{ij}^{x} h_{ij}^{y}} \left(\frac{1}{2} - \frac{4\hat{d}_{11}^{ij3}}{h_i^x h_j^y} \right)^2 \Bigg)^T.$$

Then, $\tilde{\gamma}_{12}^{ij3}, \gamma_{11}^{ij3}$ should be in the form of

$$\begin{pmatrix} \tilde{\gamma}_{12}^{ij3} \\ \gamma_{11}^{ij3} \end{pmatrix} = \lambda_{xy}^{ij3} \boldsymbol{n}_{xy1}^{ij3} + \mu_{xy}^{ij3} \boldsymbol{n}_{xy2}^{ij3} \quad \text{for some } \lambda_{xy}^{ij3}, \mu_{xy}^{ij3} \geq 0,$$

where $\boldsymbol{n}_{xy1}^{ij3} = \boldsymbol{\eta}_{xy1}^{ij3}$, if $\left(\hat{\tilde{d}}_{12}^{ij3}, \hat{d}_{11}^{ij3}\right)$ is on the curve C_{xy1}^{ij3}, otherwise $\boldsymbol{n}_{xy1}^{ij3} = (0,0)^T$; and $\boldsymbol{n}_{xy2}^{ij3} = \boldsymbol{\eta}_{xy2}^{ij3}$, if $\left(\hat{\tilde{d}}_{12}^{ij3}, \hat{d}_{11}^{ij3}\right)$ is on the curve C_{xy2}^{ij3}, otherwise $\boldsymbol{n}_{xy2}^{ij3} = (0,0)^T$.

Let

$$\boldsymbol{\eta}_{xx1}^{ij4} = \begin{pmatrix} \left(\boldsymbol{\eta}_{xx1}^{ij4}\right)_1 \\ \left(\boldsymbol{\eta}_{xx1}^{ij4}\right)_2 \\ \left(\boldsymbol{\eta}_{xx1}^{ij4}\right)_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2h_{ij}^y} \left[27(-2K)(z-K) - 3(x+y-z+3K)^2\right] \\ -\frac{1}{2h_{ij}^y} \left[27(-2K)(y+K) + 3(x+y-z+3K)^2\right], \\ \frac{1}{3h_{ij}^x}(x+y-z+3K)^2 \\ +\frac{1}{6h_{ij}^x} \left[27(-2K)(z-K) - 3(x+y-z+3K)^2\right] \\ -\frac{1}{6h_{ij}^x} \left[27(-2K)(y+K) + 3(x+y-z+3K)^2\right], \\ \frac{3}{2}(x+y-z+3K)^2 \\ +\frac{1}{4} \left[27(-2K)(z-K) - 3(x+y-z+3K)^2\right] \\ +\frac{1}{4} \left[27(-2K)(y+K) + 3(x+y-z+3K)^2\right] \end{pmatrix}^T,$$

$$\boldsymbol{\eta}_{xx2}^{ij4} = \begin{pmatrix} \left(\boldsymbol{\eta}_{xx2}^{ij4}\right)_1 \\ \left(\boldsymbol{\eta}_{xx2}^{ij4}\right)_2 \\ \left(\boldsymbol{\eta}_{xx2}^{ij4}\right)_3 \end{pmatrix} = \begin{gathered} \Big(-\tfrac{1}{2h_{ij}^y}\left[27(-2K)(z+K)+3(x+y-z-3K)^2\right] \\ -\tfrac{1}{2h_{ij}^y}\left[27(-2K)(y-K)-3(x+y-z-3K)^2\right], \\ -\tfrac{1}{3h_{ij}^x}(x+y-z-3K)^2 \\ +\tfrac{1}{6h_{ij}^x}\left[27(-2K)(z+K)+3(x+y-z-3K)^2\right] \\ -\tfrac{1}{6h_{ij}^x}\left[27(-2K)(y-K)-3(x+y-z-3K)^2\right], \\ -\tfrac{3}{2}(x+y-z-3K)^2 \\ +\tfrac{1}{4}\left[27(-2K)(z-K)-3(x+y-z-3K)^2\right] \\ +\tfrac{1}{4}\left[27(-2K)(y+K)+3(x+y-z-3K)^2\right]\Big)^T, \end{gathered}$$

$$\boldsymbol{\eta}_{xx3}^{ij4} = \begin{pmatrix} \left(\boldsymbol{\eta}_{xx3}^{ij4}\right)_1 \\ \left(\boldsymbol{\eta}_{xx3}^{ij4}\right)_2 \\ \left(\boldsymbol{\eta}_{xx3}^{ij4}\right)_3 \end{pmatrix} = \begin{gathered} \Big(\tfrac{3}{2h_{ij}^y}(y+x-z+3K)^2 \\ -\tfrac{1}{2h_{ij}^y}\left[27(-2K)(x+K)+3(y+x-z+3K)^2\right], \\ -\tfrac{1}{3h_{ij}^x}\left[27(-2K)(z-K)-3(y+x-z+3K)^2\right] \\ -\tfrac{3}{6h_{ij}^x}(x+y-z+3K)^2 \\ -\tfrac{1}{6h_{ij}^x}\left[27(-2K)(x+K)+3(y+x-z+3K)^2\right], \\ -\tfrac{1}{2}\left[27(-2K)(z-K)-3(y+x-z+3K)^2\right] \\ -\tfrac{3}{4}(y+x-z+3K)^2 \\ +\tfrac{1}{4}\left[27(-2K)(x+K)+3(y+x-z+3K)^2\right]\Big)^T, \end{gathered}$$

$$\boldsymbol{\eta}^{ij4}_{xx4} = \begin{pmatrix} \left(\boldsymbol{\eta}^{ij4}_{xx4}\right)_1 \\ \left(\boldsymbol{\eta}^{ij4}_{xx4}\right)_2 \\ \left(\boldsymbol{\eta}^{ij4}_{xx4}\right)_3 \end{pmatrix} = \begin{matrix} \Big(-\frac{3}{2h^y_{ij}}(y+x-z-3K)^2 \\ -\frac{1}{2h^y_{ij}}\left[27(-2K)(x-K)+3(y+x-z-3K)^2\right], \\ -\frac{1}{3h^x_{ij}}\left[27(-2K)(z+K)-3(y+x-z-3K)^2\right] \\ +\frac{3}{6h^x_{ij}}(x+y-z-3K)^2 \\ -\frac{1}{6h^x_{ij}}\left[27(-2K)(x-K)-3(y+x-z-3K)^2\right], \\ -\frac{1}{2}\left[27(-2K)(z+K)-3(y+x-z-3K)^2\right] \\ +\frac{3}{4}(y+x-z-3K)^2 \\ +\frac{1}{4}\left[27(-2K)(x-K)-3(y+x-z-3K)^2\right]\Big)^T, \end{matrix}$$

$$\boldsymbol{\eta}^{ij4}_{xx5} = \begin{pmatrix} \left(\boldsymbol{\eta}^{ij4}_{xx5}\right)_1 \\ \left(\boldsymbol{\eta}^{ij4}_{xx5}\right)_2 \\ \left(\boldsymbol{\eta}^{ij4}_{xx5}\right)_3 \end{pmatrix} = \begin{matrix} \Big(-\frac{1}{2h^y_{ij}}\left[27(2K)(x-K)+3(z-x-y+3K)^2\right] \\ +\frac{3}{2h^y_{ij}}(z-x-y+3K)^2, \\ -\frac{1}{3h^x_{ij}}\left[27(2K)(y-K)+3(z-x-y+3K)^2\right] \\ +\frac{1}{6h^x_{ij}}\left[27(2K)(x-K)+3(z-x-y+3K)^2\right] \\ +\frac{3}{6h^x_{ij}}(z-x-y+3K)^2, \\ -\frac{1}{2}\left[27(2K)(y-K)+3(z-x-y+3K)^2\right] \\ +\frac{1}{4}\left[27(2K)(x-K)+3(z-x-y+3K)^2\right] \\ -\frac{3}{4}(z-x-y+3K)^2\Big)^T, \end{matrix}$$

$$\boldsymbol{\eta}_{xx6}^{ij4} = \begin{pmatrix} \left(\boldsymbol{\eta}_{xx6}^{ij4}\right)_1 \\ \left(\boldsymbol{\eta}_{xx6}^{ij4}\right)_2 \\ \left(\boldsymbol{\eta}_{xx6}^{ij4}\right)_3 \end{pmatrix} = \begin{array}{c} \Big(-\frac{1}{2h_{ij}^y}\left[27(2K)(x+K) - 3(z-x-y-3K)^2\right] \\ -\frac{3}{2h_{ij}^y}(z-x-y-3K)^2, \\ -\frac{1}{3h_{ij}^x}\left[27(2K)(y+K) - 3(z-x-y-3K)^2\right] \\ +\frac{1}{6h_{ij}^x}\left[27(2K)(x+K) - 3(z-x-y-3K)^2\right] \\ -\frac{3}{6h_{ij}^x}(z-x-y-3K)^2, \\ -\frac{1}{2}\left[27(2K)(y+K) - 3(z-x-y-3K)^2\right] \\ +\frac{1}{4}\left[27(2K)(x+K) - 3(z-x-y-3K)^2\right] \\ \frac{3}{4}(z-x-y-3K)^2\Big)^T. \end{array}$$

Then, $\gamma_{21}^{ij4}\gamma_{30}^{ij4}\gamma_{20}^{ij4}$ should be in the form of

$$\begin{pmatrix} \gamma_{21}^{ij4} \\ \gamma_{30}^{ij4} \\ \gamma_{20}^{ij4} \end{pmatrix} = \lambda_{xx}^{ij4}\boldsymbol{n}_{xx1}^{ij4} + \mu_{xx}^{ij4}\boldsymbol{n}_{xx2}^{ij4} + \nu_{xx}^{ij4}\boldsymbol{n}_{xx3}^{ij4} + \tau_{xx}^{ij4}\boldsymbol{n}_{xx4}^{ij4} + \omega_{xx}^{ij4}\boldsymbol{n}_{xx5}^{ij4} + \psi_{xx}^{ij4}\boldsymbol{n}_{xx6}^{ij4}$$

$$\text{for some } \lambda_{xx}^{ij4}, \mu_{xx}^{ij4}, \nu_{xx}^{ij4}, \tau_{xx}^{ij4}, \omega_{xx}^{ij4}, \psi_{xx}^{ij4} \geq 0,$$

where $\boldsymbol{n}_{xx1}^{ij4} = \boldsymbol{\eta}_{xx1}^{ij4}$, if $\left(\hat{d}_{21}^{ij4}, \hat{d}_{30}^{ij4}, \hat{d}_{20}^{ij4}\right)$ is on the curve C_{xx1}^{ij4}, otherwise $\boldsymbol{n}_{xx1}^{ij4} = (0,0)^T$; $\boldsymbol{n}_{xx2}^{ij4} = \boldsymbol{\eta}_{xx2}^{ij4}$, if $\left(\hat{d}_{21}^{ij4}, \hat{d}_{30}^{ij4}, \hat{d}_{20}^{ij4}\right)$ is on the curve C_{xx2}^{ij4}, otherwise $\boldsymbol{n}_{xx2}^{ij4} = (0,0)^T$; $\boldsymbol{n}_{xx3}^{ij4} = \boldsymbol{\eta}_{xx3}^{ij4}$, if $\left(\hat{d}_{21}^{ij4}, \hat{d}_{30}^{ij4}, \hat{d}_{20}^{ij4}\right)$ is on the curve C_{xx3}^{ij4}, otherwise $\boldsymbol{n}_{xx3}^{ij4} = (0,0)^T$; $\boldsymbol{n}_{xx4}^{ij4} = \boldsymbol{\eta}_{xx4}^{ij4}$, if $\left(\hat{d}_{21}^{ij4}, \hat{d}_{30}^{ij4}, \hat{d}_{20}^{ij4}\right)$ is on the curve C_{xx4}^{ij4}, otherwise $\boldsymbol{n}_{xx4}^{ij4} = (0,0)^T$; $\boldsymbol{n}_{xx4}^{ij4} = \boldsymbol{\eta}_{xx5}^{ij4}$, if $\left(\hat{d}_{21}^{ij4}, \hat{d}_{30}^{ij4}, \hat{d}_{20}^{ij4}\right)$ is on the curve C_{xx5}^{ij4}, otherwise $\boldsymbol{n}_{xx5}^{ij4} = (0,0)^T$; and $\boldsymbol{n}_{xx4}^{ij4} = \boldsymbol{\eta}_{xx6}^{ij4}$, if $\left(\hat{d}_{21}^{ij4}, \hat{d}_{30}^{ij4}, \hat{d}_{20}^{ij4}\right)$ is on the curve C_{xx6}^{ij4}, otherwise $\boldsymbol{n}_{xx6}^{ij4} = (0,0)^T$.

Let

$$\boldsymbol{\eta}_{yy1}^{ij4} = \left(\frac{4}{h_{ij}^x h_{ij}^{y2}}\left(\frac{3}{2} - \frac{2\hat{d}_{03}^{ij4}}{h_i^x h_j^{y2}}\right), -\frac{6}{h_{ij}^x h_{ij}^y}\left(1 - \frac{2\hat{d}_{02}^{ij4}}{h_i^x h_j^y}\right)^2\right)^T,$$

$$\boldsymbol{\eta}_{yy2}^{ij4} = \Bigg(-\frac{4}{h_{ij}^x h_{ij}^{y2}} \left(\frac{2\hat{d}_{03}}{h_{ij}^x h_{ij}^{y2}} - \frac{6\hat{d}_{02}}{h_{ij}^x h_{ij}^y} - \frac{3}{2} \right),$$

$$\frac{12}{h_{ij}^x h_{ij}^y} \left(\frac{2\hat{d}_{03}}{h_{ij}^x h_{ij}^{y2}} - \frac{6\hat{d}_{02}}{h_{ij}^x h_{ij}^y} - \frac{3}{2} \right) + \frac{6}{h_{ij}^x h_{ij}^y} \left(1 + \frac{2\hat{d}_{02}^{ij4}}{h_i^x h_j^y} \right)^2 \Bigg)^T,$$

$$\boldsymbol{\eta}_{yy3}^{ij4} = \left(-\frac{4}{h_{ij}^x h_{ij}^{y2}} \left(\frac{3}{2} + \frac{2\hat{d}_{03}^{ij4}}{h_i^x h_j^{y2}} \right), \frac{6}{h_{ij}^x h_{ij}^y} \left(1 + \frac{2\hat{d}_{02}^{ij4}}{h_i^x h_j^y} \right)^2 \right)^T,$$

and

$$\boldsymbol{\eta}_{yy4}^{ij4} = \Bigg(-\frac{4}{h_{ij}^x h_{ij}^{y2}} \left(\frac{2\hat{d}_{03}}{h_{ij}^x h_{ij}^{y2}} - \frac{6\hat{d}_{02}}{h_{ij}^x h_{ij}^y} + \frac{3}{2} \right),$$

$$\frac{12}{h_{ij}^x h_{ij}^y} \left(\frac{2\hat{d}_{03}}{h_{ij}^x h_{ij}^{y2}} - \frac{6\hat{d}_{02}}{h_{ij}^x h_{ij}^y} + \frac{3}{2} \right) - \frac{6}{h_{ij}^x h_{ij}^y} \left(1 - \frac{2\hat{d}_{02}^{ij4}}{h_i^x h_j^y} \right)^2 \Bigg)^T.$$

Then, $\gamma_{03}^{ij4}, \gamma_{02}^{ij4}$ should be in the form of

$$\begin{pmatrix} \gamma_{03}^{ij4} \\ \gamma_{02}^{ij4} \end{pmatrix} = \lambda_{yy}^{ij4} \boldsymbol{n}_{yy1}^{ij4} + \mu_{yy}^{ij4} \boldsymbol{n}_{yy2}^{ij4} + \nu_{yy}^{ij4} \boldsymbol{n}_{yy3}^{ij4} + \tau_{yy}^{ij4} \boldsymbol{n}_{yy4}^{ij4}$$

$$\text{for some } \lambda_{yy}^{ij4}, \mu_{yy}^{ij4}, \nu_{yy}^{ij4}, \tau_{yy}^{ij4} \geq 0,$$

where $\boldsymbol{n}_{yy1}^{ij4} = \boldsymbol{\eta}_{yy1}^{ij4}$, if $\left(\hat{d}_{03}^{ij4}, \hat{d}_{02}^{ij4}\right)$ is on the curve C_{yy1}^{ij4}, otherwise $\boldsymbol{n}_{yy1}^{ij4} = (0,0)^T$; $\boldsymbol{n}_{yy2}^{ij4} = \boldsymbol{\eta}_{yy2}^{ij4}$, if $\left(\hat{d}_{03}^{ij4}, \hat{d}_{02}^{ij4}\right)$ is on the curve C_{yy2}^{ij4}, otherwise $\boldsymbol{n}_{yy2}^{ij4} = (0,0)^T$; $\boldsymbol{n}_{yy3}^{ij4} = \boldsymbol{\eta}_{yy3}^{ij4}$, if $\left(\hat{d}_{03}^{ij4}, \hat{d}_{02}^{ij4}\right)$ is on the curve C_{yy3}^{ij4}, otherwise $\boldsymbol{n}_{yy3}^{ij4} = (0,0)^T$; and $\boldsymbol{n}_{yy4}^{ij4} = \boldsymbol{\eta}_{yy4}^{ij4}$, if $\left(\hat{d}_{03}^{ij4}, \hat{d}_{02}^{ij4}\right)$ is on the curve C_{yy4}^{ij4}, otherwise $\boldsymbol{n}_{yy4}^{ij4} = (0,0)^T$.

Let

$$\boldsymbol{\eta}_{xy1}^{ij4} = \left(-\frac{12}{h_{ij}^{x2} h_{ij}^y} \left(\frac{6\hat{\tilde{d}}_{21}^{ij4}}{h_i^{x2} h_j^y} - \frac{1}{2} \right), \frac{12}{h_{ij}^x h_{ij}^y} \left(\frac{1}{2} + \frac{4\hat{d}_{11}^{ij4}}{h_i^x h_j^y} \right) \right)^T,$$

and

$$\boldsymbol{\eta}_{xy2}^{ij4} = \left(-\frac{12}{h_{ij}^{x2} h_{ij}^y} \left(\frac{6\hat{\tilde{d}}_{21}^{ij4}}{h_i^{x2} h_j^y} + \frac{1}{2} \right), -\frac{12}{h_{ij}^x h_{ij}^y} \left(\frac{1}{2} - \frac{4\hat{d}_{11}^{ij4}}{h_i^x h_j^y} \right) \right)^T.$$

Then, $\tilde{\gamma}_{21}^{ij4}, \gamma_{11}^{ij4}$ should be in the form of

$$\begin{pmatrix} \tilde{\gamma}_{21}^{ij4} \\ \\ \gamma_{11}^{ij4} \end{pmatrix} = \lambda_{xy}^{ij4} \boldsymbol{n}_{xy1}^{ij4} + \mu_{xy}^{ij4} \boldsymbol{n}_{xy2}^{ij4} \text{ for some } \lambda_{xy}^{ij4}, \mu_{xy}^{ij4} \geq 0,$$

where $\boldsymbol{n}_{xy1}^{ij4} = \boldsymbol{\eta}_{xy1}^{ij4}$, if $\left(\hat{\tilde{d}}_{21}^{ij4}, \hat{d}_{11}^{ij4}\right)$ is on the curve C_{xy1}^{ij4}, otherwise $\boldsymbol{n}_{xy1}^{ij4} = (0,0)^T$; and $\boldsymbol{n}_{xy2}^{ij4} = \boldsymbol{\eta}_{xy2}^{ij4}$, if $\left(\hat{\tilde{d}}_{21}^{ij4}, \hat{d}_{11}^{ij4}\right)$ is on the curve C_{xy2}^{ij4}, otherwise $\boldsymbol{n}_{xy2}^{ij4} = (0,0)^T$.

Therefore, we know that any $\boldsymbol{\gamma} \in \partial \mathfrak{h}(\hat{\mathbf{d}})$ must be of the following form

$$
\left\{
\begin{array}{l}
\gamma_{00}^{ij1} = z_{ij}, \gamma_{00}^{ij2} = z_{i,j+1}, \gamma_{00}^{ij3} = z_{i+1,j+1}, \gamma_{00}^{ij4} = z_{i+1,j}, \\
\gamma_{21}^{ij1} = 0, \gamma_{12}^{ij2} = 0, \gamma_{21}^{ij3} = 0, \gamma_{12}^{ij4} = 0, \gamma_{10}^{ijk}, \gamma_{01}^{ijk} \text{unrestricted}, \\
\begin{pmatrix} \gamma_{30}^{ij1} \\ \gamma_{20}^{ij1} \end{pmatrix} = \lambda_{xx}^{ij1} \boldsymbol{n}_{xx1}^{ij1} + \mu_{xx}^{ij1} \boldsymbol{n}_{xx2}^{ij1} + \nu_{xx}^{ij1} \boldsymbol{n}_{xx3}^{ij1} + \tau_{xx}^{ij1} \boldsymbol{n}_{xx4}^{ij1} \\
\begin{pmatrix} \gamma_{12}^{ij1} \\ \gamma_{03}^{ij1} \\ \gamma_{02}^{ij1} \end{pmatrix} = \begin{array}{r} \lambda_{yy}^{ij1} \boldsymbol{n}_{yy1}^{ij1} + \mu_{yy}^{ij1} \boldsymbol{n}_{yy2}^{ij1} + \nu_{yy}^{ij1} \boldsymbol{n}_{yy3}^{ij1} + \tau_{yy}^{ij1} \boldsymbol{n}_{yy4}^{ij1} \\ + \omega_{yy}^{ij1} \boldsymbol{n}_{yy5}^{ij1} + \psi_{yy}^{ij1} \boldsymbol{n}_{yy6}^{ij1} \end{array} \\
\begin{pmatrix} \tilde{\gamma}_{12}^{ij1} \\ \gamma_{11}^{ij1} \end{pmatrix} = \lambda_{xy}^{ij1} \boldsymbol{n}_{xy1}^{ij1} + \mu_{xy}^{ij1} \boldsymbol{n}_{xy2}^{ij1} \\
\begin{pmatrix} \gamma_{12}^{ij2} \\ \gamma_{03}^{ij2} \\ \gamma_{02}^{ij2} \end{pmatrix} = \begin{array}{r} \lambda_{xx}^{ij2} \boldsymbol{n}_{xx1}^{ij2} + \mu_{xx}^{ij2} \boldsymbol{n}_{xx2}^{ij2} + \nu_{xx}^{ij2} \boldsymbol{n}_{xx3}^{ij2} + \tau_{xx}^{ij2} \boldsymbol{n}_{xx4}^{ij2} \\ + \omega_{xx}^{ij2} \boldsymbol{n}_{xx5}^{ij2} + \psi_{xx}^{ij2} \boldsymbol{n}_{xx6}^{ij2} \end{array} \\
\begin{pmatrix} \gamma_{30}^{ij2} \\ \gamma_{20}^{ij2} \end{pmatrix} = \lambda_{yy}^{ij2} \boldsymbol{n}_{yy1}^{ij2} + \mu_{yy}^{ij2} \boldsymbol{n}_{yy2}^{ij2} + \nu_{yy}^{ij2} \boldsymbol{n}_{yy3}^{ij2} + \tau_{yy}^{ij2} \boldsymbol{n}_{yy4}^{ij2} \\
\begin{pmatrix} \tilde{\gamma}_{21}^{ij2} \\ \gamma_{11}^{ij2} \end{pmatrix} = \lambda_{xy}^{ij2} \boldsymbol{n}_{xy1}^{ij2} + \mu_{xy}^{ij2} \boldsymbol{n}_{xy2}^{ij2} \\
\begin{pmatrix} \gamma_{30}^{ij3} \\ \gamma_{20}^{ij3} \end{pmatrix} = \lambda_{xx}^{ij3} \boldsymbol{n}_{xx1}^{ij3} + \mu_{xx}^{ij3} \boldsymbol{n}_{xx2}^{ij3} + \nu_{xx}^{ij3} \boldsymbol{n}_{xx3}^{ij3} + \tau_{xx}^{ij3} \boldsymbol{n}_{xx4}^{ij3} \\
\begin{pmatrix} \gamma_{12}^{ij3} \\ \gamma_{03}^{ij3} \\ \gamma_{02}^{ij3} \end{pmatrix} = \begin{array}{r} \lambda_{yy}^{ij3} \boldsymbol{n}_{yy1}^{ij3} + \mu_{yy}^{ij3} \boldsymbol{n}_{yy2}^{ij3} + \nu_{yy}^{ij3} \boldsymbol{n}_{yy3}^{ij3} + \tau_{yy}^{ij3} \boldsymbol{n}_{yy4}^{ij3} \\ + \omega_{yy}^{ij3} \boldsymbol{n}_{yy5}^{ij3} + \psi_{yy}^{ij3} \boldsymbol{n}_{yy6}^{ij3} \end{array} \\
\begin{pmatrix} \tilde{\gamma}_{12}^{ij3} \\ \gamma_{11}^{ij3} \end{pmatrix} = \lambda_{xy}^{ij3} \boldsymbol{n}_{xy1}^{ij3} + \mu_{xy}^{ij3} \boldsymbol{n}_{xy2}^{ij3} \\
\begin{pmatrix} \gamma_{12}^{ij4} \\ \gamma_{03}^{ij4} \\ \gamma_{02}^{ij4} \end{pmatrix} = \begin{array}{r} \lambda_{xx}^{ij4} \boldsymbol{n}_{xx1}^{ij4} + \mu_{xx}^{ij4} \boldsymbol{n}_{xx2}^{ij4} + \nu_{xx}^{ij4} \boldsymbol{n}_{xx3}^{ij4} + \tau_{xx}^{ij4} \boldsymbol{n}_{xx4}^{ij4} \\ + \omega_{xx}^{ij4} \boldsymbol{n}_{xx5}^{ij4} + \psi_{xx}^{ij4} \boldsymbol{n}_{xx6}^{ij4} \end{array} \\
\begin{pmatrix} \gamma_{30}^{ij4} \\ \gamma_{20}^{ij4} \end{pmatrix} = \lambda_{yy}^{ij4} \boldsymbol{n}_{yy1}^{ij4} + \mu_{yy}^{ij4} \boldsymbol{n}_{yy2}^{ij4} + \nu_{yy}^{ij4} \boldsymbol{n}_{yy3}^{ij4} + \tau_{yy}^{ij4} \boldsymbol{n}_{yy4}^{ij4} \\
\begin{pmatrix} \tilde{\gamma}_{21}^{ij4} \\ \gamma_{11}^{ij4} \end{pmatrix} = \lambda_{xy}^{ij4} \boldsymbol{n}_{xy1}^{ij4} + \mu_{xy}^{ij4} \boldsymbol{n}_{xy2}^{ij4} \\
\lambda_{xx}^{ij1}, \mu_{xx}^{ij1}, \nu_{xx}^{ij1}, \tau_{xx}^{ij1}, \lambda_{yy}^{ij1}, \mu_{yy}^{ij1}, \nu_{yy}^{ij1}, \tau_{yy}^{ij1}, \omega_{yy}^{ij1}, \psi_{yy}^{ij1}, \lambda_{xy}^{ij1}, \mu_{xy}^{ij1} \geq 0 \\
\lambda_{xx}^{ij2}, \mu_{xx}^{ij2}, \nu_{xx}^{ij2}, \tau_{xx}^{ij2}, \omega_{xx}^{ij2}, \psi_{xx}^{ij2}, \lambda_{yy}^{ij2}, \mu_{yy}^{ij2}, \nu_{yy}^{ij2}, \tau_{yy}^{ij2}, \lambda_{xy}^{ij2}, \mu_{xy}^{ij2} \geq 0 \\
\lambda_{xx}^{ij3}, \mu_{xx}^{ij3}, \nu_{xx}^{ij3}, \tau_{xx}^{ij3}, \lambda_{yy}^{ij3}, \mu_{yy}^{ij3}, \nu_{yy}^{ij3}, \tau_{yy}^{ij3}, \omega_{yy}^{ij3}, \psi_{yy}^{ij3}, \lambda_{xy}^{ij3}, \mu_{xy}^{ij3} \geq 0 \\
\lambda_{xx}^{ij4}, \mu_{xx}^{ij4}, \nu_{xx}^{ij4}, \tau_{xx}^{ij4}, \omega_{xx}^{ij4}, \psi_{xx}^{ij4}, \lambda_{yy}^{ij4}, \mu_{yy}^{ij4}, \nu_{yy}^{ij4}, \tau_{yy}^{ij4}, \lambda_{xy}^{ij4}, \mu_{xy}^{ij4} \geq 0.
\end{array}
\right. \tag{3.25}
$$

On the other hand, it is easy to show that any vector $\boldsymbol{\gamma}$ of the above form is a subgradient of $\mathfrak{h}(\hat{\mathbf{d}})$, since

$$
\begin{aligned}
&\left\langle \boldsymbol{\gamma}, \mathbf{d} - \hat{\mathbf{d}} \right\rangle \\
&= \sum_{i=0}^{I-1} \sum_{j=0}^{J-1} \Big[z_{ij} \left(d_{00}^{ij1} - \hat{d}_{00}^{ij1} \right) + z_{i,j+1} \left(d_{00}^{ij2} - \hat{d}_{00}^{ij2} \right) \\
&\qquad\qquad + z_{i+1,j+1} \left(d_{00}^{ij3} - \hat{d}_{00}^{ij3} \right) + z_{i+1,j} \left(d_{00}^{ij4} - \hat{d}_{00}^{ij4} \right) \Big] \\
&\quad + \sum_{i=0}^{I-1} \sum_{j=0}^{J-1} \sum_{k=1,3} \Big\{ \Big[\gamma_{30}^{ijk} \left(d_{30}^{ijk} - \hat{d}_{30}^{ijk} \right) + \gamma_{20}^{ijk} \left(d_{20}^{ijk} - \hat{d}_{20}^{ijk} \right) \Big] + \\
&\qquad\qquad \Big[\gamma_{12}^{ijk} \left(d_{12}^{ijk} - \hat{d}_{12}^{ijk} \right) + \gamma_{03}^{ijk} \left(d_{03}^{ijk} - \hat{d}_{03}^{ijk} \right) + \gamma_{02}^{ijk} \left(d_{02}^{ijk} - \hat{d}_{02}^{ijk} \right) \Big] + \\
&\qquad\qquad \Big[\tilde{\gamma}_{12}^{ijk} \left(\tilde{d}_{12}^{ijk} - \hat{\tilde{d}}_{12}^{ijk} \right) + \gamma_{11}^{ijk} \left(d_{11}^{ijk} - \hat{d}_{11}^{ijk} \right) \Big] \Big\} + \\
&\quad + \sum_{i=0}^{I-1} \sum_{j=0}^{J-1} \sum_{k=2,4} \Big\{ \Big[\gamma_{21}^{ijk} \left(d_{21}^{ijk} - \hat{d}_{21}^{ijk} \right) + \gamma_{30}^{ijk} \left(d_{30}^{ijk} - \hat{d}_{30}^{ijk} \right) \\
&\qquad\qquad + \gamma_{20}^{ijk} \left(d_{20}^{ijk} - \hat{d}_{20}^{ijk} \right) \Big] + \\
&\qquad\qquad \Big[\gamma_{03}^{ijk} \left(d_{03}^{ijk} - \hat{d}_{03}^{ijk} \right) + \gamma_{02}^{ijk} \left(d_{02}^{ijk} - \hat{d}_{02}^{ijk} \right) \Big] + \\
&\qquad\qquad \Big[\tilde{\gamma}_{21}^{ijk} \left(\tilde{d}_{21}^{ijk} - \hat{\tilde{d}}_{21}^{ijk} \right) + \gamma_{11}^{ijk} \left(d_{11}^{ijk} - \hat{d}_{11}^{ijk} \right) \Big] \Big\} \\
&\le \mathfrak{h}(\mathbf{d}) - \mathfrak{h}(\hat{\mathbf{d}}).
\end{aligned}
$$

Thus, we have an explicit expression for $\partial\mathfrak{h}(\hat{\mathbf{d}})$ at any given dual vector $\hat{\mathbf{d}}$.

For $i = 0, \cdots, I-1$ and $j = 0, \cdots, J-1$, let

$$
\begin{aligned}
\boldsymbol{\pi}^{ij1} &= \left(\lambda_{xx}^{ij1}, \mu_{xx}^{ij1}, \nu_{xx}^{ij1}, \tau_{xx}^{ij1}, \lambda_{yy}^{ij1}, \mu_{yy}^{ij1}, \nu_{yy}^{ij1}, \tau_{yy}^{ij1}, \omega_{yy}^{ij1}, \psi_{yy}^{ij1}, \lambda_{xy}^{ij1}, \mu_{xy}^{ij1} \right)^T, \\
\boldsymbol{\pi}^{ij2} &= \left(\lambda_{xx}^{ij2}, \mu_{xx}^{ij2}, \nu_{xx}^{ij2}, \tau_{xx}^{ij2}, \omega_{xx}^{ij2}, \psi_{xx}^{ij2}, \lambda_{yy}^{ij2}, \mu_{yy}^{ij2}, \nu_{yy}^{ij2}, \tau_{yy}^{ij2}, \lambda_{xy}^{ij2}, \mu_{xy}^{ij2} \right)^T, \\
\boldsymbol{\pi}^{ij3} &= \left(\lambda_{xx}^{ij3}, \mu_{xx}^{ij3}, \nu_{xx}^{ij3}, \tau_{xx}^{ij3}, \lambda_{yy}^{ij3}, \mu_{yy}^{ij3}, \nu_{yy}^{ij3}, \tau_{yy}^{ij3}, \omega_{yy}^{ij3}, \psi_{yy}^{ij3}, \lambda_{xy}^{ij3}, \mu_{xy}^{ij3} \right)^T, \\
\boldsymbol{\pi}^{ij4} &= \left(\lambda_{xx}^{ij4}, \mu_{xx}^{ij4}, \nu_{xx}^{ij4}, \tau_{xx}^{ij4}, \omega_{xx}^{ij4}, \psi_{xx}^{ij4}, \lambda_{yy}^{ij4}, \mu_{yy}^{ij4}, \nu_{yy}^{ij4}, \tau_{yy}^{ij4}, \lambda_{xy}^{ij4}, \mu_{xy}^{ij4} \right)^T.
\end{aligned}
$$

Now putting the geometric programming primal solution $\mathbf{c}$ together with (3.25), a vector $\mathbf{c}$ is in the set $\mathfrak{C} \cap \mathfrak{X} \cap \partial\mathfrak{h}(\mathbf{d})$ if and only if there exist some $\boldsymbol{\pi}^{ijk}$, $i = 0, \cdots, I-1$ and

$j = 0, \cdots, J-1$, such that the following set $\mathbf{\Phi}$ is not empty,

$$\mathbf{\Phi} \triangleq \left\{ \begin{array}{l} \mathcal{A}\mathbf{c} = 0 \; c_{00}^{ij1} = z_{ij}, c_{00}^{ij2} = z_{i,j+1}, c_{00}^{ij3} = z_{i+1,j+1}, c_{00}^{ij4} = z_{i+1,j}, \\ c_{21}^{ij1} = 0, c_{12}^{ij2} = 0, c_{21}^{ij3} = 0, c_{12}^{ij4} = 0, c_{10}^{ijk}, c_{01}^{ijk} \text{unrestricted}, \\ \boldsymbol{\pi}^{ijk} \geq 0, \; i = 0, \cdots, I-1; \; j = 0, \cdots, J-1, \\ \begin{pmatrix} c_{30}^{ij1} \\ c_{20}^{ij1} \end{pmatrix} = \lambda_{xx}^{ij1} \boldsymbol{n}_{xx1}^{ij1} + \mu_{xx}^{ij1} \boldsymbol{n}_{xx2}^{ij1} + \nu_{xx}^{ij1} \boldsymbol{n}_{xx3}^{ij1} + \tau_{xx}^{ij1} \boldsymbol{n}_{xx4}^{ij1} \\ \begin{pmatrix} c_{12}^{ij1} \\ c_{03}^{ij1} \\ c_{02}^{ij1} \end{pmatrix} = \begin{array}{r} \lambda_{yy}^{ij1} \boldsymbol{n}_{yy1}^{ij1} + \mu_{yy}^{ij1} \boldsymbol{n}_{yy2}^{ij1} + \nu_{yy}^{ij1} \boldsymbol{n}_{yy3}^{ij1} + \tau_{yy}^{ij1} \boldsymbol{n}_{yy4}^{ij1} \\ + \omega_{yy}^{ij1} \boldsymbol{n}_{yy5}^{ij1} + \psi_{yy}^{ij1} \boldsymbol{n}_{yy6}^{ij1} \end{array} \\ \begin{pmatrix} \tilde{c}_{12}^{ij1} \\ c_{11}^{ij1} \end{pmatrix} = \lambda_{xy}^{ij1} \boldsymbol{n}_{xy1}^{ij1} + \mu_{xy}^{ij1} \boldsymbol{n}_{xy2}^{ij1} \\ \begin{pmatrix} c_{12}^{ij2} \\ c_{03}^{ij2} \\ c_{02}^{ij2} \end{pmatrix} = \begin{array}{r} \lambda_{xx}^{ij2} \boldsymbol{n}_{xx1}^{ij2} + \mu_{xx}^{ij2} \boldsymbol{n}_{xx2}^{ij2} + \nu_{xx}^{ij2} \boldsymbol{n}_{xx3}^{ij2} + \tau_{xx}^{ij2} \boldsymbol{n}_{xx4}^{ij2} \\ + \omega_{xx}^{ij2} \boldsymbol{n}_{xx5}^{ij2} + \psi_{xx}^{ij2} \boldsymbol{n}_{xx6}^{ij2} \end{array} \\ \begin{pmatrix} c_{30}^{ij2} \\ c_{20}^{ij2} \end{pmatrix} = \lambda_{yy}^{ij2} \boldsymbol{n}_{yy1}^{ij2} + \mu_{yy}^{ij2} \boldsymbol{n}_{yy2}^{ij2} + \nu_{yy}^{ij2} \boldsymbol{n}_{yy3}^{ij2} + \tau_{yy}^{ij2} \boldsymbol{n}_{yy4}^{ij2} \\ \begin{pmatrix} \tilde{c}_{21}^{ij2} \\ c_{11}^{ij2} \end{pmatrix} = \lambda_{xy}^{ij2} \boldsymbol{n}_{xy1}^{ij2} + \mu_{xy}^{ij2} \boldsymbol{n}_{xy2}^{ij2} \\ \begin{pmatrix} c_{30}^{ij3} \\ c_{20}^{ij3} \end{pmatrix} = \lambda_{xx}^{ij3} \boldsymbol{n}_{xx1}^{ij3} + \mu_{xx}^{ij3} \boldsymbol{n}_{xx2}^{ij3} + \nu_{xx}^{ij3} \boldsymbol{n}_{xx3}^{ij3} + \tau_{xx}^{ij3} \boldsymbol{n}_{xx4}^{ij3} \\ \begin{pmatrix} c_{12}^{ij3} \\ c_{03}^{ij3} \\ c_{02}^{ij3} \end{pmatrix} = \begin{array}{r} \lambda_{yy}^{ij3} \boldsymbol{n}_{yy1}^{ij3} + \mu_{yy}^{ij3} \boldsymbol{n}_{yy2}^{ij3} + \nu_{yy}^{ij3} \boldsymbol{n}_{yy3}^{ij3} + \tau_{yy}^{ij3} \boldsymbol{n}_{yy4}^{ij3} \\ + \omega_{yy}^{ij3} \boldsymbol{n}_{yy5}^{ij3} + \psi_{yy}^{ij3} \boldsymbol{n}_{yy6}^{ij3} \end{array} \\ \begin{pmatrix} \tilde{c}_{12}^{ij3} \\ c_{11}^{ij3} \end{pmatrix} = \lambda_{xy}^{ij3} \boldsymbol{n}_{xy1}^{ij3} + \mu_{xy}^{ij3} \boldsymbol{n}_{xy2}^{ij3} \\ \begin{pmatrix} c_{12}^{ij4} \\ c_{03}^{ij4} \\ c_{02}^{ij4} \end{pmatrix} = \begin{array}{r} \lambda_{xx}^{ij4} \boldsymbol{n}_{xx1}^{ij4} + \mu_{xx}^{ij4} \boldsymbol{n}_{xx2}^{ij4} + \nu_{xx}^{ij4} \boldsymbol{n}_{xx3}^{ij4} + \tau_{xx}^{ij4} \boldsymbol{n}_{xx4}^{ij4} \\ + \omega_{xx}^{ij4} \boldsymbol{n}_{xx5}^{ij4} + \psi_{xx}^{ij4} \boldsymbol{n}_{xx6}^{ij4} \end{array} \\ \begin{pmatrix} c_{30}^{ij4} \\ c_{20}^{ij4} \end{pmatrix} = \lambda_{yy}^{ij4} \boldsymbol{n}_{yy1}^{ij4} + \mu_{yy}^{ij4} \boldsymbol{n}_{yy2}^{ij4} + \nu_{yy}^{ij4} \boldsymbol{n}_{yy3}^{ij4} + \tau_{yy}^{ij4} \boldsymbol{n}_{yy4}^{ij4} \\ \begin{pmatrix} \tilde{c}_{21}^{ij4} \\ c_{11}^{ij4} \end{pmatrix} = \lambda_{xy}^{ij4} \boldsymbol{n}_{xy1}^{ij4} + \mu_{xy}^{ij4} \boldsymbol{n}_{xy2}^{ij4}. \end{array} \right. \tag{3.26}$$

In this dissertation, we solve (3.26) by solving the following optimization problem

$$(\mathbf{T}) \qquad \begin{cases} \min\limits_{\mathbf{c},\boldsymbol{\pi}^{ijk}} \ |\mathbf{c}|_1 \\ \\ s.t. \ \mathbf{c}, \boldsymbol{\pi}^{ijk} \in \boldsymbol{\Phi} \end{cases} \tag{3.27}$$

where $| \bullet |_1$ is the vector 1-norm. The objective function of (3.27) is the regulation term as mentioned in [22]. Of course, other different objective functions could also be considered.

The above optimization problem can be transformed to a linear programming problem easily.

3.3 Computational experiments

We tested seven examples for the generalized geometric programming framework. For each example, we solve problem (3.18) to obtain a dual optimal solution $\mathbf{d}^*$. After that, we solve problem (3.27) to obtain a corresponding primal optimal solution $\mathbf{x}^*$. The solver we used to obtain the solutions is MINOS 5.5, which runs on parallel workstations with more than 20 CPUs located in Northwestern University, Evanston, Illinois.

Example 3.3.1 (Single peak 1) *The data set is* $I = 5, J = 5,\ x_i = i,\ i = 0, 1, \ldots, I-1,$ $y_j = j,\ j = 0, 1, \ldots, J-1,$

$$\{z_{ij}\} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

MINOS took 1680 *iterations to solve the dual problem (3.18)and* 1608 *iterations to solve the dual to primal transformation problem (3.27).*

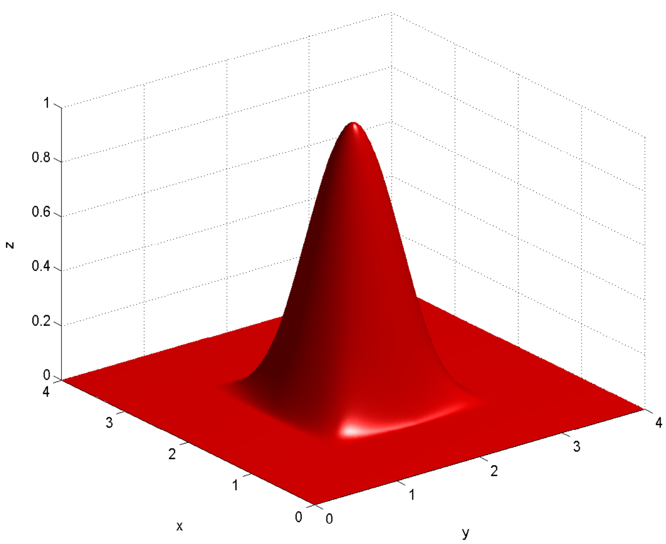


Figure 3.1: Example 1.

Example 3.3.2 (Single peak 2) *The data set is $I = 5, J = 5$, $x_i = i$, $i = 0, 1, \ldots, I-1$, $y_j = j$, $j = 0, 1, \ldots, J-1$,*

$$\{z_{ij}\} = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 8 & 7 & 8 \\ 6 & 7 & 8 & 9 & 10 \\ 8 & 9 & 10 & 11 & 12 \end{bmatrix}.$$

MINOS took 3484 *iterations to solve the dual problem (3.18)and* 1216 *iterations to solve the dual to primal transformation problem (3.27).*

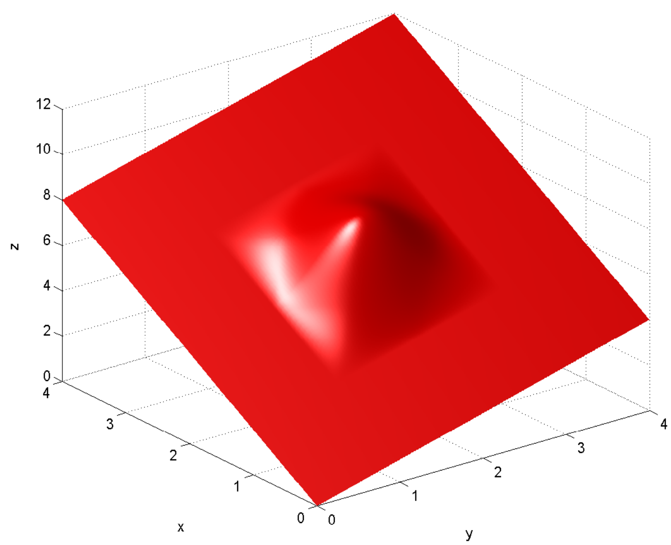


Figure 3.2: Example 2.

Example 3.3.3 (Step jump) *The data set is* $I = 5, J = 5,\ x_i = i,\ i = 0, 1, \ldots, I-1,$ $y_j = j,\ j = 0, 1, \ldots, J-1,$

$$\{z_{ij}\} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

MINOS took 5676 *iterations to solve the dual problem (3.18)and* 1296 *iterations to solve the dual to primal transformation problem (3.27).*

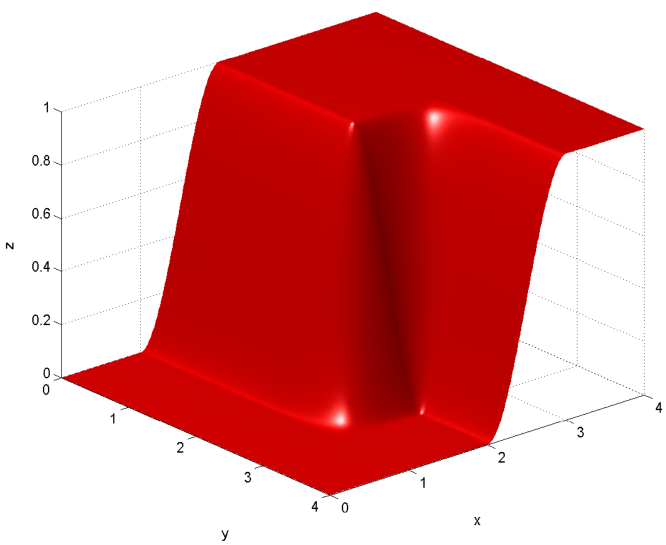


Figure 3.3: Example 3.

Example 3.3.4 (Step jump, mirror reflect) *The data set is* $I = 5, J = 5$, $x_i = i$, $i = 0, 1, \ldots, I-1$, $y_j = j$, $j = 0, 1, \ldots, J-1$,

$$\{z_{ij}\} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

MINOS took 2554 *iterations to solve the dual problem (3.18)and* 2007 *iterations to solve the dual to primal transformation problem (3.27).*

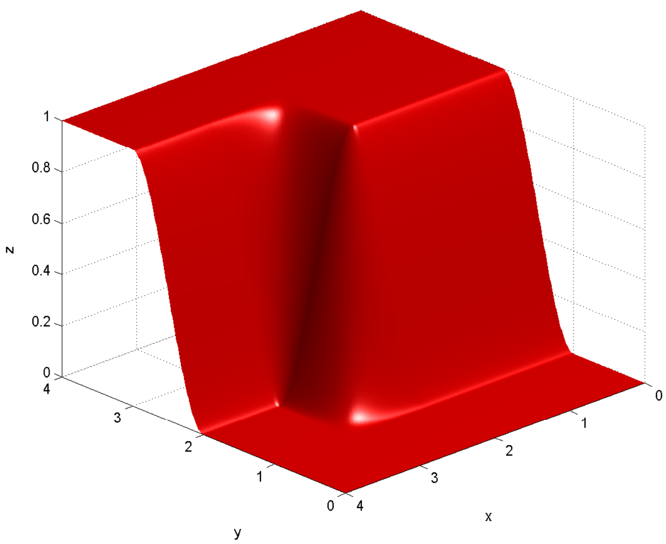


Figure 3.4: Example 4.

Example 3.3.5 (Bowl) *The data set is* $I = 5, J = 5$, $x_i = i$, $i = 0, 1, \ldots, I-1$, $y_j = j$, $j = 0, 1, \ldots, J-1$,

$$\{z_{ij}\} = \begin{bmatrix} 8 & 5 & 4 & 5 & 8 \\ 5 & 2 & 1 & 2 & 5 \\ 4 & 1 & 0 & 1 & 4 \\ 5 & 2 & 1 & 2 & 5 \\ 8 & 5 & 4 & 5 & 8 \end{bmatrix}.$$

MINOS took 8354 *iterations to solve the dual problem (3.18)and* 2516 *iterations to solve the dual to primal transformation problem (3.27).*

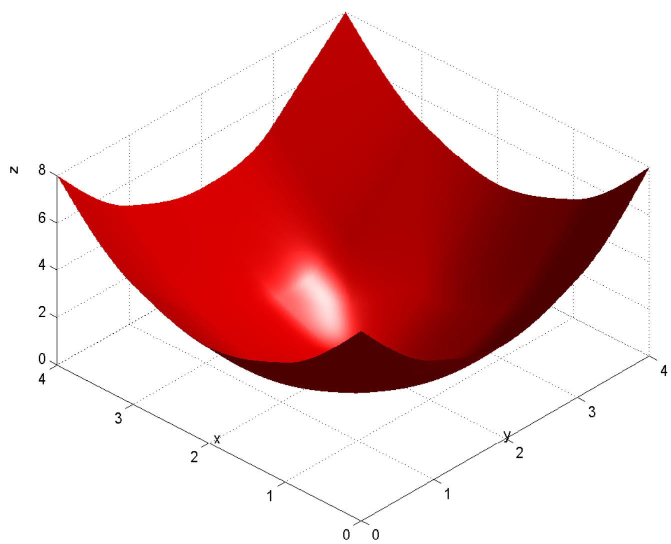


Figure 3.5: Example 5.

Example 3.3.6 (Portion of a ellipsoid) *The data set is* $I = 5, J = 5,\ x_i = i,\ i = 0, 1, \ldots, I-1,\ y_j = j,\ j = 0, 1, \ldots, J-1,$

$$\{z_{ij}\} = \begin{bmatrix} 16 & 18 & 24 & 34 & 48 \\ 9 & 11 & 17 & 27 & 41 \\ 4 & 6 & 12 & 22 & 36 \\ 1 & 3 & 9 & 19 & 33 \\ 0 & 2 & 8 & 18 & 32 \end{bmatrix}.$$

MINOS took 8519 *iterations to solve the dual problem (3.18)and* 2203 *iterations to solve the dual to primal transformation problem (3.27).*

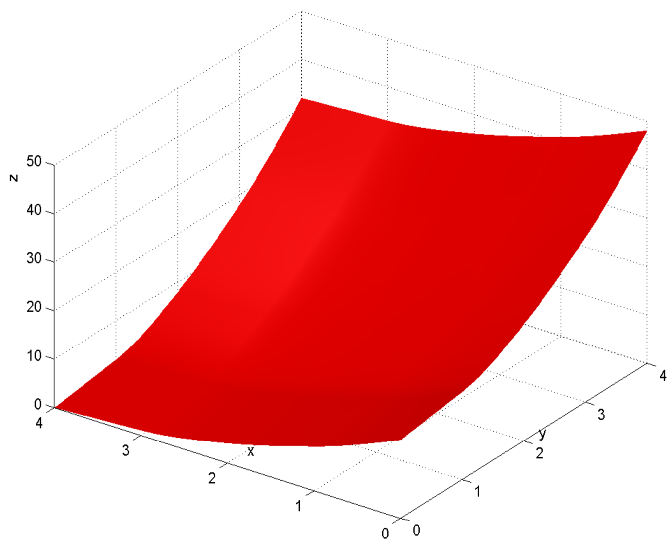


Figure 3.6: Example 6.

Example 3.3.7 (Pillar) *The data set is* $I = 6, J = 6,$ $x_i = i,$ $i = 0, 1, \ldots, I-1,$ $y_j = j,$ $j = 0, 1, \ldots, J-1,$

$$\{z_{ij}\} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

MINOS took 4686 *iterations to solve the dual problem (3.18)and* 2103 *iterations to solve the dual to primal transformation problem (3.27).*

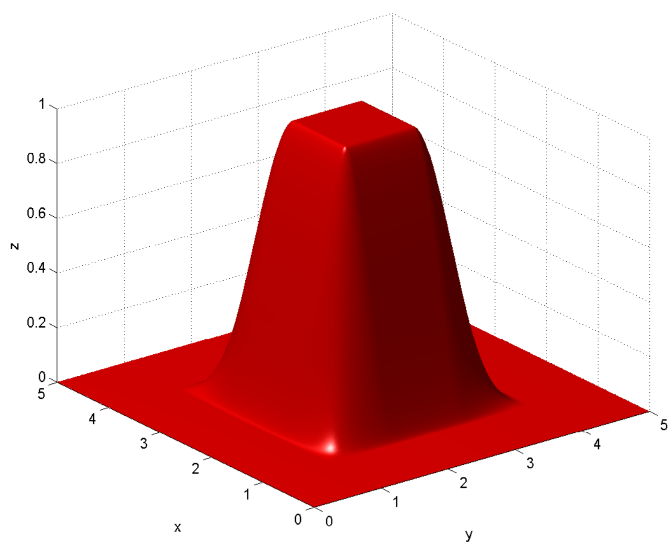


Figure 3.7: Example 7.

Numerically, the outcomes of the geometric programming framework for these seven examples are exact solutions of corresponding bivariate cubic L_1 splines. Visually, the splines generated from the geometric programming framework keep shape well.

Example	Dual variables	Dual constraints	# of iters	D-to-P iters	Time (sec)
3.3.1	1000	2620	1680	1608	73
3.3.2	1000	2620	3484	1216	212
3.3.3	1000	2620	5676	1296	351
3.3.4	1000	2620	2554	2007	143
3.3.5	1000	2620	8354	2516	560
3.3.6	1000	2620	8519	2203	570
3.3.7	1440	3765	4686	2103	2102

Table 3.1: Summary of numerical experiments of GP framework

The performance of the numerical experiments is summarized in table 3.1. From the table, it is easy to see that since the problem size is very large even for small grid size such as 5×5 and 6×6, it is prohibitively hard to solve the problem using a general purpose solver like MINOS.

3.4 Conclusion

We have developed a generalized geometric programming model for generating bivariate cubic L_1 splines. We have also shown that solving a bivariate cubic L_1 spline problem is equivalent to solving a convex programming problem whose objective function is linear and constraints are differentiable almost everywhere.

The geometric programming formulation provides a theoretical framework for calculating the coefficients of bivariate cubic L_1 splines. Based on this framework, we may as well derive various properties of the bivariate cubic L_1 splines.

In this chapter, we use the general-purposed solver, MINOS, to solve seven examples to verify the geometric programming framework. We note that it is almost impossible for MINOS to solve a bivariate cubic L_1 spline problem with a scale larger than 10×10. However, most of real-life applications are in very large scale. Therefore, we need to develop special algorithms for this problem.

Chapter 4

Linearity Preservation of Bivariate Cubic L_1 Splines

4.1 Introduction

As mentioned in Chapter 1, conventional polynomial splines are important and very successful interpolating functions [11, 17, 23, 32, 34]. They are easy to manipulate, evaluate and implement on a computer. Their coefficients can be efficiently calculated. However, experience has shown that conventional splines do not preserve "shape" well. Especially for arbitrary, "multiscale" data, that is, data with irregular values and irregular knot spacings, conventional splines may exhibit extraneous oscillation [3].

There is no widely accepted definition of shape preservation. Shape preservation is usually considered as the ability of an interplant of discrete data to preserve what observers perceive to be the "shape" of the data. Shape preservation has often been associated with preserving various properties, such as linearity and convexity/concavity [2, 24, 26] and eliminating extraneous "nonphysical" oscillation. Both variational and rule-based approaches [18] have had considerable success in preserving shape for "piecewise smooth data," that is, data on piecewise smooth curves/surfaces such as automobile bodies, ship hulls and smoothly machined mechanical objects. The creation of "fair" or visually pleasing curves [11, 31] is a closely related topic in piecewise smooth curve/surface representation. Many of these approaches involve Bézier curves and polynomial and rational B-splines. However, none of these approaches are capable of preserving shape for arbitrary, generically nonsmooth, multiscale data representing phenomena such as terrain, geological features and biological objects.

Cubic L_1 splines [21, 20, 22] have been shown to be promising for user-input-free, shape-preserving interpolation. As we have seen, computational experience indicates that bivariate cubic L_1 splines are capable of providing C^1-smooth shape-preserving interpolation for arbitrary multiscale data without additional constraints or user interaction. Cheng *et al* have proved that univariate cubic L_1 splines preserve linearity, convexity and concavity [6]. However, up to the present, the evidence that bivariate cubic L_1 splines preserve the shape of data has come mainly from computational experience. In this chapter, we use the geometric programming framework for bivariate cubic L_1 splines, proposed in previous chapter, to theoretically prove that bivariate cubic L_1 splines preserve linearity.

4.2 Geometric programming framework

Recall from last chapter, by Theorem 3.1.3, a primal optimal solution can be obtained from a dual optimal solution $\mathbf{d}^*$ of problem by satisfying the optimality conditions

$$
\begin{aligned}
&\text{(I)} && \mathbf{c}^* \in \mathfrak{C} \cap \mathfrak{X}, \mathbf{d}^* \in \mathfrak{D} \cap \mathfrak{Y} \\
&\text{(II)} && \langle \mathbf{c}^*, \mathbf{d}^* \rangle = 0 \\
&\text{(III)} && \mathbf{c}^* \in \partial \mathfrak{h}(\mathbf{d}^*).
\end{aligned}
$$

By the definition of $\mathfrak{X}$ and $\mathfrak{Y}$, condition (II) is automatically satisfied. Thus the primal optimal solution $\mathbf{c}^*$ is a vector such that

$$\mathbf{c}^* \in \mathfrak{C} \cap \mathfrak{X} \cap \partial \mathfrak{h}(\mathbf{d}^*). \tag{4.1}$$

For $i = 0, \ldots, I-1$, $j = 0, \ldots, J-1$, denote

$$
\begin{aligned}
\boldsymbol{\pi}^{ij1} &= \left(\lambda_{xx}^{ij1}, \mu_{xx}^{ij1}, \nu_{xx}^{ij1}, \tau_{xx}^{ij1}, \lambda_{yy}^{ij1}, \mu_{yy}^{ij1}, \nu_{yy}^{ij1}, \tau_{yy}^{ij1}, \omega_{yy}^{ij1}, \psi_{yy}^{ij1}, \lambda_{xy}^{ij1}, \mu_{xy}^{ij1}\right)^T, \\
\boldsymbol{\pi}^{ij2} &= \left(\lambda_{xx}^{ij2}, \mu_{xx}^{ij2}, \nu_{xx}^{ij2}, \tau_{xx}^{ij2}, \omega_{xx}^{ij2}, \psi_{xx}^{ij2}, \lambda_{yy}^{ij2}, \mu_{yy}^{ij2}, \nu_{yy}^{ij2}, \tau_{yy}^{ij2}, \lambda_{xy}^{ij2}, \mu_{xy}^{ij2}\right)^T, \\
\boldsymbol{\pi}^{ij3} &= \left(\lambda_{xx}^{ij3}, \mu_{xx}^{ij3}, \nu_{xx}^{ij3}, \tau_{xx}^{ij3}, \lambda_{yy}^{ij3}, \mu_{yy}^{ij3}, \nu_{yy}^{ij3}, \tau_{yy}^{ij3}, \omega_{yy}^{ij3}, \psi_{yy}^{ij3}, \lambda_{xy}^{ij3}, \mu_{xy}^{ij3}\right)^T, \\
\boldsymbol{\pi}^{ij4} &= \left(\lambda_{xx}^{ij4}, \mu_{xx}^{ij4}, \nu_{xx}^{ij4}, \tau_{xx}^{ij4}, \omega_{xx}^{ij4}, \psi_{xx}^{ij4}, \lambda_{yy}^{ij4}, \mu_{yy}^{ij4}, \nu_{yy}^{ij4}, \tau_{yy}^{ij4}, \lambda_{xy}^{ij4}, \mu_{xy}^{ij4}\right)^T.
\end{aligned}
$$

Then vector $\mathbf{c}^*$ is in the set $\mathfrak{C} \cap \mathfrak{X} \cap \partial \mathfrak{h}(\mathbf{d}^*)$ if and only if there exist $\boldsymbol{\pi}^{ijk}$, $i = 0, \ldots, I-1$ and $j = 0, \ldots, J-1$, such that the following set $\boldsymbol{\Phi}$ is not empty.

$$
\mathbf{\Phi} \triangleq \left\{
\begin{array}{l}
\mathcal{A}\mathbf{c} = 0 \ c_{00}^{ij1} = z_{ij}, c_{00}^{ij2} = z_{i,j+1}, c_{00}^{ij3} = z_{i+1,j+1}, c_{00}^{ij4} = z_{i+1,j}, \\
c_{21}^{ij1} = 0, c_{12}^{ij2} = 0, c_{21}^{ij3} = 0, c_{12}^{ij4} = 0, c_{10}^{ijk}, c_{01}^{ijk} \text{unrestricted}, \\
\boldsymbol{\pi}^{ijk} \geq 0,\ i = 0, \ldots, I-1;\ j = 0, \ldots, J-1, \\
\begin{pmatrix} c_{30}^{ij1} \\ c_{20}^{ij1} \end{pmatrix} = \lambda_{xx}^{ij1} \boldsymbol{n}_{xx1}^{ij1} + \mu_{xx}^{ij1} \boldsymbol{n}_{xx2}^{ij1} + \nu_{xx}^{ij1} \boldsymbol{n}_{xx3}^{ij1} + \tau_{xx}^{ij1} \boldsymbol{n}_{xx4}^{ij1} \\
\begin{pmatrix} c_{12}^{ij1} \\ c_{03}^{ij1} \\ c_{02}^{ij1} \end{pmatrix} = \begin{array}{r} \lambda_{yy}^{ij1} \boldsymbol{n}_{yy1}^{ij1} + \mu_{yy}^{ij1} \boldsymbol{n}_{yy2}^{ij1} + \nu_{yy}^{ij1} \boldsymbol{n}_{yy3}^{ij1} + \tau_{yy}^{ij1} \boldsymbol{n}_{yy4}^{ij1} \\ +\omega_{yy}^{ij1} \boldsymbol{n}_{yy5}^{ij1} + \psi_{yy}^{ij1} \boldsymbol{n}_{yy6}^{ij1} \end{array} \\
\begin{pmatrix} \tilde{c}_{12}^{ij1} \\ c_{11}^{ij1} \end{pmatrix} = \lambda_{xy}^{ij1} \boldsymbol{n}_{xy1}^{ij1} + \mu_{xy}^{ij1} \boldsymbol{n}_{xy2}^{ij1} \\
\begin{pmatrix} c_{21}^{ij2} \\ c_{30}^{ij2} \\ c_{20}^{ij2} \end{pmatrix} = \begin{array}{r} \lambda_{xx}^{ij2} \boldsymbol{n}_{xx1}^{ij2} + \mu_{xx}^{ij2} \boldsymbol{n}_{xx2}^{ij2} + \nu_{xx}^{ij2} \boldsymbol{n}_{xx3}^{ij2} + \tau_{xx}^{ij2} \boldsymbol{n}_{xx4}^{ij2} \\ +\omega_{xx}^{ij2} \boldsymbol{n}_{xx5}^{ij2} + \psi_{xx}^{ij2} \boldsymbol{n}_{xx6}^{ij2} \end{array} \\
\begin{pmatrix} c_{03}^{ij2} \\ c_{02}^{ij2} \end{pmatrix} = \lambda_{yy}^{ij2} \boldsymbol{n}_{yy1}^{ij2} + \mu_{yy}^{ij2} \boldsymbol{n}_{yy2}^{ij2} + \nu_{yy}^{ij2} \boldsymbol{n}_{yy3}^{ij2} + \tau_{yy}^{ij2} \boldsymbol{n}_{yy4}^{ij2} \\
\begin{pmatrix} \tilde{c}_{21}^{ij2} \\ c_{11}^{ij2} \end{pmatrix} = \lambda_{xy}^{ij2} \boldsymbol{n}_{xy1}^{ij2} + \mu_{xy}^{ij2} \boldsymbol{n}_{xy2}^{ij2} \\
\begin{pmatrix} c_{30}^{ij3} \\ c_{20}^{ij3} \end{pmatrix} = \lambda_{xx}^{ij3} \boldsymbol{n}_{xx1}^{ij3} + \mu_{xx}^{ij3} \boldsymbol{n}_{xx2}^{ij3} + \nu_{xx}^{ij3} \boldsymbol{n}_{xx3}^{ij3} + \tau_{xx}^{ij3} \boldsymbol{n}_{xx4}^{ij3} \\
\begin{pmatrix} c_{12}^{ij3} \\ c_{03}^{ij3} \\ c_{02}^{ij3} \end{pmatrix} = \begin{array}{r} \lambda_{yy}^{ij3} \boldsymbol{n}_{yy1}^{ij3} + \mu_{yy}^{ij3} \boldsymbol{n}_{yy2}^{ij3} + \nu_{yy}^{ij3} \boldsymbol{n}_{yy3}^{ij3} + \tau_{yy}^{ij3} \boldsymbol{n}_{yy4}^{ij3} \\ +\omega_{yy}^{ij3} \boldsymbol{n}_{yy5}^{ij3} + \psi_{yy}^{ij3} \boldsymbol{n}_{yy6}^{ij3} \end{array} \\
\begin{pmatrix} \tilde{c}_{12}^{ij3} \\ c_{11}^{ij3} \end{pmatrix} = \lambda_{xy}^{ij3} \boldsymbol{n}_{xy1}^{ij3} + \mu_{xy}^{ij3} \boldsymbol{n}_{xy2}^{ij3} \\
\begin{pmatrix} c_{21}^{ij4} \\ c_{30}^{ij4} \\ c_{20}^{ij4} \end{pmatrix} = \begin{array}{r} \lambda_{xx}^{ij4} \boldsymbol{n}_{xx1}^{ij4} + \mu_{xx}^{ij4} \boldsymbol{n}_{xx2}^{ij4} + \nu_{xx}^{ij4} \boldsymbol{n}_{xx3}^{ij4} + \tau_{xx}^{ij4} \boldsymbol{n}_{xx4}^{ij4} \\ +\omega_{xx}^{ij4} \boldsymbol{n}_{xx5}^{ij4} + \psi_{xx}^{ij4} \boldsymbol{n}_{xx6}^{ij4} \end{array} \\
\begin{pmatrix} c_{03}^{ij4} \\ c_{02}^{ij4} \end{pmatrix} = \lambda_{yy}^{ij4} \boldsymbol{n}_{yy1}^{ij4} + \mu_{yy}^{ij4} \boldsymbol{n}_{yy2}^{ij4} + \nu_{yy}^{ij4} \boldsymbol{n}_{yy3}^{ij4} + \tau_{yy}^{ij4} \boldsymbol{n}_{yy4}^{ij4} \\
\begin{pmatrix} \tilde{c}_{21}^{ij4} \\ c_{11}^{ij4} \end{pmatrix} = \lambda_{xy}^{ij4} \boldsymbol{n}_{xy1}^{ij4} + \mu_{xy}^{ij4} \boldsymbol{n}_{xy2}^{ij4}.
\end{array}
\right. \tag{4.2}
$$

where the definition of $\boldsymbol{n}$ can be found in Chapter 3.

A solution of (4.2) can be found by solving the optimization problem

$$(\mathbf{T}) \qquad \begin{cases} \min_{\mathbf{c},\boldsymbol{\pi}^{ijk}} |\mathbf{c}|_1 \\ s.t.\ \mathbf{c},\boldsymbol{\pi}^{ijk} \in \boldsymbol{\Phi} \end{cases}, \tag{4.3}$$

where $|\bullet|_1$ represents the ℓ_1 norm of a vector.

4.3 Linearity

By examining the relationship between the coefficients of bivariate cubic L_1 splines and the partial derivatives at the corner points of each rectangle, the coefficients of bivariate cubic L_1 splines can be expressed in terms of the partial derivatives at the corner points as follows,

$$\begin{aligned} c_{00}^{ij1} &= z_{ij} \\ c_{00}^{ij2} &= z_{i,j+1} \\ c_{10}^{ij1} &= z_{ij}^{x} \\ c_{01}^{ij1} &= z_{ij}^{y} \\ c_{10}^{ij2} &= z_{i,j+1}^{x} \\ c_{01}^{ij2} &= z_{i,j+1}^{y} \\ c_{00}^{ij4} &= z_{i+1,j} \\ c_{00}^{ij3} &= z_{i+1,j+1} \\ c_{10}^{ij4} &= z_{i+1,j}^{x} \\ c_{01}^{ij4} &= z_{i+1,j}^{y} \end{aligned} \tag{4.4}$$

$$
\begin{aligned}
c_{10}^{ij3} &= z_{i+1,j+1}^{x} \\
c_{01}^{ij3} &= z_{i+1,j+1}^{y} \\
c_{21}^{ij1} &= 0 \\
c_{12}^{ij4} &= 0 \\
c_{21}^{ij3} &= 0 \\
c_{12}^{ij2} &= 0 \\
(\Delta x_i)^2 c_{20}^{ij1} &= 3z_{i+1,j} - \Delta x_{i+1} z_{i+1,j}^{x} - 2\Delta x_i z_{ij}^{x} - 3z_{ij} \\
(\Delta x_i)^3 c_{30}^{ij1} &= -2z_{i+1,j} + \Delta x_i z_{ij}^{x} + 2z_{ij} + \Delta x_{i+1} z_{i+1,j}^{x} \\
(\Delta y_j)^2 c_{02}^{ij4} &= -3z_{i+1,j} - 2\Delta y_j z_{i+1,j}^{y} - \Delta y_{j+1} z_{i+1,j+1}^{y} + 3z_{i+1,j+1} \\
(\Delta y_j)^3 c_{03}^{ij4} &= 2z_{i+1,j} + \Delta y_j z_{i+1,j}^{y} + \Delta y_{j+1} z_{i+1,j+1}^{y} - 2z_{i+1,j+1} \\
(\Delta x_i)^2 c_{20}^{ij3} &= 3z_{i,j+1} + \Delta x_i z_{i,j+1}^{x} + 2\Delta x_{i+1} z_{i+1,j+1}^{x} - 3z_{i+1,j+1} \\
(\Delta x_i)^3 c_{30}^{ij3} &= 2z_{i,j+1} + \Delta x_i z_{i,j+1}^{x} + \Delta x_{i+1} z_{i+1,j+1}^{x} - 2z_{i+1,j+1} \\
(\Delta y_j)^2 c_{02}^{ij2} &= -3z_{i,j+1} + 2\Delta y_{j+1} z_{i,j+1}^{y} + \Delta y_j z_{ij}^{y} + 3z_{ij} \\
(\Delta y_j)^3 c_{03}^{ij2} &= -2z_{i,j+1} + \Delta y_{j+1} z_{i,j+1}^{y} + \Delta y_j z_{ij}^{y} + 2z_{ij} \\
\Delta x_i \Delta y_j c_{11}^{ij1} &= \Delta y_j z_{i+1,j}^{y} - \Delta y_j z_{ij}^{y} \\
\Delta x_i \Delta y_j c_{11}^{ij4} &= \Delta x_{i+1} z_{i+1,j+1}^{x} - \Delta x_{i+1} z_{i+1,j}^{x} \\
\Delta x_i \Delta y_j c_{11}^{ij3} &= \Delta y_{j+1} z_{i+1,j+1}^{y} - \Delta y_{j+1} z_{i,j+1}^{y} \\
\Delta x_i \Delta y_j c_{11}^{ij2} &= \Delta x_i z_{i,j+1}^{x} - \Delta x_i z_{ij}^{x} \\
(\Delta x_i)^2 c_{20}^{ij2} &= 3z_{i+1,j} - \Delta x_{i+1} z_{i+1,j}^{x} - \tfrac{3}{2}\Delta x_i z_{ij}^{x} - 3z_{ij} \\
&+ \tfrac{1}{2}\Delta y_j z_{i+1,j}^{y} - \tfrac{1}{2}\Delta y_j z_{ij}^{y} - \tfrac{1}{2}\Delta x_i z_{i,j+1}^{x} \\
(\Delta y_j)^2 c_{02}^{ij1} &= \tfrac{1}{2}\Delta x_i z_{i,j+1}^{x} - \tfrac{1}{2}\Delta x_i z_{ij}^{x} + 3z_{i,j+1} - \Delta y_{j+1} z_{i,j+1}^{y} \\
&- \tfrac{3}{2}\Delta y_j z_{ij}^{y} - 3z_{ij} - \tfrac{1}{2}\Delta y_j z_{i+1,j}^{y}
\end{aligned} \tag{4.5}
$$

$$
\begin{aligned}
(\Delta x_i)^2\Delta y_j c_{21}^{ij2} &= -3z_{i,j+1} - \Delta x_i z^x_{i,j+1} - \Delta x_{i+1} z^x_{i+1,j+1} + 3z_{i+1,j+1} - \tfrac{1}{2}\Delta y_{j+1} z^y_{i+1,j+1} \\
&+ \tfrac{1}{2}\Delta y_{j+1} z^y_{i,j+1} + \Delta x_i z^x_{ij} - 3z_{i+1,j} + \Delta x_{i+1} z^x_{i+1,j} + 3z_{ij} - \tfrac{1}{2}\Delta y_j z^y_{i+1,j} \\
&+ \tfrac{1}{2}\Delta y_j z^y_{ij} \\
(\Delta x_i)^2 c_{20}^{ij4} &= \tfrac{3}{2}\Delta x_{i+1} z^x_{i+1,j+1} + \tfrac{1}{2}\Delta x_{i+1} z^x_{i+1,j} + 3z_{i,j+1} + \Delta x_i z^x_{i,j+1} - 3z_{i+1,j+1} \\
&+ \tfrac{1}{2}\Delta y_{j+1} z^y_{i+1,j+1} - \tfrac{1}{2}\Delta y_{j+1} z^y_{i,j+1} \\
(\Delta y_j)^2 c_{02}^{ij3} &= \tfrac{1}{2}\Delta x_{i+1} z^x_{i+1,j+1} - \tfrac{1}{2}\Delta x_{i+1} z^x_{i+1,j} - 3z_{i+1,j+1} + \tfrac{3}{2}\Delta y_{j+1} z^y_{i+1,j+1} \\
&+ \tfrac{1}{2}\Delta y_{j+1} z^y_{i,j+1} + 3z_{i+1,j} + \Delta y_j z^y_{i+1,j} \\
\Delta x_i(\Delta y_j)^2 c_{12}^{ij1} &= -\tfrac{1}{2}\Delta x_{i+1} z^x_{i+1,j+1} + \tfrac{1}{2}\Delta x_{i+1} z^x_{i+1,j} - \Delta y_{j+1} z^y_{i+1,j+1} - \tfrac{1}{2}\Delta x_i z^x_{i,j+1} \\
&+ \tfrac{1}{2}\Delta x_i z^x_{ij} - 3z_{i,j+1} + \Delta y_{j+1} z^y_{i,j+1} + \Delta y_j z^y_{ij} + 3z_{ij} \\
&- \Delta y_j z^y_{i+1,j} - 3z_{i+1,j} + 3z_{i+1,j+1} \\
(\Delta x_i)^3 c_{30}^{ij4} &= -\tfrac{1}{2}\Delta x_{i+1} z^x_{i+1,j+1} - \tfrac{1}{2}\Delta x_{i+1} z^x_{i+1,j} - z_{i,j+1} - \tfrac{1}{2}\Delta x_i z^x_{i,j+1} + z_{i+1,j+1} \\
&- \tfrac{1}{2}\Delta x_i z^x_{ij} - z_{ij} + z_{i+1,j} \\
(\Delta x_i)^2\Delta y_j c_{21}^{ij4} &= -3z_{i,j+1} - \Delta x_i z^x_{i,j+1} - \Delta x_{i+1} z^x_{i+1,j+1} + 3z_{i+1,j+1} - \tfrac{1}{2}\Delta y_{j+1} z^y_{i+1,j+1} \\
&+ \tfrac{1}{2}\Delta y_{j+1} z^y_{i,j+1} + \Delta x_i z^x_{ij} - 3z_{i+1,j} + \Delta x_{i+1} z^x_{i+1,j} + 3z_{ij} - \tfrac{1}{2}\Delta y_j z^y_{i+1,j} \\
&+ \tfrac{1}{2}\Delta y_j z^y_{ij} \\
(\Delta y_j)^3 c_{03}^{ij1} &= -z_{i,j+1} - z_{i+1,j+1} + \tfrac{1}{2}\Delta y_{j+1} z^y_{i+1,j+1} + \tfrac{1}{2}\Delta y_{j+1} z^y_{i,j+1} + z_{i+1,j} + z_{ij} \\
&+ \tfrac{1}{2}\Delta y_j z^y_{i+1,j} + \tfrac{1}{2}\Delta y_j z^y_{ij} \\
(\Delta x_i)^3 c_{30}^{ij2} &= z_{ij} + \tfrac{1}{2}\Delta x_i z^x_{ij} - z_{i+1,j+1} + \tfrac{1}{2}\Delta x_{i+1} z^x_{i+1,j+1} - z_{i+1,j} + \tfrac{1}{2}\Delta x_{i+1} z^x_{i+1,j} \\
&+ z_{i,j+1} + \tfrac{1}{2}\Delta x_i z^x_{i,j+1}
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
\Delta x_i(\Delta y_j)^2 c_{12}^{ij3} &= -\tfrac{1}{2}\Delta x_{i+1} z_{i+1,j+1}^x + \tfrac{1}{2}\Delta x_{i+1} z_{i+1,j}^x - \Delta y_{j+1} z_{i+1,j+1}^y - \tfrac{1}{2}\Delta x_i z_{i,j+1}^x \\
&+ \tfrac{1}{2}\Delta x_i z_{ij}^x - 3z_{i,j+1} + \Delta y_{j+1} z_{i,j+1}^y + \Delta y_j z_{ij}^y + 3z_{ij} \\
&- \Delta y_j z_{i+1,j}^y - 3z_{i+1,j} + 3z_{i+1,j+1} \\
(\Delta y_j)^3 c_{03}^{ij3} &= -z_{ij} - \tfrac{1}{2}\Delta y_j z_{ij}^y + z_{i+1,j+1} - \tfrac{1}{2}\Delta y_{j+1} z_{i+1,j+1}^y - z_{i+1,j} - \tfrac{1}{2}\Delta y_j z_{i+1,j}^y \\
&+ z_{i,j+1} - \tfrac{1}{2}\Delta y_j z_{i,j+1}^y,
\end{aligned}
\tag{4.7}
$$

where $z_{ij}^x = \left.\frac{\partial z(x,y)}{\partial x}\right|_{(x_i,y_j)}$, $z_{ij}^y = \left.\frac{\partial z(x,y)}{\partial y}\right|_{(x_i,y_j)}$ are the partial derivatives of the bivariate cubic L_1 spline $z(x,y)$ at point (x_i, y_j).

Denote the slope with respect to x and y at point (x_i, y_j) as

$$
\begin{aligned}
\nabla z_x^{ij} &= \frac{z_{i+1,j} - z_{i,j}}{\Delta x_i}, \\
\nabla z_y^{ij} &= \frac{z_{i,j+1} - z_{i,j}}{\Delta y_j}.
\end{aligned}
$$

Then based on this observation, we can prove the following theorem.

Theorem 4.3.1 (Linearity) *If the four data points $(x_i, y_j, z_{i,j})$, $(x_{i+1}, y_j, z_{i+1,j})$, $(x_i, y_{j+1}, z_{i,j+1})$ and $(x_{i+1}, y_{j+1}, z_{i+1,j+1})$ lie on a plane, then the bivariate cubic L_1 spline $z(x,y)$ preserves linearity over rectangular area $D_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ if and only if the x-derivatives and y-derivatives of $z(x,y)$ at the four corner points are equal to Δz_x^{ij} and Δz_y^{ij} respectively, i.e. $z_{ij}^x = z_{i+1,j}^x = z_{i,j+1}^x = z_{i+1,j+1}^x = \nabla z_x^{ij}$ and $z_{ij}^y = z_{i+1,j}^y = z_{i,j+1}^y = z_{i+1,j+1}^y = \nabla z_y^{ij}$.*

Proof: Suppose the x-derivatives and y-derivatives of bivariate cubic L_1 spline $z(x,y)$ at the four corner points of D_{ij} are equal to ∇z_x^{ij} and ∇z_y^{ij} respectively, i.e.,

$$
z_{ij}^x = z_{i+1,j}^x = z_{i,j+1}^x = z_{i+1,j+1}^x = \nabla z_x^{ij}
$$

$$
z_{ij}^y = z_{i+1,j}^y = z_{i,j+1}^y = z_{i+1,j+1}^y = \nabla z_y^{ij}.
$$

Plugging above equations into (4.4) and (4.5), we get

$$\begin{array}{llllllllllll} c_{00}^{ij1} &=& z_{ij} & c_{00}^{ij2} &=& z_{i,j+1} & c_{00}^{ij3} &=& z_{i+1,j+1} & c_{00}^{ij4} &=& z_{i+1,j} \\ c_{10}^{ij1} &=& \nabla z_x^{ij} & c_{10}^{ij2} &=& \nabla z_x^{ij} & c_{10}^{ij3} &=& \nabla z_x^{ij} & c_{10}^{ij4} &=& \nabla z_x^{ij} \\ c_{01}^{ij1} &=& \nabla z_y^{ij} & c_{01}^{ij2} &=& \nabla z_y^{ij} & c_{01}^{ij4} &=& \nabla z_y^{ij} & c_{01}^{ij3} &=& \nabla z_y^{ij}. \end{array}$$

Since other coefficients are all zero, the bivariate cubic L_1 spline $z(x,y)$ over D_{ij} becomes

$$\begin{aligned} z^{ij1}(x,y) &= \nabla z_x^{ij}(x-x_i) + \nabla z_y^{ij}(y-y_j) + z_{ij} \\ z^{ij2}(x,y) &= \nabla z_x^{ij}(x-x_i) + \nabla z_y^{ij}(y-y_{j+1}) + z_{i,j+1} \\ z^{ij3}(x,y) &= \nabla z_x^{ij}(x-x_{i+1}) + \nabla z_y^{ij}(y-y_{j+1}) + z_{i+1,j+1} \\ z^{ij4}(x,y) &= \nabla z_x^{ij}(x-x_{i+1}) + \nabla z_y^{ij}(y-y_j) + z_{i+1,j}, \end{aligned} \tag{4.8}$$

which is a piece-wise linear function. Moreover since $z(x,y)$ is smooth, $z(x,y)$ is linear over D_{ij}.

Conversely, if the bivariate cubic L_1 spline $z(x,y)$ is linear over D_{ij}, then $z^{ij1}(x,y), \cdots, z^{ij4}(x,y)$ can be expressed as (4.8). By (4.4), we have $z_{ij}^x = z_{i+1,j}^x = z_{i,j+1}^x = z_{i+1,j+1}^x = \nabla z_x^{ij}$ and $z_{ij}^y = z_{i+1,j}^y = z_{i,j+1}^y = z_{i+1,j+1}^y = \nabla z_y^{ij}$. $\square$

Based on Theorem 4.3.1 and the generalized geometric programming framework, we have the following corollary.

Corollary 4.3.1 *Given a dual optimal solution* $\mathbf{d}^*$ *of (3.18), if* $(d_{30}^{ij1^*}, d_{20}^{ij1^*})$, $(d_{03}^{ij2^*}, d_{02}^{ij2^*})$, $(d_{03}^{ij3^*}, d_{02}^{ij3^*})$ *and* $(d_{03}^{ij4^*}, d_{02}^{ij4^*})$ *are interior points of the dual domains* Ω_{xx}^{ij1}, Ω_{yy}^{ij2}, Ω_{xx}^{ij3} *and* Ω_{yy}^{ij4} *respectively, then under the same condition of Theorem 4.3.1, the bivariate cubic* L_1 *spline preserves linearity over rectangular area* $D_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$.

Proof: If $(d_{30}^{ij1^*}, d_{20}^{ij1^*})$ is an interior point of Ω_{xx}^{ij1}, then the normal vectors $\boldsymbol{n}_{xx1}, \cdots, \boldsymbol{n}_{xx4}$

in the dual to primal transformation (4.2) are all zero vector. Thus

$$c_{20}^{ij1} = 0 \quad \text{and} \quad c_{30}^{ij1} = 0. \tag{4.9}$$

Similarly,

$$\begin{aligned} c_{02}^{ij2} = 0 \quad &\text{and} \quad c_{03}^{ij2} = 0 \\ c_{20}^{ij3} = 0 \quad &\text{and} \quad c_{30}^{ij3} = 0 \\ c_{02}^{ij4} = 0 \quad &\text{and} \quad c_{03}^{ij4} = 0. \end{aligned} \tag{4.10}$$

From (4.5), we have

$$\begin{aligned} \Delta x_i c_{20}^{ij1} &= -z_{i+1,j}^x - 2z_{ij}^x + 3\nabla z_x^{ij} \\ (\Delta x_i)^2 c_{30}^{ij1} &= z_{i+1,j}^x + z_{ij}^x - 2\nabla z_x^{ij} \\ \Delta y_j c_{02}^{ij2} &= 2z_{i,j+1}^y + z_{ij}^y - 3\nabla z_y^{ij} \\ (\Delta y_j)^2 c_{03}^{ij2} &= z_{i,j+1}^y + z_{ij}^y - 2\nabla z_y^{ij} \\ \Delta x_i c_{20}^{ij3} &= z_{i,j+1}^x + 2z_{i+1,j+1}^x - 3\nabla z_x^{i,j+1} \\ (\Delta x_i)^2 c_{30}^{ij3} &= z_{i,j+1}^x + z_{i+1,j+1}^x - 2\nabla z_x^{i,j+1} \\ \Delta y_j c_{02}^{ij4} &= -2z_{i+1,j}^y - z_{i+1,j+1}^y + 3\nabla z_y^{i+1,j} \\ (\Delta y_j)^2 c_{03}^{ij4} &= z_{i+1,j}^y + z_{i+1,j+1}^y - 2\nabla z_y^{i+1,j}. \end{aligned} \tag{4.11}$$

By (4.9), (4.10) and (4.11), we have the following linear system,

$$\begin{aligned}
-z^x_{i+1,j} - 2z^x_{ij} + 3\nabla z^{ij}_x &= 0 \\
z^x_{i+1,j} + z^x_{ij} - 2\nabla z^{ij}_x &= 0 \\
2z^y_{i,j+1} + z^y_{ij} - 3\nabla z^{ij}_y &= 0 \\
z^y_{i,j+1} + z^y_{ij} - 2\nabla z^{ij}_y &= 0 \\
z^x_{i,j+1} + 2z^x_{i+1,j+1} - 3\nabla z^{i,j+1}_x &= 0 \\
z^x_{i,j+1} + z^x_{i+1,j+1} - 2\nabla z^{i,j+1}_x &= 0 \\
-2z^y_{i+1,j} - z^y_{i+1,j+1} + 3\nabla z^{i+1,j}_y &= 0 \\
z^y_{i+1,j} + z^y_{i+1,j+1} - 2\nabla z^{i+1,j}_y &= 0,
\end{aligned}$$

which has a unique solution

$$\begin{aligned}
z^x_{ij} &= \nabla z^{ij}_x, & z^y_{ij} &= \nabla z^{ij}_y \\
z^x_{i+1,j} &= \nabla z^{ij}_x, & z^y_{i+1,j} &= \nabla z^{i+1,j}_y \\
z^x_{i,j+1} &= \nabla z^{i,j+1}_x, & z^y_{i,j+1} &= \nabla z^{ij}_y \\
z^x_{i+1,j+1} &= \nabla z^{i,j+1}_x, & z^y_{i+1,j+1} &= \nabla z^{i+1,j}_y.
\end{aligned}$$

Since the four data points $(x_i, y_j, z_{i,j})$, $(x_{i+1}, y_j, z_{i+1,j})$, $(x_i, y_{j+1}, z_{i,j+1})$ and $(x_{i+1}, y_{j+1}, z_{i+1,j+1})$ lie on a plane, $\nabla z^{ij}_x = \nabla z^{i,j+1}_x$ and $\nabla z^{ij}_y = \nabla z^{i+1,j}_y$. Thus

$$\begin{aligned}
z^x_{ij} &= \nabla z^{ij}_x, & z^y_{ij} &= \nabla z^{ij}_y \\
z^x_{i+1,j} &= \nabla z^{ij}_x, & z^y_{i+1,j} &= \nabla z^{ij}_y \\
z^x_{i,j+1} &= \nabla z^{ij}_x, & z^y_{i,j+1} &= \nabla z^{ij}_y \\
z^x_{i+1,j+1} &= \nabla z^{ij}_x, & z^y_{i+1,j+1} &= \nabla z^{ij}_y,
\end{aligned}$$

i.e., the x-derivatives and y-derivatives of $z(x, y)$ at the four corner points of $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ are equal to Δz^{ij}_x and Δz^{ij}_y, respectively. Then following Theorem 4.3.1, the bivariate cubic L_1 spline preserves linearity over rectangular area $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$. $\square$

The following example illustrates Corollary 4.3.1.

Example 4.3.1 *The data set is* $I = 6, J = 6,\ x_i = i,\ i = 0, 1, \ldots, I-1,\ y_j = j,\ j = 0, 1, \ldots, J-1,$

$$\{z_{ij}\} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 15 & 16 & 7 & 8 \\ 4 & 5 & 16 & 17 & 8 & 9 \\ 5 & 6 & 7 & 8 & 9 & 10 \\ 6 & 7 & 8 & 9 & 10 & 11 \end{bmatrix}.$$

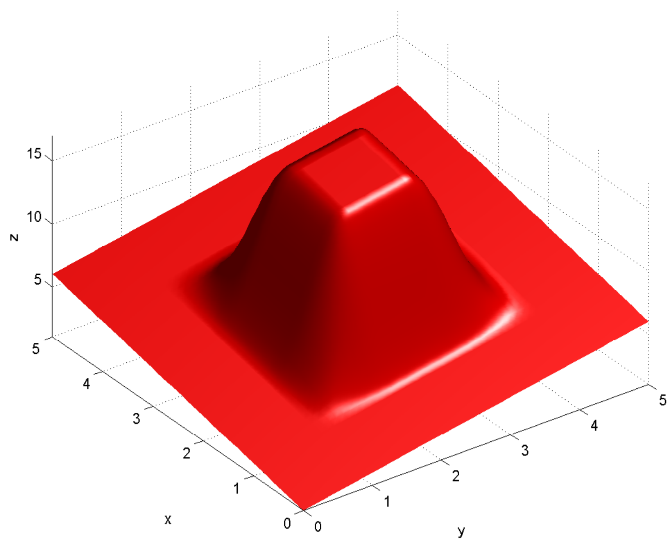


Figure 4.1: Example 7.

The optimal dual solution of GP framework lies in the interior of the feasible domain. The resulted bivariate cubic L_1 *spline preserves linearity.*

4.4 Conclusion

We have proved that, under appropriate conditions, bivariate cubic L_1 splines preserve linearity of data. L_1 splines do not always and everywhere preserve linearity but, even when they do not, L_1 splines still tend to preserve shape. The fact that L_1 splines can preserve shape without strictly preserving linearity raises questions about the widespread strategy of trying to achieve shape preservation by strict, rule-based enforcement of linearity. Widespread strategy plays a fundamental role in domain decomposition [35], a widely used method to create L_1 splines on large scale date set by combining L_1 splines over the smaller overlapped sub-regions that compose entire grid (domain). Further investigation on this strategy needs to be conducted.

Based on the generalized geometric programming framework, we may in the future prove some other shape-preserving properties that have been proved for univariate L_1 splines, such as convexity and concavity.

Chapter 5

Tensor-product Approach

In this chapter, an efficient active set algorithm for univariate cubic L_1 spline is introduced. Based on this active set algorithm, a tensor-product approximation approach for bivariate cubic L_1 splines is proposed. The computational results are reported at the end of this chapter.

5.1 Introduction

In Chapter 3, we proposed a generalized geometric programming framework for bivariate cubic L_1 splines. This framework includes a primal problem, a dual program, which is a convex programming problem with a linear objective function and cubic convex constraints, and a linear programming problem, which transfers dual optimal solution to a corresponding primal optimal solution. This framework provides us a platform for theoretical treatment of the bivariate cubic L_1 splines. Since there is no specific algorithm has been designed, we have to use a general purpose nonlinear solver to handle it. However, because of the inde-composability of the constraint structure and the large number of variables and constraints, it is too difficult to generate large scale bivariate cubic L_1 splines by using a general-purpose nonlinear solver.

Notice that the coefficients of a bivariate cubic L_1 spline on a tensor-product grid can be determined by the interpolation values and the first order information at each knot [14]. We may use the first partial derivatives obtained from the univariate cubic L_1 splines at each knot as an approximation for the first order information of true bivariate cubic L_1 splines. In this way, we can approximate the corresponding bivariate cubic L_1 spline. (See Section 5.4 for details.) If we can generate these univariate splines quickly, the approximation is expected to be efficient. Cheng *et al.* [5] proposed an active set method for univariate cubic L_1 splines, which can find the exact solution efficiently. Therefore, we may use this algorithm to generate univariate cubic L_1 splines. Based on this idea, we develop a tensor-product approximation approach for bivariate cubic L_1 splines in this chapter.

5.2 Tensor-product grids and Sibson elements

In this chapter we introduce a new set of notations. The ordered sets $\{x_i\}_{i=0}^{I}$ and $\{y_j\}_{j=0}^{J}$ are strictly monotonic partitions of the finite real intervals $[a,b]$ and $[\tilde{a},\tilde{b}]$, i.e

$$a = x_0 < x_1 < \cdots < x_{I-1} < x_I = b$$

$$\tilde{a} = y_0 < y_1 < \cdots < y_{J-1} < y_J = \tilde{b},$$

which forms a tensor-product grid $\Delta^2 = \{x_i, y_j\}$ for $i = 0, \ldots, I$ and $j = 0, \ldots, J$. Denote the rectangle area $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ as D_{ij}. The data to be interpolated are given at the knots (x_i, y_j), $i = 0, \ldots, I, j = 0, \ldots, J$. In contrast to the conventional requirement of slowly changing knot spacing, the spacing of these knots in the x and y directions can be completely arbitrary, with closely spaced knots intermingled with knots spaced far from each other.

The Sibson element [16] used for the splines in this chapter are C^1 smooth, piecewise cubic polynomials. For convenience, we restate the definition of Sibson element as follows.

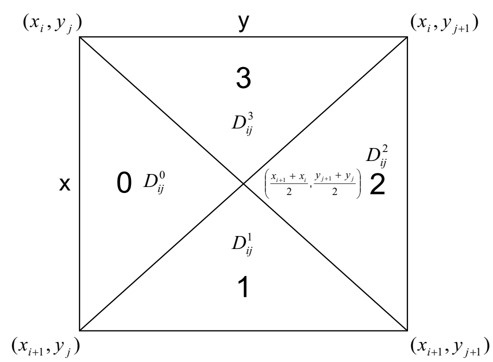


Figure 5.1: The Sibson split on rectangle D_{ij}.

Definition 5.2.1 (Sibson element) *Given a rectangle* $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ *in a tensor product grid* $\Delta^2 = \{x_i, y_j\}$*, for* $i = 0, \ldots, I$ *and* $j = 0, \ldots, J$*, which is divided into four triangles by drawing the two diagonals of the rectangle. The Sibson element is a shape function* $z(x, y)$ *that has the following properties: it is* (i) *cubic in each triangle,* (ii) C^1 *smooth at the lines separating the four triangles,* (iii) C^1 *smooth with the Sibson elements in the adjacent rectangles,* (iv) *it has derivative* $\frac{\partial z(x,y)}{\partial x}$ *that is linear along the edges* $x = x_i$*, and* $x = x_{i+1}$ *of the rectangle, and* (v) *it has derivative* $\frac{\partial z(x,y)}{\partial y}$ *that is linear along the edges* $y = y_i$ *and* $y = y_{i+1}$ *of the rectangle.*

The Sibson element $z(x, y)$ in a given rectangle depends only on the values of $z(x, y)$, $\frac{\partial z(x,y)}{\partial x}$ and $\frac{\partial z(x,y)}{\partial y}$ at the four corners of that rectangle (12 parameters per rectangle) [14]. More specifically, We consider the rectangle $D_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ in Figure 5.1. To create the Sibson element on D_{ij}, it is divided into four triangles, D_{ij}^k, $k = 0, \ldots, 3$. Cubic polynomials are defined on each triangle in the following two forms. For triangles indexed by $k = 0, 3$ the bivariate polynomials are written as

$$\begin{aligned} z_k(x,y) =& c_{30}^k(x-x_i)^3 + c_{21}^k(x-x_i)^2(y-y_j) \\ &+ c_{12}^k(x-x_i)(y-y_j)^2 + c_{03}^k(y-y_j)^3 \\ &+ c_{20}^k(x-x_i)^2 + c_{11}^k(x-x_i)(y-y_j) + c_{02}^k(y-y_j)^2 \\ &+ c_{10}^k(x-x_i) + c_{01}^k(y-y_j) + c_{00}^k \end{aligned} \tag{5.1}$$

and for $k = 1, 2$ the bivariate polynomials are written as

$$\begin{aligned} z_k(x,y) =& c_{30}^k(x_{i+1}-x)^3 + c_{21}^k(x_{i+1}-x)^2(y_{j+1}-y) \\ &+ c_{12}^k(x_{i+1}-x)(y_{j+1}-y)^2 + c_{03}^k(y_{j+1}-y)^3 \\ &+ c_{20}^k(x_{i+1}-x)^2 + c_{11}^k(x_{i+1}-x)(y_{j+1}-y) + c_{02}^k(y_{j+1}-y)^2 \\ &+ c_{10}^k(x_{i+1}-x) + c_{01}^k(y_{j+1}-y) + c_{00}^k \end{aligned} \tag{5.2}$$

Suppose that we are given elevation and derivative values $z_{ij} = z(x_i, y_j)$, $z_{ij}^x = \frac{\partial z(x,y)}{\partial x}\Big|_{x=x_i, y=y_j}$ and $z_{ij}^y = \frac{\partial z(x,y)}{\partial y}\Big|_{x=x_i, y=y_j}$ at each corner point of rectangle $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$. Let $\Delta x_i = x_{i+1} - x_i$ and $\Delta y_j = y_{j+1} - y_j$. Define

$$r = \frac{x - x_i}{\Delta x_i}, \quad s = \frac{y - y_j}{\Delta y_j}$$

Then these four cubic polynomials on D_{ij} can be expressed by the elevation and derivative values as follows [14]:

$$
\begin{aligned}
z^{ij0}(x,y) &= \left(1 - 3s^2 - 3r^2 + s^3 + 2r^3 + 3rs^2\right) z_{ij} \\
&+ \Delta x_i \left(r^3 + \frac{1}{2}rs^2 - 2r^2 - \frac{1}{2}s^2 + r\right) z^x_{ij} \\
&+ \Delta y_j \left(rs^2 + \frac{1}{2}s^3 - rs - \frac{3}{2}s^2 + s\right) z^y_{ij} + \left(-2r^3 - 3rs^2 + s^3 + 3r^2\right) z_{i+1,j} \\
&+ \Delta x_i \left(r^3 + \frac{1}{2}rs^2 - r^2\right) z^x_{i+1,j} + \Delta y_j \left(-rs^2 + \frac{1}{2}s^3 + rs - \frac{1}{2}s^2\right) z^y_{i+1,j} \\
&+ \left(-3rs^2 - s^3 + 3s^2\right) z_{i,j+1} + \Delta x_i \left(-\frac{1}{2}rs^2 + \frac{1}{2}s^2\right) z^x_{i,j+1} \\
&+ \Delta y_j \left(\frac{1}{2}s^3 - s^2 + rs^2\right) z^y_{i,j+1} + \left(3rs^2 - s^3\right) z_{i+1,j+1} \\
&+ \Delta x_i \left(\frac{1}{2}rs^2\right) z^x_{i+1,j+1} + \Delta y_j \left(-rs^2 + \frac{1}{2}s^3\right) z^y_{i+1,j+1}
\end{aligned}
\tag{5.3}
$$

$$
\begin{aligned}
z^{ij1}(x,y) &= \left(-(1-r)^3 + 3(1-r)^2(1-s)\right) z_{ij} \\
&+ \Delta x_i \left(\frac{1}{2}(1-r)^3 + (1-r)^2(1-s)\right) z^x_{ij} + \Delta y_j \left(\frac{1}{2}(1-r)^2(1-s)\right) z^y_{ij} \\
&+ \left((1-r)^3 - 3(1-r)^2(1-s) - 2(1-s)^3 + 3(1-s)^2\right) z_{i+1,j} \\
&+ \Delta x_i \left(-\frac{1}{2}(1-r)^3 + (1-r)^2(1-s) - \frac{1}{2}(1-r)^2 - (1-r)(1-s)\right) z^x_{i+1,j} \\
&+ \Delta y_j \left(-\frac{1}{2}(1-r)^2(1-s) - (1-s)^3 + (1-s)^2\right) z^y_{i+1,j} \\
&+ \left(-(1-r)^3 - 3(1-r)^2(1-s) + 3(1-r)^2\right) z_{i,j+1} \\
&+ \Delta x_i \left(-\frac{1}{2}(1-r)^2 - (1-r)^2(1-s) + (1-r)^2\right) z^x_{i,j+1} \\
&+ \Delta y_j \left(\frac{1}{2}(1-r)^2(1-s) - \frac{1}{2}(1-r)^2\right) z^y_{i,j+1} \\
&+ \left((1-r)^3 + 3(1-r)^2(1-s) + 2(1-s)^3 - 3(1-r)^2 - 3(1-s)^2 + 1\right) z_{i+1,j+1} \\
&+ \Delta x_i \left(-\frac{1}{2}(1-r)^3 + 3(1-r)^2(1-s) + \frac{3}{2}(1-r)^2 + (1-r)(1-s) - (1-r)\right) z^x_{i+1,j+1} \\
&+ \Delta y_j \left(-\frac{1}{2}(1-r)^2(1-s) - (1-s)^3 + \frac{1}{2}(1-r)^2 + 2(1-s)^2 - (1-s)\right) z^y_{i+1,j+1}
\end{aligned}
\tag{5.4}
$$

$$
\begin{aligned}
z^{ij2}(x,y) &= \left(3(1-r)(1-s)^2 - (1-s)^3\right) z_{ij} \\
&+ \Delta x_i \left(\frac{1}{2}(1-r)(1-s)^2\right) z^x_{ij} + \Delta y_j \left(\frac{1}{2}(1-r)(1-s)^2 - \frac{1}{2}(1-s)^3\right) z^y_{ij} \\
&+ \left(-3(1-r)(1-s)^2 - (1-s)^3 + 3(1-s)^2\right) z_{i+1,j} \\
&+ \Delta x_i \left(\frac{1}{2}(1-r)(1-s)^2 - \frac{1}{2}(1-r)^2\right) z^x_{i+1,j} \\
&+ \Delta y_j \left(-(1-r)(1-s)^2 - \frac{1}{2}(1-s)^3 + (1-s)^2\right) z^y_{i+1,j} \\
&+ \left(-2(1-r)^3 - 3(1-r)(1-s)^2 + (1-s)^3 + 3(1-r)^2\right) z_{i,j+1} \\
&+ \Delta x_i \left(-(1-r)^3 - \frac{1}{2}(1-r)(1-s)^2 + (1-r)^2\right) z^x_{i,j+1} \\
&+ \Delta y_j \left((1-r)(1-s)^2 - \frac{1}{2}(1-s)^2 - (1-r)(1-s) + \frac{1}{2}(1-s)^2\right) z^y_{i,j+1} \\
&+ \left(2(1-r)^3 + 3(1-r)(1-s)^2 + (1-s)^3 - 3(1-r)^2 - 3(1-s)^2 + 1\right) z_{i+1,j+1} \\
&+ \Delta x_i \left(-(1-r)^3 - \frac{1}{2}(1-r)(1-s)^2 + 2(1-r)^2 + \frac{1}{2}(1-s)^2 - (1-r)\right) z^x_{i+1,j+1} \\
&+ \Delta y_j \left(-(1-r)(1-s)^2 - \frac{1}{2}(1-s)^3 + (1-r)(1-s) + \frac{3}{2}(1-s)^2 - (1-s)\right) z^y_{i+1,j+1}
\end{aligned}
\tag{5.5}
$$

$$
\begin{aligned}
z^{ij3}(x,y) &= \left(r^3 + 3r^2 s + 2s^2 - 3r^2 - 3s^2 + 1\right) z_{ij} \\
&+ \Delta x_i \left(\frac{1}{2}r^3 + r^2 s - \frac{3}{2}r^2 - rs + r\right) z^x_{ij} \\
&+ \Delta y_j \left(\frac{1}{2}r^2 s + s^3 - \frac{1}{2}r^2 - 2s^2 + s\right) z^y_{ij} + \left(-r^3 - 3r^2 s + 3r^2\right) z_{i+1,j} \\
&+ \Delta x_i \left(\frac{1}{2}r^3 + r^2 s - r^2\right) z^x_{i+1,j} + \Delta y_j \left(-\frac{1}{2}r^2 s + \frac{1}{2}r^2\right) z^y_{i+1,j} \\
&+ \left(r^3 - 3r^2 s - 2s^3 + 3s^2\right) z_{i,j+1} + \Delta x_i \left(\frac{1}{2}r^3 - r^2 s - \frac{1}{2}r^2 + rs\right) z^x_{i,j+1} \\
&+ \Delta y_j \left(\frac{1}{2}r^2 s + s^3 - s^2\right) z^y_{i,j+1} + \left(-r^3 + 3r^2 s\right) z_{i+1,j+1} \\
&+ \Delta x_i \left(\frac{1}{2}r^3 - r^2 s\right) z^x_{i+1,j+1} + \Delta y_j \left(-\frac{1}{2}r^2 s\right) z^y_{i+1,j+1}
\end{aligned}
\tag{5.6}
$$

5.3 Univariate and bivariate cubic L_1 splines

As mentioned in Chapter 2, the d variate polynomial space of total degree m is defined as

$$\mathbb{P}_m^{(d)} = \text{span}\left\{ \prod_{i=1}^{d} x_i^{\alpha_i} \;\middle|\; \mathbf{x} = (x_1, \ldots, x_d)^T \in R^d,\ \sum_{i=1}^{d} \alpha_i < m,\ \ \alpha_i \in Z_+,\ i = 1, \ldots, d \right\}$$

where Z_+ is the set of all the nonnegative integers. Let $\Delta^1 = \{x_i\}_{i=0}^n$ be a strictly monotonic partition of a finite real interval $[a, b]$ such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

Let Δ^2 be the tensor-product grid of rectangle area $[a, b] \times [\tilde{a}, \tilde{b}]$ defined in the beginning of Section 4.2. We define the space of piecewise univariate polynomials of order m with knots $x_1, \ldots, x_{n-1}$ as

$$\mathbb{PP}_m^{(1)}(\Delta^1) = \Big\{ f(\mathbf{x}) | f(\mathbf{x}) = p_i(\mathbf{x}) \text{ for } \mathbf{x} \in I_i \overset{\triangle}{=} [x_i, x_{i+1}]\,, \\ p_i(\mathbf{x}) \in \mathbb{P}_m^{(1)},\ i = 0, \ldots, n-1 \Big\},$$

Define the space of piecewise bivariate polynomials of total order m on a tensor-product grid with Sibson split over knots (x_i, y_j), $i = 0, \cdots, I, j = 0, \cdots, J$ as

$$\mathbb{PP}_m^{(2)}(\Delta^2) = \Big\{ f(\mathbf{x}) | f(\mathbf{x}) = p_{ij}^k(\mathbf{x}) \text{ for } \mathbf{x} \in D_{ij}^k,\ \ p_{ij}^k(\mathbf{x}) \in \mathbb{P}_m^{(2)}, \\ i = 0, \cdots, I-1; j = 0, \cdots, J-1; k = 0, \cdots, 3 \Big\}$$

Let $C^r\,[a, b] \times [\tilde{a}, \tilde{b}]$ be the space of functions whose first r^{th} derivatives are continuous on $[a, b] \times [\tilde{a}, \tilde{b}]$, for a given positive integer r.

Definition 5.3.1 (Univariate cubic L_1 splines) *Let $\Delta^1 = \{x_i\}_{i=0}^n$ be a partition of the interval $[a, b]$, let $\{(x_i, z_i)\}_{i=0}^n$ be the given data set. A piecewise univariate cubic polynomial*

$\mathcal{S}(x)$ is called a univariate cubic L_1 spline, if,

$$\mathcal{S}(x) = \underset{s(x)}{\operatorname{argmin}}\left\{ \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \left[\left|s''(x)\right|\right] dx \,\middle|\, s(x) \in \mathbb{PP}_4^{(1)}(\Delta^1) \cap C^1[a,b] \right.$$
$$\left. \text{and } s(x_i) = z_i,\ i = 0, \cdots, n-1 \right\},$$

Definition 5.3.2 (Bivariate cubic L_1 splines) *Let $\Delta^2 = \{(x_i, y_j)\},\ i = 0, \cdots, I, j = 0, \cdots, J$, be a partition of the rectangle $D \triangleq [a,b] \times [\tilde{a}, \tilde{b}]$, $D_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ be the triangle partitioned by Δ^2 over D, and $\{(x_i, y_j, z_{ij})\},\ i = 0, \cdots, I, j = 0, \cdots, J$, be the given data set. A piecewise cubic polynomial $\mathcal{Z}(x,y)$ is called a cubic L_1 spline, if*

$$\mathcal{Z}(x,y) = \underset{z(x,y)}{\operatorname{argmin}}\left\{ \sum_{i=0}^{I-1}\sum_{j=0}^{J-1} \iint_{(x,y)\in D_{ij}} \left[\left|\frac{\partial^2 z(x,y)}{\partial x^2}\right| + 2\left|\frac{\partial^2 z(x,y)}{\partial x \partial y}\right| + \left|\frac{\partial^2 z(x,y)}{\partial y^2}\right|\right] dxdy \right.$$
$$\left| \, z(x,y) \in \mathbb{PP}_4^{(2)}(\Delta^2) \cap C^1[a,b] \times [\tilde{a}, \tilde{b}],\ z(x,y) \text{ is Sibson element on } D_{ij}, \right.$$
$$\left. \text{and } z(x_i, y_j) = z_{ij},\ i = 0, \cdots, I, j = 0, \cdots, J \right\},$$

Recently, Cheng *et al.* [5] proposed an active set method for univariate cubic L_1 splines, which can find the exact coefficients for the L_1 splines pretty efficiently. We outline the algorithm here. For details, please refer to [5].

Algorithm 5.3.1 [Active set algorithm for finding cubic L_1 splines]

Step 0. Choose an initial dual feasible solution $\boldsymbol{\beta}^{(0)}$ and get its segmentation $\Sigma^{(0)} = \left\{\sigma_1^{(0)}, \ldots, \sigma_{M_0}^{(0)}\right\}$. Let $k = 0$. Mark all segments unprocessed.

Step 1. If all segments are marked as processed, stop. Otherwise, pick one unprocessed segment $\sigma_m^{(k)}$, $m = 1, \ldots, M_k$.

Step 2. Obtain an optimal solution of Dual problem for $\sigma_m^{(k)}$.

Step 3. Update the segmentation to $\Sigma^{(k+1)}$ and update markers of segments.

Step 4. Increase k by 1 and go to Step 1.

5.4 Tensor-product approach for bivariate cubic L_1 splines

As stated in section 4.2, a bivariate cubic L_1 spline is determined by interpolation values and first partial derivatives at each knot, i.e. z_{ij}, z_{ij}^x and z_{ij}^y. In order to generate bivariate cubic splines by using (5.3)-(5.6) , we need the first order information, z_{ij}^x and z_{ij}^y, $i = 0, \ldots, I, j = 0, \ldots, J$. Suppose we generate the univariate cubic L_1 spline $S_j^x(x)$ which passes through $\{(x_0, z_{0j}), (x_1, z_{1j}), \cdots, (x_I, z_{Ij})\}$, $j = 0, \ldots, J$, and univariate cubic L_1 spline $S_i^y(y)$ which passes through $\{(y_0, z_{i0}), (y_1, z_{i1}), \cdots, (y_J, z_{iJ})\}$, $i = 0, \ldots, I$, by the active set algorithm introduced in the last section. Then z_{ij}^x can be easily approximated by taking the first derivative of the univariate cubic L_1 spline $S_j^x(x)$ at point x_i. Similarly, z_{ij}^y can be approximated by taking the first derivative of the univariate cubic L_1 spline $S_i^y(y)$ at point y_j. Thus, we can use formulas (5.3)-(5.6) and z_{ij}, z_{ij}^x, z_{ij}^y to generate a bivariate cubic spline which approximates the corresponding bivariate cubic L_1 spline. Based on this idea, we propose the following tensor-product approximation approach for bivariate cubic L_1 splines.

Algorithm 5.4.1 [Tensor-product approach for bivariate cubic L_1 splines] *Give a tensor-product* $\Delta^2 = \{(x_i, y_j)\}, \ i = 0, \ldots, I, j = 0, \ldots, J,$ *over the rectangle* $D \triangleq [a, b] \times [\tilde{a}, \tilde{b}]$ *and data set* $\{(x_i, y_j, z_{ij})\}, \ i = 0, \cdots, I, j = 0, \cdots, J.$

Step 1. For $j = 0, \ldots, J$:

Use the active set method (Algorithm 5.3.1) to generate the univariate cubic L_1 *spline* $S_j^x(x)$ *passing through the points* $\{(x_0, z_{0j}), (x_1, z_{1j}), \cdots, (x_I, z_{Ij})\}$;

Step 2. *For* $i = 0, \ldots, I$:

Use the active set method (Algorithm 5.3.1) to generate the univariate cubic L_1 *splines* $S_i^y(y)$ *passing through the points* $\{(y_0, z_{i0}), (y_1, z_{i1}), \cdots, (y_J, z_{iJ})\}$;

Step 3. *For* $i = 0, \ldots, I$ *and* $j = 0, \ldots, J$:

Take the first derivative of $S_j^x(x)$ *and* $S_i^y(y)$ *at each knot* (x_i, y_j), *denote as* $\tilde{z}_{ij}^x = \left.\frac{dS_j^x(x)}{dx}\right|_{x=x_i}$ *and* $\tilde{z}_{ij}^y(y) = \left.\frac{dS_i^y(y)}{dy}\right|_{y=y_j}$ *respectively;*

Step 4. *For* $i = 0, \ldots, I-1$ *and* $j = 0, \ldots, J-1$:

Based on $(\tilde{z}_{i,j}^x, \tilde{z}_{i,j}^y, z_{i,j})$, $(\tilde{z}_{i,j+1}^x, \tilde{z}_{i,j+1}^y, z_{i,j+1})$, $(\tilde{z}_{i+1,j}^x, \tilde{z}_{i+1,j}^y, z_{i+1,j})$ *and* $(\tilde{z}_{i+1,j+1}^x, \tilde{z}_{i+1,j+1}^y, z_{i+1,j+1})$, *use formula (5.3)-(5.6) to generate the four cubic polynomials* $z^{ij0}(x,y)$, $z^{ij1}(x,y)$, $z^{ij2}(x,y)$ *and* $z^{ij3}(x,y)$ *of the bivariate cubic* L_1 *spline on rectangle* $[x_i, y_j] \times [x_{i+1}, y_{j+1}]$.

This tensor-product approximation approach is expected to be a good approximation to bivariate cubic L_1 splines, since it is based on the exact univariate cubic L_1 splines which preserve the one-dimensional shape well.

5.5 Computational results

In order to test the performance of the tensor-product approximation approach, we wrote a MATLAB code to implement Algorithm 5.4.1. The code was run on a Pentium 4 computer with 1.4GHz CPU and 128M memory.

Example 5.5.1 (Single peak 1) *The data set is* $I = 5, J = 5$, $x_i = i$, $i = 0, 1, \ldots, I-1$, $y_j = j$, $j = 0, 1, \ldots, J-1$,

$$\{z_{ij}\} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

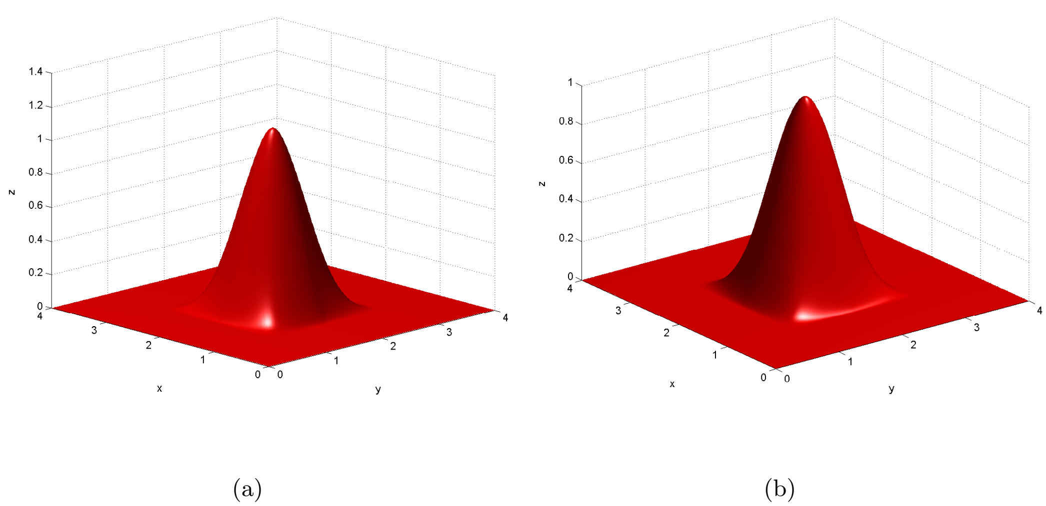


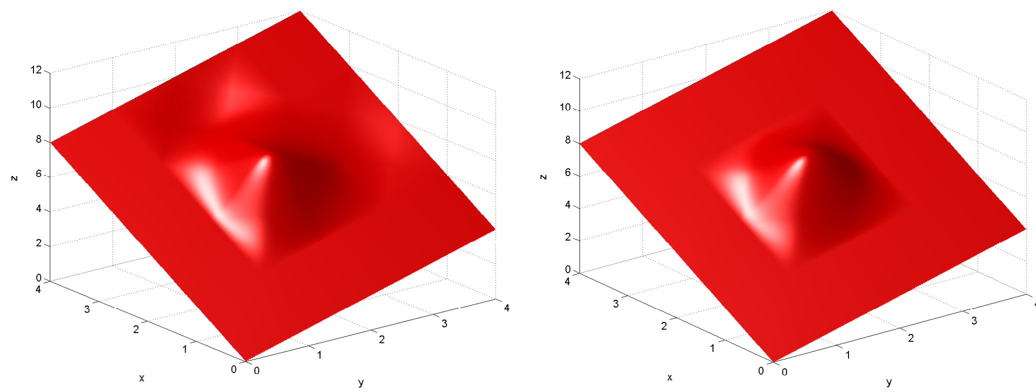
Figure 5.2: Comparison of approximated cubic L_1 splines (a) and true L_1 splines (b)

The difference between the highest points of 5.2(a) and 5.2(b) is 0.0990.

Example 5.5.2 (Single peak 2) *The data set is* $I = 5, J = 5$, $x_i = i$, $i = 0, 1, \ldots, I-1$,

$y_j = j$, $j = 0, 1, \ldots, J-1$,

$$\{z_{ij}\} = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 8 & 7 & 8 \\ 6 & 7 & 8 & 9 & 10 \\ 8 & 9 & 10 & 11 & 12 \end{bmatrix}.$$



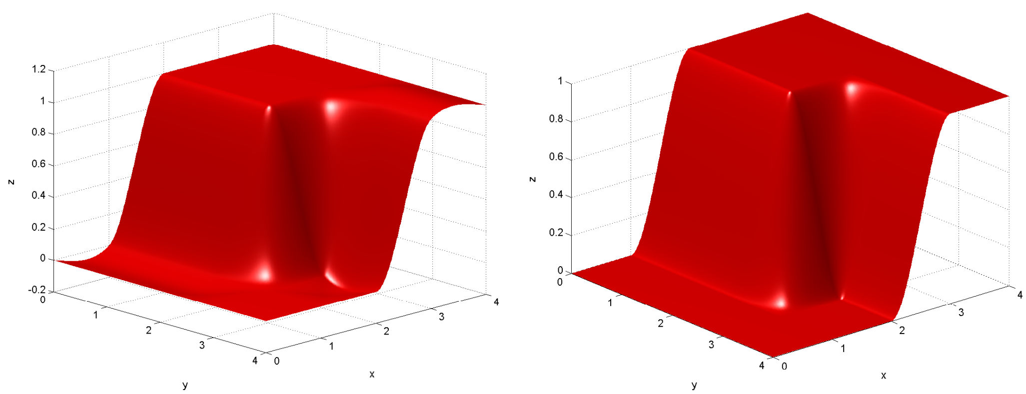
(a) (b)

Figure 5.3: Comparison of approximated cubic L_1 splines (a) and true L_1 splines (b)

The difference between the highest points of 5.3(a) and 5.3(b) is 0.

Example 5.5.3 (Step jump) *The data set is* $I = 5, J = 5,\ x_i = i,\ i = 0, 1, \ldots, I-1,$ $y_j = j,\ j = 0, 1, \ldots, J-1,$

$$\{z_{ij}\} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$



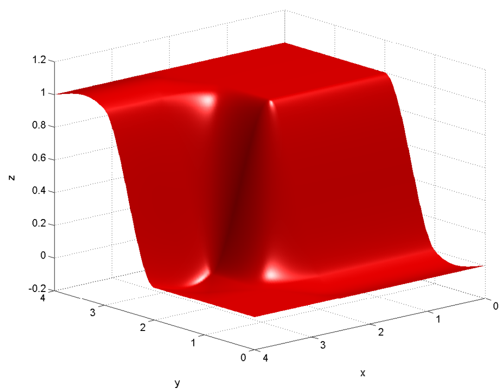
(a) (b)

Figure 5.4: Comparison of approximated cubic L_1 splines (a) and true L_1 splines (b)

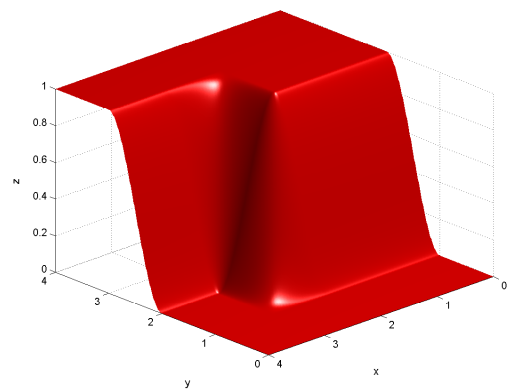
The difference between the highest points of 5.4(a) and 5.4(b) is 0.0501*.*

Example 5.5.4 (Step jump, mirror reflect) *The data set is* $I = 5, J = 5$, $x_i = i$, $i = 0, 1, \ldots, I-1$, $y_j = j$, $j = 0, 1, \ldots, J-1$,

$$\{z_{ij}\} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$



(a)



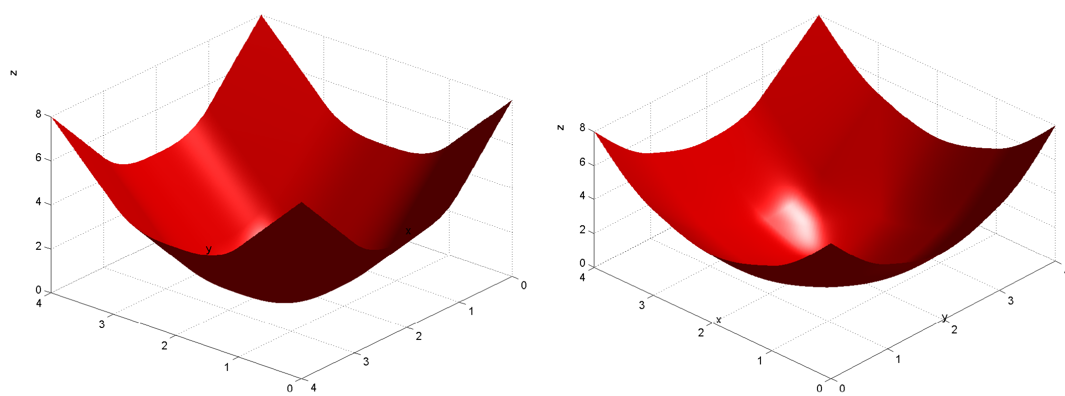
(b)

Figure 5.5: Comparison of approximated cubic L_1 splines (a) and true L_1 splines (b)

The difference between the highest points of 5.5(a) and 5.5(b) is 0.0501*.*

Example 5.5.5 (Bowl) *The data set is* $I = 5, J = 5,\ x_i = i,\ i = 0, 1, \ldots, I-1,\ y_j = j,\ j = 0, 1, \ldots, J-1,$

$$\{z_{ij}\} = \begin{bmatrix} 8 & 5 & 4 & 5 & 8 \\ 5 & 2 & 1 & 2 & 5 \\ 4 & 1 & 0 & 1 & 4 \\ 5 & 2 & 1 & 2 & 5 \\ 8 & 5 & 4 & 5 & 8 \end{bmatrix}.$$



(a) (b)

Figure 5.6: Comparison of approximated cubic L_1 splines (a) and true L_1 splines (b)

The difference between the highest points of 5.6(a) and 5.6(b) is 0*.*

Example 5.5.6 (Portion of a ellipsoid) *The data set is* $I = 5, J = 5,\ x_i = i,\ i = 0, 1, \ldots, I-1,\ y_j = j,\ j = 0, 1, \ldots, J-1,$

$$\{z_{ij}\} = \begin{bmatrix} 16 & 18 & 24 & 34 & 48 \\ 9 & 11 & 17 & 27 & 41 \\ 4 & 6 & 12 & 22 & 36 \\ 1 & 3 & 9 & 19 & 33 \\ 0 & 2 & 8 & 18 & 32 \end{bmatrix}.$$

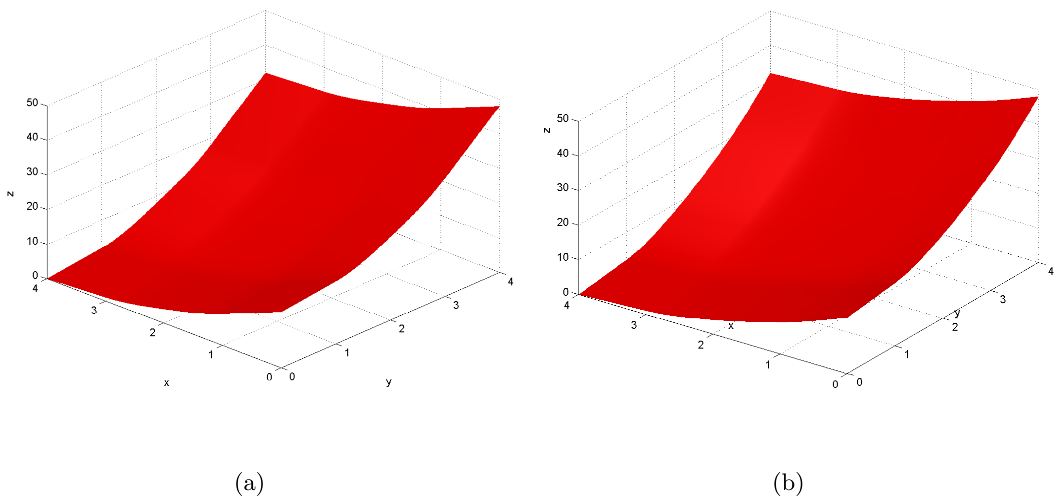
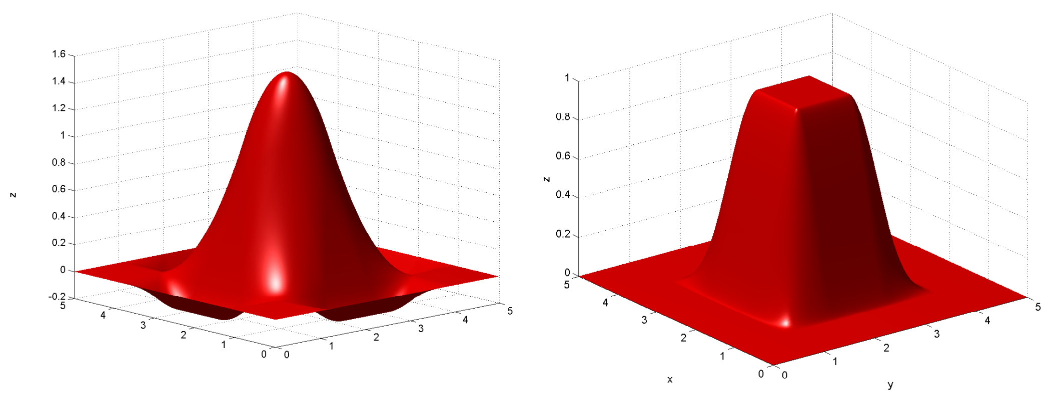


Figure 5.7: Comparison of approximated cubic L_1 splines (a) and true L_1 splines (b)

The difference between the highest points of 5.7(a) and 5.7(b) is 0.

Example 5.5.7 (Pillar) *The data set is* $I = 6, J = 6,\ x_i = i,\ i = 0, 1, \ldots, I-1,\ y_j = j,\ j = 0, 1, \ldots, J-1,$

$$\{z_{ij}\} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$



(a) (b)

Figure 5.8: Comparison of approximated cubic L_1 splines (a) and true L_1 splines (b)

The difference between the highest points of 5.8(a) and 5.8(b) is 0.5000.

Example	Difference between top points	Difference/height	Computing Time(sec)
5.5.1	0.0990	9.90%	0.06325
5.5.2	0.0000	0.00%	0.06299
5.5.3	0.0501	5.01%	0.07127
5.5.4	0.0501	5.01%	0.06902
5.5.5	0.0000	0.00%	0.06865
5.5.6	0.0000	0.00%	0.08594
5.5.7	0.5000	50.00%	0.13858

Table 5.1: Summary of numerical experiments of tensor-product approach

The performance of the approximation approach is summarized in Table 5.1. From the table, it shows that the tensor-product approximation approach is very efficient and there is not much difficulty to handle large scale problem by using this approach. From the table, we can also see in many cases, the tensor-product approach can keep the shape well. But in some cases, this approach may introduce extraneous oscillations. For example, in Example 5.5.7, the error is quite large (50% of the height). That is because the corresponding bivariate cubic L_1 spline cannot be described by the information obtained from univariate cubic L_1 splines.

5.6 Conclusion

The computational results show that, using univariate cubic L_1 splines, the proposed tensor-product approach can approximate bivariate cubic L_1 splines very efficiently. Since it is based on an efficient active set algorithm that can solve large size problems without much difficulty, this approximation approach can generate solutions for large scale problems quickly.

Since the proposed tensor-product approximation approach only uses the information obtained from univariate cubic L_1 splines along two different directions to approximate bivariate cubic L_1 splines, much useful information may be lost when the shape can not be exactly decomposed by sampling in two different directions. In some extreme cases, this approximation approach may misrepresent the bivariate cubic L_1 splines badly.

Although the proposed approximation approach may not preserve shape perfectly, it presents an efficient way to generate a 3-dimensional surface close to the one generated by the true bivariate cubic L_1 spline and it can handle large scale bivariate cubic L_1 splines without much difficulty.

Chapter 6

Interior Point Methods for Discretized Bivariate Cubic L_1 Splines

6.1 Introduction

The primal problems for cubic L_1 splines are nonsmooth convex optimization problems. As described in Chapter 3, using generalized geometric programming theory [4, 37], the bivariate cubic L_1 splines can be formulated into a smooth convex programming framework. For univariate cubic L_1 splines, geometric programming theory leads to an efficient active set algorithm [5]. For bivariate cubic L_1 splines, an active set algorithm will have to deal with a large number of constraints makes it impractical. Bivariate cubic L_1 splines can be generated using general-purpose nonlinear convex programming solvers, such as MINOS, SNOPT and MOSEK. However, such solvers are only efficient for small size problems.

In Chapter 5, we proposed a tensor-product approach to approximate bivariate cubic L_1 splines. Although it is amazingly efficient for large scale problems, in some extreme cases, the error between the resulted approximation and the true L_1 splines can be large.

Minimizing the continuum L_1 spline functional is a nonlinear program. Designing robust and efficient methods for this nonlinear program is not easy at all. Therefore, we restrict our attention to the task of minimizing the discretized L_1 spline functional, which is a linear program. In this chapter, two interior-point methods, the primal affine method and primal-dual methods, are proposed to generate discretized bivariate cubic L_1 splines.

This chapter is organized as follows: in Section 6.2, we review the tensor-product grids and bivariate cubic L_1 splines. In Section 6.3, a discretization scheme and a linear program formulation are introduced. In Section 6.4, we propose two interior point methods for the discretized bivariate cubic L_1 splines. In Section 6.5, we report the computational results, and in the last section, we make some conclusions.

6.2 Bivariate spines on tensor-product grids

For the convenience of statement, let us review the definition of bivariate L_1 splines that will be used in this chapter. Let there be given two strictly monotonic but otherwise arbitrary sets $\{x_i\}_{i=0}^{I}$ and $\{y_j\}_{j=0}^{J}$ that partition the finite real intervals $[a, b]$ and $[\bar{a}, \bar{b}]$, respectively, that is,

$$a = x_0 < x_1 < \cdots < x_{I-1} < x_I = b,$$

$$\bar{a} = y_0 < y_1 < \cdots < y_{J-1} < y_J = \bar{b}.$$

The knots of the splines that we consider will be the knots of the tensor-product grid $\Delta^2 = \{(x_i, y_j) \,|\, i = 0, 1, \ldots, I,\ j = 0, 1, \ldots, J\}$. D_{ij} will denote the rectangle $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$.

At each knot (x_i, y_j), $i = 0, 1, \ldots, I, j = 0, 1, \ldots, J$, let real-number data z_{ij} be given. We wish to find a piecewise cubic function $z(x, y)$ that interpolates the data (x_i, y_j, z_{ij}), $i = 0, 1, \ldots, I$, $j = 0, 1, \ldots, J$ and preserves the "shape" of the data.

Bivariate cubic L_1 splines are calculated by minimizing expressions involving the L_1 norm of the second derivatives of Sibson elements that interpolate the given data.

Definition 6.2.1 (Bivariate cubic L_1 spline) *A function $\mathcal{Z} = \mathcal{Z}(x, y)$ is called a bivariate cubic L_1 spline if*

$$\mathcal{Z} = \underset{z}{\operatorname{argmin}} \Bigg\{ \Pi = \sum_{i=0}^{I-1} \sum_{j=0}^{J-1} \iint_{(x,y)\in D_{ij}} \left[\left| \frac{\partial^2 z(x,y)}{\partial x^2} \right| + \left| 2\frac{\partial^2 z(x,y)}{\partial x \partial y} \right| + \left| \frac{\partial^2 z(x,y)}{\partial y^2} \right| \right] dxdy$$

$$\Big|\, z(x,y) \text{ is a Sibson element in each rectangle } D_{ij} \text{ of } \Delta^2$$

$$\text{and } z(x_i, y_j) = z_{ij},\ i = 0, 1, \cdots, I, j = 0, 1, \cdots, J \Bigg\},$$

where, the Sibson element $z(x, y)$ in a given rectangle depends only on the values of $z(x, y)$, $\frac{\partial z(x,y)}{\partial x}$ and $\frac{\partial z(x,y)}{\partial y}$ at the four corners of that rectangle (12 parameters per rectangle). The values of $\partial z/\partial x$ and $\partial z/\partial y$ at node (x_i, y_j) will be denoted by z_{ij}^x and z_{ij}^y, respectively.

Bivariate cubic L_1 splines always exist, but they need not be unique because Π is not necessarily strictly convex [22]. When there are several candidates for an L_1 spline, the candidate with the smallest absolute values of z_{ij}^x and z_{ij}^y is the choice of many users and the choice that we adopt here. We compute L_1 splines by minimizing not Π alone but Π with an added "regularization" term,

$$\begin{aligned}
&\Pi + \epsilon \sum_{i=0}^{I} \sum_{j=0}^{J} \left[\left| z_{ij}^x \right| + \left| z_{ij}^y \right| \right] \\
=\ & \sum_{i=0}^{I-1} \sum_{j=0}^{J-1} \iint_{(x,y)\in D_{ij}} \left[\left| \frac{\partial^2 z}{\partial x^2} \right| + \left| 2\frac{\partial^2 z}{\partial x \partial y} \right| + \left| \frac{\partial^2 z}{\partial y^2} \right| \right] dxdy + \epsilon \sum_{i=0}^{I} \sum_{j=0}^{J} \left[\left| z_{ij}^x \right| + \left| z_{ij}^y \right| \right], \quad (6.1)
\end{aligned}$$

where the regularization parameter ϵ is a small positive number. Calculating the coefficients of a cubic L_1 spline problem is equivalent to solving a nonsmooth optimization problem.

6.3 Discretization and linear programming formulation

Minimization of functional (6.1) is a nonsmooth nonlinear program. Although there exist some classical treatments, such as semidefinite programming, subgradient based algorithm and smoothing techniques, all these methods involve large overhead and turn out to be inefficient for generating large scale L_1 splines. Developing a computationally efficient nonlinear programming algorithm for minimizing (6.1) is a challenging and unsolved task.

We focus on the minimization of a discretization of functional (6.1). For convenience purpose, we call the Sibson-element interplant of the given set of data that minimizes a given discretization of functional (6.1) as the bivariate cubic L_1 spline. To generate bivariate L_1 splines on $\bar{D} = [x_0, x_I] \times [y_0, y_J]$, we discretize the integrals in (6.1) in the following manner. First, express the integral over $\bar{D}$ as the sum of the integrals over the rectangles $(x_i, x_{i+1}) \times (y_j, y_{j+1})$, $i = 0, 1, \ldots, I-1, j = 0, 1, \ldots, J-1$. Let K be an integer ≥ 2. Divide each rectangle by $1/[2K(K-1)]$ times the sum of the $2K(K-1)$ values of the integrand at the midpoints of the sides of the subrectangles that are in the interior of the rectangle. We choose this discretization because it uses values of the integrand only in the interiors (and not on the boundaries) of the four triangles that make up each rectangle. Hence the second derivative on each discretized point is well defined. Other discretization may work equally well.

More specifically, each interval of $[x_i, x_{i+1}]$ and $[y_j, y_{j+1}]$ is discretized into K equal subintervals. Let x_i^u and y_j^v be the midpoint of the u^{th} subinterval in interval $[x_i, x_{i+1}]$ and the midpoint of the v^{th} subinterval in interval $[y_j, y_{j+1}]$, respectively, where $u, v = 1, \ldots, K$. Denote the right end point of the s^{th} subinterval in interval $[x_i, x_{i+1}]$ and the t^{th} subinterval in interval $[y_j, y_{j+1}]$ as $\bar{x}_i^s = x_i + s\left(\frac{x_{i+1}-x_i}{K}\right)$ and $\bar{y}_j^t = y_j + t\left(\frac{y_{j+1}-y_j}{K}\right)$, $s, t = 1, \cdots, K-1$ accordingly. Then minimizing (6.1) can be approximated by minimizing

$$
\begin{aligned}
\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\sum_{u=1}^{K}\sum_{t=1}^{K-1}\Bigg\{ & \left|\frac{\partial^2 z_{ij}(x_i^u, \bar{y}_j^t)}{\partial x^2}\right| \frac{(x_{i+1}-x_i)(y_{j+1}-y_j)}{2K(K-1)} + \\
& 2\left|\frac{\partial^2 z_{ij}(x_i^u, \bar{y}_j^t)}{\partial x \partial y}\right| \frac{(x_{i+1}-x_i)(y_{j+1}-y_j)}{2K(K-1)} + \\
& \left|\frac{\partial^2 z_{ij}(x_i^u, \bar{y}_j^t)}{\partial y^2}\right| \frac{(x_{i+1}-x_i)(y_{j+1}-y_j)}{2K(K-1)} + \\
& \left|\frac{\partial^2 z_{ij}(\bar{x}_i^t, y_j^u)}{\partial x^2}\right| \frac{(x_{i+1}-x_i)(y_{j+1}-y_j)}{2K(K-1)} + \\
& 2\left|\frac{\partial^2 z_{ij}(\bar{x}_i^t, y_j^u)}{\partial x \partial y}\right| \frac{(x_{i+1}-x_i)(y_{j+1}-y_j)}{2K(K-1)} + \\
& \left|\frac{\partial^2 z_{ij}(\bar{x}_i^t, y_j^u)}{\partial y^2}\right| \frac{(x_{i+1}-x_i)(y_{j+1}-y_j)}{2K(K-1)} \Bigg\},
\end{aligned}
\tag{6.2}
$$

which is equivalent to minimize the l_1 norm of the residual of the following overdetermined system:

$$
\begin{aligned}
\frac{(x_{i+1}-x_i)(y_{j+1}-y_j)}{2K(K-1)} \left.\frac{\partial^2 z_{ij}(x,y)}{\partial x^2}\right|_{(x,y)=(x_i^u, \bar{y}_j^t)} &= 0 \\
2\frac{(x_{i+1}-x_i)(y_{j+1}-y_j)}{2K(K-1)} \left.\frac{\partial^2 z_{ij}(x,y)}{\partial x \partial y}\right|_{(x,y)=(x_i^u, \bar{y}_j^t)} &= 0 \\
\frac{(x_{i+1}-x_i)(y_{j+1}-y_j)}{2K(K-1)} \left.\frac{\partial^2 z_{ij}(x,y)}{\partial y^2}\right|_{(x,y)=(x_i^u, \bar{y}_j^t)} &= 0 \\
\frac{(x_{i+1}-x_i)(y_{j+1}-y_j)}{2K(K-1)} \left.\frac{\partial^2 z_{ij}(x,y)}{\partial x^2}\right|_{(x,y)=(\bar{x}_i^t, y_j^u)} &= 0 \\
2\frac{(x_{i+1}-x_i)(y_{j+1}-y_j)}{2K(K-1)} \left.\frac{\partial^2 z_{ij}(x,y)}{\partial x \partial y}\right|_{(x,y)=(\bar{x}_i^t, y_j^u)} &= 0 \\
\frac{(x_{i+1}-x_i)(y_{j+1}-y_j)}{2K(K-1)} \left.\frac{\partial^2 z_{ij}^k(x,y)}{\partial y^2}\right|_{(x,y)=(\bar{x}_i^t, y_j^u)} &= 0
\end{aligned}
\tag{6.3}
$$

where $i = 0, \cdots, I-1,\ j = 0, \cdots, J-1, u = 1, \cdots, K,\ t = 1, \cdots, K-1$.

Together with the regularization equations

$$\epsilon z_{ij}^{x} = 0, \quad \epsilon z_{ij}^{y} = 0, \qquad i = 0, \cdots, I,\ j = 0, \cdots, J, \tag{6.4}$$

the system (6.3) can be written in the following matrix form

$$\mathbf{A}\mathbf{x} = \mathbf{b}. \tag{6.5}$$

Here, $\mathbf{A}$ is a known, banded matrix with $2(I+1)(J+1)$ columns and $6IJK(K-1) + 2(I+1)(J+1)$ rows, $\mathbf{x}$ is a $2(I+1)(J+1)$-dimensional vector of the unknowns z_{ij}^{x} and z_{ij}^{y}, $i = 0, \cdots, I$, $j = 0, \cdots, J$, and $\mathbf{b}$ is a known right-hand-side vector.

Solving the overdetemined linear system (6.5) in the discrete l_1 norm is equivalent to solve the optimization problem

$$\underset{\mathbf{x}}{\text{minimize}} \left\| \mathbf{A}\mathbf{x} - \mathbf{b} \right\|_1, \tag{6.6}$$

where $\| \bullet \|_1$ is the vector 1-norm. It is not difficult to verify that the optimization problem (6.6) can be translated into the following linear program,

$$\begin{aligned} \min \quad & \bar{\mathbf{c}}\mathbf{x} \\ \text{s.t.} \quad & \bar{\mathbf{A}}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned} \tag{6.7}$$

where $\bar{\mathbf{c}} = [\mathbf{0}, \mathbf{0}, \mathbf{e}, \mathbf{e}] \in \mathbb{R}^{12IJK(K-1)+8(I+1)(J+1)}$, $\bar{\mathbf{A}} = [\mathbf{A}, -\mathbf{A}, -\mathbf{I}, \mathbf{I}]$ a $(6IJK(K-1) + 2(I+1)(J+1)) \times (12IJK(K-1) + 8(I+1)(J+1))$ matrix, and $\mathbf{e} = (1, \cdots, 1) \in \mathbb{R}^{6IJK(K-1)+2(I+1)(J+1)}$.

Thus, generating a discretized bivariate cubic L_1 spline is equivalent to solving a linear program (6.7).

6.4 Interior-point methods

Two interior-point methods are proposed to solve linear program (6.7).

6.4.1 Primal affine scaling method

The basic idea behind primal affine scaling algorithm is to apply an appropriate transformation to the solution space such that the current interior solution is placed near the center of the transformed solution space. Then, moving along the steepest descent direction with an appropriate step length to obtain a new point with reduced objective value [36].

Let $\mathbf{x}^k \in \mathbb{R}^\rho$, where $\rho = 12IJK(K-1) + 8(I+1)(J+1)$, be an interior point of the nonnegative orthant $\mathbb{R}^\rho_+$, i.e., $x_i^k > 0$ for $i = 1, \ldots, \rho$. We define an $\rho \times \rho$ diagonal matrix

$$\mathbf{X}_k = \operatorname{diag}(\mathbf{x}^k) = \begin{bmatrix} x_1^k & 0 & \ldots & 0 \\ 0 & x_2^k & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & x_\rho^k \end{bmatrix}. \tag{6.8}$$

It is easily seen that $\mathbf{X}_k$ is nonsingular with an inverse matrix $\mathbf{X}_k^{-1}$, which is also a diagonal matrix with $1/x_i^k$ as its i^{th} diagonal element for $i = 1, \ldots, \rho$.

Algorithm 6.4.1 [Primal Affine Scaling Algorithm]

Step 1. *(Initialization): Set* $k = 0$ *and find* $\mathbf{x}^0 > 0$ *such that* $\bar{\mathbf{A}}\mathbf{x}^0 = \mathbf{b}$.

Step 2. *(computation of dual estimates): Compute the vector of dual estimates*

$$\mathbf{w}^k = (\bar{\mathbf{A}}\mathbf{X}_k^2\bar{\mathbf{A}}^T)^{-1}\bar{\mathbf{A}}\mathbf{X}_k^2\bar{\mathbf{c}}$$

where $\mathbf{X}_k$ *is a diagonal matrix whose diagonal elements are the components of* $\mathbf{x}^k$*.*

Step 3. *(computation of reduced costs): Calculate the reduced costs vector*

$$\mathbf{r}^k = \bar{\mathbf{c}} - \bar{\mathbf{A}}^T\mathbf{w}^k.$$

Step 4. *(check for optimality): If* $\mathbf{r}^k \geq 0$ *and* $\mathbf{e}^T\mathbf{X}_k\mathbf{r}^k \leq \epsilon$*(a given small positive number), then STOP.* $\mathbf{x}^k$ *is primal optimal and* $\mathbf{w}^k$ *is dual optimal. Otherwise, go to the next step.*

Step 5. *(obtain the direction of translation): Compute the direction*

$$\mathbf{d}_y^k = -\mathbf{X}_k\mathbf{r}^k.$$

Step 6. *(checking for unboundedness and constant objective value): If* $\mathbf{d}_y^k > 0$*, then STOP. The problem is unbounded. If* $\mathbf{d}_y^k = 0$*, then also STOP.* $\mathbf{x}^k$ *is primal optimal. Otherwise go to step 7.*

Step 7. *(compute step-length): Compute the step-length*

$$\alpha_k = \min_i \left\{ \frac{\alpha}{-d_{y_i}^k} \middle| (d_y^k)_i < 0 \right\} \qquad 0 < \alpha < 1$$

Step 8. *(move to a new solution): Perform the translation*

$$\mathbf{x}^{k+1} = \mathbf{x}^k + \alpha_k\mathbf{X}_k\mathbf{d}_y^k.$$

Reset $k \leftarrow k+1$ *and go to step 2.*

This primal affine scaling algorithm requires a feasible initial interior point $\mathbf{x}^0 > 0$

satisfying $\bar{\mathbf{A}}\mathbf{x}^0 = \mathbf{b}$ (c.f. *Step 1*). Such an initial point can be obtained by using the 2-phase or big-M methods. However, these two methods will either increase the computational effort dramatically or destroy the sparse structure of the problem. Therefore, constructing a feasible initial interior point becomes practically important. Set $\hat{\mathbf{x}}$ to be

$$\hat{\mathbf{x}} = \begin{bmatrix} 2\mathbf{e} \\ \mathbf{e} \\ (\mathbf{Ae}-\mathbf{b})^{+}+\mathbf{e} \\ (\mathbf{Ae}-\mathbf{b})^{-}+\mathbf{e} \end{bmatrix}, \tag{6.9}$$

where

$$(Ae-b)_i^{+} = \begin{cases} A_ie-b_i, & A_ie-b_i>0 \\ 0, & A_ie-b_i\leq 0 \end{cases},$$

$$(Ae-b)_i^{-} = \begin{cases} 0, & A_ie-b_i\geq 0 \\ b_i-A_ie, & A_ie-b_i<0 \end{cases},$$

and A_i is the i^{th} row of A. It is easy to verify $\hat{\mathbf{x}} > \mathbf{0}$ and $\bar{\mathbf{A}}\hat{\mathbf{x}} = \mathbf{b}$. Therefore $\hat{\mathbf{x}}$ can be served as an initial point for the primal affine scaling method.

6.4.2 Primal-dual method

In this subsection, we introduce a primal-dual interior-point algorithm [10] which is generally more robust than the primal affine scaling algorithm, to solve linear program (6.7).

The primal-dual algorithm uses the idea of approximating the central path by taking Newton steps in both the primal and the dual spaces. In the following primal-dual interior-point algorithm, $\mathbf{x}^k \in \mathbb{R}^\rho$, $\mathbf{w}^k, \mathbf{s}^k \in \mathbb{R}^{6IJK(K-1)+2(I+1)(J+1)}$.

Algorithm 6.4.2 [Primal-Dual Interior-Point Algorithm]

Step 1. *(starting the algorithm): Set* $k = 0$*. Choose an arbitrary* $(\mathbf{x}^0; \mathbf{w}^0; \mathbf{s}^0)$ *with* $\mathbf{x}^0 > \mathbf{0}$ *and* $\mathbf{s}^0 > \mathbf{0}$*, and choose sufficiently small positive numbers* ϵ_1*,* ϵ_2 *and* ϵ_3*.*

Step 2. *(intermediate computations): Compute*

$$\mu^k = \frac{(\mathbf{x}^k)^T \mathbf{s}^k}{n}$$

$\mathbf{t}^k = \mathbf{b} - \bar{\mathbf{A}}\mathbf{x}^k$, $\mathbf{u}^k = \bar{\mathbf{c}} - \bar{\mathbf{A}}^T\mathbf{w}^k - \mathbf{s}^k$, $\mathbf{v}^k = \mu^k\mathbf{e} - \mathbf{X}_k\mathbf{S}_k\mathbf{e}$, $\mathbf{p}^k = \mathbf{X}_k^{-1}\mathbf{v}^k$, *and* $\hat{\mathbf{D}}_k^2 = \mathbf{X}_k\mathbf{S}_k^{-1}$*, where* $\mathbf{X}_k$ *and* $\mathbf{S}_k$ *are diagonal matrices whose diagonal entries are* x_i^k *and* s_i^k*, respectively.*

Step 3. *(checking for optimality): If*

$$\mu^k < \epsilon_1, \quad \frac{\|\mathbf{t}\|}{\|\mathbf{b}\| + 1} < \epsilon_2, \quad \text{and} \quad \frac{\|\mathbf{u}\|}{\|\bar{\mathbf{c}}\| + 1} < \epsilon_3$$

then STOP. The solution is optimal. Otherwise go to the next stop.

[Note: $\|\mathbf{u}\|$ *and* $\|\bar{\mathbf{c}}\|$ *are computed only when the dual constraints are violated. If* $\mathbf{u} \geq \mathbf{0}$*, then there is no need to compute this measure of optimality.]*

Step 4. *(calculating directions of translation): Compute*

$$\begin{aligned}
\mathbf{d}_w^k &= \left(\bar{\mathbf{A}}\hat{\mathbf{D}}_k^2\bar{\mathbf{A}}^T\right)^{-1}\left(\bar{\mathbf{A}}\hat{\mathbf{D}}_k^2\left(\mathbf{u}^k - \mathbf{p}^k\right) + \mathbf{t}^k\right) \\
\mathbf{d}_s^k &= \mathbf{u}^k - \bar{\mathbf{A}}^T\mathbf{d}_w^k \\
\mathbf{d}_x^k &= \hat{\mathbf{D}}_k^2\left(\mathbf{p}^k - \mathbf{d}_s^k\right)
\end{aligned}$$

Step 5. *(checking for unboundedness): If*

$$\mathbf{t}^k = \mathbf{0},\ \mathbf{d}_x^k > \mathbf{0} \quad \textit{and} \quad \bar{\mathbf{c}}^T \mathbf{d}_x^k < \mathbf{0}$$

then the primal problem is unbounded. If

$$\mathbf{u}^k = \mathbf{0},\ \mathbf{d}_s^k > \mathbf{0} \quad \textit{and} \quad \mathbf{b}^T \mathbf{d}_w^k > \mathbf{0}$$

then the dual problem is unbounded. If either of these cases happens, STOP.

Otherwise go to the next step.

Step 6. *(finding step-lengths): Compute the primal and dual step-lengths*

$$\beta_P = \frac{1}{\max\left\{1, -d_{x_i}^k / \alpha x_i^k\right\}}$$

and

$$\beta_D = \frac{1}{\max\left\{1, -d_{s_i}^k / \alpha s_i^k\right\}}$$

where $\alpha < 1$ *(say 0.99).*

Step 7. *(moving to a new point): Update the solution vectors*

$$\begin{aligned}
\mathbf{x}^{k+1} &\leftarrow \mathbf{x}^k + \beta_P \mathbf{d}_x^k \\
\mathbf{w}^{k+1} &\leftarrow \mathbf{w}^k + \beta_D \mathbf{d}_w^k \\
\mathbf{s}^{k+1} &\leftarrow \mathbf{s}^k + \beta_D \mathbf{d}_s^k
\end{aligned}$$

Set $k \leftarrow k+1$ *and go to* ***Step 2****.*

Note that this algorithm does not require a phase one procedure to generate an initial feasible solution.

6.5 Computational results

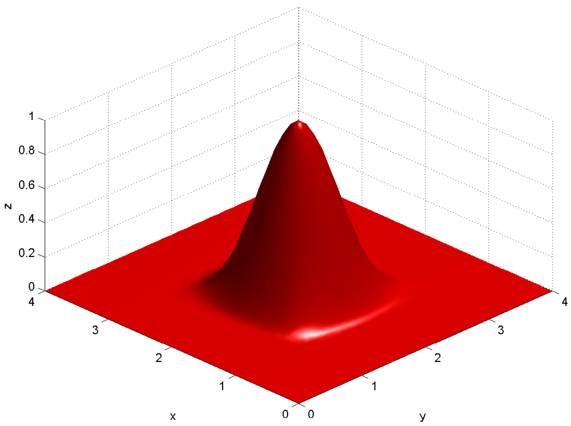
The two proposed interior-point methods for generating discretied bivariate cubic L_1 splines have been tested for eight sets of data. Algorithm 6.4.1 and Algorithm 6.4.2 are both implemented in MATLAB. In the computational experiments, we set K to be 3 and the regularization parameter ϵ to be 10^{-4}. Both algorithms stop when the primal infeasibility and dual infeasibility are less than 10^{-5} and duality gap is less than 10^{-3}. For the primal affine method, (6.9) is used as the initial point. All computational experiments were run on a Pentium 4 computer with 1.4GHz CPU and 512M memory.

For each instance, we present the data set first. Then the plots of the resulting bivariate cubic L_1 spline, the convergence trajectories of the duality gap, the primal and the dual infeasibility versus iterations (horizontal axis) for the primal affine scaling method[1] and the primal-dual method are illustrated in three subfigures (a), (b) and (c) for each example.

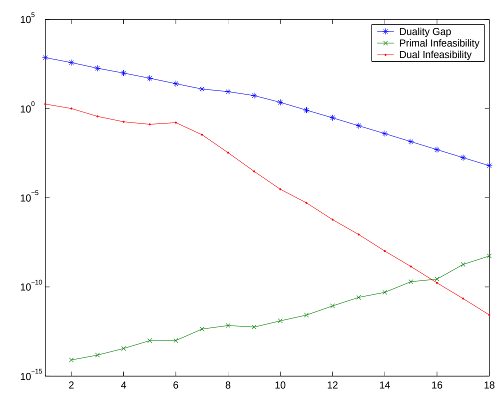
[1]Since the initial starting point for primal affine scaling method is feasible, the point representing the infeasibility of the first iteration, which is 0 therefore, does not appear in the corresponding figure.

Example 6.5.1 (Single peak on flat surface) *Data:* $I = 4, J = 4$, $x_i = i$, $i = 0, 1, \ldots, I$, $y_j = j$, $j = 0, 1, \ldots, J$,

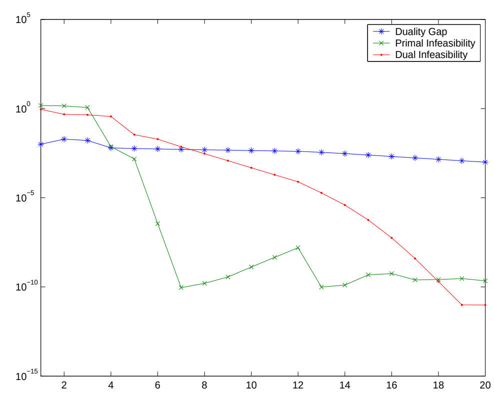
$$\mathbf{z} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$



(a) Bivariate cubic L_1 spline interpolation



(b) Convergence trajectories of duality gap, primal and dual infeasibilities of primal affine scaling method

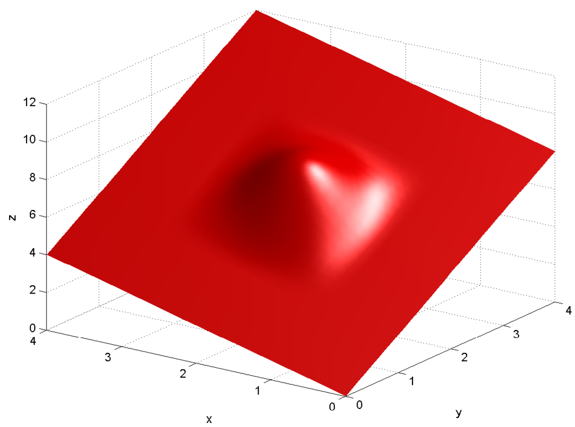


(c) Convergence trajectories of duality gap, primal and dual infeasibilities of primal-dual method

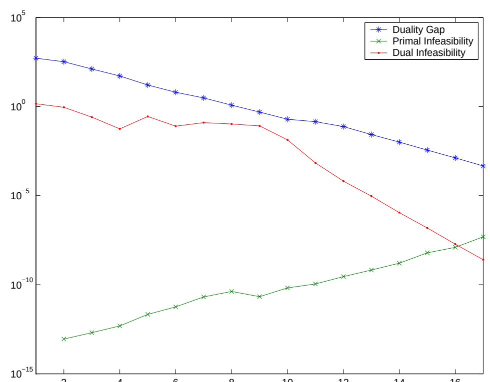
Figure 6.1: Splines for data representing single peak on flat surface

Example 6.5.2 (Single peak on slanted surface) *Data:* $I = 4, J = 4,\ x_i = i, i = 0, 1, \ldots, I,\ y_j = j,\ j = 0, 1, \ldots, J,$

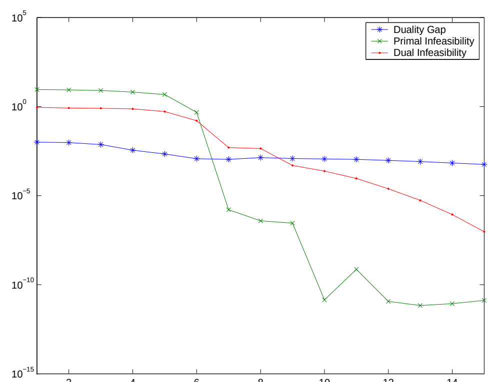
$$\mathbf{z} = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 8 & 7 & 8 \\ 6 & 7 & 8 & 9 & 10 \\ 8 & 9 & 10 & 11 & 12 \end{bmatrix}.$$



(a) Bivariate cubic L_1 spline interpolation



(b) Convergence trajectories of duality gap, primal and dual infeasibilities of primal affine scaling method

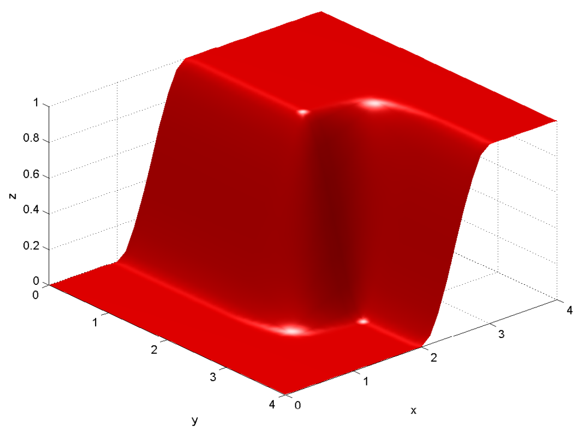


(c) Convergence trajectories of duality gap, primal and dual infeasibilities of primal-dual method

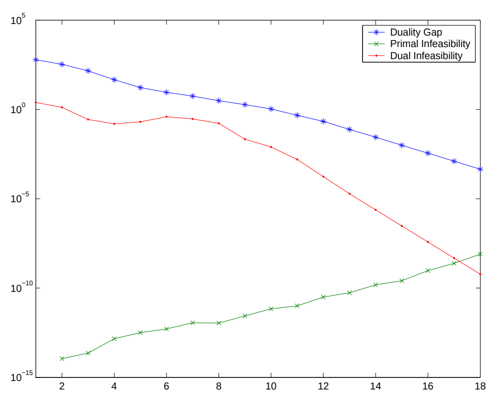
Figure 6.2: Splines for data representing single peak on slanted surface

Example 6.5.3 (Step discontinuity) *Data:* $I = 4, J = 4$, $x_i = i$, $i = 0, 1, \ldots, I$, $y_j = j$, $j = 0, 1, \ldots, J$,

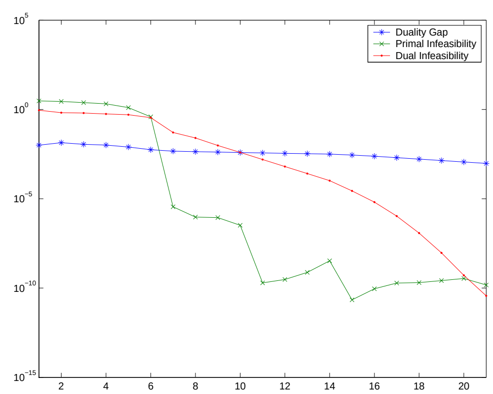
$$\mathbf{z} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$



(a) Bivariate cubic L_1 spline interpolation



(b) Convergence trajectories of duality gap, primal and dual infeasibilities of primal affine scaling method

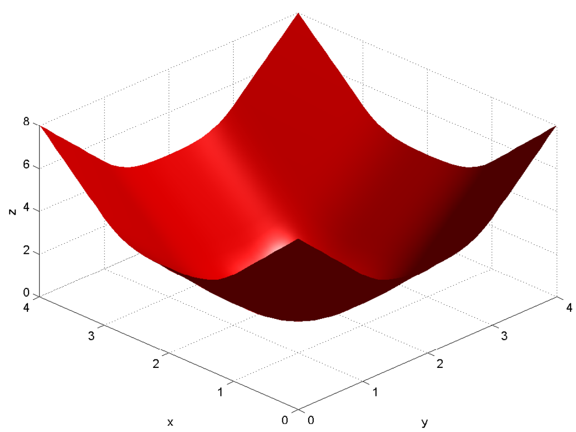


(c) Convergence trajectories of duality gap, primal and dual infeasibilities of primal-dual method

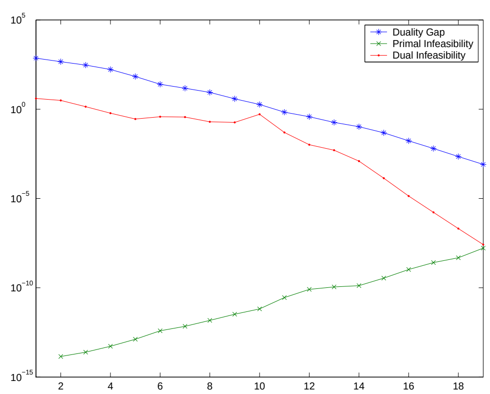
Figure 6.3: Splines for data representing step discontinuity

Example 6.5.4 (Bowl of a paraboloid) *Data:* $I = 4, J = 4$, $x_i = i$, $i = 0, 1, \ldots, I$, $y_j = j$, $j = 0, 1, \ldots, J$,

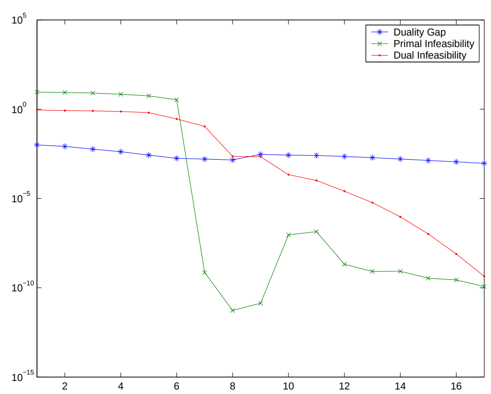
$$\mathbf{z} = \begin{bmatrix} 8 & 5 & 4 & 5 & 8 \\ 5 & 2 & 1 & 2 & 5 \\ 4 & 1 & 0 & 1 & 4 \\ 5 & 2 & 1 & 2 & 5 \\ 8 & 5 & 4 & 5 & 8 \end{bmatrix}.$$



(a) Bivariate cubic L_1 spline interpolation



(b) Convergence trajectories of duality gap, primal and dual infeasibilities of primal affine scaling method

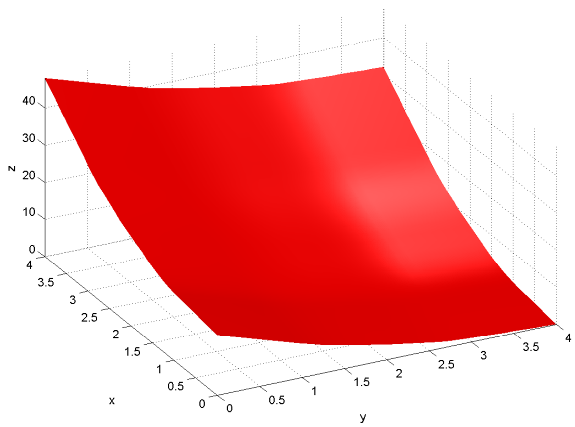


(c) Convergence trajectories of duality gap, primal and dual infeasibilities of primal-dual method

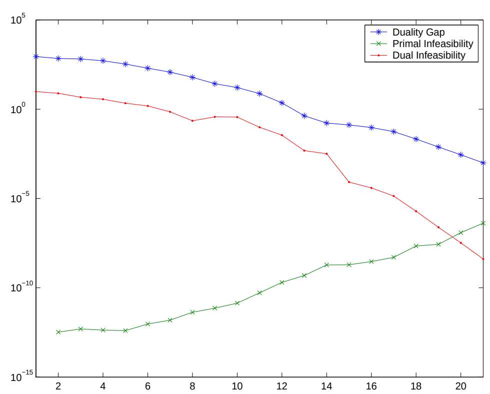
Figure 6.4: Splines for data representing bowl of a paraboloid

Example 6.5.5 (Side of a paraboloid) *Data:* $I = 4, J = 4,\ x_i = i,\ i = 0, 1, \ldots, I,$ $y_j = j,\ j = 0, 1, \ldots, J,$

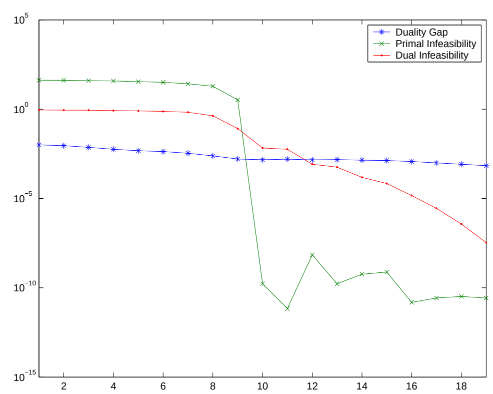
$$\mathbf{z} = \begin{bmatrix} 16 & 18 & 24 & 34 & 48 \\ 9 & 11 & 17 & 27 & 41 \\ 4 & 6 & 12 & 22 & 36 \\ 1 & 3 & 9 & 19 & 33 \\ 0 & 2 & 8 & 18 & 32 \end{bmatrix}.$$



(a) Bivariate cubic L_1 spline interpolation



(b) Convergence trajectories of duality gap, primal and dual infeasibilities of primal affine scaling method

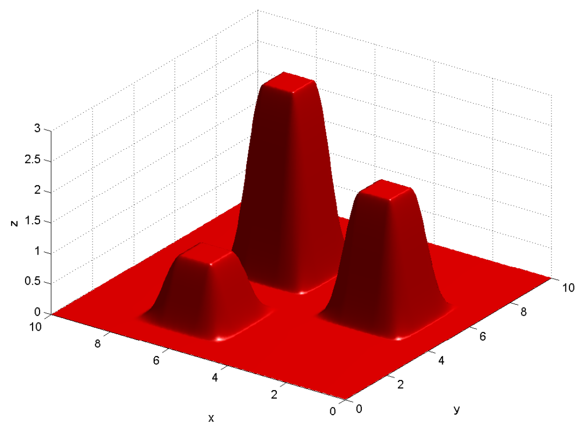


(c) Convergence trajectories of duality gap, primal and dual infeasibilities of primal-dual method

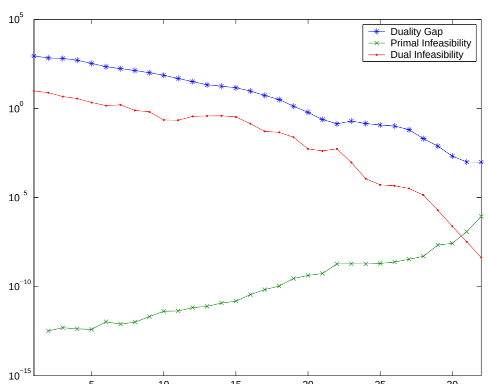
Figure 6.5: Splines for data representing side of a paraboloid

Example 6.5.6 (4-point peaks on flat surface) *Data:* $I = 10, J = 10$, $x_i = i$, $i = 0, 1, \ldots, I$, $y_j = j$, $j = 0, 1, \ldots, J$, *The* $\mathbf{z}$ *matrix is calculated by setting* $z_{ij} = 0$ *for* $i = 0, 1, \ldots, I$ *and* $j = 0, 1, \ldots, J$ *and then changing*

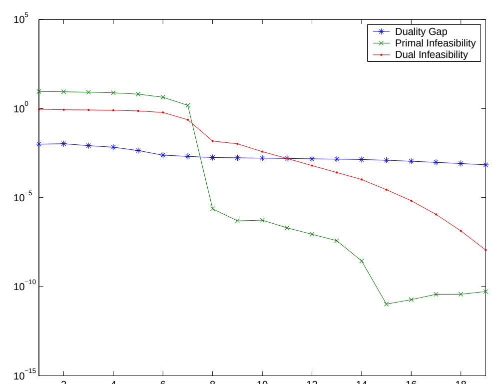
$$\begin{array}{llllll} z_{2,6} = 1 & z_{2,7} = 1 & z_{5,2} = 2 & z_{5,3} = 2 & z_{6,6} = 3 & z_{6,7} = 3 \\ z_{3,6} = 1 & z_{3,7} = 1 & z_{6,2} = 2 & z_{6,3} = 2 & z_{7,6} = 3 & z_{7,7} = 3 \end{array}$$



(a) Bivariate cubic L_1 spline interpolation



(b) Convergence trajectories of duality gap, primal and dual infeasibilities of primal affine scaling method

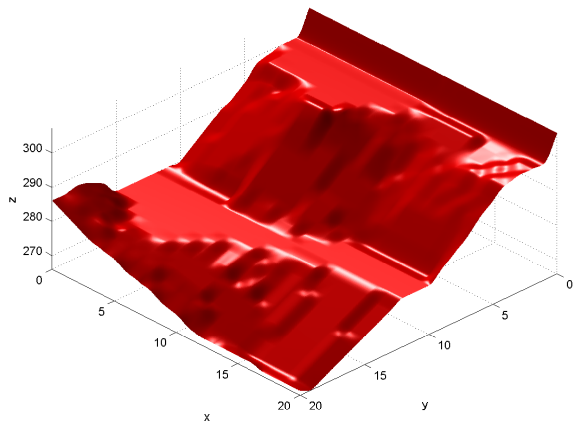


(c) Convergence trajectories of duality gap, primal and dual infeasibilities of primal-dual method

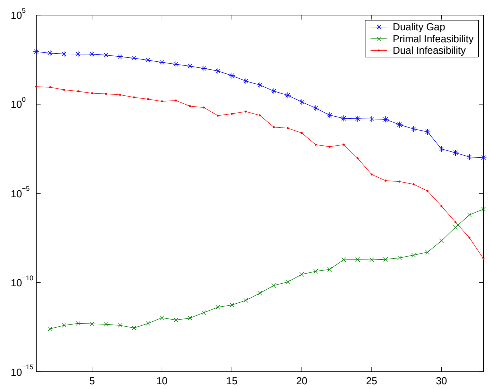
Figure 6.6: Splines for data representing 4-point peaks on flat surface

Example 6.5.7 (Terrain) *Data:* $I = 20, J = 20,\ x_i = i,\ i = 0, 1, \ldots, I,\ y_j = j,\ j = 0, 1, \ldots, J,$

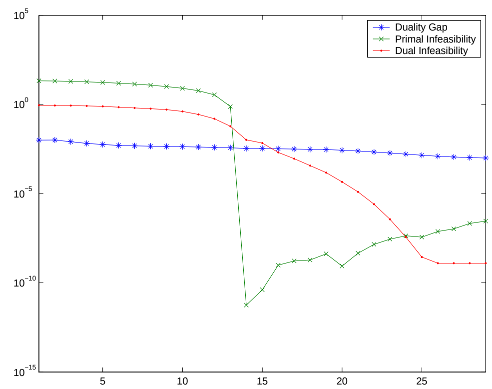
z:	307	307	307	307	307	307	307	307	307	307	307	307	307	307	307	307	307	307	307	307	307
	300	300	300	300	300	300	300	300	300	300	300	300	300	300	300	300	300	300	300	300	300
	300	300	300	300	300	300	300	300	300	300	300	300	300	300	300	300	299	299	299	300	300
	299	300	300	300	300	300	300	300	300	300	300	299	298	298	298	298	298	298	299	299	300
	298	300	300	300	300	300	300	300	300	299	298	297	296	295	296	296	297	298	298	299	299
	294	296	297	297	299	300	300	300	300	298	296	295	293	292	293	294	294	295	295	296	296
	291	293	294	296	297	299	300	300	297	295	293	292	290	290	290	291	291	291	292	292	293
	289	290	292	293	294	295	296	295	293	292	290	289	288	287	287	288	288	288	288	288	289
	286	287	289	290	291	292	292	291	290	289	287	286	285	284	284	284	284	284	284	284	286
	283	285	286	287	288	288	289	288	287	286	285	284	282	281	280	280	280	280	280	280	283
	280	282	283	284	284	284	286	285	284	283	283	282	281	280	280	280	280	280	280	280	280
	280	280	280	280	280	280	283	282	282	281	281	280	280	280	280	280	280	280	280	280	280
	280	280	280	280	280	280	280	280	280	280	280	280	280	280	280	280	280	280	280	280	280
	280	280	280	280	280	280	280	280	280	280	280	280	280	280	280	279	279	279	279	278	278
	280	280	280	280	280	280	280	280	280	280	280	279	279	278	278	277	277	277	277	276	276
	280	280	280	280	280	280	280	280	280	279	279	278	277	276	276	276	275	275	275	274	274
	284	282	280	280	280	280	280	280	279	278	278	276	276	274	274	274	273	273	273	273	272
	286	284	282	280	280	280	280	279	278	277	276	275	274	273	272	272	271	271	271	271	270
	287	285	283	281	281	280	279	278	277	276	275	273	272	271	270	269	269	269	269	269	268
	286	285	283	282	280	279	278	277	275	274	273	272	270	269	268	267	267	267	267	267	266
	286	284	282	280	279	278	276	275	274	272	271	269	268	268	268	268	268	267	267	267	267



(a) Bivariate cubic L_1 spline interpolation



(b) Convergence trajectories of duality gap, primal and dual infeasibilities of primal affine scaling method

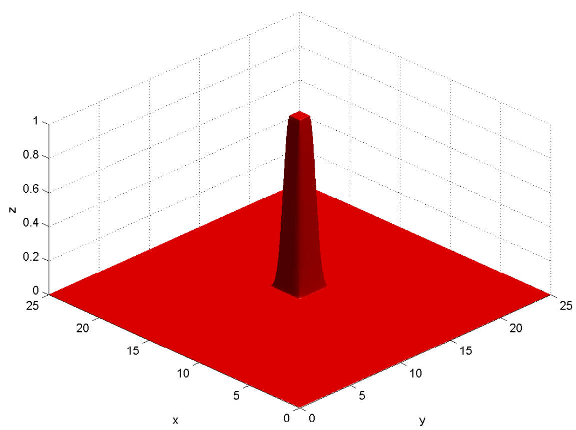


(c) Convergence trajectories of duality gap, primal and dual infeasibilities of primal-dual method

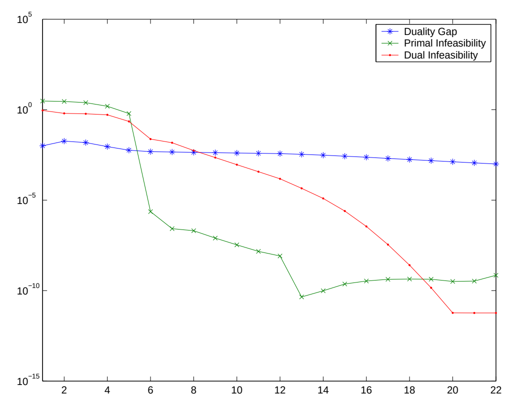
Figure 6.7: Splines for data representing terrain

Example 6.5.8 (Single 4-point peak on flat surface) *Data:* $I = 25, J = 25,\ x_i = i,\ i = 0, 1, \ldots, I,\ y_j = j,\ j = 0, 1, \ldots, J,$ *The* $\mathbf{z}$ *matrix is calculated by setting* $z_{ij} = 0$ *for* $i = 0, 1, \ldots, I$ *and* $j = 0, 1, \ldots, J$ *and then changing*

$$z_{13,13} = 1 \quad z_{13,14} = 1$$
$$z_{14,13} = 1 \quad z_{14,14} = 1.$$



(a) Bivariate cubic L_1 spline interpolation



(b) Convergence trajectories of duality gap, primal and dual infeasibilities of primal-dual method

Figure 6.8: Splines for data representing single 4-point peak on flat surface

From the plots of the eight examples, one can see that there is no extraneous oscillation involved in the generated L_1 splines. This fact illustrates the shape-preservation of bivariate cubic L_1 splines.

The number of variables and the number of constraints in the corresponding linear programming formulation for the above eight examples are listed in Table 6.1. It shows that the size of the linear program increases rapidly with respect to the grid size, I and J.

In order to solve these large scale problems, we need to resort to numerical techniques. Otherwise, we cannot even solve the problem with $I = 10$ and $J = 10$. The bottleneck of

I	J	Number of variables	Number of constraints
4	4	1352	626
10	10	8168	3842
20	20	32328	15282
25	25	50408	23852

Table 6.1: The scales of the testing problems with $K = 3$.

these two interior-point methods is to compute the inverse of matrix

$$\bar{\mathbf{A}}\mathbf{X}_k\mathbf{S}_k^{-1}\bar{\mathbf{A}}^T \text{ or } \bar{\mathbf{A}}\mathbf{X}_k^2\bar{\mathbf{A}}^T. \tag{6.10}$$

Our experiments show that the computation of the inverse of matrix (6.10) accounts for about 75% of the total running time. In order to speed up the algorithm and reduce the usage of memory, we used reverse Cuthill-Mckee algorithm [13] to transfer sparse matrix (6.10) into a block-diagonal matrix with narrow band-width. Then some factorization techniques, such as Cholesky and QR factorization [15], is applied to the matrix (6.10) to compute the inverse matrix. By using these techniques, we can handle the problems with sizes up to $I = 28$ and $J = 28$.

The statistics of computational experiments for the primal affine scaling method and the primal-dual interior-point method are summarized in Table 6.2 and Table 6.3 respectively. In these tables, for each testing example, the total number of iterations, the computing time (the time to generate the coefficients for cubic L_1 splines), the duality gap, the primal infeasibility and the dual infeasibility of the solution are reported.

Example	Iters	Time(sec)	Duality Gap	Primal infeasibility	Dual infeasibility
6.5.1	18	1.118	6.2760E-4	5.4981E-9	2.6750E-12
6.5.2	17	1.185	4.6433E-4	4.8787E-8	2.5167E-9
6.5.3	18	1.127	4.4855E-4	7.9698E-9	6.0023E-10
6.5.4	19	1.257	8.0700E-4	1.6568E-8	2.6248E-8
6.5.5	21	1.543	9.8909E-4	4.1408E-7	4.9015E-9
6.5.6	32	61.754	9.7325E-4	8.7402E-7	4.2906E-9
6.5.7	33	1794.650	9.9096E-4	1.3181E-6	2.1611E-9
6.5.8	n/a[2]	n/a	n/a	n/a	n/a

Table 6.2: Statistics of computational experiments for primal affine scaling method

Example	Iters	Time(sec)	Duality Gap	Primal infeasibility	Dual infeasibility
6.5.1	20	1.482	9.9484E-4	9.1165E-11	9.6989E-12
6.5.2	15	1.147	5.6573E-4	6.7821E-12	9.4225E-8
6.5.3	21	1.586	9.5071E-4	2.1884E-11	3.6732E-11
6.5.4	17	1.284	9.3197E-4	5.3184E-12	4.4298E-10
6.5.5	19	1.375	6.8522E-4	6.8594E-12	3.3915E-8
6.5.6	19	35.884	6.9763E-4	1.0506E-11	1.1324E-8
6.5.7	29	1558.926	9.9827E-4	5.6701E-12	1.2541E-9
6.5.8	22	3672.985	9.9903E-4	4.4611E-11	5.8524E-12

Table 6.3: Statistics of computational experiments for primal-dual method

As shown in Table 6.2, primal affine scaling method converges in 35 iterations for example 6.5.1 – 6.5.7. But it fails to converge for example 6.5.8 due to the large problem size. Table 6.3 shows that the primal-dual interior-point method converges to the solution in 30 iterations for all the examples.

Observing the figures relating to the primal affine method, one can find that the dual

[2] After 500 iterations, the duality gap cannot be reduced less than 10^{-3}.

infeasibility decreases faster than the duality gap and the primal infeasibility tends to increase gradually. In contrast, the convergence plots of the primal-dual method show the speed of the primal infeasibility is in general faster than that of the dual infeasibility and the convergence speed of duality gap is relatively slow compared to the convergence of infeasibilities.

In order to measure the performance of these two methods, we compare the number of iterations, the primal infeasibility and the duality gap of these two methods in Table 6.4.

Example	Iterations		Primal Infeasibility		Duality Gap	
	P[3]	P-D	P	P-D	P	P-D
6.5.1	18	20	5.4981E-9	9.1165E-11	6.2760E-4	9.9484E-4
6.5.2	17	18	4.8787E-8	6.7821E-12	4.6433E-4	5.6573E-4
6.5.3	18	21	7.9698E-9	2.1884E-11	4.4855E-4	9.5071E-4
6.5.4	19	19	1.6568E-8	5.3184E-12	8.0700E-4	9.3197E-4
6.5.5	21	19	4.1408E-7	6.8594E-12	9.8909E-4	6.8522E-4
6.5.6	32	19	8.7402E-7	1.0506E-11	9.7325E-4	6.9763E-4
6.5.7	33	29	1.3181E-6	5.6701E-12	9.9096E-4	9.9827E-4
6.5.8	n/a	22	n/a	4.4611E-11	n/a	9.9903E-4

Table 6.4: Comparison between primal affine scaling and primal-dual method

As shown in Table 6.4, the primal affine scaling method fails to reduce the duality gap of example 6.5.8 down to 10^{-3} in 500 iterations. The primal-dual method converges to the optimal solution within 30 iterations for all the examples. But for small size examples, the primal affine scaling method may take fewer iterations than the primal-dual method. In contrast, the primal-dual method converges faster for larger size examples. Another observation is that, the solution obtained by the primal-dual method has smaller primal

[3] "P" represents "Primal affine scaling method"; "P-D" represents "Primal-dual method"

infeasibility than the primal affine scaling method when the duality gap is reduced to 10^{-3}. Therefore, under the same discretization framework, the primal-dual method may be more preferable to the primal affine scaling method in terms of robustness and fast convergence.

6.6 Conclusion and discussion

In this chapter, we have proposed two interior-point methods, the primal affine scaling method and the primal-dual method, for generating discretized bivariate cubic L_1 splines. Computational results show that for small size problems, both methods converge and the primal affine scaling method may converge faster than the primal-dual method. However, for larger size problems, the primal-dual method demonstrates its advantages of having fast convergence and robustness over the primal affine scaling method.

Although the primal-dual method shows the robustness and efficiency, it is not ideal for solving large-scale problems. There is a compressed version of the primal affine method [22], which can solve large-scale problem. However, this compressed method is not robust as the uncompressed version. The compression scheme cannot be used directly for the primal-dual method because of the structure of the central path equations in the primal-dual method. In next chapter, we will develop a compressed version of the primal-dual method to handle large-scale problems.

Chapter 7

Compressed Primal-dual Interior Point Method for Discretized Bivariate Cubic L_1 Splines

7.1 Introduction

The interior point methods proposed in Chapter 6 generate bivariate cubic L_1 splines efficiently for small and medium size data set. However, inverting a $[6IJK(K-1)+2(I+1)(J+1)] \times [6IJK(K-1)+2(I+1)(J+1)]$ matrix may pervent them from handling large scale problems.

In 2001, Lavery proposed a compressed primal affine scaling method [22] to generate discretized bivariate cubic L_1 splines. This method can efficiently solve large scale problems in most cases. However, it fails to converge for some test problems. In this chapter we develop a compressed primal-dual method to generate large scale bivariate cubic L_1 splines.

7.2 Compressed primal-dual method

As described in Section 6.3, we need to solve the following optimization problem for generating discretized bivariate cubic L_1 splines:

$$\min_{\mathbf{x}} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_1 , \tag{7.1}$$

where $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{x} \in \mathbb{R}^n$, m is the number of sample points and $n = 2(I+1)(J+1)$ is the number of variables. (Usually $m \gg n$.)

(7.1) can be transformed as the following LP problem,

$$\begin{array}{llrcrcrcl} & \min & & & \mathbf{e}^T\mathbf{r}_+ & + & \mathbf{e}^T\mathbf{r}_- & & \\ \text{(Primal)} & s.t. & \mathbf{A}\mathbf{x} & - & \mathbf{r}_+ & + & \mathbf{r}_- & = & \mathbf{b} \\ & & & & \mathbf{r}_+ & , & \mathbf{r}_- & \geq & \mathbf{0}. \end{array} \tag{7.2}$$

The corresponding dual problem of (7.2) is

$$\begin{array}{llrcrcl} & \max & & & \mathbf{b}^T\mathbf{y} & & \\ \text{(Dual)} & s.t. & & & \mathbf{A}^T\mathbf{y} & = & \mathbf{0} \\ & & -\mathbf{e} & \leq & \mathbf{y} & \leq & \mathbf{e}. \end{array} \tag{7.3}$$

Applying the logarithmic barrier function technique to (7.2), we have the following nonlinear programming formulation

$$\begin{array}{llrclcrcl} & \min & & & \mathbf{e}^T\mathbf{r}_+ & + & \mathbf{e}^T\mathbf{r}_- & + & \mu\sum_{j=1}^{m}(\ln(r_+)_j + \ln(r_-)_j) \\ (\mathrm{P}_\mu) & s.t. & \mathbf{A}\mathbf{x} & - & \mathbf{r}_+ & + & \mathbf{r}_- & = & \mathbf{b} \\ & & & & \mathbf{r}_+ & , & \mathbf{r}_- & > & \mathbf{0}. \end{array} \tag{7.4}$$

The Karush-Kuhn-Tucker (KKT) conditions of above nonlinear program can be written as

$$\begin{aligned} \mathbf{A}^T\mathbf{y} &= \mathbf{0}, \qquad -\mathbf{e} < \mathbf{y} < \mathbf{e}, \\ \mathbf{A}\mathbf{x} - \mathbf{r}_+ + \mathbf{r}_- &= \mathbf{b}, \qquad \mathbf{r}_+ > 0, \mathbf{r}_- > 0, \\ (\mathbf{I}+\mathbf{Y})\mathbf{R}_+\mathbf{e} &= \mu\mathbf{e}, \\ (\mathbf{I}-\mathbf{Y})\mathbf{R}_-\mathbf{e} &= \mu\mathbf{e}, \end{aligned} \tag{7.5}$$

where $\mathbf{Y}$, $\mathbf{R}$ are the diagonal matrices formed by $\mathbf{y}$, $\mathbf{r}$, respectively.

The Newton direction of this system is determined by

$$\begin{bmatrix} \mathbf{A}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} & -\mathbf{I} & \mathbf{I} \\ \mathbf{R}_+^k & \mathbf{0} & (\mathbf{I}+\mathbf{Y}^k) & \mathbf{0} \\ -\mathbf{R}_-^k & \mathbf{0} & \mathbf{0} & (\mathbf{I}-\mathbf{Y}^k) \end{bmatrix} \begin{bmatrix} \mathbf{d}_y^k \\ \mathbf{d}_x^k \\ \mathbf{d}_{r_+}^k \\ \mathbf{d}_{r_-}^k \end{bmatrix} = - \begin{bmatrix} \mathbf{A}^T\mathbf{y}^k \\ \mathbf{A}\mathbf{x}^k - \mathbf{r}_+^k + \mathbf{r}_-^k - \mathbf{b} \\ (\mathbf{I}+\mathbf{Y}^k)\mathbf{R}_+^k\mathbf{e} - \mu^k\mathbf{e} \\ (\mathbf{I}-\mathbf{Y}^k)\mathbf{R}_-^k\mathbf{e} - \mu^k\mathbf{e} \end{bmatrix}, \tag{7.6}$$

where $\mathbf{Y}^k$, $\mathbf{R}_+^k$ and $\mathbf{R}_-^K$ are the diagonal matrices formed by vector $\mathbf{y}$, $\mathbf{r}_+^k$ and $\mathbf{r}_-^k$, respectively. Multiplying it out, we have

$$\mathbf{A}^T\mathbf{d}_y^k = \mathbf{t}^k, \tag{7.7}$$

$$\mathbf{A}\mathbf{d}_x^k - \mathbf{d}_{r_+}^k + \mathbf{d}_{r_-}^k = \mathbf{u}^k, \tag{7.8}$$

$$\mathbf{R}_+^k\mathbf{d}_y^k + (\mathbf{I}+\mathbf{Y}^k)\mathbf{d}_{r_+}^k = \mathbf{v}_+^k, \tag{7.9}$$

$$-\mathbf{R}_-^k\mathbf{d}_y^k + (\mathbf{I}-\mathbf{Y}^k)\mathbf{d}_{r_-}^k = \mathbf{v}_-^k, \tag{7.10}$$

where

$$\mathbf{t}^k = -\mathbf{A}^T\mathbf{y}^k, \tag{7.11}$$

$$\mathbf{u}^k = \mathbf{b} - \mathbf{A}\mathbf{x}^k + \mathbf{r}_+^k - \mathbf{r}_-^k, \tag{7.12}$$

$$\mathbf{v}_+^k = \mu^k\mathbf{e} - (\mathbf{I}+\mathbf{Y}^k)\mathbf{R}_+^k\mathbf{e}, \tag{7.13}$$

$$\mathbf{v}_-^k = \mu^k\mathbf{e} - (\mathbf{I}-\mathbf{Y}^k)\mathbf{R}_-^k\mathbf{e}. \tag{7.14}$$

From (7.9) and (7.10), we have,

$$\mathbf{d}_{r_+}^k = \mathbf{p}^k - (\mathbf{I} + \mathbf{Y}_k)^{-1}\mathbf{R}_+^k \mathbf{d}_y^k, \tag{7.15}$$

$$\mathbf{d}_{r_-}^k = \mathbf{q}^k + (\mathbf{I} - \mathbf{Y}_k)^{-1}\mathbf{R}_-^k \mathbf{d}_y^k, \tag{7.16}$$

where

$$\mathbf{p}^k = (\mathbf{I} + \mathbf{Y}^k)^{-1}\mathbf{v}_+^k, \quad \mathbf{q}^k = (\mathbf{I} - \mathbf{Y}^k)^{-1}\mathbf{v}_-^k. \tag{7.17}$$

Plug (7.15) and (7.16) into (7.8), then we have

$$\mathbf{A}\mathbf{d}_x^k - \mathbf{p}^k + \mathbf{q}^k + \mathbf{S}^k\mathbf{d}_y^k = \mathbf{u}^k, \tag{7.18}$$

where

$$\mathbf{S}^k = (\mathbf{I} + \mathbf{Y}_k)^{-1}\mathbf{R}_+^k + (\mathbf{I} - \mathbf{Y}_k)^{-1}\mathbf{R}_-^k \tag{7.19}$$

is a diagonal matrix.

Multiply both sides of the above equation by $\mathbf{A}^T(\mathbf{S}^k)^{-1}$,

$$\mathbf{A}^T(\mathbf{S}^k)^{-1}\mathbf{A}\mathbf{d}_x^k = \mathbf{A}^T(\mathbf{S}^k)^{-1}(\mathbf{p}^k - \mathbf{q}^k + \mathbf{u}^k) - \mathbf{A}^T\mathbf{d}_y^k. \tag{7.20}$$

From (7.7),

$$\mathbf{d}_x^k = [\mathbf{A}^T(\mathbf{S}^k)^{-1}\mathbf{A}]^{-1}[\mathbf{A}^T(\mathbf{S}^k)^{-1}(\mathbf{p}^k - \mathbf{q}^k + \mathbf{u}^k) - \mathbf{t}^k]. \tag{7.21}$$

From (7.18),

$$\mathbf{d}_y^k = (\mathbf{S}^k)^{-1}(\mathbf{p}^k - \mathbf{q}^k + \mathbf{u}^k - \mathbf{A}\mathbf{d}_x^k). \tag{7.22}$$

From (7.15) and (7.16), we can calculate $\mathbf{d}_{r_+}^k$ and $\mathbf{d}_{r_-}^k$.

Thus, for $(\mathbf{x}^k, \mathbf{y}^k, \mathbf{r}_+^k, \mathbf{r}_-^k)$, where $-\mathbf{e} < \mathbf{y}^k < \mathbf{e}$ and $\mathbf{r}_+^k > 0,\ \mathbf{r}_-^k > 0$,

$$\begin{aligned}
\mathbf{d}_x^k &= [\mathbf{A}^T(\mathbf{S}^k)^{-1}\mathbf{A}]^{-1}[\mathbf{A}^T(\mathbf{S}^k)^{-1}(\mathbf{p}^k - \mathbf{q}^k + \mathbf{u}^k) - \mathbf{t}^k], \\
\mathbf{d}_y^k &= (\mathbf{S}^k)^{-1}(\mathbf{p}^k - \mathbf{q}^k + \mathbf{u}^k - \mathbf{A}\mathbf{d}_x^k), \\
\mathbf{d}_{r_+}^k &= \mathbf{p}^k - (\mathbf{I} + \mathbf{Y}_k)^{-1}\mathbf{R}_+^k \mathbf{d}_y^k, \\
\mathbf{d}_{r_-}^k &= \mathbf{q}^k + (\mathbf{I} - \mathbf{Y}_k)^{-1}\mathbf{R}_-^k \mathbf{d}_y^k.
\end{aligned}$$

After obtaining a Newton direction at the kth iteration, the compressed primal-dual algorithm iterates to a new point according to the following translation:

$$\mathbf{x}^{k+1} = \mathbf{x}^k + \beta^k \mathbf{d}_x^k \tag{7.23}$$

$$\mathbf{y}^{k+1} = \mathbf{y}^k + \beta^k \mathbf{d}_y^k \tag{7.24}$$

$$\mathbf{r}_+^{k+1} = \mathbf{r}_+^k + \beta^k \mathbf{d}_{r_+}^k \tag{7.25}$$

$$\mathbf{r}_-^{k+1} = \mathbf{r}_-^k + \beta^k \mathbf{d}_{r_-}^k \tag{7.26}$$

with an appropriately chosen step-length β^k at the kth iteration such that $-\mathbf{e} < \mathbf{y}^{k+1} < \mathbf{e}$ and $\mathbf{r}_+^{k+1} > \mathbf{0}$, $\mathbf{r}_-^{k+1} > \mathbf{0}$.

Let

$$(r_+^0)_i = \begin{cases} 1 - b_i, & -b_i > 0 \\ 1, & -b_i \le 0 \end{cases}, \qquad (r_-^0)_i = \begin{cases} 1, & -b_i \ge 0 \\ 1 + b_i, & -b_i < 0 \end{cases}. \tag{7.27}$$

It is not difficult to verify that

$$\boldsymbol{\chi} = (\mathbf{x}^0 = \mathbf{0}, \mathbf{y}^0 = \mathbf{0}, \mathbf{r}_+^0 > \mathbf{0}, \mathbf{r}_-^0 > \mathbf{0})^T$$

is a feasible interior point for (7.5). Therefore, the proposed algorithm may start from $\boldsymbol{\chi}$.

Based on above arguments, we summarize the compressed primal-dual interior-point method as follows.

Algorithm 7.2.1 [Compressed primal-dual interior-point algorithm]

Step 1. *(starting the algorithm): Set* $k = 0$. *Choose initial point* $\boldsymbol{\chi} = (\mathbf{x}^0 = \mathbf{0}, \mathbf{y}^0 = \mathbf{0},\ \mathbf{r}_+^0 > \mathbf{0}, \mathbf{r}_-^0 > \mathbf{0})^T$ *where* $\mathbf{r}_+^0$, $\mathbf{r}_-^0$ *are defined by (7.27), and choose sufficiently small positive numbers* ϵ_1, ϵ_2 *and* ϵ_3.

Step 2. *(intermediate computations): Compute*

$$\mu^k = \frac{(\mathbf{e} + \mathbf{y}^k)^T \mathbf{r}_+^k + (e - \mathbf{y}^k)^T \mathbf{r}_-^k}{2m},$$

$\mathbf{t}^k = -\mathbf{A}^T \mathbf{y}^k$, $\mathbf{u}^k = \mathbf{b} - \mathbf{A}\mathbf{x}^k + \mathbf{r}_+^k - \mathbf{r}_-^k$, $\mathbf{v}_+^k = \mu^k \mathbf{e} - (\mathbf{I} + \mathbf{Y}^k)\mathbf{R}_+^k \mathbf{e}$, $\mathbf{v}_-^k = \mu^k \mathbf{e} - (\mathbf{I} - \mathbf{Y}^k)\mathbf{R}_-^k \mathbf{e}$. $\mathbf{p}^k = (\mathbf{I} + \mathbf{Y}^k)^{-1}\mathbf{v}_+^k$, $\mathbf{q}^k = (\mathbf{I} - \mathbf{Y}^k)^{-1}\mathbf{v}_-^k$. $\mathbf{S}^k = (\mathbf{I} + \mathbf{Y}^k)^{-1}\mathbf{R}_+^k + (\mathbf{I} - \mathbf{Y}^k)^{-1}\mathbf{R}_-^k$,

where $\mathbf{Y}^k$, $\mathbf{R}_+^k$ *and* $\mathbf{R}_-^k$ *are diagonal matrices whose diagonal entries are* y_i^k, $(r_+^k)_i$ *and* $(r_-^k)_i$, *respectively.*

Step 3. *(checking for optimality): If*

$$\mu^k < \epsilon_1, \quad \frac{\|\mathbf{t}\|}{\|\mathbf{b}\| + 1} < \epsilon_2, \quad \text{and} \quad \frac{\|\mathbf{u}\|}{\|\bar{\mathbf{c}}\| + 1} < \epsilon_3$$

then STOP. The solution is optimal. Otherwise go to the next stop, where $\bar{\mathbf{c}} \triangleq (\mathbf{0}, \mathbf{0}, \mathbf{e}, \mathbf{e})^T$.

Step 4. *(calculating directions of translation): Compute*

$$\begin{aligned}
\mathbf{d}_x^k &= [\mathbf{A}^T(\mathbf{S}^k)^{-1}\mathbf{A}]^{-1}[\mathbf{A}^T(\mathbf{S}^k)^{-1}(\mathbf{p}^k - \mathbf{q}^k + \mathbf{u}^k) - \mathbf{t}^k], \\
\mathbf{d}_y^k &= (\mathbf{S}^k)^{-1}(\mathbf{p}^k - \mathbf{q}^k + \mathbf{u}^k - \mathbf{A}\mathbf{d}_x^k), \\
\mathbf{d}_{r_+}^k &= \mathbf{p}^k - (\mathbf{I} + \mathbf{Y}_k)^{-1}\mathbf{R}_+^k \mathbf{d}_y^k, \\
\mathbf{d}_{r_-}^k &= \mathbf{q}^k + (\mathbf{I} - \mathbf{Y}_k)^{-1}\mathbf{R}_-^k \mathbf{d}_y^k.
\end{aligned}$$

Step 5. *(finding step-lengths): Compute the primal and dual step-lengths*

$$\beta_y = \min_i \left\{ \alpha \max \left\{ \frac{1 - y_i^k}{(d_y^k)_i}, \frac{-1 - y_i^k}{(d_y^k)_i} \right\} \right\}$$

and

$$\beta_r = \max \left\{ \frac{1}{\max \left\{ 1, -(d_{r_+}^k)_i / \alpha (r_+^k)_i \right\}}, \frac{1}{\max \left\{ 1, -(d_{r_-}^k)_i / \alpha (r_-^k)_i \right\}} \right\}$$

where $\alpha < 1$ *(say 0.99).*

Step 6. *(moving to a new point): Update the solution vectors*

$$\begin{aligned}
\mathbf{x}^{k+1} &\leftarrow \mathbf{x}^k + \beta_r \mathbf{d}_x^k \\
\mathbf{y}^{k+1} &\leftarrow \mathbf{y}^k + \beta_y \mathbf{d}_y^k \\
\mathbf{r}_+^{k+1} &\leftarrow \mathbf{r}_+^k + \beta_r \mathbf{d}_{r_+}^k \\
\mathbf{r}_-^{k+1} &\leftarrow \mathbf{r}_-^k + \beta_r \mathbf{d}_{r_-}^k .
\end{aligned}$$

Set $k \leftarrow k + 1$ *and go to Step 2.* □

Remark: Since the objective value of (7.2) is always nonnegative, this algorithm does not need to check the unboundedness in each iteration.

The most expensive step in the compressed primal-dual method is to invert the matrix $\mathbf{A}^T (\mathbf{S}^k)^{-1} \mathbf{A}$ of dimension $2[(I+1)(J+1)] \times 2[(I+1)(J+1)]$. Compared to the methods proposed in Chapter 6, the computational bottleneck (inverting a $[6IJK(K-1) + 2(I+1)(J+1)] \times [6IJK(K-1) + 2(I+1)(J+1)]$ matrix) and storage requirement are dramatically reduced.

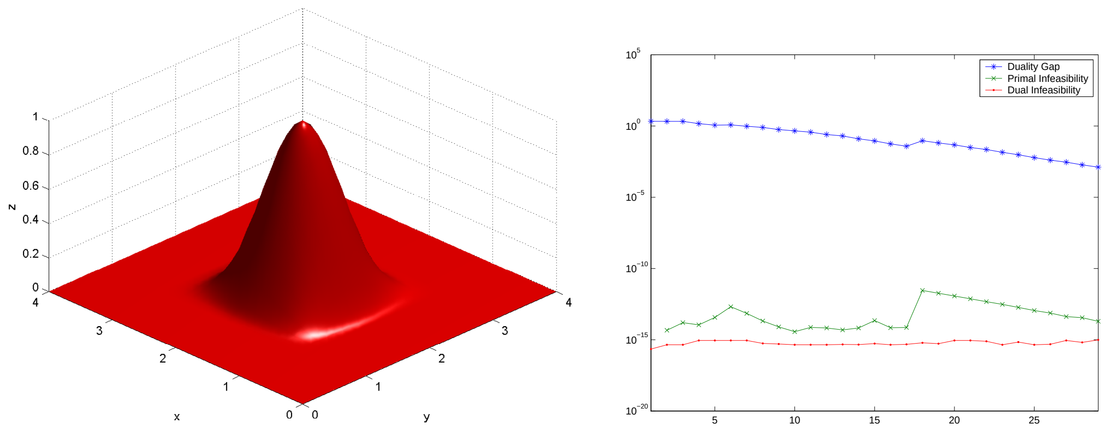
7.3 Computational results

We have tested the proposed compressed primal-dual interior point method on the same data sets used in Chapter 6. Algorithm 7.2.1 is implemented in MATLAB. In the computational experiments, we set K to be 3. The algorithm stops when the primal infeasibility and dual infeasibility are less than 10^{-5} and duality gap is less than 10^{-3}. α in Step 5 of Algorithm 7.2.1 is set to be 0.40. All computational experiments run on a Pentium 4 computer with 1.4GHz CPU and 512M memory.

For each instance, we present the data set first. Then the plots of the resulted bivariate cubic L_1 spline as well as the corresponding convergence trajectories of the duality gap, the primal and the dual infeasibility versus iterations (horizontal axis) are illustrated in subfigures (a) and (b) for each example.

Example 7.3.1 (Single peak on flat surface) *Data:* $I = 4, J = 4, x_i = i, i = 0, 1, \ldots, I,$ $y_j = j, j = 0, 1, \ldots, J,$

$$\mathbf{z} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$



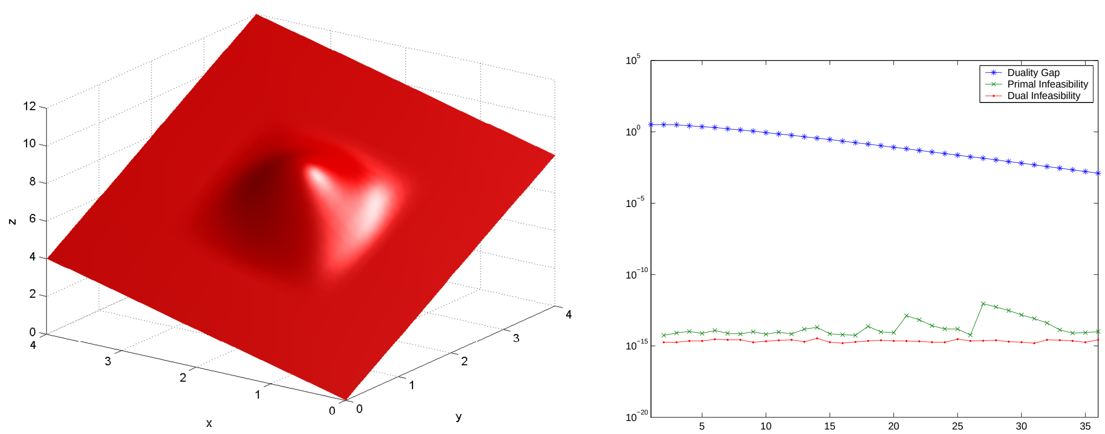
(a) Bivariate cubic L_1 spline interpolation

(b) Convergence trajectories of duality gap, primal and dual infeasibilities

Figure 7.1: Splines for data representing single peak on flat surface

Example 7.3.2 (Single peak on slanted surface) *Data:* $I = 4, J = 4,\ x_i = i,\ i = 0, 1, \ldots, I,\ y_j = j,\ j = 0, 1, \ldots, J,$

$$\mathbf{z} = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 8 & 7 & 8 \\ 6 & 7 & 8 & 9 & 10 \\ 8 & 9 & 10 & 11 & 12 \end{bmatrix}.$$



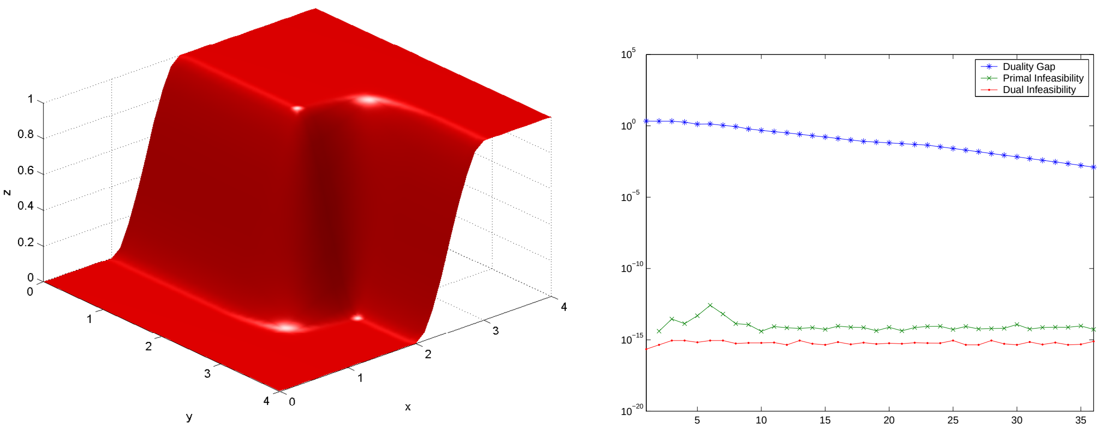
(a) Bivariate cubic L_1 spline interpolation

(b) Convergence trajectories of duality gap, primal and dual infeasibilities

Figure 7.2: Splines for data representing single peak on slanted surface

Example 7.3.3 (Step discontinuity) *Data:* $I = 4, J = 4,\ x_i = i,\ i = 0, 1, \ldots, I,\ y_j = j,\ j = 0, 1, \ldots, J,$

$$\mathbf{z} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$



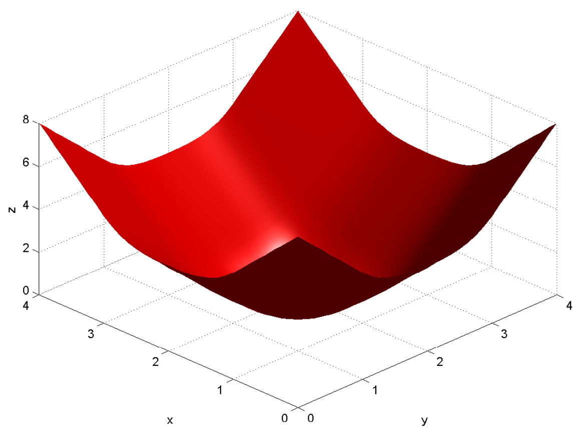
(a) Bivariate cubic L_1 spline interpolation

(b) Convergence trajectories of duality gap, primal and dual infeasibilities

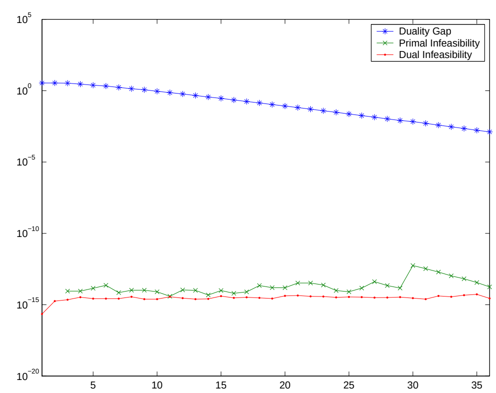
Figure 7.3: Splines for data representing step discontinuity

Example 7.3.4 (Bowl of a paraboloid) *Data:* $I = 4, J = 4,\ x_i = i,\ i = 0, 1, \ldots, I,$ $y_j = j,\ j = 0, 1, \ldots, J,$

$$\mathbf{z} = \begin{bmatrix} 8 & 5 & 4 & 5 & 8 \\ 5 & 2 & 1 & 2 & 5 \\ 4 & 1 & 0 & 1 & 4 \\ 5 & 2 & 1 & 2 & 5 \\ 8 & 5 & 4 & 5 & 8 \end{bmatrix}.$$



(a) Bivariate cubic L_1 spline interpolation

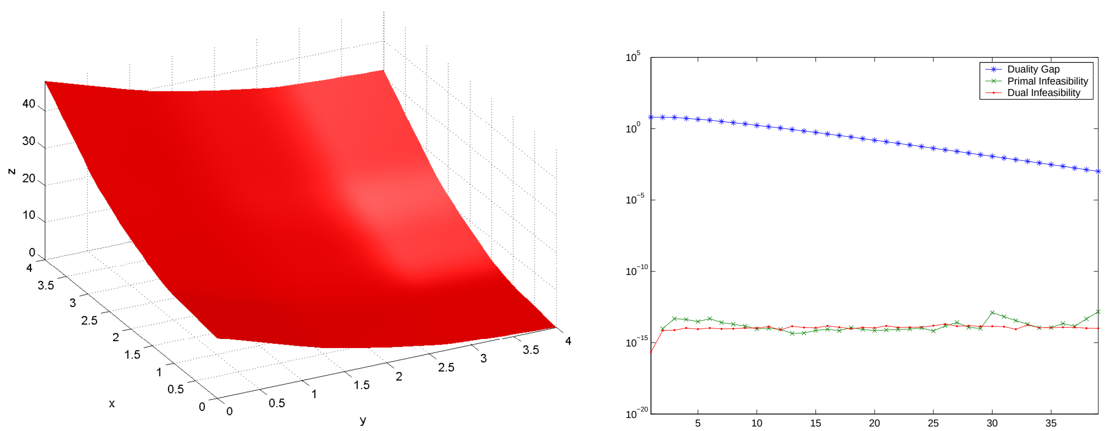


(b) Convergence trajectories of duality gap, primal and dual infeasibilities

Figure 7.4: Splines for data representing bowl of a paraboloid

Example 7.3.5 (Side of a paraboloid) *Data:* $I = 4, J = 4$, $x_i = i$, $i = 0, 1, \ldots, I$, $y_j = j$, $j = 0, 1, \ldots, J$,

$$\mathbf{z} = \begin{bmatrix} 16 & 18 & 24 & 34 & 48 \\ 9 & 11 & 17 & 27 & 41 \\ 4 & 6 & 12 & 22 & 36 \\ 1 & 3 & 9 & 19 & 33 \\ 0 & 2 & 8 & 18 & 32 \end{bmatrix}.$$



(a) Bivariate cubic L_1 spline interpolation

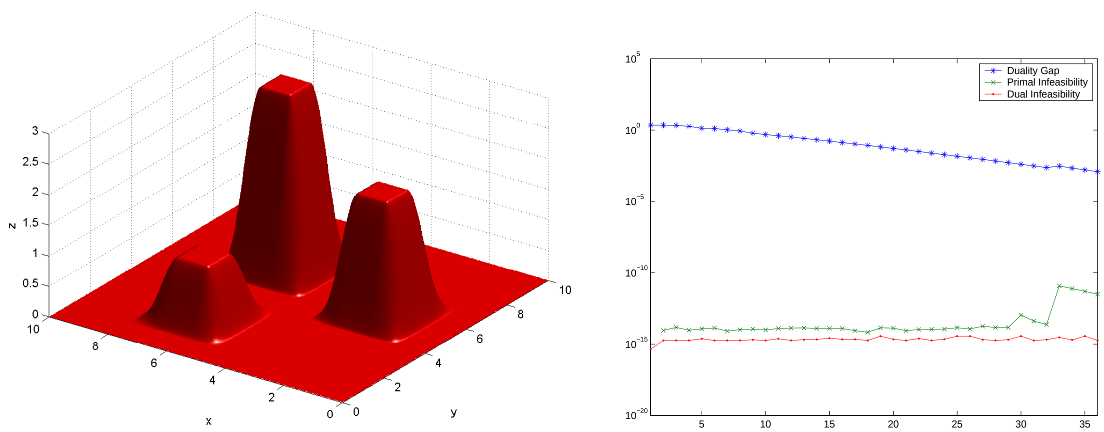
(b) Convergence trajectories of duality gap, primal and dual infeasibilities

Figure 7.5: Splines for data representing side of a paraboloid

Example 7.3.6 (4-point peaks on flat surface) *Data:* $I = 10, J = 10$, $x_i = i$, $i = 0, 1, \ldots, I$, $y_j = j$, $j = 0, 1, \ldots, J$, *The* $\mathbf{z}$ *matrix is calculated by setting* $z_{ij} = 0$ *for* $i = 0, 1, \ldots, I$ *and* $j = 0, 1, \ldots, J$ *and then changing*

$$z_{2,6} = 1 \quad z_{2,7} = 1 \qquad z_{5,2} = 2 \quad z_{5,3} = 2 \qquad z_{6,6} = 3 \quad z_{6,7} = 3$$

$$z_{3,6} = 1 \quad z_{3,7} = 1 \qquad z_{6,2} = 2 \quad z_{6,3} = 2 \qquad z_{7,6} = 3 \quad z_{7,7} = 3$$



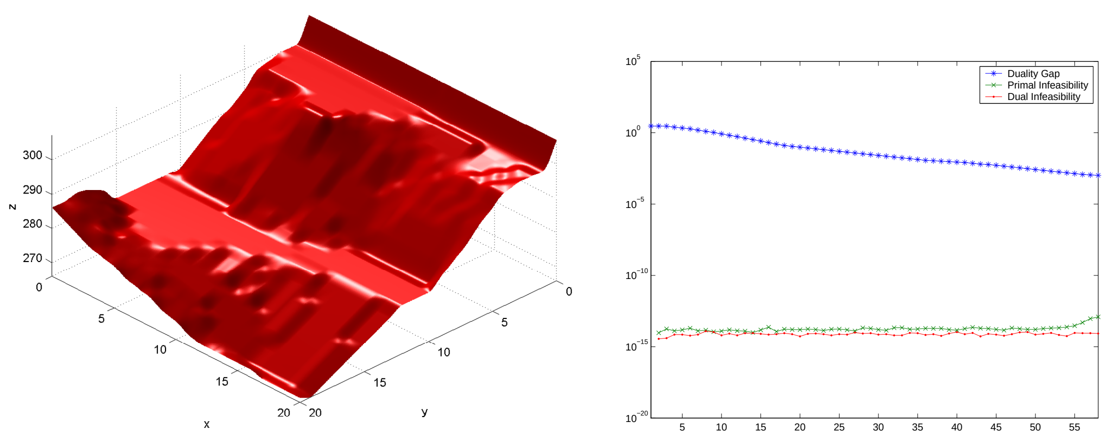
(a) Bivariate cubic L_1 spline interpolation

(b) Convergence trajectories of duality gap, primal and dual infeasibilities

Figure 7.6: Splines for data representing 4-point peaks on flat surface

Example 7.3.7 (Terrain) *Data:* $I = 20, J = 20,\ x_i = i,\ i = 0, 1, \ldots, I,\ y_j = j,\ j = 0, 1, \ldots, J,$

z:	307	307	307	307	307	307	307	307	307	307	307	307	307	307	307	307	307	307	307	307	307
	300	300	300	300	300	300	300	300	300	300	300	300	300	300	300	300	300	300	300	300	300
	300	300	300	300	300	300	300	300	300	300	300	300	300	300	300	300	299	299	299	300	300
	299	300	300	300	300	300	300	300	300	300	300	299	298	298	298	298	298	298	299	299	300
	298	300	300	300	300	300	300	300	300	299	298	297	296	295	296	296	297	298	298	299	299
	294	296	297	297	299	300	300	300	300	298	296	295	293	292	293	294	294	295	295	296	296
	291	293	294	296	297	299	300	300	297	295	293	292	290	290	290	291	291	291	292	292	293
	289	290	292	293	294	295	296	295	293	292	290	289	288	287	287	288	288	288	288	288	289
	286	287	289	290	291	292	292	291	290	289	287	286	285	284	284	284	284	284	284	284	286
	283	285	286	287	288	288	289	288	287	286	285	284	282	281	280	280	280	280	280	280	283
	280	282	283	284	284	284	286	285	284	283	283	282	281	280	280	280	280	280	280	280	280
	280	280	280	280	280	280	283	282	282	281	281	280	280	280	280	280	280	280	280	280	280
	280	280	280	280	280	280	280	280	280	280	280	280	280	280	280	280	280	280	280	280	280
	280	280	280	280	280	280	280	280	280	280	280	280	280	280	280	279	279	279	279	278	278
	280	280	280	280	280	280	280	280	280	280	280	279	279	278	278	277	277	277	277	276	276
	280	280	280	280	280	280	280	280	280	279	279	278	277	276	276	276	275	275	275	274	274
	284	282	280	280	280	280	280	280	279	278	278	276	276	274	274	274	273	273	273	273	272
	286	284	282	280	280	280	280	279	278	277	276	275	274	273	272	272	271	271	271	271	270
	287	285	283	281	281	280	279	278	277	276	275	273	272	271	270	269	269	269	269	269	268
	286	285	283	282	280	279	278	277	275	274	273	272	270	269	268	267	267	267	267	267	266
	286	284	282	280	279	278	276	275	274	272	271	269	268	268	268	268	268	267	267	267	267



(a) Bivariate cubic L_1 spline interpolation

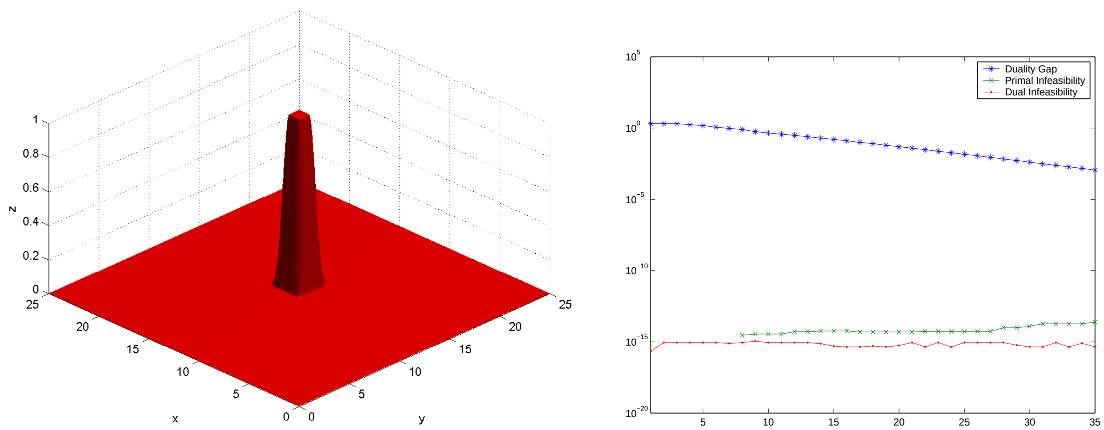
(b) Convergence trajectories of duality gap, primal and dual infeasibilities

Figure 7.7: Splines for data representing terrain

Example 7.3.8 (Single 4-point peak on flat surface) *Data:* $I = 25, J = 25$, $x_i = i$, $i = 0, 1, \ldots, I$, $y_j = j$, $j = 0, 1, \ldots, J$, *The* $\mathbf{z}$ *matrix is calculated by setting* $z_{ij} = 0$ *for* $i = 0, 1, \ldots, I$ *and* $j = 0, 1, \ldots, J$ *and then changing*

$$z_{13,13} = 1 \quad z_{13,14} = 1$$

$$z_{14,13} = 1 \quad z_{14,14} = 1.$$



(a) Bivariate cubic L_1 spline interpolation

(b) Convergence trajectories of duality gap, primal and dual infeasibilities

Figure 7.8: Splines for data representing single 4-point peak on flat surface

We list the total number of iterations, computing time and duality gap for each example in the Table 7.1. It shows that all the proposed method converges fast and correctly for all the examples.

Example	No. of Iterations	Computing Time (Sec)	Duality Gap
7.3.1	29	0.2810	9.5961E-4
7.3.2	36	0.2960	9.5282E-4
7.3.3	36	0.2970	9.5978E-4
7.3.4	36	0.3280	9.6885E-4
7.3.5	39	0.3290	7.7171E-4
7.3.6	36	8.8923	8.8923E-4
7.3.7	58	18.3430	9.5403E-4
7.3.8	35	22.3440	8.7820E-4

Table 7.1: Statistics of computational experiments for compressed primal-dual method

Example	No. of Iterations		Computing Time		Duality Gap	
	CP-D[1]	P-D	CP-D	P-D	CP-D	P-D
7.3.1	29	20	0.281	1.482	9.5961E-4	9.9484E-4
7.3.2	36	15	0.296	1.147	9.5282E-4	5.6573E-4
7.3.3	36	21	0.297	1.586	9.5978E-4	9.5071E-4
7.3.4	36	17	0.328	1.284	9.6885E-4	9.3197E-4
7.3.5	39	19	0.329	1.375	7.7171E-4	6.8522E-4
7.3.6	36	19	8.892	35.884	8.8923E-4	6.9763E-4
7.3.7	58	29	18.343	1558.926	9.5403E-4	9.9827E-4
7.3.8	35	22	22.344	3672.985	8.7820E-4	9.9903E-4

Table 7.2: Comparison between compressed primal-dual method and primal-dual interior point method

In order to judge the performance of the proposed method, we compare the statistics of the compressed primal-dual method with the primal-dual method proposed in Chapter 6 and summarize the comparison in Table 7.1. Table 7.1 shows that although the compressed primal-dual method needs more iterations to converge, the total computing time used by

[1] "CP-D" represents "Compressed Primal-dual method"; "P-D" represents "Primal-dual method".

the compressed primal-dual method is much less than the primal-dual method.

The compressed primal-dual method also demonstrates its robustness. For example 7.3.7, it reduces the duality gap down to 7.8624E-15 in 182 iterations, and for example 7.3.8 it reduces the duality gap down to 8.5067E-15 in 125 iterations.

7.4 Conclusion

In this chapter, we proposed a compressed primal-dual method for generating discretized bivariate cubic L_1 splines. The proposed method tremendously reduces the storage and computational effort for generating discretized bivariate cubic L_1 splines. Numerical results show that the compress primal-dual method generates discretized bivariate cubic L_1 splines efficiently and robustly for large scale problems.

Chapter 8

Conclusion and Future Research

Bivariate cubic L_1 splines have been shown by empirical experience to have much better shape-preserving capability than conventional smooth splines. However, theoretical analysis and efficient algorithms for finding exact bivariate cubic L_1 splines have not been fully explored. This dissertation is directed to correct these deficiencies.

Generating a bivariate cubic L_1 spline requires solving a nonsmooth convex optimization problem, which is difficult to solve exactly. In order to find exact solution and perform theoretical analysis, we have formulated the bivariate cubic L_1 splines as a generalized geometric programming problem and derived a geometric dual program with a linear program for dual-to-primal transformation. The coefficients of a bivariate cubic L_1 spline are determined by a dual optimal solution. In this framework, we have shown that bivariate cubic L_1 splines preserve linearity for multi-scale data under some mild conditions.

A tensor-product approach has been proposed to obtain bivariate cubic L_1 splines over large scale data set. This approach can efficiently generate good approximation of bivariate cubic L_1 splines with large size data size in most cases.

We have also developed a primal-dual method for obtaining the solution for discretized bivariate cubic L_1 splines. This method converges to the optimal solution correctly. It also has the advantages of fast convergence and numerical robustness for small and medium size problems. In order to efficiently and robustly generate large-scale bivariate cubic L_1 splines, a "compressed" primal-dual method has been developed.

In the future, we expect to extend our work in the following directions:

(i) In this dissertation, we have developed a geometric programming framework for bivariate cubic L_1 splines over the tensor-product grid. However, the data sets for most of real life applications are based on irregular grids, called triangulated irregular networks (TINs). We would like to establish a geometric programming model for bivariate cubic L_1 splines based on TINs for theoretical analysis and algorithm development.

(ii) We have shown that bivariate cubic L_1 splines preserving linearity. But other shape-preserving properties, such as convexity and concavity, are still missing for further investigation.

(iii) In Chapter 7, we proposed a compressed primal-dual method. An implementation for real applications and a comparison with the compressed primal affine scaling method need to be conducted.

(iv) Based on the geometric programming framework for bivariate cubic L_1 splines proposed in this dissertation, we would like to develop a continuum-based algorithm to obtain the exact solutions.

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Appendix A

Appendix A: Calculation of Conjugate Transfrom for $\mathfrak{h}(y) : \mathfrak{D}$

We attach the calculations of the conjugate transforms on triangle $(ij1)$ for

$$\mathfrak{g}_{xx}^{ij1}(\mathbf{c}^{ij1}) = \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} \left| 6c_{30}^{ij1}(x - x_i) + 2c_{20}^{ij1} \right| dxdy$$

$$\mathfrak{g}_{yy}^{ij1}(\mathbf{c}^{ij1}) = \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} \left| 2c_{12}^{ij1}(x - x_i) + 6c_{03}^{ij1}(y - y_j) + 2c_{02}^{ij1} \right| dxdy$$

$$\frac{1}{2}\mathfrak{g}_{xy}^{ij1}(\mathbf{c}^{ij1}) = \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} \left| 2c_{12}^{ij1}(y - y_j) + c_{11}^{ij1} \right| dxdy$$

Lemma A.0.1 (Properities of Conjugate Transform) *Given that* $\mathfrak{g}(\boldsymbol{x}) : \boldsymbol{x} \in \mathfrak{C} \subset R^n$ *has a known conjugate transform* $\mathfrak{h}(\boldsymbol{y}) : \boldsymbol{y} \in \mathfrak{D} \subset R^n$*, then*

1. *for a given scalar s, the function $\mathfrak{g}(\boldsymbol{x}) + s : \boldsymbol{x} \in \mathfrak{C}$ has s conjugate transform $\mathfrak{h}(\boldsymbol{y}) - s : \boldsymbol{y} \in \mathfrak{D}$,*

2. *for a given vector $\boldsymbol{u} \in R^n$, the function $\mathfrak{g}(\boldsymbol{x}) + \boldsymbol{u} : \boldsymbol{x} + \boldsymbol{u} \in \mathfrak{C}$ has a conjugate transform $\mathfrak{h}(\boldsymbol{y}) - \langle \boldsymbol{u}, \boldsymbol{y} \rangle : \boldsymbol{y} \in \mathfrak{D}$,*

3. *for a given scalar $\lambda > 0$, the function $\lambda\mathfrak{g}(\boldsymbol{x}) : \boldsymbol{x} \in \mathfrak{C}$ has a conjuate transform $\lambda\mathfrak{h}(\boldsymbol{y}/\lambda) : \lambda\boldsymbol{y} \in \mathfrak{D}$.*

If we denote the conjuagte transform function and domain of $\frac{1}{2}\mathfrak{g}_{xy}^{ij1}(\mathbf{c}^{ij1})$ as $\mathfrak{h}_{\frac{1}{2}xy}^{ij1}(\mathbf{d}^{ij1})$ and $\Omega_{\frac{1}{2}xy}^{ij1}$ respectively, then according to the 3rd property in Lemma A.0.1, the conjugate transform function and domain of $\mathfrak{g}_{xy}^{ij1}(\mathbf{c}^{ij1}) = 2\frac{1}{2}\mathfrak{g}_{xy}^{ij1}(\mathbf{c}^{ij1})$ are $2\mathfrak{h}_{\frac{1}{2}xy}^{ij1}(\mathbf{d}^{ij1}/2)$ and $\Omega_{xy}^{ij1} =$

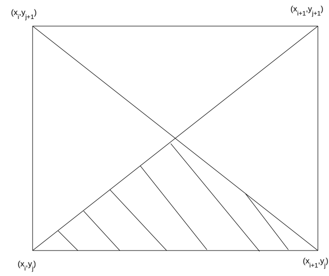


Figure A.1: Region D

$\left\{\mathbf{d}^{ij1}\middle|2\mathbf{d}^{ij1} \in \Omega^{ij1}_{\frac{1}{2}xy}\right\}$. So we can easily get the conjugate transform function and domain of $\mathfrak{g}^{ij1}_{xy}(\mathbf{c}^{ij1})$ by changing variables.

Since the conjugate transforms on the other three triangles i.e. triangle $(ij2)$, $(ij3)$ and $(ij4)$, are symmetric to triangle $(ij1)$, we can get them similarly as we did for those on triangle $(ij1)$.

A.1 Notation

One rectangle $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ is divided into four triangles. Consider one triangle as illustrated by shaded area in Figure A.1, denote it as (D). The bi-variate cubic interpolating function defined over this triangle is

$$\begin{aligned} z(x,y) &= c_{30}(x-x_i)^3 + c_{21}(x-x_i)^2(y-y_j) + c_{12}(x-x_i)(y-y_j)^2 + c_{03}(y-y_j)^3 \\ &\quad + c_{20}(x-x_i)^2 + c_{11}(x-x_i)(y-y_j) + c_{02}(y-y_j)^2 \\ &\quad + c_{10}(x-x_i) + c_{01}(y-y_j) + c_{00}, \end{aligned} \tag{A.1}$$

where the ten coefficients are unknown variables. The partial derivatibes of $z(x,y)$ are

$$\begin{aligned} \frac{\partial z}{\partial x} &= 3c_{30}(x-x_i)^2 + 2c_{21}(x-x_i)(y-y_j) + c_{12}(y-y_j)^2 \\ &\quad + 2c_{20}(x-x_i) + c_{11}(y-y_j) + c_{10}, \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial y} &= c_{21}(x-x_i)^2 + 2c_{12}(x-x_i)(y-y_j) + 3c_{03}(y-y_j)^2 \\ &\quad + c_{11}(x-x_i) + 2c_{02}(y-y_j) + c_{01}, \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} &= 6c_{30}(x-x_i) + 2c_{21}(y-y_j) + 2c_{20}, \\ \frac{\partial^2 z}{\partial y^2} &= 2c_{12}(x-x_i) + 6c_{03}(y-y_j) + 2c_{02}, \\ \frac{\partial^2 z}{\partial x \partial y} &= 2c_{21}(x-x_i) + 2c_{12}(y-y_j) + c_{11}. \end{aligned}$$

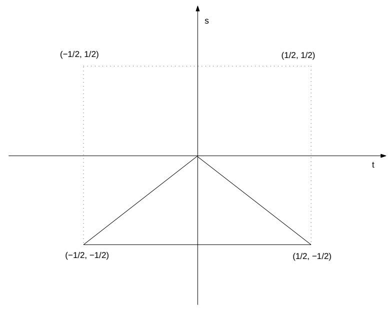


Figure A.2: Region DI

By the constructing of Sibson element, $\frac{\partial z}{\partial y}$ is linear along the edge $y = y_j$, i.e.,

$$\left.\frac{\partial z}{\partial y}\right|_{y=y_j} = c_{21}\left(x - x_i\right)^2 + c_{11}\left(x - x_i\right) + c_{01}$$

is a linear function. Hence,

$$c_{21} = 0. \tag{A.2}$$

It is obviously that

$$c_{00} = z_{00}. \tag{A.3}$$

Therefore, the second partial derivative of $z\left(x, y\right)$ are

$$\frac{\partial^2 z}{\partial x^2} = 6c_{30}\left(x - x_i\right) + 2c_{20}, \tag{A.4}$$

$$\frac{\partial^2 z}{\partial y^2} = 2c_{12}\left(x - x_i\right) + 6c_{03}\left(y - y_j\right) + 2c_{02}, \tag{A.5}$$

$$\frac{\partial^2 z}{\partial x \partial y} = 2c_{12}\left(y - y_j\right) + c_{11}. \tag{A.6}$$

Denote

$$h_i^x = x_{i+1} - x_i, \tag{A.7}$$

and

$$h_j^y = y_{j+1} - y_j. \tag{A.8}$$

By a 1-1 onto transformation

$$t = \frac{x - x_i}{h_i^x} - \frac{1}{2} \tag{A.9}$$

$$s = \frac{y - y_j}{h_j^y} - \frac{1}{2} \tag{A.10}$$

the rectangle $(x, y) \in [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ is mapped to $(t, s) \in \left[-\frac{1}{2}, \frac{1}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right]$, and the triangle (D) is mapped to the triangle (DI) as illusted in Figure A.2.

A.2 Conjugate transform of $\left\|\frac{\partial^2 z}{\partial x^2}\right\|_1$

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2} &= 6c_{30}\left(x - x_i\right) + 2c_{20} \\ &= 6c_{30}h_i^x t + \left(3c_{30}h_i^x + 2c_{20}\right), \quad t \in \left[-\frac{1}{2}, \frac{1}{2}\right].\end{aligned}$$

Let

$$\begin{cases} A = 3c_{30}h_i^x, \\ B = 3c_{30}h_i^x + 2c_{20}, \end{cases}$$

then

$$\begin{cases} c_{30} = \frac{1}{3h_i^x}A, \\ c_{20} = -\frac{1}{2}A + \frac{1}{2}B. \end{cases}$$

Let

$$\begin{aligned}f\left(t\right) &= 6c_{30}h_i^x t + \left(3c_{30}h_i^x + 2c_{20}\right) \\ &= 2At + B, \quad t \in \left[-\frac{1}{2}, \frac{1}{2}\right]\end{aligned}$$

which is a linear function of t. Let

$$\begin{aligned}&F\left(c_{30}, c_{20}\right) \\ &= \left\|\frac{\partial^2 z}{\partial x^2}\right\|_1 \\ &= \int_D \left|\frac{\partial^2 z}{\partial x^2}\right| dxdy \\ &= h_i^x h_j^y \int_{DI} \left|6c_{30}h_i^x t + \left(3c_{30}h_i^x + 2c_{20}\right)\right| dtds \\ &= h_i^x h_j^y \int_{DI} \left|2At + B\right| dtds.\end{aligned}$$

A.2.1 Case 1

Case 1: $f\left(t\right) \geq 0, \forall t \in \left[-\frac{1}{2}, \frac{1}{2}\right]$.

Since $f\left(t\right)$ is a linear function, this is the case that

$$\begin{aligned}f\left(-\frac{1}{2}\right) &= -A + B \geq 0, \\ f\left(\frac{1}{2}\right) &= A + B \geq 0,\end{aligned}$$

Therefore, this situation is equivalent to

$$|A| \leq B.$$

i.e.,

$$-1 \le \frac{A}{B} \le 1, B \ge 0.$$

Hence,

$$\begin{aligned}
&F\left(c_{30}, c_{20}\right) \\
&= h_i^x h_j^y \int_{DI} |2At + B|\, dtds \\
&= h_i^x h_j^y \int_{-\frac{1}{2}}^{0} \int_{-\frac{1}{2}}^{t} [2At + B]\, dsdt + h_i^x h_j^y \int_{0}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{-t} [2At + B]\, dsdt \\
&= \frac{1}{4} h_i^x h_j^y B. \qquad \text{(A.11)}
\end{aligned}$$

In this situation, the conjugate transform of $F\left(c_{30}, c_{20}\right)$ is

$$\begin{aligned}
&G\left(\xi, \eta\right) \\
&= \sup\left\{\xi c_{30} + \eta c_{20} - F\left(c_{30}, c_{20}\right)\right\} \\
&= \sup_{|A| \le B} \left\{\xi \frac{1}{3h_i^x} A + \eta\left(-\frac{1}{2}A + \frac{1}{2}B\right) - \frac{1}{4} h_i^x h_j^y B\right\} \\
&= \sup_{|A| \le B} \left\{\left(\frac{\xi}{3h_i^x} - \frac{\eta}{2}\right) A + \left(\frac{\eta}{2} - \frac{h_i^x h_j^y}{4}\right) B\right\}.
\end{aligned}$$

Let

$$t = \frac{A}{B}, -1 \le t \le 1,$$

then

$$\begin{aligned}
&\left(\frac{\xi}{3h_i^x} - \frac{\eta}{2}\right) A + \left(\frac{\eta}{2} - \frac{h_i^x h_j^y}{4}\right) B \\
&= \left(\frac{\xi}{3h_i^x} - \frac{\eta}{2}\right) tB + \left(\frac{\eta}{2} - \frac{h_i^x h_j^y}{4}\right) B \\
&= \left[\left(\frac{\xi}{3h_i^x} - \frac{\eta}{2}\right) t + \left(\frac{\eta}{2} - \frac{h_i^x h_j^y}{4}\right)\right] B \\
&= h\left(t\right) B, -1 \le t \le 1
\end{aligned}$$

where $h\left(t\right)$ is a linear function. Since $B \ge 0$, $G\left(\xi, \eta\right)$ is finite if and only if $h\left(t\right) \le 0$, which is equivalent to $h\left(-1\right) \le 0$ and $h\left(1\right) \le 0$. Hence,

$$\begin{aligned}
&h\left(-1\right) \\
&= -\left(\frac{\xi}{3h_i^x} - \frac{\eta}{2}\right) + \left(\frac{\eta}{2} - \frac{h_i^x h_j^y}{4}\right) \\
&= -\frac{\xi}{3h_i^x} + \eta - \frac{h_i^x h_j^y}{4} \le 0
\end{aligned}$$

implies

$$\eta \leq \frac{\xi}{3h_i^x} + \frac{h_i^x h_j^y}{4} = \frac{1}{3h_i^x}\left(\xi + \frac{3}{4}h_i^{x2}h_j^y\right).$$

And

$$\begin{aligned} & h(1) \\ &= \left(\frac{\xi}{3h_i^x} - \frac{\eta}{2}\right) + \left(\frac{\eta}{2} - \frac{h_i^x h_j^y}{4}\right) \\ &= \frac{\xi}{3h_i^x} - \frac{h_i^x h_j^y}{4} \leq 0 \end{aligned}$$

implies

$$\xi \leq \frac{3}{4}h_i^{x2}h_j^y.$$

Therefore, in Case 1, the conjugate transform of $F(c_{30}, c_{20})$ is

$$G(\xi, \eta) = 0,$$

where

$$(\xi, \eta) \in \Omega_1 = \left\{\xi \leq \frac{3}{4}h_i^{x2}h_j^y, \eta \leq \frac{1}{3h_i^x}\left(\xi + \frac{3}{4}h_i^{x2}h_j^y\right)\right\}. \tag{A.12}$$

A.2.2 Case 2

Case 2: $f(t) \leq 0, \forall t \in \left[-\frac{1}{2}, \frac{1}{2}\right]$.

This case is symmatric with case 1. Since $f(t)$ is a linear function, this is the case that

$$\begin{aligned} f\left(-\frac{1}{2}\right) &= -A + B \leq 0, \\ f\left(\frac{1}{2}\right) &= A + B \leq 0, \end{aligned}$$

Therefore, this situation is equivalent to

$$|A| \leq -B = |B|.$$

i.e.,

$$-1 \leq -\frac{A}{B} \leq 1, B \leq 0.$$

Hence,

$$\begin{aligned} & F(c_{30}, c_{20}) \\ &= h_i^x h_j^y \int_{DI} |2At + B|\, dtds \\ &= -\frac{1}{4}h_i^x h_j^y B. \end{aligned} \tag{A.13}$$

In this situation, the conjugate transform of $F(c_{30}, c_{20})$ is

$$\begin{aligned}
&G(\xi, \eta) \\
&= \sup\{\xi c_{30} + \eta c_{20} - F(c_{30}, c_{20})\} \\
&= \sup_{|A| \le -B} \left\{ \xi \frac{1}{3h_i^x} A + \eta \left(-\frac{1}{2} A + \frac{1}{2} B \right) + \frac{1}{4} h_i^x h_j^y B \right\} \\
&= \sup_{|A| \le -B} \left\{ \left(\frac{\xi}{3h_i^x} - \frac{\eta}{2} \right) A + \left(\frac{\eta}{2} + \frac{h_i^x h_j^y}{4} \right) B \right\}.
\end{aligned}$$

Let

$$t = -\frac{A}{B}, -1 \le t \le 1,$$

then

$$\begin{aligned}
&\left(\frac{\xi}{3h_i^x} - \frac{\eta}{2} \right) A + \left(\frac{\eta}{2} + \frac{h_i^x h_j^y}{4} \right) B \\
&= -\left(\frac{\xi}{3h_i^x} - \frac{\eta}{2} \right) tB + \left(\frac{\eta}{2} + \frac{h_i^x h_j^y}{4} \right) B \\
&= \left[\left(-\frac{\xi}{3h_i^x} + \frac{\eta}{2} \right) t + \left(\frac{\eta}{2} + \frac{h_i^x h_j^y}{4} \right) \right] B \\
&= h(t) B, -1 \le t \le 1
\end{aligned}$$

where $h(t)$ is a linear function. Since $B \le 0$, $G(\xi, \eta)$ is finite if and only if $h(t) \ge 0$, which is equivalent to $h(-1) \ge 0$ and $h(1) \ge 0$. Hence,

$$\begin{aligned}
&h(-1) \\
&= -\left(-\frac{\xi}{3h_i^x} + \frac{\eta}{2} \right) + \left(\frac{\eta}{2} + \frac{h_i^x h_j^y}{4} \right) \\
&= \frac{\xi}{3h_i^x} + \frac{h_i^x h_j^y}{4} \ge 0
\end{aligned}$$

implies

$$\xi \ge -\frac{3}{4} h_i^{x2} h_j^y.$$

And

$$\begin{aligned}
&h(1) \\
&= \left(-\frac{\xi}{3h_i^x} + \frac{\eta}{2} \right) + \left(\frac{\eta}{2} + \frac{h_i^x h_j^y}{4} \right) \\
&= -\frac{\xi}{3h_i^x} + \eta + \frac{h_i^x h_j^y}{4} \ge 0
\end{aligned}$$

implies

$$\eta \ge \frac{\xi}{3h_i^x} - \frac{h_i^x h_j^y}{4} = \frac{1}{3h_i^x} \left(\xi - \frac{3}{4} h_i^{x2} h_j^y \right).$$

Therefore, in Case 2, the conjugate transform of $F(c_{30}, c_{20})$ is

$$G(\xi, \eta) = 0,$$

where

$$(\xi, \eta) \in \Omega_2 = \left\{ \xi \geq -\frac{3}{4} h_i^{x2} h_j^y, \eta \geq \frac{1}{3h_i^x} \left(\xi - \frac{3}{4} h_i^{x2} h_j^y \right) \right\}. \tag{A.14}$$

A.2.3 Case 3

Case 3: $f\left(-\frac{1}{2}\right) < 0$ and $f\left(\frac{1}{2}\right) > 0$.
this is the case that

$$f\left(-\frac{1}{2}\right) = -A + B < 0,$$
$$f\left(\frac{1}{2}\right) = A + B > 0,$$

whihch is equivalent to

$$|B| < A, A > 0.$$

Let $\hat{t}$ be the root of $f(t)$, i.e.,

$$f(\hat{t}) = 0, \hat{t} \in \left[-\frac{1}{2}, \frac{1}{2}\right].$$

case 3.1 $B > 0$

this is equivalent to

$$-\frac{1}{2} < \hat{t} < 0, A > 0, B > 0.$$

$$F(c_{30}, c_{20}) = \left\| \frac{\partial^2 z}{\partial x^2} \right\|_1$$
$$= h_i^x h_j^y \left\{ \int_{-\frac{1}{2}}^{\hat{t}} \int_{-\frac{1}{2}}^{t} (-2At - B)\, dsdt + \int_{\hat{t}}^{0} \int_{-\frac{1}{2}}^{t} (2At + B)\, dsdt + \int_{0}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{-t} (2At + B)\, dsdt \right\}$$
$$= h_i^x h_j^y \left\{ -\frac{1}{12} \frac{B^3}{A^2} + \frac{1}{4} \frac{B^2}{A} + \frac{1}{12} A \right\}.$$

Let

$$t = \frac{B}{A}, 0 < t < 1,$$

In this situation, the conjugate transform of $F(c_{30}, c_{20})$ is

$$
\begin{aligned}
&G(\xi,\eta)\\
&= \sup\left\{\xi c_{30} + \eta c_{20} - F(c_{30}, c_{20})\right\}\\
&= \sup_{|B|<A, A>0, B>0}\left\{\xi\frac{1}{3h_i^x}A + \eta\left(-\frac{1}{2}A + \frac{1}{2}B\right) - h_i^x h_j^y\left[-\frac{1}{12}\frac{B^3}{A^2} + \frac{1}{4}\frac{B^2}{A} + \frac{1}{12}A\right]\right\}\\
&= \sup_{0<t<1, A>0}\left\{\xi\frac{1}{3h_i^x}A + \eta\left(-\frac{1}{2}A + \frac{1}{2}tA\right) - h_i^x h_j^y\left[-\frac{1}{12}t^3A + \frac{1}{4}t^2A + \frac{1}{12}A\right]\right\}\\
&= \sup_{0<t<1, A>0}\left\{\frac{h_i^x h_j^y}{12}t^3 - \frac{h_i^x h_j^y}{4}t^2 + \frac{\eta}{2}t + \left(\frac{\xi}{3h_i^x} - \frac{\eta}{2} - \frac{h_i^x h_j^y}{12}\right)\right\}A\\
&= \sup_{0<t<1, A>0}\frac{h_i^x h_j^y}{12}\left\{t^3 - 3t^2 + \frac{6\eta}{h_i^x h_j^y}t + \left(\frac{4\xi}{h_i^{x2}h_j^y} - \frac{6\eta}{h_i^x h_j^y} - 1\right)\right\}A\\
&= \sup_{0<t<1, A>0}\left\{\frac{h_i^x h_j^y}{12}g(t)A\right\}
\end{aligned}
$$

Since $A > 0$, $G(\xi,\eta)$ is finite if and only if

$$g(t) = t^3 - 3t^2 + \frac{6\eta}{h_i^x h_j^y}t + \left(\frac{4\xi}{h_i^{x2}h_j^y} - \frac{6\eta}{h_i^x h_j^y} - 1\right) \le 0, t \in (0,1).$$

Let

$$t = x + 1,$$

then

$$\tilde{g}(x) = x^3 + \left(\frac{6\eta}{h_i^x h_j^y} - 3\right)x + \left(\frac{4\xi}{h_i^{x2}h_j^y} - 3\right), x \in (-1,0).$$

$$
\begin{aligned}
\tilde{g}'(x) &= 3x^2 + \left(\frac{6\eta}{h_i^x h_j^y} - 3\right)\\
&= 3\left[x^2 + \left(\frac{2\eta}{h_i^x h_j^y} - 1\right)\right]
\end{aligned}
$$

Case 1 of $\tilde{g}(x)$: If $\frac{2\eta}{h_i^x h_j^y} - 1 \ge 0$, i.e., $\eta \ge \frac{h_i^x h_j^y}{2}$, then $\tilde{g}(x)$ is monotomically increasing, and $\tilde{g}(x) \le 0, x \in (-1,0)$, if and only if

$$\tilde{g}(0) = \frac{4\xi}{h_i^{x2}h_j^y} - 3 \le 0,$$

i.e.,

$$\xi \le \frac{3}{4}h_i^{x2}h_j^y.$$

Therefore, in this case, $G(\xi,\eta)$ is finite if and only if

$$\xi \le \frac{3}{4}h_i^{x2}h_j^y, \eta \ge \frac{h_i^x h_j^y}{2}. \tag{A.15}$$

Case 2 of $\tilde{g}(x)$: If $\frac{2\eta}{h_i^x h_j^y} - 1 < 0$, i.e., $\eta < \frac{h_i^x h_j^y}{2}$, then there exists a x^* such that $\tilde{g}'(x^*) = 0$. We have

$$x^* = -\sqrt{1 - \frac{2\eta}{h_i^x h_j^y}} < 0.$$

Now we consider the situation that

$$-1 \le x^* < 0.$$

Hence, $\tilde{g}(x^*)$ is the maximum value of $\tilde{g}(x)$. $\tilde{g}(x) \le 0$ if and only if $\tilde{g}(x^*) \le 0$.

$$-1 \le x^* < 0 \Longrightarrow 0 \le \eta < \frac{h_i^x h_j^y}{2}.$$

$$\begin{aligned}
&\tilde{g}(x^*) \\
&= -\left(1 - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{3}{2}} - \left(\frac{6\eta}{h_i^x h_j^y} - 3\right)\left(1 - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{1}{2}} + \left(\frac{4\xi}{h_i^{x2} h_j^y} - 3\right) \\
&= 2\left(1 - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{3}{2}} + \left(\frac{4\xi}{h_i^{x2} h_j^y} - 3\right)
\end{aligned}$$

Hence, $\tilde{g}(x^*) \le 0$ implies

$$\begin{aligned}
\left(1 - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{3}{2}} &\le \frac{3}{2} - \frac{2\xi}{h_i^{x2} h_j^y} \\
\left(1 - \frac{2\eta}{h_i^x h_j^y}\right)^{3} &\le \left(\frac{3}{2} - \frac{2\xi}{h_i^{x2} h_j^y}\right)^2 \\
\eta &\ge \frac{h_i^x h_j^y}{2} - \frac{h_i^x h_j^y}{2}\left(\frac{3}{2} - \frac{2\xi}{h_i^{x2} h_j^y}\right)^{\frac{2}{3}}.
\end{aligned}$$

Since $1 - \frac{2\eta}{h_i^x h_j^y} > 0$, above formula also implies that

$$\frac{3}{2} - \frac{2\xi}{h_i^{x2} h_j^y} \ge 0$$

which is equivalent to

$$\xi \le \frac{3}{4} h_i^{x2} h_j^y.$$

In summary, case 2 is the situation such that

$$\begin{aligned}
\frac{2\eta}{h_i^x h_j^y} - 1 &< 0 \\
-1 \le x^* &< 0 \\
\tilde{g}(x^*) &\le 0
\end{aligned}$$

i.e.,

$$0 \le \eta < \frac{h_i^x h_j^y}{2}, \xi \le \frac{3}{4} h_i^{x2} h_j^y, \eta \ge \frac{h_i^x h_j^y}{2} - \frac{h_i^x h_j^y}{2}\left(\frac{3}{2} - \frac{2\xi}{h_i^{x2} h_j^y}\right)^{\frac{2}{3}}. \tag{A.16}$$

Property: From function

$$\eta = \frac{h_i^x h_j^y}{2} - \frac{h_i^x h_j^y}{2}\left(\frac{3}{2} - \frac{2\xi}{h_i^{x2} h_j^y}\right)^{\frac{2}{3}}$$

we can derive that

$$\frac{d\eta}{d\xi} = \frac{2}{3h_i^x}\left(\frac{3}{2} - \frac{2\xi}{h_i^{x2} h_j^y}\right)^{-\frac{1}{3}} > 0$$

$$\frac{d^2\eta}{d\xi^2} = \frac{4}{9h_i^{x2} h_j^y}\left(\frac{3}{2} - \frac{2\xi}{h_i^{x2} h_j^y}\right)^{-\frac{4}{3}} > 0$$

Hence, (A.16) defines a convex set, and

$$\left.\frac{d\eta}{d\xi}\right|_{(\xi,\eta)=\left(\frac{3}{4}h_i^{x2}h_j^y, \frac{1}{2}h_i^x h_j^y\right)} = +\infty$$

$$\left.\frac{d\eta}{d\xi}\right|_{(\xi,\eta)=\left(\frac{1}{4}h_i^{x2}h_j^y, 0\right)} = \frac{2}{3h_i^x}.$$

Case 3 of $\tilde{g}(x)$: If $\frac{2\eta}{h_i^x h_j^y} - 1 < 0$ and $x^* \le -1$, then $\tilde{g}(x)$ is monotonically decreasing in $(-1, 0)$. Hence, $\tilde{g}(x) \le 0$ if and only if $\tilde{g}(-1) \le 0$.

$$\begin{aligned}
&\tilde{g}(-1) \\
&= -1 - \frac{6\eta}{h_i^x h_j^y} + 3 + \frac{4\xi}{h_i^{x2} h_j^y} - 3 \\
&= -\frac{6\eta}{h_i^x h_j^y} + \frac{4\xi}{h_i^{x2} h_j^y} - 1
\end{aligned}$$

$\tilde{g}(-1) \le 0$ implies that

$$\eta \ge \frac{2}{3h_i^x}\xi - \frac{h_i^x h_j^y}{6} = \frac{2}{3h_i^x}\left(\xi - \frac{1}{4}h_i^{x2} h_j^y\right).$$

$$x^* = -\sqrt{1 - \frac{2\eta}{h_i^x h_j^y}} \le -1$$

implies

$$\eta \le 0.$$

Furthermore,

$$0 \ge \eta \ge \frac{2}{3h_i^x}\left(\xi - \frac{1}{4}h_i^{x2} h_j^y\right)$$

implies

$$\xi \le \frac{1}{4} h_i^{x2} h_j^y.$$

In summary, case 3 is the situation that

$$\begin{aligned} \frac{2\eta}{h_i^x h_j^y} - 1 < 0 \\ x^* \le -1 \\ \tilde{g}(-1) \le 0 \end{aligned}$$

i.e.,

$$\xi \le \frac{1}{4} h_i^{x2} h_j^y, \eta \le 0, \eta \ge \frac{2}{3h_i^x} \left(\xi - \frac{1}{4} h_i^{x2} h_j^y \right) \tag{A.17}$$

At point $\left(\frac{1}{4} h_i^{x2} h_j^y, 0\right)$, the derivative $\frac{d\eta}{d\xi} = \frac{2}{3h_i^x}$.

Conclusion: in the situation of $B > 0$, the conojugate transform of $F(c_{30}, c_{20})$ is

$$G(\xi, \eta) = 0,$$

where (ξ, η) is defined over the union of sets (A.15), (A.16), and (A.17), i.e.,

$$\Omega_{31} = \left\{ \begin{array}{c} \xi \le \frac{3}{4} h_i^{x2} h_j^y \text{ if } \eta \ge \frac{h_i^x h_j^y}{2}, \\ \xi \le \frac{3}{4} h_i^{x2} h_j^y, \eta \ge \frac{h_i^x h_j^y}{2} - \frac{h_i^x h_j^y}{2} \left(\frac{3}{2} - \frac{2\xi}{h_i^{x2} h_j^y} \right)^{\frac{2}{3}} \text{ if } 0 \le \eta < \frac{h_i^x h_j^y}{2}, \\ \xi \le \frac{1}{4} h_i^{x2} h_j^y, \eta \ge \frac{2}{3h_i^x} \left(\xi - \frac{1}{4} h_i^{x2} h_j^y \right) \text{ if } \eta \le 0. \end{array} \right\}. \tag{A.18}$$

This is a convex set. The boundary of this set is C^1-smooth.

case 3.2 $B < 0$

this is equivalent to

$$0 < \hat{t} < \frac{1}{2}, A > 0, B < 0.$$

$$\begin{aligned} & \left\| \frac{\partial^2 z}{\partial x^2} \right\|_1 \\ &= h_i^x h_j^y \left\{ \int_{-\frac{1}{2}}^{0} \int_{-\frac{1}{2}}^{t} (-2At - B)\, ds dt + \int_{0}^{\hat{t}} \int_{-\frac{1}{2}}^{-t} (-2At - B)\, ds dt + \int_{\hat{t}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{-t} (2At + B)\, ds dt \right\} \\ &= h_i^x h_j^y \left\{ \frac{1}{12} \frac{B^3}{A^2} + \frac{1}{4} \frac{B^2}{A} + \frac{1}{12} A \right\}. \end{aligned}$$

Let

$$t = \frac{B}{A}, -1 < t < 0,$$

In this situation, the conjugate transform of $F\left(c_{30}, c_{20}\right)$ is

$$
\begin{aligned}
& G(\xi, \eta) \\
& =\sup \left\{\xi c_{30}+\eta c_{20}-F\left(c_{30}, c_{20}\right)\right\} \\
& =\sup _{|B|<A, A>0, B<0}\left\{\xi \frac{1}{3 h_i^x} A+\eta\left(-\frac{1}{2} A+\frac{1}{2} B\right)-h_i^x h_j^y\left[\frac{1}{12} \frac{B^3}{A^2}+\frac{1}{4} \frac{B^2}{A}+\frac{1}{12} A\right]\right\} \\
& =\sup _{-1<t<0, A>0}\left\{\xi \frac{1}{3 h_i^x} A+\eta\left(-\frac{1}{2} A+\frac{1}{2} t A\right)-h_i^x h_j^y\left[\frac{1}{12} t^3 A+\frac{1}{4} t^2 A+\frac{1}{12} A\right]\right\} \\
& =\sup _{-1<t<0, A>0}\left\{-\frac{h_i^x h_j^y}{12} t^3-\frac{h_i^x h_j^y}{4} t^2+\frac{\eta}{2} t+\left(\frac{\xi}{3 h_i^x}-\frac{\eta}{2}-\frac{h_i^x h_j^y}{12}\right)\right\} A \\
& =\sup _{-1<t<0, A>0} \frac{h_i^x h_j^y}{12}\left\{-t^3-3 t^2+\frac{6 \eta}{h_i^x h_j^y} t+\left(\frac{4 \xi}{h_i^{x 2} h_j^y}-\frac{6 \eta}{h_i^x h_j^y}-1\right)\right\} A \\
& =\sup _{-1<t<0, A>0}\left\{-\frac{h_i^x h_j^y}{12} g(t) A\right\}
\end{aligned}
$$

Since $A>0$, $G(\xi, \eta)$ is finite if and only if

$$
g(t)=t^3+3 t^2-\frac{6 \eta}{h_i^x h_j^y} t+\left(-\frac{4 \xi}{h_i^{x 2} h_j^y}+\frac{6 \eta}{h_i^x h_j^y}+1\right) \geq 0, t \in(-1,0) .
$$

Let

$$
t=x-1,
$$

then

$$
\tilde{g}(x)=x^3+\left(-\frac{6 \eta}{h_i^x h_j^y}-3\right) x+\left(-\frac{4 \xi}{h_i^{x 2} h_j^y}+\frac{12 \eta}{h_i^x h_j^y}+3\right), x \in(0,1) .
$$

$$
\begin{aligned}
\tilde{g}^{\prime}(x) & =3 x^2+\left(-\frac{6 \eta}{h_i^x h_j^y}-3\right) \\
& =3\left[x^2+\left(-\frac{2 \eta}{h_i^x h_j^y}-1\right)\right]
\end{aligned}
$$

Case 1 of $\tilde{g}(x)$: If $-\frac{2 \eta}{h_i^x h_j^y}-1 \geq 0$, i.e., $\eta \leq-\frac{h_i^x h_j^y}{2}$, then $\tilde{g}(x)$ is monotomically increasing, and $\tilde{g}(x) \geq 0, x \in(0,1)$, if and only if

$$
\tilde{g}(0)=-\frac{4 \xi}{h_i^{x 2} h_j^y}+\frac{12 \eta}{h_i^x h_j^y}+3 \geq 0,
$$

i.e.,

$$
\eta \geq \frac{1}{3 h_i^x}\left(\xi-\frac{3}{4} h_i^{x 2} h_j^y\right) .
$$

Furthermore,

$$
-\frac{h_i^x h_j^y}{2} \geq \eta \geq \frac{1}{3 h_i^x}\left(\xi-\frac{3}{4} h_i^{x 2} h_j^y\right)
$$

implies that

$$\xi \leq -\frac{3}{4} h_i^{x2} h_j^y.$$

Therefore, in this case, $G(\xi, \eta)$ is finite if and only if

$$\xi \leq -\frac{3}{4} h_i^{x2} h_j^y, \eta \leq -\frac{h_i^x h_j^y}{2}, \eta \geq \frac{1}{3 h_i^x} \left(\xi - \frac{3}{4} h_i^{x2} h_j^y \right). \tag{A.19}$$

At point $(\xi, \eta) = \left(-\frac{3}{4} h_i^{x2} h_j^y, -\frac{h_i^x h_j^y}{2} \right)$, we have $\frac{d\xi}{d\eta} = 3 h_i^x$.

Case 2 of $\tilde{g}(x)$: If $-\frac{2\eta}{h_i^x h_j^y} - 1 < 0$, i.e., $\eta > -\frac{h_i^x h_j^y}{2}$, then there exists a $x^* \in (0, 1)$ such that $\tilde{g}'(x^*) = 0$. We have

$$x^* = \sqrt{1 + \frac{2\eta}{h_i^x h_j^y}} > 0.$$

Now we consider the situation that

$$0 < x^* < 1.$$

Hence, $\tilde{g}(x^*)$ is the minimum value of $\tilde{g}(x)$. $\tilde{g}(x) \geq 0$ if and only if $\tilde{g}(x^*) \geq 0$.

$$0 < x^* < 1 \Longrightarrow -\frac{h_i^x h_j^y}{2} < \eta < 0.$$

$$\begin{aligned}
&\tilde{g}(x^*) \\
&= \left(1 + \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{3}{2}} + \left(-\frac{6\eta}{h_i^x h_j^y} - 3\right)\left(1 + \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{1}{2}} + \left(-\frac{4\xi}{h_i^{x2} h_j^y} + \frac{12\eta}{h_i^x h_j^y} + 3\right) \\
&= -2\left(1 + \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{3}{2}} + \left(-\frac{4\xi}{h_i^{x2} h_j^y} + \frac{12\eta}{h_i^x h_j^y} + 3\right)
\end{aligned}$$

Hence, $\tilde{g}(x^*) \geq 0$ implies

$$\begin{aligned}
-\left(\frac{2\eta}{h_i^x h_j^y} + 1\right)^{\frac{3}{2}} &\geq \frac{2\xi}{h_i^{x2} h_j^y} - \frac{6\eta}{h_i^x h_j^y} - \frac{3}{2} \\
\left(\frac{2\eta}{h_i^x h_j^y} + 1\right)^{3} &\leq \left(\frac{2\xi}{h_i^{x2} h_j^y} - \frac{6\eta}{h_i^x h_j^y} - \frac{3}{2}\right)^2 \\
\xi &\leq 3 h_i^x \eta - \frac{1}{2} h_i^{x2} h_j^y \left(\frac{2\eta}{h_i^x h_j^y} + 1\right)^{\frac{3}{2}} + \frac{3}{4} h_i^{x2} h_j^y
\end{aligned}$$

In summary, case 2 is the situation such that

$$\begin{aligned}
-\frac{2\eta}{h_i^x h_j^y} - 1 &< 0 \\
0 < x^* &< 1 \\
\tilde{g}(x^*) &\geq 0
\end{aligned}$$

i.e.,

$$-\frac{h_i^x h_j^y}{2} < \eta < 0, \xi \leq 3h_i^x \eta - \frac{1}{2} h_i^{x2} h_j^y \left(\frac{2\eta}{h_i^x h_j^y} + 1\right)^{\frac{3}{2}} + \frac{3}{4} h_i^{x2} h_j^y$$

Property: From function

$$\xi = 3h_i^x \eta - \frac{1}{2} h_i^{x2} h_j^y \left(\frac{2\eta}{h_i^x h_j^y} + 1\right)^{\frac{3}{2}} + \frac{3}{4} h_i^{x2} h_j^y,$$

we can derive that

$$\begin{aligned}
\frac{d\xi}{d\eta} &= 3h_i^x - \frac{1}{2} h_i^{x2} h_j^y \frac{3}{2} \left(\frac{2\eta}{h_i^x h_j^y} + 1\right)^{\frac{1}{2}} \frac{2}{h_i^x h_j^y} \\
&= 3h_i^x - \frac{3}{2} h_i^x \left(\frac{2\eta}{h_i^x h_j^y} + 1\right)^{\frac{1}{2}} \\
&> 3h_i^x - \frac{3}{2} h_i^x \\
&= \frac{3}{2} h_i^x > 0
\end{aligned}$$

$$\frac{d^2\xi}{d\eta^2} = -\frac{3}{2h_j^y} \left(\frac{2\eta}{h_i^x h_j^y} + 1\right)^{-\frac{1}{2}} < 0$$

$$\left.\frac{d\xi}{d\eta}\right|_{(\xi,\eta)=\left(-\frac{3}{4} h_i^{x2} h_j^y, -\frac{h_i^x h_j^y}{2}\right)} = 3h_i^x$$

$$\left.\frac{d\xi}{d\eta}\right|_{(\xi,\eta)=\left(\frac{1}{4} h_i^{x2} h_j^y, 0\right)} = \frac{3}{2} h_i^x.$$

Furthermore, ξ is a monotonically increasing function of η, and

$$\xi \leq 3h_i^x \eta - \frac{1}{2} h_i^{x2} h_j^y \left(\frac{2\eta}{h_i^x h_j^y} + 1\right)^{\frac{3}{2}} + \left.\frac{3}{4} h_i^{x2} h_j^y\right|_{\eta=0} = \frac{1}{4} h_i^{x2} h_j^y$$

Hence, (ξ, η) is defined over the set

$$\xi \leq \frac{1}{4} h_i^{x2} h_j^y, -\frac{h_i^x h_j^y}{2} < \eta < 0, \xi \leq 3h_i^x \eta - \frac{1}{2} h_i^{x2} h_j^y \left(\frac{2\eta}{h_i^x h_j^y} + 1\right)^{\frac{3}{2}} + \frac{3}{4} h_i^{x2} h_j^y \qquad (A.20)$$

which is a convex set.

Case 3 of $\tilde{g}(x)$: If $-\frac{2\eta}{h_i^x h_j^y} - 1 < 0$ and $x^* \geq 1$, then $\tilde{g}(x)$ is monotonically decreasing in $(0, 1)$. Hence, $\tilde{g}(x) \geq 0$ if and only if $\tilde{g}(1) \geq 0$.

$$\begin{aligned}
&\tilde{g}(1) \\
&= 1 - \frac{6\eta}{h_i^x h_j^y} - 3 - \frac{4\xi}{h_i^{x2} h_j^y} + \frac{12\eta}{h_i^x h_j^y} + 3 \\
&= -\frac{4\xi}{h_i^{x2} h_j^y} + \frac{6\eta}{h_i^x h_j^y} + 1
\end{aligned}$$

$\tilde{g}(1) \geq 0$ implies that

$$\eta \geq \frac{2}{3h_i^x}\xi - \frac{h_i^x h_j^y}{6} = \frac{2}{3h_i^x}\left(\xi - \frac{1}{4}h_i^{x2}h_j^y\right).$$

$$x^* = \sqrt{1 + \frac{2\eta}{h_i^x h_j^y}} \geq 1.$$

implies

$$\eta \geq 0.$$

In summary, case 3 is the situation that

$$-\frac{2\eta}{h_i^x h_j^y} - 1 < 0$$
$$x^* \geq 1$$
$$\tilde{g}(1) \geq 0$$

i.e.,

$$\eta \geq 0, \eta \geq \frac{2}{3h_i^x}\left(\xi - \frac{1}{4}h_i^{x2}h_j^y\right) \tag{A.21}$$

At point $\left(\frac{1}{4}h_i^{x2}h_j^y, 0\right)$, the derivative $\frac{d\xi}{d\eta} = \frac{3}{2}h_i^x$.

Conclusion: in the situation of $B < 0$, the conojugate transform of $F(c_{30}, c_{20})$ is

$$G(\xi, \eta) = 0,$$

where (ξ, η) is defined over the union of sets (A.19), (A.20), and (A.21), i.e.,

$$\Omega_{32} = \left\{ \begin{array}{c} \eta \geq \frac{2}{3h_i^x}\left(\xi - \frac{1}{4}h_i^{x2}h_j^y\right) \text{ if } \eta \geq 0, \\ \xi \leq \frac{1}{4}h_i^{x2}h_j^y, \xi \leq 3h_i^x\eta - \frac{1}{2}h_i^{x2}h_j^y\left(\frac{2\eta}{h_i^x h_j^y} + 1\right)^{\frac{3}{2}} + \frac{3}{4}h_i^{x2}h_j^y \text{ if } -\frac{h_i^x h_j^y}{2} < \eta < 0, \\ \xi \leq -\frac{3}{4}h_i^{x2}h_j^y, \eta \geq \frac{1}{3h_i^x}\left(\xi - \frac{3}{4}h_i^{x2}h_j^y\right) \text{ if } \eta \leq -\frac{h_i^x h_j^y}{2}. \end{array} \right\}. \tag{A.22}$$

This is a convex set. The boundary of this set is C^1-smooth.

Conclusion of case 3

In the situation of Case 3, the conjugate transform of $F(c_{30}, c_{20})$ is

$$G(\xi, \eta) = 0,$$

where (ξ, η) is defined over the intersection of Ω_{31} and Ω_{32}. i.e.,

$$\Omega_3 = \Omega_{31} \cap \Omega_{32}.$$

We have shown that the boundary curves of both Ω_{31} and Ω_{32} are convex and C^1-smooth. They intersect at the point $(\xi, \eta) = \left(\frac{1}{4}h_i^{x2}h_j^y, 0\right)$. Furthermore, theire tangent function at point $\left(\frac{1}{4}h_i^{x2}h_j^y, 0\right)$ is $\eta = \frac{2}{3h_i^x}\left(\xi - \frac{1}{4}h_i^{x2}h_j^y\right)$. Hence, the constraint $\eta \geq \frac{2}{3h_i^x}\left(\xi - \frac{1}{4}h_i^{x2}h_j^y\right)$ is redundant. So we have

$$\Omega_3 = \left\{ \begin{array}{c} \xi \leq \frac{3}{4}h_i^{x2}h_j^y \text{ if } \eta \geq \frac{h_i^x h_j^y}{2}, \\ \xi \leq \frac{3}{4}h_i^{x2}h_j^y, \eta \geq \frac{h_i^x h_j^y}{2} - \frac{h_i^x h_j^y}{2}\left(\frac{3}{2} - \frac{2\xi}{h_i^{x2}h_j^y}\right)^{\frac{2}{3}} \text{ if } 0 \leq \eta < \frac{h_i^x h_j^y}{2}, \\ \xi \leq \frac{1}{4}h_i^{x2}h_j^y, \xi \leq 3h_i^x\eta - \frac{1}{2}h_i^{x2}h_j^y\left(\frac{2\eta}{h_i^x h_j^y} + 1\right)^{\frac{3}{2}} + \frac{3}{4}h_i^{x2}h_j^y \text{ if } -\frac{h_i^x h_j^y}{2} < \eta < 0, \\ \xi \leq -\frac{3}{4}h_i^{x2}h_j^y, \eta \geq \frac{1}{3h_i^x}\left(\xi - \frac{3}{4}h_i^{x2}h_j^y\right) \text{ if } \eta \leq -\frac{h_i^x h_j^y}{2}. \end{array} \right\}$$

[WE ONLY NEED TO CONSIDER THE FEASIBLE SET OVER $|\xi| \leq \frac{3}{4}h_i^{x2}h_j^y$, $|\eta| \leq \frac{h_i^x h_j^y}{2}$, WHICH IS CONSTRAINED BY PIECE-WISE CUBIC FUNCTIONS. NEXT WE GET RID OF THE PIECE-WISE, TO SHOW THAT IT IS THE INTERSECTION OF THESE TWO CUBIC FUNCTIONS]

The function

$$\eta \geq \frac{h_i^x h_j^y}{2} - \frac{h_i^x h_j^y}{2}\left(\frac{3}{2} - \frac{2\xi}{h_i^{x2}h_j^y}\right)^{\frac{2}{3}}$$

can be re-stated as

$$\xi \leq \frac{3}{4}h_i^{x2}h_j^y - \frac{1}{2}h_i^{x2}h_j^y\left(1 - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{3}{2}}.$$

Consider two functions

$$f(\eta) = \frac{3}{4}h_i^{x2}h_j^y - \frac{1}{2}h_i^{x2}h_j^y\left(1 - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{3}{2}} \tag{A.23}$$

and

$$g(\eta) = 3h_i^x\eta - \frac{1}{2}h_i^{x2}h_j^y\left(\frac{2\eta}{h_i^x h_j^y} + 1\right)^{\frac{3}{2}} + \frac{3}{4}h_i^{x2}h_j^y \tag{A.24}$$

defined over the set $\eta \in \left[-\frac{h_i^x h_j^y}{2}, \frac{h_i^x h_j^y}{2}\right]$. Let

$$\begin{aligned} H(\eta) &= f(\eta) - g(\eta) \\ &= \frac{1}{2}h_i^{x2}h_j^y\left(1 + \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{3}{2}} - \frac{1}{2}h_i^{x2}h_j^y\left(1 - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{3}{2}} - 3h_i^x\eta, \eta \in \left[-\frac{h_i^x h_j^y}{2}, \frac{h_i^x h_j^y}{2}\right]. \end{aligned}$$

Then

$$H(0) = 0,$$

$$H'(\eta) = \frac{3}{2}h_i^x\left(1 + \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{1}{2}} + \frac{3}{2}h_i^x\left(1 - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{1}{2}} - 3h_i^x,$$

$$H'(0) = 0,$$

$$H''(\eta) = \frac{3}{2h_{jk}^{y}} \left\{ \left(1 + \frac{2\eta}{h_i^x h_j^y}\right)^{-\frac{1}{2}} - \left(1 - \frac{2\eta}{h_i^x h_j^y}\right)^{-\frac{1}{2}} \right\}.$$

For $-\frac{h_i^x h_j^y}{2} \leq \eta \leq 0$, $H''(\eta) \geq 0$. Hence, $H'(\eta)$ is monotonically increasing, and

$$H'(\eta) \leq H'(0) = 0.$$

Therefore, $H(\eta)$ is monotonically decreasing and

$$H(\eta) \geq H(0) = 0,$$

i.e.,

$$f(\eta) \geq g(\eta), -\frac{h_i^x h_j^y}{2} \leq \eta \leq 0.$$

The constraint $f(\eta) \geq \xi$ is automatically satisfied when $g(\eta) \geq \xi$.

Following the same logic, we have

$$f(\eta) \leq g(\eta), 0 \leq \eta \leq \frac{h_i^x h_j^y}{2}.$$

The constraint $g(\eta) \geq \xi$ is automatically satisfied when $f(\eta) \geq \xi$.

The feasible set Ω_3 is simplified as

$$\Omega_3 = \left\{ \begin{array}{c} \xi \leq \frac{3}{4} h_i^{x2} h_j^y \text{ if } \eta \geq \frac{h_i^x h_j^y}{2}, \\ \xi \leq \frac{3}{4} h_i^{x2} h_j^y, \eta \geq \frac{h_i^x h_j^y}{2} - \frac{h_i^x h_j^y}{2} \left(\frac{3}{2} - \frac{2\xi}{h_i^{x2} h_j^y}\right)^{\frac{2}{3}} \text{ and} \\ \xi \leq 3h_i^x \eta - \frac{1}{2} h_i^{x2} h_j^y \left(\frac{2\eta}{h_i^x h_j^y} + 1\right)^{\frac{3}{2}} + \frac{3}{4} h_i^{x2} h_j^y \text{ if } -\frac{h_i^x h_j^y}{2} \leq \eta \leq \frac{h_i^x h_j^y}{2}, \\ \xi \leq -\frac{3}{4} h_i^{x2} h_j^y, \eta \geq \frac{1}{3h_i^x}\left(\xi - \frac{3}{4} h_i^{x2} h_j^y\right) \text{ if } \eta \leq -\frac{h_i^x h_j^y}{2}. \end{array} \right\} \tag{A.25}$$

A.2.4 Case 4

Case 4: $f\left(-\frac{1}{2}\right) > 0$ and $f\left(\frac{1}{2}\right) < 0$

This is the case that

$$f\left(-\frac{1}{2}\right) = -A + B > 0,$$
$$f\left(\frac{1}{2}\right) = A + B < 0,$$

whihch is equivalent to

$$|B| < -A, A < 0.$$

Let $\hat{t}$ be the root of $f(t)$, i.e.,

$$f\left(\hat{t}\right) = 0, \hat{t} \in \left[-\frac{1}{2}, \frac{1}{2}\right].$$

case 4.1 $B > 0$

This is equivalent to

$$0 < \hat{t} = -\frac{B}{2A} < \frac{1}{2}, A < 0, B > 0.$$

$$\begin{aligned}
&F(c_{30}, c_{20}) = \left\| \frac{\partial^2 z}{\partial x^2} \right\|_1 \\
&= h_i^x h_j^y \left\{ \int_{-\frac{1}{2}}^{0} \int_{-\frac{1}{2}}^{t} (2At + B)\, dsdt + \int_{0}^{\hat{t}} \int_{-\frac{1}{2}}^{-t} (2At + B)\, dsdt + \int_{\hat{t}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{-t} (-2At - B)\, dsdt \right\} \\
&= -h_i^x h_j^y \left\{ \frac{1}{12}\frac{B^3}{A^2} + \frac{1}{4}\frac{B^2}{A} + \frac{1}{12}A \right\}.
\end{aligned}$$

Let

$$t = \frac{B}{A}, -1 < t < 0,$$

In this situation, the conjugate transform of $F(c_{30}, c_{20})$ is

$$\begin{aligned}
&G(\xi, \eta) \\
&= \sup\left\{\xi c_{30} + \eta c_{20} - F(c_{30}, c_{20})\right\} \\
&= \sup_{|B| < -A, A < 0, B > 0} \left\{ \xi \frac{1}{3h_i^x} A + \eta \left(-\frac{1}{2}A + \frac{1}{2}B \right) + h_i^x h_j^y \left[\frac{1}{12}\frac{B^3}{A^2} + \frac{1}{4}\frac{B^2}{A} + \frac{1}{12}A \right] \right\} \\
&= \sup_{-1 < t < 0, A < 0} \left\{ \xi \frac{1}{3h_i^x} A + \eta \left(-\frac{1}{2}A + \frac{1}{2}tA \right) + h_i^x h_j^y \left[\frac{1}{12}t^3 A + \frac{1}{4}t^2 A + \frac{1}{12}A \right] \right\} \\
&= \sup_{-1 < t < 0, A < 0} \left\{ \frac{h_i^x h_j^y}{12} t^3 + \frac{h_i^x h_j^y}{4} t^2 + \frac{\eta}{2} t + \left(\frac{\xi}{3h_i^x} - \frac{\eta}{2} + \frac{h_i^x h_j^y}{12} \right) \right\} A \\
&= \sup_{-1 < t < 0, A < 0} \frac{h_i^x h_j^y}{12} \left\{ t^3 + 3t^2 + \frac{6\eta}{h_i^x h_j^y} t + \left(\frac{4\xi}{h_i^{x2} h_j^y} - \frac{6\eta}{h_i^x h_j^y} + 1 \right) \right\} A \\
&= \sup_{-1 < t < 0, A < 0} \left\{ \frac{h_i^x h_j^y}{12} g(t)\, A \right\}
\end{aligned}$$

Since $A < 0$, $G(\xi, \eta)$ is finite if and only if

$$g(t) = t^3 + 3t^2 + \frac{6\eta}{h_i^x h_j^y} t + \left(+ \frac{4\xi}{h_i^{x2} h_j^y} - \frac{6\eta}{h_i^x h_j^y} + 1 \right) \geq 0, t \in (-1, 0).$$

Let

$$t = x - 1,$$

then

$$\tilde{g}(x) = x^3 + \left(\frac{6\eta}{h_i^x h_j^y} - 3 \right) x + \left(\frac{4\xi}{h_i^{x2} h_j^y} - \frac{12\eta}{h_i^x h_j^y} + 3 \right), x \in (0, 1).$$

$$
\begin{aligned}
\tilde{g}'(x) &= 3x^2 + \left(\frac{6\eta}{h_i^x h_j^y} - 3\right) \\
&= 3\left[x^2 + \left(\frac{2\eta}{h_i^x h_j^y} - 1\right)\right]
\end{aligned}
$$

Case 1 of $\tilde{g}(x)$: If $\frac{2\eta}{h_i^x h_j^y} - 1 \geq 0$, i.e., $\eta \geq \frac{h_i^x h_j^y}{2}$, then $\tilde{g}(x)$ is monotomically increasing, and $\tilde{g}(x) \geq 0, x \in (0,1)$, if and only if

$$
\tilde{g}(0) = \frac{4\xi}{h_i^{x2} h_j^y} - \frac{12\eta}{h_i^x h_j^y} + 3 \geq 0,
$$

i.e.,

$$
\eta \leq \frac{1}{3h_i^x}\left(\xi + \frac{3}{4} h_i^{x2} h_j^y\right).
$$

Furthermore,

$$
\frac{h_i^x h_j^y}{2} \leq \eta \leq \frac{1}{3h_i^x}\left(\xi + \frac{3}{4} h_i^{x2} h_j^y\right)
$$

implies that

$$
\xi \geq \frac{3}{4} h_i^{x2} h_j^y.
$$

Therefore, in this case, $G(\xi, \eta)$ is finite if and only if

$$
\xi \geq \frac{3}{4} h_i^{x2} h_j^y, \eta \geq \frac{h_i^x h_j^y}{2}, \eta \leq \frac{1}{3h_i^x}\left(\xi + \frac{3}{4} h_i^{x2} h_j^y\right). \tag{A.26}
$$

Case 2 of $\tilde{g}(x)$: If $\frac{2\eta}{h_i^x h_j^y} - 1 < 0$, i.e., $\eta < \frac{h_i^x h_j^y}{2}$, then there exists a $x^* \in (0,1)$ such that $\tilde{g}'(x^*) = 0$. We have

$$
x^* = \sqrt{1 - \frac{2\eta}{h_i^x h_j^y}} > 0.
$$

Now we consider the situation that

$$
0 < x^* < 1.
$$

Hence, $\tilde{g}(x^*)$ is the minimum value of $\tilde{g}(x)$. $\tilde{g}(x) \geq 0$ if and only if $\tilde{g}(x^*) \geq 0$.

$$
0 < x^* < 1 \Longrightarrow 0 < \eta < \frac{h_i^x h_j^y}{2}.
$$

$$
\begin{aligned}
&\tilde{g}(x^*) \\
&= \left(1 - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{3}{2}} + \left(\frac{6\eta}{h_i^x h_j^y} - 3\right)\left(1 - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{1}{2}} + \left(\frac{4\xi}{h_i^{x2} h_j^y} - \frac{12\eta}{h_i^x h_j^y} + 3\right) \\
&= -2\left(1 - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{3}{2}} + \left(\frac{4\xi}{h_i^{x2} h_j^y} - \frac{12\eta}{h_i^x h_j^y} + 3\right)
\end{aligned}
$$

Hence, $\tilde{g}(x^*) \geq 0$ implies

$$\left(1 - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{3}{2}} \leq \frac{2\xi}{h_i^{x2} h_j^y} - \frac{6\eta}{h_i^x h_j^y} + \frac{3}{2}$$

$$\left(1 - \frac{2\eta}{h_i^x h_j^y}\right)^{3} \leq \left(\frac{2\xi}{h_i^{x2} h_j^y} - \frac{6\eta}{h_i^x h_j^y} + \frac{3}{2}\right)^2$$

$$\xi \geq 3h_i^x \eta + \frac{1}{2} h_i^{x2} h_j^y \left(1 - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{3}{2}} - \frac{3}{4} h_i^{x2} h_j^y$$

In summary, case 2 is the situation such that

$$\frac{2\eta}{h_i^x h_j^y} - 1 < 0$$

$$0 < x^* < 1$$

$$\tilde{g}(x^*) \geq 0$$

i.e.,

$$0 < \eta < \frac{h_i^x h_j^y}{2}, \xi \geq 3h_i^x \eta + \frac{1}{2} h_i^{x2} h_j^y \left(1 - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{3}{2}} - \frac{3}{4} h_i^{x2} h_j^y$$

Property: From function

$$\xi = 3h_i^x \eta + \frac{1}{2} h_i^{x2} h_j^y \left(1 - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{3}{2}} - \frac{3}{4} h_i^{x2} h_j^y,$$

we can derive that

$$\begin{aligned}
\frac{d\xi}{d\eta} &= 3h_i^x - \frac{1}{2} h_i^{x2} h_j^y \frac{3}{2} \left(1 - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{1}{2}} \frac{2}{h_i^x h_j^y} \\
&= 3h_i^x - \frac{3}{2} h_i^x \left(1 - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{1}{2}} \\
&> 3h_i^x - \frac{3}{2} h_i^x \\
&= \frac{3}{2} h_i^x > 0
\end{aligned}$$

$$\frac{d^2\xi}{d\eta^2} = \frac{3}{2h_j^y} \left(1 - \frac{2\eta}{h_i^x h_j^y}\right)^{-\frac{1}{2}} > 0$$

$$\left.\frac{d\xi}{d\eta}\right|_{(\xi,\eta)=\left(\frac{3}{4} h_i^{x2} h_j^y, \frac{h_i^x h_j^y}{2}\right)} = 3h_i^x$$

$$\left.\frac{d\xi}{d\eta}\right|_{(\xi,\eta)=\left(-\frac{1}{4} h_i^{x2} h_j^y, 0\right)} = \frac{3}{2} h_i^x.$$

Furthermore, ξ is a monotonically increasing function of η, and

$$\xi \geq 3h_i^x\eta + \frac{1}{2}h_i^{x2}h_j^y\left(1-\frac{2\eta}{h_i^xh_j^y}\right)^{\frac{3}{2}} - \frac{3}{4}h_i^{x2}h_j^y\Bigg|_{\eta=0} = -\frac{1}{4}h_i^{x2}h_j^y$$

Hence, (ξ, η) is defined over the set

$$\xi \geq -\frac{1}{4}h_i^{x2}h_j^y, 0 < \eta < \frac{h_i^xh_j^y}{2}, \xi \geq 3h_i^x\eta + \frac{1}{2}h_i^{x2}h_j^y\left(1-\frac{2\eta}{h_i^xh_j^y}\right)^{\frac{3}{2}} - \frac{3}{4}h_i^{x2}h_j^y \tag{A.27}$$

which is a convex set.

Case 3 of $\tilde{g}(x)$: If $\frac{2\eta}{h_i^xh_j^y} - 1 < 0$ and $x^* \geq 1$, then $\tilde{g}(x)$ is monotonically decreasing in $(0,1)$. Hence, $\tilde{g}(x) \geq 0$ if and only if $\tilde{g}(1) \geq 0$.

$$\begin{aligned}
&\tilde{g}(1) \\
&= 1 + \frac{6\eta}{h_i^xh_j^y} - 3 + \frac{4\xi}{h_i^{x2}h_j^y} - \frac{12\eta}{h_i^xh_j^y} + 3 \\
&= \frac{4\xi}{h_i^{x2}h_j^y} - \frac{6\eta}{h_i^xh_j^y} + 1
\end{aligned}$$

$\tilde{g}(1) \geq 0$ implies that

$$\eta \leq \frac{2}{3h_i^x}\xi + \frac{h_i^xh_j^y}{6} = \frac{2}{3h_i^x}\left(\xi + \frac{1}{4}h_i^{x2}h_j^y\right).$$

$$x^* = \sqrt{1-\frac{2\eta}{h_i^xh_j^y}} \geq 1.$$

implies

$$\eta \leq 0.$$

In summary, case 3 is the situation that

$$\begin{aligned}
\frac{2\eta}{h_i^xh_j^y} - 1 &< 0 \\
x^* &\geq 1 \\
\tilde{g}(1) &\geq 0
\end{aligned}$$

i.e.,

$$\eta \leq 0, \eta \leq \frac{2}{3h_i^x}\left(\xi + \frac{1}{4}h_i^{x2}h_j^y\right) \tag{A.28}$$

At point $\left(-\frac{1}{4}h_i^{x2}h_j^y, 0\right)$, the derivative $\frac{d\xi}{d\eta} = \frac{3}{2}h_i^x$.

Conclusion: in the situation of $B > 0$, the conojugate transform of $F(c_{30}, c_{20})$ is

$$G(\xi, \eta) = 0,$$

where (ξ, η) is defined over the union of sets (A.26), (A.27), and (A.28), i.e.,

$$\Omega_{41} = \left\{ \begin{array}{c} \eta \leq \frac{2}{3h_i^x}\left(\xi + \frac{1}{4}h_i^{x2}h_j^y\right) \text{ if } \eta \leq 0, \\ \xi \geq -\frac{1}{4}h_i^{x2}h_j^y, \xi \geq 3h_i^x\eta + \frac{1}{2}h_i^{x2}h_j^y\left(1 - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{3}{2}} - \frac{3}{4}h_i^{x2}h_j^y \text{ if } 0 < \eta < \frac{h_i^x h_j^y}{2}, \\ \xi \geq \frac{3}{4}h_i^{x2}h_j^y, \eta \leq \frac{1}{3h_i^x}\left(\xi + \frac{3}{4}h_i^{x2}h_j^y\right) \text{ if } \eta \geq \frac{h_i^x h_j^y}{2}. \end{array} \right\}. \tag{A.29}$$

This is a convex set. The boundary of this set is C^1-smooth.

case 4.2 $B < 0$

this is equivalent to

$$-\frac{1}{2} < \hat{t} < 0, A < 0, B < 0.$$

$$\begin{aligned} F(c_{30}, c_{20}) &= \left\|\frac{\partial^2 z}{\partial x^2}\right\|_1 \\ &= h_i^x h_j^y \left\{\int_{-\frac{1}{2}}^{\hat{t}}\int_{-\frac{1}{2}}^{t}(2At + B)\,dsdt + \int_{\hat{t}}^{0}\int_{-\frac{1}{2}}^{t}(-2At - B)\,dsdt + \int_{0}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{-t}(-2At - B)\,dsdt\right\} \\ &= h_i^x h_j^y \left\{\frac{1}{12}\frac{B^3}{A^2} - \frac{1}{4}\frac{B^2}{A} - \frac{1}{12}A\right\}. \end{aligned}$$

Let

$$t = \frac{B}{A}, 0 < t < 1,$$

In this situation, the conjugate transform of $F(c_{30}, c_{20})$ is

$$\begin{aligned} &G(\xi, \eta) \\ &= \sup\{\xi c_{30} + \eta c_{20} - F(c_{30}, c_{20})\} \\ &= \sup_{|B| < -A, A<0, B<0}\left\{\xi\frac{1}{3h_i^x}A + \eta\left(-\frac{1}{2}A + \frac{1}{2}B\right) - h_i^x h_j^y\left[\frac{1}{12}\frac{B^3}{A^2} - \frac{1}{4}\frac{B^2}{A} - \frac{1}{12}A\right]\right\} \\ &= \sup_{0<t<1, A<0}\left\{\xi\frac{1}{3h_i^x}A + \eta\left(-\frac{1}{2}A + \frac{1}{2}tA\right) - h_i^x h_j^y\left[\frac{1}{12}t^3A - \frac{1}{4}t^2A - \frac{1}{12}A\right]\right\} \\ &= \sup_{0<t<1, A<0}\left\{-\frac{h_i^x h_j^y}{12}t^3 + \frac{h_i^x h_j^y}{4}t^2 + \frac{\eta}{2}t + \left(\frac{\xi}{3h_i^x} - \frac{\eta}{2} + \frac{h_i^x h_j^y}{12}\right)\right\}A \\ &= \sup_{0<t<1, A<0} -\frac{h_i^x h_j^y}{12}\left\{t^3 - 3t^2 - \frac{6\eta}{h_i^x h_j^y}t - \left(\frac{4\xi}{h_i^{x2}h_j^y} - \frac{6\eta}{h_i^x h_j^y} + 1\right)\right\}A \\ &= \sup_{0<t<1, A<0}\left\{-\frac{h_i^x h_j^y}{12}g(t)A\right\} \end{aligned}$$

Since $A < 0$, $G(\xi, \eta)$ is finite if and only if

$$g(t) = t^3 - 3t^2 - \frac{6\eta}{h_i^x h_j^y}t - \left(\frac{4\xi}{h_i^{x2}h_j^y} - \frac{6\eta}{h_i^x h_j^y} + 1\right) \leq 0, t \in (0, 1).$$

Let

$$t = x + 1,$$

then

$$\tilde{g}(x) = x^3 - \left(\frac{6\eta}{h_i^x h_j^y} + 3\right) x - \left(\frac{4\xi}{h_i^{x2} h_j^y} + 3\right), x \in (-1, 0).$$

$$\begin{aligned}\tilde{g}'(x) &= 3x^2 - \left(\frac{6\eta}{h_i^x h_j^y} + 3\right) \\ &= 3\left[x^2 - \left(\frac{2\eta}{h_i^x h_j^y} + 1\right)\right]\end{aligned}$$

Case 1 of $\tilde{g}(x)$: If $-\left(\frac{2\eta}{h_i^x h_j^y} + 1\right) \geq 0$, i.e., $\eta \leq -\frac{h_i^x h_j^y}{2}$, then $\tilde{g}(x)$ is monotomically increasing, and $\tilde{g}(x) \leq 0, x \in (-1, 0)$, if and only if

$$\tilde{g}(0) = -\frac{4\xi}{h_i^{x2} h_j^y} - 3 \leq 0,$$

i.e.,

$$\xi \geq -\frac{3}{4} h_i^{x2} h_j^y.$$

Therefore, in this case, $G(\xi, \eta)$ is finite if and only if

$$\xi \geq -\frac{3}{4} h_i^{x2} h_j^y, \eta \leq -\frac{h_i^x h_j^y}{2}. \tag{A.30}$$

Case 2 of $\tilde{g}(x)$: If $-\left(\frac{2\eta}{h_i^x h_j^y} + 1\right) < 0$, i.e., $\eta > -\frac{h_i^x h_j^y}{2}$, then there exists a x^* such that $\tilde{g}'(x^*) = 0$. We have

$$x^* = -\sqrt{1 + \frac{2\eta}{h_i^x h_j^y}} < 0.$$

Now we consider the situation that

$$-1 \leq x^* < 0.$$

Hence, $\tilde{g}(x^*)$ is the maximum value of $\tilde{g}(x)$. $\tilde{g}(x) \leq 0$ if and only if $\tilde{g}(x^*) \leq 0$.

$$-1 \leq x^* < 0 \Longrightarrow -\frac{h_i^x h_j^y}{2} < \eta \leq 0.$$

$$\begin{aligned}&\tilde{g}(x^*) \\ &= -\left(1 + \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{3}{2}} + \left(\frac{6\eta}{h_i^x h_j^y} + 3\right)\left(1 + \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{1}{2}} - \left(\frac{4\xi}{h_i^{x2} h_j^y} + 3\right) \\ &= 2\left(1 + \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{3}{2}} - \left(\frac{4\xi}{h_i^{x2} h_j^y} + 3\right)\end{aligned}$$

Hence, $\tilde{g}(x^*) \le 0$ implies

$$\begin{aligned}\left(1+\frac{2\eta}{h_i^x h_j^y}\right)^{\frac{3}{2}} &\le \frac{3}{2}+\frac{2\xi}{h_i^{x2}h_j^y}\\ \left(1+\frac{2\eta}{h_i^x h_j^y}\right)^{3} &\le \left(\frac{3}{2}+\frac{2\xi}{h_i^{x2}h_j^y}\right)^2\\ \eta &\le -\frac{h_i^x h_j^y}{2}+\frac{h_i^x h_j^y}{2}\left(\frac{3}{2}+\frac{2\xi}{h_i^{x2}h_j^y}\right)^{\frac{2}{3}}.\end{aligned}$$

Since $-\left(1+\frac{2\eta}{h_i^x h_j^y}\right) < 0$, above formula also implies that

$$\frac{3}{2}+\frac{2\xi}{h_i^{x2}h_j^y} > 0$$

which is equivalent to

$$\xi \ge -\frac{3}{4}h_i^{x2}h_j^y.$$

In summary, case 2 is the situation such that

$$\begin{aligned}-\left(\frac{2\eta}{h_i^x h_j^y}+1\right) &< 0\\ -1 \le x^* &< 0\\ \tilde{g}(x^*) &\le 0\end{aligned}$$

i.e.,

$$-\frac{h_i^x h_j^y}{2} < \eta \le 0, \xi > -\frac{3}{4}h_i^{x2}h_j^y, \eta \le -\frac{h_i^x h_j^y}{2}+\frac{h_i^x h_j^y}{2}\left(\frac{3}{2}+\frac{2\xi}{h_i^{x2}h_j^y}\right)^{\frac{2}{3}}. \tag{A.31}$$

Property: From function

$$\eta = -\frac{h_i^x h_j^y}{2}+\frac{h_i^x h_j^y}{2}\left(\frac{3}{2}+\frac{2\xi}{h_i^{x2}h_j^y}\right)^{\frac{2}{3}}$$

we can derive that

$$\begin{aligned}\frac{d\eta}{d\xi} &= \frac{2}{3h_i^x}\left(\frac{3}{2}+\frac{2\xi}{h_i^{x2}h_j^y}\right)^{-\frac{1}{3}} > 0\\ \frac{d^2\eta}{d\xi^2} &= -\frac{4}{9h_i^{x2}h_j^y}\left(\frac{3}{2}+\frac{2\xi}{h_i^{x2}h_j^y}\right)^{-\frac{4}{3}} < 0\end{aligned}$$

Hence, (A.31) defines a convex set, and

$$\left.\frac{d\eta}{d\xi}\right|_{(\xi,\eta)=\left(-\frac{3}{4}h_i^{x2}h_j^y,-\frac{1}{2}h_i^x h_j^y\right)} = +\infty$$

$$\left.\frac{d\eta}{d\xi}\right|_{(\xi,\eta)=\left(-\frac{1}{4}h_i^{x2}h_j^y,0\right)}=\frac{2}{3h_i^x}.$$

Case 3 of $\tilde{g}(x)$: If $-\left(\frac{2\eta}{h_i^x h_j^y}+1\right)<0$ and $x^*\leq -1$, then $\tilde{g}(x)$ is monotonically decreasing in $(-1,0)$. Hence, $\tilde{g}(x)\leq 0$ if and only if $\tilde{g}(-1)\leq 0$.

$$\begin{aligned}&\tilde{g}(-1)\\&=-1+\frac{6\eta}{h_i^x h_j^y}+3-\frac{4\xi}{h_i^{x2}h_j^y}-3\\&=\frac{6\eta}{h_i^x h_j^y}-\frac{4\xi}{h_i^{x2}h_j^y}-1\end{aligned}$$

$\tilde{g}(-1)\leq 0$ implies that

$$\eta\leq\frac{2}{3h_i^x}\xi+\frac{h_i^x h_j^y}{6}=\frac{2}{3h_i^x}\left(\xi+\frac{1}{4}h_i^{x2}h_j^y\right).$$

$$x^*=-\sqrt{1+\frac{2\eta}{h_i^x h_j^y}}\leq -1$$

implies

$$\eta\geq 0.$$

Furthermore,

$$0\leq\eta\leq\frac{2}{3h_i^x}\left(\xi+\frac{1}{4}h_i^{x2}h_j^y\right)$$

implies

$$\xi\geq-\frac{1}{4}h_i^{x2}h_j^y.$$

In summary, case 3 is the situation that

$$\begin{aligned}-\left(\frac{2\eta}{h_i^x h_j^y}+1\right)&<0\\x^*&\leq -1\\\tilde{g}(-1)&\leq 0\end{aligned}$$

i.e.,

$$\xi\geq-\frac{1}{4}h_i^{x2}h_j^y,\eta\geq 0,\eta\leq\frac{2}{3h_i^x}\left(\xi+\frac{1}{4}h_i^{x2}h_j^y\right)\tag{A.32}$$

At point $\left(-\frac{1}{4}h_i^{x2}h_j^y,0\right)$, the derivative $\frac{d\eta}{d\xi}=\frac{2}{3h_i^x}$.

Conclusion: in the situation of $B<0$, the conojugate transform of $F(c_{30},c_{20})$ is

$$G(\xi,\eta)=0,$$

where (ξ,η) is defined over the union of sets (A.30), (A.31), and (A.32), i.e.,

$$\Omega_{42}=\left\{\begin{array}{c}\xi\geq-\frac{3}{4}h_i^{x2}h_j^y \text{ if } \eta\leq-\frac{h_i^xh_j^y}{2},\\ \xi>-\frac{3}{4}h_i^{x2}h_j^y,\eta\leq-\frac{h_i^xh_j^y}{2}+\frac{h_i^xh_j^y}{2}\left(\frac{3}{2}+\frac{2\xi}{h_i^{x2}h_j^y}\right)^{\frac{2}{3}} \text{ if } -\frac{h_i^xh_j^y}{2}<\eta\leq 0,\\ \xi\geq-\frac{1}{4}h_i^{x2}h_j^y,\eta\leq\frac{2}{3h_i^x}\left(\xi+\frac{1}{4}h_i^{x2}h_j^y\right) \text{ if } \eta\geq 0.\end{array}\right\}. \tag{A.33}$$

This is a convex set. The boundary of this set is C^1-smooth.

Conclusion of case 4

[This part is similar to Case 3, but we need to go through the details to verify it.]

In the situation of Case 4, the conjugate transform of $F(c_{30},c_{20})$ is

$$G(\xi,\eta)=0,$$

where (ξ,η) is defined over the intersection of Ω_{41} and Ω_{42}. i.e.,

$$\Omega_4=\Omega_{41}\cap\Omega_{42}.$$

We have shown that the boundary curves of both Ω_{41} and Ω_{42} are convex and C^1-smooth. They intersect at the point $(\xi,\eta)=\left(-\frac{1}{4}h_i^{x2}h_j^y,0\right)$. Furthermore, theire tangent function at point $\left(-\frac{1}{4}h_i^{x2}h_j^y,0\right)$ is $\eta=\frac{2}{3h_i^x}\left(\xi+\frac{1}{4}h_i^{x2}h_j^y\right)$. Hence, the constraint $\eta\leq\frac{2}{3h_i^x}\left(\xi+\frac{1}{4}h_i^{x2}h_j^y\right)$ is redundant. So we have

$$\Omega_4=\left\{\begin{array}{c}\xi\geq-\frac{3}{4}h_i^{x2}h_j^y \text{ if } \eta\leq-\frac{h_i^xh_j^y}{2},\\ \xi>-\frac{3}{4}h_i^{x2}h_j^y,\eta\leq-\frac{h_i^xh_j^y}{2}+\frac{h_i^xh_j^y}{2}\left(\frac{3}{2}+\frac{2\xi}{h_i^{x2}h_j^y}\right)^{\frac{2}{3}} \text{ if } -\frac{h_i^xh_j^y}{2}<\eta<0,\\ \xi\geq-\frac{1}{4}h_i^{x2}h_j^y,\xi\geq 3h_i^x\eta+\frac{1}{2}h_i^{x2}h_j^y\left(1-\frac{2\eta}{h_i^xh_j^y}\right)^{\frac{3}{2}}-\frac{3}{4}h_i^{x2}h_j^y \text{ if } 0\leq\eta<\frac{h_i^xh_j^y}{2},\\ \xi\geq\frac{3}{4}h_i^{x2}h_j^y,\eta\leq\frac{1}{3h_i^x}\left(\xi+\frac{3}{4}h_i^{x2}h_j^y\right) \text{ if } \eta\geq\frac{h_i^xh_j^y}{2}.\end{array}\right\}$$

[WE ONLY NEED TO CONSIDER THE FEASIBLE SET OVER $|\xi|\leq\frac{3}{4}h_i^{x2}h_j^y$, $|\eta|\leq\frac{h_i^xh_j^y}{2}$, WHICH IS CONSTRAINED BY PIECE-WISE CUBIC FUNCTIONS. NEXT WE GET RID OF THE PIECE-WISE, TO SHOW THAT IT IS THE INTERSECTION OF THESE TWO CUBIC FUNCTIONS]

The function

$$\eta\leq-\frac{h_i^xh_j^y}{2}+\frac{h_i^xh_j^y}{2}\left(\frac{3}{2}+\frac{2\xi}{h_i^{x2}h_j^y}\right)^{\frac{2}{3}}$$

can be re-stated as

$$\xi\geq-\frac{3}{4}h_i^{x2}h_j^y+\frac{1}{2}h_i^{x2}h_j^y\left(1+\frac{2\eta}{h_i^xh_j^y}\right)^{\frac{3}{2}}.$$

Consider two functions

$$f(\eta)=-\frac{3}{4}h_i^{x2}h_j^y+\frac{1}{2}h_i^{x2}h_j^y\left(1+\frac{2\eta}{h_i^xh_j^y}\right)^{\frac{3}{2}} \tag{A.34}$$

and

$$g(\eta) = 3h_i^x\eta + \frac{1}{2}h_i^{x2}h_j^y\left(1 - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{3}{2}} - \frac{3}{4}h_i^{x2}h_j^y \tag{A.35}$$

defined over the set $\eta \in \left[-\frac{h_i^x h_j^y}{2}, \frac{h_i^x h_j^y}{2}\right]$. Let

$$\begin{aligned} H(\eta) &= f(\eta) - g(\eta) \\ &= \frac{1}{2}h_i^{x2}h_j^y\left(1 + \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{3}{2}} - \frac{1}{2}h_i^{x2}h_j^y\left(1 - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{3}{2}} - 3h_i^x\eta, \eta \in \left[-\frac{h_i^x h_j^y}{2}, \frac{h_i^x h_j^y}{2}\right]. \end{aligned}$$

Then the $H(\eta)$ is the same we got in Case 3. Following the same logic we get the following results:

$$f(\eta) \le g(\eta), -\frac{h_i^x h_j^y}{2} \le \eta \le 0.$$

The constraint $f(\eta) \le 0$ is automatically satisfied when $g(\eta) \le 0$.

$$f(\eta) \ge g(\eta), 0 \le \eta \le \frac{h_i^x h_j^y}{2}.$$

The constraint $g(\eta) \le 0$ is automatically satisfied when $f(\eta) \le 0$.

The feasible set Ω_4 is simplified as

$$\Omega_4 = \left\{ \begin{array}{c} \xi \ge -\frac{3}{4}h_i^{x2}h_j^y \text{ if } \eta \le -\frac{h_i^x h_j^y}{2}, \\ \xi \ge -\frac{3}{4}h_i^{x2}h_j^y, \eta \le -\frac{h_i^x h_j^y}{2} + \frac{h_i^x h_j^y}{2}\left(\frac{3}{2} + \frac{2\xi}{h_i^{x2}h_j^y}\right)^{\frac{2}{3}} \text{ and} \\ \xi \ge 3h_i^x\eta + \frac{1}{2}h_i^{x2}h_j^y\left(1 - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{3}{2}} - \frac{3}{4}h_i^{x2}h_j^y \text{ if } -\frac{h_i^x h_j^y}{2} \le \eta \le \frac{h_i^x h_j^y}{2}, \\ \xi \ge \frac{3}{4}h_i^{x2}h_j^y, \eta \le \frac{1}{3h_i^x}\left(\xi + \frac{3}{4}h_i^{x2}h_j^y\right) \text{ if } \eta \ge \frac{h_i^x h_j^y}{2}. \end{array} \right\} \tag{A.36}$$

A.2.5 Conclusion of conjugate transform of $\left\|\frac{\partial^2 z}{\partial x^2}\right\|_1$

The conjugate transform of $\left\|\frac{\partial^2 z}{\partial x^2}\right\|_1$ is

$$G(\xi, \eta) = 0,$$

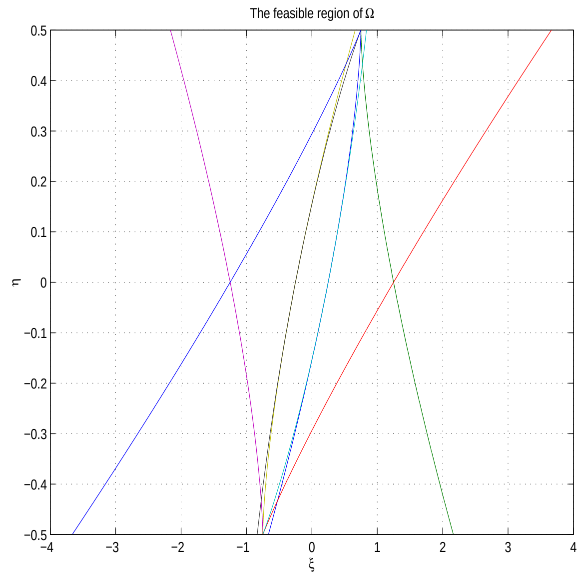


Figure A.3: The feasible region Ω of $\left\| \frac{\partial^2 z}{\partial x^2} \right\|_1$

where (ξ, η) is defined over the set

$$\begin{aligned} &\Omega \\ &= \Omega_1 \cap \Omega_2 \cap \Omega_3 \cap \Omega_4 \\ &= \left\{ \begin{array}{c} \left(1 - \frac{2\eta}{h_i^x h_j^y}\right)^3 \leq \left(\frac{3}{2} - \frac{2\xi}{h_i^{x2} h_j^y}\right)^2, \\ \left(1 + \frac{2\eta}{h_i^x h_j^y}\right)^3 \leq \left(\frac{2\xi}{h_i^{x2} h_j^y} - \frac{6\eta}{h_i^x h_j^y} - \frac{3}{2}\right)^2, \\ \left(1 + \frac{2\eta}{h_i^x h_j^y}\right)^3 \leq \left(\frac{3}{2} + \frac{2\xi}{h_i^{x2} h_j^y}\right)^2, \\ \left(1 - \frac{2\eta}{h_i^x h_j^y}\right)^3 \leq \left(\frac{2\xi}{h_i^{x2} h_j^y} - \frac{6\eta}{h_i^x h_j^y} + \frac{3}{2}\right)^2, \end{array} \right\}, \end{aligned}$$

which is a convex set bounded by four cubic functions developed in Case 3 and Case 4 (See Figure A.3).

A.3 Conjugate transform of $\left\| \frac{\partial^2 z}{\partial y^2} \right\|_1$

$$\begin{aligned}
& f(t,s) \\
& = \frac{\partial^2 z}{\partial y^2} \\
& = 2c_{12}(x - x_i) + 6c_{03}(y - y_j) + 2c_{02} \\
& = 2c_{12}h_i^x\left(t + \frac{1}{2}\right) + 6c_{03}h_j^y\left(s + \frac{1}{2}\right) + 2c_{02} \\
& = 2c_{12}h_i^x t + 6c_{03}h_j^y s + c_{12}h_i^x + 3c_{03}h_j^y + 2c_{02}, \\
(t,s) \in DI.
\end{aligned}$$

The set DI is a triangle with vertex $(0,0)$, $\left(-\frac{1}{2}, -\frac{1}{2}\right)$, and $\left(\frac{1}{2}, -\frac{1}{2}\right)$. $\left\| \frac{\partial^2 z}{\partial y^2} \right\|_1$ is a piecewise linear function of $(c_{12}, c_{03}, c_{02}) \in R^3$. Depending on the sign of $f(t,s)$ at three vertex of DI, the feasible set of $\left\| \frac{\partial^2 z}{\partial y^2} \right\|_1$ is divided into 8 regions E_k, $1 \le k \le 8$. The conjugate transform of $\left\| \frac{\partial^2 z}{\partial y^2} \right\|_1$ can be calculated over each one of these 8 regions separately. Let

$$\begin{aligned}
& F(c_{12}, c_{03}, c_{02}) \\
& = \left\| \frac{\partial^2 z}{\partial y^2} \right\|_1 \\
& = h_i^x h_j^y \int_{DI} \left| 2c_{12}h_i^x t + 6c_{03}h_j^y s + c_{12}h_i^x + 3c_{03}h_j^y + 2c_{02} \right| dtds.
\end{aligned}$$

Let the conjugate transform of $F(c_{12}, c_{03}, c_{02})$ be

$$\begin{aligned}
& G(\xi, \eta, \gamma) \\
& = \sup_{(c_{12}, c_{03}, c_{02}) \in R^3} \{\xi c_{12} + \eta c_{03} + \gamma c_{02} - F(c_{12}, c_{03}, c_{02})\} \\
& = \max_{1 \le k \le 8} \left\{ \sup_{(c_{12}, c_{03}, c_{02}) \in E_k} \{\xi c_{12} + \eta c_{03} + \gamma c_{02} - F(c_{12}, c_{03}, c_{02})\} \right\} \\
& = \max_{1 \le k \le 8} G_k(\xi, \eta, \gamma),
\end{aligned}$$

where $G_k(\xi, \eta, \gamma)$ is defined over the set Ω_k. Hence, $G(\xi, \eta, \gamma)$ is defined over

$$\Omega = \bigcap_{k=1}^{8} \Omega_k.$$

Let

$$\begin{cases}
x = \frac{\eta}{3h_j^y}, \\
y = \frac{\xi}{2h_i^x} - \frac{\eta}{6h_j^y}, \\
z = \frac{\xi}{2h_i^x} + \frac{\eta}{6h_j^y} - \frac{\gamma}{2},
\end{cases}$$

then

$$\begin{cases} \eta = 3h_j^y x, \\ \xi = h_i^x (x + 2y), \\ \gamma = 2x + 2y - 2z, \end{cases}$$

Let

$$K = \frac{1}{12} h_i^x h_j^y.$$

A.3.1 Case 1

Case 1:$(c_{12}, c_{03}, c_{02}) \in E_1 : f(t,s) \geq 0, \forall (t,s) \in DI$.

Let

$$\begin{cases} A = h_i^x c_{12} + 3h_j^y c_{03}, \\ B = h_i^x c_{12} - 3h_j^y c_{03}, \\ C = h_i^x c_{12} + 3h_j^y c_{03} + 2c_{02}, \end{cases}$$

then

$$\begin{cases} c_{12} = \frac{1}{2h_i^x} (A + B), \\ c_{03} = \frac{1}{6h_j^y} (A - B), \\ c_{02} = -\frac{1}{2} A + \frac{1}{2} C, \end{cases}$$

and

$$\begin{aligned} & f(t,s) \\ & = 2c_{12} h_i^x t + 6c_{03} h_j^y s + c_{12} h_i^x + 3c_{03} h_j^y + 2c_{02} \\ & = (A + B) t + (A - B) s + C. \end{aligned}$$

$f(t,s) \geq 0, \forall (t,s) \in DI$ is equivalent to

$$\begin{aligned} f(0,0) &= C \geq 0, \\ f\left(-\frac{1}{2}, -\frac{1}{2}\right) &= -A + C \geq 0, \\ f\left(\frac{1}{2}, -\frac{1}{2}\right) &= B + C \geq 0. \end{aligned}$$

$$\begin{aligned} & F(c_{12}, c_{03}, c_{02}) \\ & = h_i^x h_j^y \int_{DI} |(A + B) t + (A - B) s + C| \, dtds \\ & = h_i^x h_j^y \left\{ \int_{-\frac{1}{2}}^{0} \int_{-\frac{1}{2}}^{t} [(A + B) t + (A - B) s + C] \, dsdt + \int_{0}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{-t} [(A + B) t + (A - B) s + C] \, dsdt \right\} \\ & = h_i^x h_j^y \left\{ -\frac{1}{12} A + \frac{1}{12} B + \frac{1}{4} C \right\}. \end{aligned}$$

$$
\begin{aligned}
&G_1(\xi,\eta,\gamma)\\
&= \sup_{(c_{12},c_{03},c_{02})\in E_1} \{\xi c_{12} + \eta c_{03} + \gamma c_{02} - F(c_{12},c_{03},c_{02})\}\\
&= \sup_{\substack{C\geq 0,\\ -A+C\geq 0,\\ B+C\geq 0}} \left\{\xi \frac{1}{2h_i^x}(A+B) + \eta \frac{1}{6h_j^y}(A-B) + \gamma\left(-\frac{1}{2}A + \frac{1}{2}C\right) - h_i^x h_j^y \left[-\frac{1}{12}A + \frac{1}{12}B + \frac{1}{4}C\right]\right\}\\
&= \sup_{\substack{C\geq 0,\\ -A+C\geq 0,\\ B+C\geq 0}} \left\{\left(\frac{\xi}{2h_i^x} + \frac{\eta}{6h_j^y} - \frac{\gamma}{2} + \frac{h_i^x h_j^y}{12}\right)A + \left(\frac{\xi}{2h_i^x} - \frac{\eta}{6h_j^y} - \frac{h_i^x h_j^y}{12}\right)B + \left(\frac{\gamma}{2} - \frac{h_i^x h_j^y}{4}\right)C\right\}.
\end{aligned}
$$

Let

$$
t = \frac{A}{C}, -\infty < t \leq 1,
$$

and

$$
s = \frac{B}{C}, -1 \leq s < +\infty,
$$

then

$$
\begin{aligned}
&G_1(\xi,\eta,\gamma)\\
&= \sup_{\substack{C\geq 0,\\ t\in(-\infty,1],\\ s\in[-1,+\infty)}} \left\{\left(\frac{\xi}{2h_i^x} + \frac{\eta}{6h_j^y} - \frac{\gamma}{2} + \frac{h_i^x h_j^y}{12}\right)t + \left(\frac{\xi}{2h_i^x} - \frac{\eta}{6h_j^y} - \frac{h_i^x h_j^y}{12}\right)s + \left(\frac{\gamma}{2} - \frac{h_i^x h_j^y}{4}\right)\right\}C\\
&= \sup_{\substack{C\geq 0,\\ t\in(-\infty,1],\\ s\in[-1,+\infty)}} \{g(t,s)\,C\} = 0.
\end{aligned}
$$

Since $C \geq 0$, $G_1(\xi,\eta,\gamma)$ is finite if and only if $g(t,s) \leq 0$. Furhtermore, $g(t,s)$ is a linear function of t and s. For any fixed s, $g(t,s) \leq 0$ as t goes to $-\infty$ requires that the coefficient of t is non-negative. Similarly, for any fixed t, $g(t,s) \leq 0$ as s goes to ∞ requires that the coefficient of s is non-positive. When the coefficient of t is non-negative and the coefficient of s is non-positive, $g(t,s)$ is increasing about t and decreasing about s. Hence, $g(t,s)$ achieves its maximum value at point $(t,s) = (1,-1)$.

$$
\begin{aligned}
&g(t=1,s=-1)\\
&= \left(\frac{\xi}{2h_i^x} + \frac{\eta}{6h_j^y} - \frac{\gamma}{2} + \frac{h_i^x h_j^y}{12}\right) - \left(\frac{\xi}{2h_i^x} - \frac{\eta}{6h_j^y} - \frac{h_i^x h_j^y}{12}\right) + \left(\frac{\gamma}{2} - \frac{h_i^x h_j^y}{4}\right)\\
&= \frac{\eta}{3h_j^y} - \frac{h_i^x h_j^y}{12}.
\end{aligned}
$$

Therefore, $g(t,s) \leq 0$ is equivalent to

$$
\begin{cases}
\frac{\xi}{2h_i^x} + \frac{\eta}{6h_j^y} - \frac{\gamma}{2} + \frac{h_i^x h_j^y}{12} \geq 0,\\
\frac{\xi}{2h_i^x} - \frac{\eta}{6h_j^y} - \frac{h_i^x h_j^y}{12} \leq 0,\\
g(t=1,s=-1) \leq 0,
\end{cases}
$$

i.e.,

$$\begin{cases} \frac{\xi}{2h_i^x} + \frac{\eta}{6h_j^y} - \frac{\gamma}{2} + \frac{h_i^x h_j^y}{12} \geq 0, \\ \frac{\xi}{2h_i^x} - \frac{\eta}{6h_j^y} - \frac{h_i^x h_j^y}{12} \leq 0, \\ \eta \leq \frac{1}{4} h_i^x h_j^{y2}. \end{cases}$$

representing by x, y, and z

$$\begin{cases} x \leq K, \\ y \leq K, \\ z \geq -K. \end{cases} \tag{A.37}$$

A.3.2 Case 2

Case 2: $(c_{12}, c_{03}, c_{02}) \in E_2 : f(t,s) \leq 0, \forall (t,s) \in DI$.

Let A, B, and C be the same as defined in Case 1. This is the situation that

$$f(0,0) = C \leq 0,$$
$$f\left(-\frac{1}{2}, -\frac{1}{2}\right) = -A + C \leq 0,$$
$$f\left(\frac{1}{2}, -\frac{1}{2}\right) = B + C \leq 0.$$

$$\begin{aligned} & F(c_{12}, c_{03}, c_{02}) \\ & = h_i^x h_j^y \int_{DI} |(A+B)t + (A-B)s + C| \, dtds \\ & = -h_i^x h_j^y \left\{ \int_{-\frac{1}{2}}^{0} \int_{-\frac{1}{2}}^{t} [(A+B)t + (A-B)s + C] \, dsdt + \int_{0}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{-t} [(A+B)t + (A-B)s + C] \, dsdt \right\} \\ & = h_i^x h_j^y \left\{ \frac{1}{12} A - \frac{1}{12} B - \frac{1}{4} C \right\}. \end{aligned}$$

Let

$$t = \frac{A}{C}, -\infty < t \leq 1,$$

and

$$s = \frac{B}{C}, -1 \leq s < +\infty,$$

then

$$
\begin{aligned}
&G_2\left(\xi,\eta,\gamma\right)\\
&=\sup_{(c_{12},c_{03},c_{02})\in E_2}\left\{\xi c_{12}+\eta c_{03}+\gamma c_{02}-F\left(c_{12},c_{03},c_{02}\right)\right\}\\
&=\sup_{\substack{C\le 0,\\ -A+C\le 0,\\ B+C\le 0}}\left\{\xi\frac{1}{2h_i^x}\left(A+B\right)+\eta\frac{1}{6h_j^y}\left(A-B\right)+\gamma\left(-\frac{1}{2}A+\frac{1}{2}C\right)-h_i^xh_j^y\left[\frac{1}{12}A-\frac{1}{12}B-\frac{1}{4}C\right]\right\}\\
&=\sup_{\substack{C\le 0,\\ -A+C\le 0,\\ B+C\le 0}}\left\{\left(\frac{\xi}{2h_i^x}+\frac{\eta}{6h_j^y}-\frac{\gamma}{2}-\frac{h_i^xh_j^y}{12}\right)A+\left(\frac{\xi}{2h_i^x}-\frac{\eta}{6h_j^y}+\frac{h_i^xh_j^y}{12}\right)B+\left(\frac{\gamma}{2}+\frac{h_i^xh_j^y}{4}\right)C\right\}\\
&=\sup_{\substack{C\le 0,\\ t\in(-\infty,1],\\ s\in[-1,+\infty)}}\left\{\left(\frac{\xi}{2h_i^x}+\frac{\eta}{6h_j^y}-\frac{\gamma}{2}-\frac{h_i^xh_j^y}{12}\right)t+\left(\frac{\xi}{2h_i^x}-\frac{\eta}{6h_j^y}+\frac{h_i^xh_j^y}{12}\right)s+\left(\frac{\gamma}{2}+\frac{h_i^xh_j^y}{4}\right)\right\}C\\
&=\sup_{\substack{C\le 0,\\ t\in(-\infty,1],\\ s\in[-1,+\infty)}}\left\{g\left(t,s\right)C\right\}.
\end{aligned}
$$

Since $C\le 0$, $G_2\left(\xi,\eta,\gamma\right)$ is finite if and only if $g\left(t,s\right)\ge 0$, which is equivalent to

$$
\begin{cases}
\frac{\xi}{2h_i^x}+\frac{\eta}{6h_j^y}-\frac{\gamma}{2}-\frac{h_i^xh_j^y}{12}\le 0,\\
\frac{\xi}{2h_i^x}-\frac{\eta}{6h_j^y}+\frac{h_i^xh_j^y}{12}\ge 0,\\
g\left(t=1,s=-1\right)\ge 0,
\end{cases}
$$

i.e.,

$$
\begin{cases}
\frac{\xi}{2h_i^x}+\frac{\eta}{6h_j^y}-\frac{\gamma}{2}-\frac{h_i^xh_j^y}{12}\le 0,\\
\frac{\xi}{2h_i^x}-\frac{\eta}{6h_j^y}+\frac{h_i^xh_j^y}{12}\ge 0,\\
\eta\ge -\frac{1}{4}h_i^xh_j^{y2}.
\end{cases}
$$

$$
\begin{cases}
x\ge -K,\\
y\ge -K,\\
z\le K.
\end{cases}
\tag{A.38}
$$

A.3.3 Case 3

Case 3: $(c_{12},c_{03},c_{02})\in E_3 : f\left(0,0\right)\le 0,\ f\left(-\frac{1}{2},-\frac{1}{2}\right)\ge 0,\ f\left(\frac{1}{2},-\frac{1}{2}\right)\ge 0$

Let A, B, and C be the same as defined in Case 1. This is the situation that

$$
\begin{aligned}
f\left(0,0\right)&=C\le 0,\\
f\left(-\frac{1}{2},-\frac{1}{2}\right)&=-A+C\ge 0,\\
f\left(\frac{1}{2},-\frac{1}{2}\right)&=B+C\ge 0.
\end{aligned}
$$

$$f(t,s) = (A+B)t + (A-B)s + C$$

$$\begin{aligned}
&F(c_{12}, c_{03}, c_{02}) \\
&= h_i^x h_j^y \int_{-\frac{1}{2}}^{-\frac{C}{2A}} \int_{-\frac{1}{2}}^{t} f(t,s)\,dsdt + h_i^x h_j^y \int_{-\frac{C}{2A}}^{-\frac{C}{2B}} \int_{-\frac{1}{2}}^{-\frac{A+B}{A-B}t-\frac{C}{A-B}} f(t,s)\,dsdt + h_i^x h_j^y \int_{-\frac{C}{2B}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{-t} f(t,s)\,dsdt \\
&- h_i^x h_j^y \int_{-\frac{C}{2A}}^{0} \int_{-\frac{A+B}{A-B}t-\frac{C}{A-B}}^{t} f(t,s)\,dsdt - h_i^x h_j^y \int_{0}^{-\frac{C}{2B}} \int_{-\frac{A+B}{A-B}t-\frac{C}{A-B}}^{-t} f(t,s)\,dsdt \\
&= \frac{1}{12} h_i^x h_j^y \left\{ -A + B + 3C + \frac{2C^3}{AB} \right\}.
\end{aligned}$$

Let

$$t = \frac{A}{C}, 1 \le t < +\infty,$$

and

$$s = \frac{B}{C}, -\infty < s \le -1,$$

then

$$\begin{aligned}
&G_3(\xi,\eta,\gamma) \\
&= \sup_{(c_{12},c_{03},c_{02})\in E_3} \left\{ \xi c_{12} + \eta c_{03} + \gamma c_{02} - F(c_{12},c_{03},c_{02}) \right\} \\
&= \sup_{\substack{C\le 0,\\ -A+C\ge 0,\\ B+C\ge 0}} \left\{ \xi \frac{1}{2h_i^x}(A+B) + \eta \frac{1}{6h_j^y}(A-B) + \gamma\left(-\frac{1}{2}A + \frac{1}{2}C\right) - \frac{1}{12} h_i^x h_j^y \left[-A + B + 3C + \frac{2C^3}{AB} \right] \right\} \\
&= \sup_{\substack{C\le 0,\\ -A+C\ge 0,\\ B+C\ge 0}} \left\{ \left(\frac{\xi}{2h_i^x} + \frac{\eta}{6h_j^y} - \frac{\gamma}{2} + \frac{h_i^x h_j^y}{12} \right) A + \left(\frac{\xi}{2h_i^x} - \frac{\eta}{6h_j^y} - \frac{h_i^x h_j^y}{12} \right) B + \left(\frac{\gamma}{2} - \frac{h_i^x h_j^y}{4} \right) C - \frac{h_i^x h_j^y}{6} \frac{C^3}{AB} \right\} \\
&= \sup_{\substack{C\le 0,\\ t\in[1,+\infty),\\ s\in(-\infty,-1]}} \left\{ \left(\frac{\xi}{2h_i^x} + \frac{\eta}{6h_j^y} - \frac{\gamma}{2} + \frac{h_i^x h_j^y}{12} \right) t + \left(\frac{\xi}{2h_i^x} - \frac{\eta}{6h_j^y} - \frac{h_i^x h_j^y}{12} \right) s + \left(\frac{\gamma}{2} - \frac{h_i^x h_j^y}{4} \right) - \frac{h_i^x h_j^y}{6} \frac{1}{ts} \right\} C \\
&= \sup_{\substack{C\le 0,\\ t\in[1,+\infty),\\ s\in(-\infty,-1]}} \left\{ g(t,s)\, C \right\}.
\end{aligned}$$

Since $C \le 0$, $G_3(\xi,\eta,\gamma)$ is finite if and only if $g(t,s) \ge 0$ for any .$t \in [1,+\infty)$ and $s \in (-\infty,-1]$.

Consider a function

$$g(t,s) = at + bs + c + \frac{d}{ts}, t \in [1,+\infty), s \in (-\infty,-1].$$

In this case, we have

$$a = \frac{\xi}{2h_i^x} + \frac{\eta}{6h_j^y} - \frac{\gamma}{2} + \frac{h_i^x h_j^y}{12} = z + K,$$
$$b = \frac{\xi}{2h_i^x} - \frac{\eta}{6h_j^y} - \frac{h_i^x h_j^y}{12} = y - K,$$
$$c = \frac{\gamma}{2} - \frac{h_i^x h_j^y}{4} = x + y - z - 3K,$$
$$d = -\frac{h_i^x h_j^y}{6} = -2K < 0,$$

$$g\left(t, s\right) \geq 0 \text{ as } t \to +\infty$$

implies that

$$a \geq 0 \Longleftrightarrow z \geq -K. \tag{A.39}$$

Similarly,

$$g\left(t, s\right) \geq 0 \text{ as } s \to -\infty$$

implies that

$$b \leq 0 \Longleftrightarrow y \leq K. \tag{A.40}$$

The first order derivatives of g are

$$\frac{\partial g}{\partial t} = a - \frac{d}{t^2 s},$$
$$\frac{\partial g}{\partial s} = b - \frac{d}{t s^2},$$
$$\nabla g\left(t^*, s^*\right) = 0$$

implies that

$$d = at^2 s = bts^2,$$

hence,

$$s = \frac{a}{b} t,$$

and

$$t^* = \sqrt[3]{\frac{bd}{a^2}},$$
$$s^* = \sqrt[3]{\frac{ad}{b^2}},$$

The second order derivatives of g are

$$\frac{\partial^2 g}{\partial t^2} = \frac{2d}{t^3 s},$$
$$\frac{\partial^2 g}{\partial t \partial s} = \frac{d}{t^2 s^2},$$
$$\frac{\partial^2 g}{\partial s^2} = \frac{2d}{t s^3},$$

the Hessian is

$$H = \begin{bmatrix} \frac{2d}{t^3 s} & \frac{d}{t^2 s^2} \\ \frac{d}{t^2 s^2} & \frac{2d}{t s^3} \end{bmatrix},$$

$$|H| = \frac{4d^2}{t^4 s^4} - \frac{d^2}{t^4 s^4} = \frac{3d^2}{t^4 s^4}.$$

In this case, we have $d < 0$, $t > 0$, and $s < 0$. Hence, $g(t, s)$ is strictly convex over its feasible set $[1, +\infty) \times (-\infty, -1]$. Therefore, $(t^*, s^*) = \left(\sqrt[3]{\frac{bd}{a^2}}, \sqrt[3]{\frac{ad}{b^2}}\right)$ is the unique solution of $\nabla g = 0$.

Next we derive the conditions of $g(t, s) \geq 0$ in different cases.

Case 3.1

Case 3.1: (t^*, s^*) is an interior point of $[1, +\infty) \times (-\infty, -1]$. Then $g(t, s)$ achieves its minimum value at the stationary point (t^*, s^*). We need

$$\begin{aligned} t^* &> 1, \\ s^* &< -1, \\ g(t^*, s^*) &\geq 0. \end{aligned}$$

where $t^* = \sqrt[3]{\frac{bd}{a^2}} > 1$ implies that

$$bd > a^2,$$

i.e.,

$$y - K < -\frac{1}{2K}(z + K)^2. \tag{A.41}$$

Similarly, $s^* = \sqrt[3]{\frac{ad}{b^2}} < -1$ implies that

$$ad < -b^2,$$

i.e.,

$$z + K > \frac{1}{2K}(y - K)^2. \tag{A.42}$$

$g(t^*, s^*) \geq 0$ implies that

$$\begin{aligned} & g(t^*, s^*) \\ &= a\sqrt[3]{\frac{bd}{a^2}} + b\sqrt[3]{\frac{ad}{b^2}} + c + \frac{d}{\sqrt[3]{\frac{bd}{a^2}}\sqrt[3]{\frac{ad}{b^2}}} \\ &= 3\sqrt[3]{abd} + c \geq 0. \end{aligned}$$

i.e.,

$$x \geq -y + z + 3K - 3\sqrt[3]{(-2K)(y - K)(z + K)}. \tag{A.43}$$

(A.41) and (A.42) create a convex set in y-z space. The boundary of them intersect in two points $(y, z) = (K, -K)$ and $(y, z) = (-K, K)$.

Case 3.2

Assume that the minimum value of $g(t,s)$ is achieved at $\bar{t} > 1$, $\bar{s} = -1$.

From the optimality condition

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t}(t-\bar{t}) + \frac{\partial g(\bar{t},\bar{s})}{\partial s}(s-\bar{s}) \geq 0,$$

let $s = -1$, since $t - \bar{t}$ can be either positive or negative, we have

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t} = 0.$$

Similarly, let $t = \bar{t}$, since $s - \bar{s} \leq 0$, we have

$$\frac{\partial g(\bar{t},\bar{s})}{\partial s} \leq 0.$$

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t} = a + \frac{d}{\bar{t}^2} = 0 \Rightarrow \bar{t} = \sqrt{\frac{-d}{a}}.$$

Furthermore,

$$\bar{t} > 1 \Rightarrow a < -d \Rightarrow z < K. \tag{A.44}$$

$$\frac{\partial g(\bar{t},\bar{s})}{\partial s} = b - \frac{d}{\bar{t}} = b + \sqrt{-ad} \leq 0$$

implies that

$$\begin{gathered} y \leq K \\ (y-K) + \sqrt{2K(z+K)} \leq 0 \\ z + k \leq \frac{1}{2K}(y-K)^2. \end{gathered} \tag{A.45}$$

Finally,

$$\begin{aligned} & g(t,s) \\ & \geq g(\bar{t},\bar{s}) \\ & = a\sqrt{\frac{-d}{a}} - b + c - \frac{d}{\sqrt{\frac{-d}{a}}} \\ & = 2\sqrt{-ad} - b + c \geq 0 \end{aligned}$$

implies that

$$\begin{gathered} \sqrt{8K(z+K)} \geq -x + z + 2K. \\ x \geq z + 2K - \sqrt{8K(z+K)}. \end{gathered} \tag{A.46}$$

Case 3.3

Assume that the minimum value of $g(t,s)$ is achieved at $\bar{t}=1$, $\bar{s}<-1$.

From the optimality condition

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t}(t-\bar{t})+\frac{\partial g(\bar{t},\bar{s})}{\partial s}(s-\bar{s})\geq 0,$$

similar as the previous subsection, we have

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t}\geq 0,$$
$$\frac{\partial g(\bar{t},\bar{s})}{\partial s}=0.$$

$$\frac{\partial g(\bar{t},\bar{s})}{\partial s}=b-\frac{d}{\bar{s}^2}=0\Rightarrow \bar{s}=-\sqrt{\frac{-d}{-b}}$$

Furthermore,

$$\bar{s}<-1\Rightarrow b>d\Rightarrow y>-K. \tag{A.47}$$

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t}=a-\frac{d}{\bar{s}}=a-\sqrt{bd}\geq 0$$

implies that

$$\begin{aligned} z&\geq -K\\ z+K&\geq\sqrt{(-2K)(y-K)}\\ y-K&\geq-\frac{1}{2K}(z+K)^2. \end{aligned}\tag{A.48}$$

Finally,

$$\begin{aligned} &g(t,s)\\ &\geq g(\bar{t},\bar{s})\\ &=2\sqrt{bd}+a+c\geq 0 \end{aligned}$$

implies that

$$\begin{aligned} \sqrt{-8K(y-K)}&\geq -x-y+2K\\ x&\geq -y+2K-\sqrt{-8K(y-K)}. \end{aligned}\tag{A.49}$$

Case 3.4

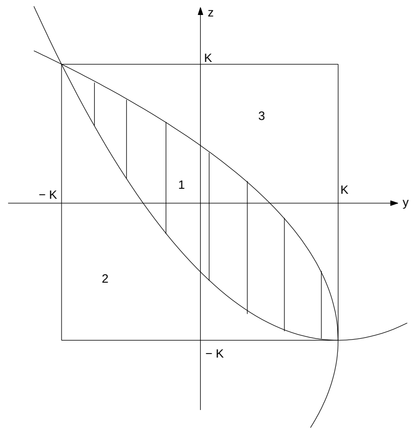
Assume that the minimum value of $g(t,s)$ is achieved at $\bar{t}=1$, $\bar{s}=-1$.

From the optimality condition

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t}(t-\bar{t})+\frac{\partial g(\bar{t},\bar{s})}{\partial s}(s-\bar{s})\geq 0,$$

similar as the previous subsection, we have

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t}\geq 0,$$
$$\frac{\partial g(\bar{t},\bar{s})}{\partial s}\leq 0.$$



$$\frac{\partial g(\bar{t},\bar{s})}{\partial t} = a + d \geq 0$$

implies that

$$z \geq K. \tag{A.50}$$

$$\frac{\partial g(\bar{t},\bar{s})}{\partial s} = b - d \leq 0$$

implies that

$$y \leq -K. \tag{A.51}$$

Furthermore,

$$\begin{aligned} & g(t,s) \\ & \geq g(\bar{t}=1,\bar{s}=-1) \\ & = a - b + c - d \\ & = x + K \geq 0. \end{aligned}$$

implies that

$$x \geq -K. \tag{A.52}$$

However, (A.50) and (A.51) define a set that does not intersect the set defined by case 1 and case 2. Hence, we can discard case 3.4.

Discussion of case 3

In this subsection, we need to show that the point (x, y, z) satisfying $g(t,s) \geq 0$ forms a convex set. Furthermore, the boundary surface of this convex set is C^1 smooth.

The set

$$\{(y,z) : y \in [-K,K], z \in [-K,K]\}$$

is partitioned into three regions according to the first three cases of case 3. In region i, point (x, y, z) should satisfy

$$x \geq h_i(y,z)$$

to ensure $g(t,s) \geq 0$. We have derived in (A.43), (A.46) and (A.49) that

$$h_1(y,z) = -y + z + 3K - 3\sqrt[3]{(-2K)(y-K)(z+K)}$$
$$h_2(y,z) = z + 2K - \sqrt{8K(z+K)}$$
$$h_3(y,z) = -y + 2K - \sqrt{-8K(y-K)}$$

First check the convexity of these three functions.

$$\frac{\partial h_1}{\partial y} = -1 + 2K(z+K)\left[(-2K)(y-K)(z+K)\right]^{-\frac{2}{3}}$$
$$\frac{\partial h_1}{\partial z} = 1 + 2K(y-K)\left[(-2K)(y-K)(z+K)\right]^{-\frac{2}{3}}$$
$$\frac{\partial^2 h_1}{\partial y^2} = \frac{2}{3}(2K)^2(z+K)^2\left[(-2K)(y-K)(z+K)\right]^{-\frac{5}{3}} \geq 0$$
$$\frac{\partial^2 h_1}{\partial y \partial z} = 2K\left[(-2K)(y-K)(z+K)\right]^{-\frac{2}{3}} + \frac{2}{3}(2K)^2(y-K)(z+K)\left[(-2K)(y-K)(z+K)\right]^{-\frac{5}{3}}$$
$$= -\frac{1}{3}(2K)^2(y-K)(z+K)\left[(-2K)(y-K)(z+K)\right]^{-\frac{5}{3}}$$
$$\frac{\partial^2 h_1}{\partial z^2} = \frac{2}{3}(2K)^2(y-K)^2\left[(-2K)(y-K)(z+K)\right]^{-\frac{5}{3}}$$

The determant of Hessian of h_1 is

$$|H| = \frac{\partial^2 h_1}{\partial y^2}\frac{\partial^2 h_1}{\partial z^2} - \left(\frac{\partial^2 h_1}{\partial y \partial z}\right)^2$$
$$= \frac{1}{3}\left\{(2K)^2(y-K)(z+K)\left[(-2K)(y-K)(z+K)\right]^{-\frac{5}{3}}\right\}^2 \geq 0$$

hence, $h_1(y,z)$ is a convex function.

$$\frac{\partial h_2}{\partial z} = 1 - 2K\left[2K(z+K)\right]^{-\frac{1}{2}}$$
$$\frac{\partial^2 h_2}{\partial z^2} = 2K^2\left[2K(z+K)\right]^{-\frac{3}{2}} \geq 0$$

$$\frac{\partial h_3}{\partial y} = -1 + 2K\left[-2K(y-K)\right]^{-\frac{1}{2}}$$
$$\frac{\partial^2 h_3}{\partial y^2} = 2K^2\left[-2K(y-K)\right]^{-\frac{3}{2}} \geq 0$$

So both $h_2(y,z)$ and $h_3(y,z)$ are convex functions.

Secondly check the smoothness of the boundary surface of the set.

Region 1 and region 2 intersect at the curve $z + K = \frac{1}{2K}(y-K)^2$. On this curve, we have

$$(-2K)(y-K)(z+K) = -(y-K)^3$$

$$h_1 = 2y + z$$
$$h_2 = 2y + z$$

$$\frac{\partial h_1}{\partial y} = -1 + 2K(z+K)\left[-(y-K)^3\right]^{-\frac{2}{3}} = 0$$
$$\frac{\partial h_1}{\partial z} = 1 + 2K(y-K)\left[-(y-K)^3\right]^{-\frac{2}{3}} = 1 - 2K\left[2K(z+K)\right]^{-\frac{1}{2}} = \frac{\partial h_2}{\partial z}$$

hence, h_1 and h_2 are C^1 smooth along their boundary.

Region 1 and region 3 intersect at the curve $y - K = -\frac{1}{2K}(z+K)^2$. On this curve, we have

$$(-2K)(y-K)(z+K) = (z+K)^3$$

$$h_1 = -y - 2z$$
$$h_3 = -y - 2z$$

$$\frac{\partial h_1}{\partial y} = -1 + \frac{2K}{z+K} = -1 + 2K\left[-2K(y-K)\right]^{-\frac{1}{2}} = \frac{\partial h_3}{\partial y}$$
$$\frac{\partial h_1}{\partial z} = 1 + 2K(y-K)(z+K)^{-2} = 0$$

hence, h_1 and h_3 are C^1 smooth along their boundary.

Thirdly, we need to show that

$$\text{in region 1 } h_1(y,z) \geq h_2(y,z) \text{ and } h_1(y,z) \geq h_3(y,z)$$
$$\text{in region 2 } h_2(y,z) \geq h_1(y,z) \text{ and } h_2(y,z) \geq h_3(y,z)$$
$$\text{in region 3 } h_3(y,z) \geq h_1(y,z) \text{ and } h_3(y,z) \geq h_2(y,z)$$

To show $h_1(y,z) \geq h_2(y,z)$ in region 1 is only need to show $\frac{\partial h_1}{\partial y} \geq 0$. To show $h_1(y,z) \geq h_3(y,z)$ in region 1 is only need to show $\frac{\partial h_1}{\partial z} \leq 0$. Notice that region 1 is defined by (A.45) and (A.48), i.e,

$$(y-K) + \sqrt{2K(z+K)} \leq 0$$
$$z + K \geq \sqrt{(-2K)(y-K)}$$

Hence,

$$\begin{aligned}\frac{\partial h_1}{\partial y} &= -1 + 2K(z+K)\left[(-2K)(y-K)(z+K)\right]^{-\frac{2}{3}} \\ &\geq -1 + 2K(z+K)\left[(-2K)(z+K)(-1)\sqrt{2K(z+K)}\right]^{-\frac{2}{3}} \\ &= 0\end{aligned}$$

$$\begin{aligned}\frac{\partial h_1}{\partial z} &= 1 + 2K(y-K)\left[(-2K)(y-K)(z+K)\right]^{-\frac{2}{3}} \\ &\leq 1 + 2K(y-K)\left[(-2K)(y-K)\sqrt{(-2K)(y-K)}\right]^{-\frac{2}{3}} \\ &= 0\end{aligned}$$

The other conditions can be checked by the same way.

The last thing need to check is

$$-K \leq h_i \leq K \text{ in region i}$$

Notice that $\frac{\partial h_1}{\partial y} \geq 0$, $\frac{\partial h_1}{\partial z} \leq 0$, $\frac{\partial h_2}{\partial z} \leq 0$, and $\frac{\partial h_3}{\partial y} \geq 0$, hence the maximum value of the boundary surface is achieved at $(y, z) = (K, -K)$, which is K. And the minimum value is achieved at $(y, z) = (-K, K)$, which is $-K$.

Conclusion of case 3 (combine with case 1 and 2):

$$\begin{aligned}(x, y, z) \in \{&|x| \leq K, |y| \leq K, |z| \leq K,\\ &x \geq -y + z + 3K - 3\sqrt[3]{(-2K)(y-K)(z+K)},\\ &x \geq z + 2K - \sqrt{8K(z+K)},\\ &\quad x \geq -y + 2K - \sqrt{(-8K)(y-K)}\}\end{aligned}$$

A.3.4 Case 4

Case 4: $(c_{12}, c_{03}, c_{02}) \in E_3 : f(0,0) \geq 0,\ f\left(-\frac{1}{2}, -\frac{1}{2}\right) \leq 0,\ f\left(\frac{1}{2}, -\frac{1}{2}\right) \leq 0$

Let A, B, and C be the same as defined in Case 1. This is the situation that

$$\begin{aligned} f(0,0) &= C \geq 0,\\ f\left(-\frac{1}{2}, -\frac{1}{2}\right) &= -A + C \leq 0,\\ f\left(\frac{1}{2}, -\frac{1}{2}\right) &= B + C \leq 0. \end{aligned}$$

$$f(t, s) = (A+B)t + (A-B)s + C$$

$$\begin{aligned} &F(c_{12}, c_{03}, c_{02})\\ &= -h_i^x h_j^y \int_{-\frac{1}{2}}^{-\frac{C}{2A}} \int_{-\frac{1}{2}}^{t} f(t,s)\,dsdt - h_i^x h_j^y \int_{-\frac{C}{2A}}^{-\frac{C}{2B}} \int_{-\frac{1}{2}}^{-\frac{A+B}{A-B}t - \frac{C}{A-B}} f(t,s)\,dsdt - h_i^x h_j^y \int_{-\frac{C}{2B}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{-t} f(t,s)\,dsdt\\ &+ h_i^x h_j^y \int_{-\frac{C}{2A}}^{0} \int_{-\frac{A+B}{A-B}t - \frac{C}{A-B}}^{t} f(t,s)\,dsdt + h_i^x h_j^y \int_{0}^{-\frac{C}{2B}} \int_{-\frac{A+B}{A-B}t - \frac{C}{A-B}}^{-t} f(t,s)\,dsdt\\ &= \frac{1}{12} h_i^x h_j^y \left\{A - B - 3C - \frac{2C^3}{AB}\right\}. \end{aligned}$$

Let

$$t = \frac{A}{C}, 1 \leq t < +\infty,$$

and

$$s = \frac{B}{C}, -\infty < s \leq -1,$$

then

$$
\begin{aligned}
&G_4(\xi,\eta,\gamma)\\
&= \sup_{(c_{12},c_{03},c_{02})\in E_3} \left\{\xi c_{12}+\eta c_{03}+\gamma c_{02}-F(c_{12},c_{03},c_{02})\right\}\\
&= \sup_{\substack{C\geq 0,\\ -A+C\leq 0,\\ B+C\leq 0}} \left\{\xi\frac{1}{2h_i^x}(A+B)+\eta\frac{1}{6h_j^y}(A-B)+\gamma\left(-\frac{1}{2}A+\frac{1}{2}C\right)-\frac{1}{12}h_i^xh_j^y\left[A-B-3C-\frac{2C^3}{AB}\right]\right\}\\
&= \sup_{\substack{C\geq 0,\\ -A+C\leq 0,\\ B+C\leq 0}} \left\{\left(\frac{\xi}{2h_i^x}+\frac{\eta}{6h_j^y}-\frac{\gamma}{2}-\frac{h_i^xh_j^y}{12}\right)A+\left(\frac{\xi}{2h_i^x}-\frac{\eta}{6h_j^y}+\frac{h_i^xh_j^y}{12}\right)B+\left(\frac{\gamma}{2}+\frac{h_i^xh_j^y}{4}\right)C+\frac{h_i^xh_j^y}{6}\frac{C^3}{AB}\right\}\\
&= \sup_{\substack{C\geq 0,\\ t\in[1,+\infty),\\ s\in(-\infty,-1]}} \left\{\left(\frac{\xi}{2h_i^x}+\frac{\eta}{6h_j^y}-\frac{\gamma}{2}-\frac{h_i^xh_j^y}{12}\right)t+\left(\frac{\xi}{2h_i^x}-\frac{\eta}{6h_j^y}+\frac{h_i^xh_j^y}{12}\right)s+\left(\frac{\gamma}{2}+\frac{h_i^xh_j^y}{4}\right)+\frac{h_i^xh_j^y}{6}\frac{1}{ts}\right\}C\\
&= \sup_{\substack{C\geq 0,\\ t\in[1,+\infty),\\ s\in(-\infty,-1]}} \left\{g(t,s)\,C\right\}.
\end{aligned}
$$

Since $C\geq 0$, $G_4(\xi,\eta,\gamma)$ is finite if and only if $g(t,s)\leq 0$ for any $.t\in[1,+\infty)$ and $s\in(-\infty,-1]$.

Consider a function

$$g(t,s)=at+bs+c+\frac{d}{ts}, t\in[1,+\infty), s\in(-\infty,-1].$$

In this case, we have

$$
\begin{aligned}
a &= \frac{\xi}{2h_i^x}+\frac{\eta}{6h_j^y}-\frac{\gamma}{2}-\frac{h_i^xh_j^y}{12}=z-K,\\
b &= \frac{\xi}{2h_i^x}-\frac{\eta}{6h_j^y}+\frac{h_i^xh_j^y}{12}=y+K,\\
c &= \frac{\gamma}{2}+\frac{h_i^xh_j^y}{4}=x+y-z+3K,\\
d &= \frac{h_i^xh_j^y}{6}=2K>0,
\end{aligned}
$$

$$g(t,s)\leq 0 \text{ as } t\to+\infty$$

implies that

$$a\leq 0\Longleftrightarrow z\leq K. \tag{A.53}$$

Similarly,

$$g(t,s)\leq 0 \text{ as } s\to-\infty$$

implies that

$$b\geq 0\Longleftrightarrow y\geq -K. \tag{A.54}$$

In this case, $g(t,s)$ is strictly concave over its feasible set $[1,+\infty)\times(-\infty,-1]$. Therefore, $(t^*,s^*) = \left(\sqrt[3]{\frac{bd}{a^2}}, \sqrt[3]{\frac{ad}{b^2}}\right)$ is the unique solution of $\nabla g = 0$.

Next we derive the conditions of $g(t,s) \le 0$ in different cases.

Case 4.1

Case 4.1: (t^*,s^*) is an interior point of $[1,+\infty)\times(-\infty,-1]$. Then $g(t,s)$ achieves its maximum value at the stationary point (t^*,s^*). We need

$$\begin{aligned} t^* &> 1, \\ s^* &< -1, \\ g(t^*,s^*) &\le 0. \end{aligned}$$

where $t^* = \sqrt[3]{\frac{bd}{a^2}} > 1$ implies that

$$bd > a^2,$$

i.e.,

$$y + K > \frac{1}{2K}(z-K)^2. \tag{A.55}$$

Similarly, $s^* = \sqrt[3]{\frac{ad}{b^2}} < -1$ implies that

$$ad < -b^2,$$

i.e.,

$$z - K < -\frac{1}{2K}(y+K)^2. \tag{A.56}$$

$g(t^*,s^*) \le 0$ implies that

$$\begin{aligned} & g(t^*,s^*) \\ & = a\sqrt[3]{\frac{bd}{a^2}} + b\sqrt[3]{\frac{ad}{b^2}} + c + \frac{d}{\sqrt[3]{\frac{bd}{a^2}}\sqrt[3]{\frac{ad}{b^2}}} \\ & = 3\sqrt[3]{abd} + c \le 0. \end{aligned}$$

i.e.,

$$x \le -y + z - 3K + 3\sqrt[3]{(-2K)(y+K)(z-K)}. \tag{A.57}$$

(A.55) and (A.56) create a convex set in y-z space. The boundary of them intersect in two points $(y,z) = (K,-K)$ and $(y,z) = (-K,K)$.

Case 4.2

Assume that the maximum value of $g(t,s)$ is achieved at $\bar{t} > 1$, $\bar{s} = -1$.

From the optimality condition

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t}(t-\bar{t}) + \frac{\partial g(\bar{t},\bar{s})}{\partial s}(s-\bar{s}) \le 0,$$

let $s=-1$, since $t-\bar{t}$ can be either positive or negative, we have

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t}=0.$$

Similarly, let $t=\bar{t}$, since $s-\bar{s}\leq 0$, we have

$$\frac{\partial g(\bar{t},\bar{s})}{\partial s}\geq 0.$$

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t}=a+\frac{d}{\bar{t}^2}=0\Rightarrow \bar{t}=\sqrt{\frac{d}{-a}}.$$

Furthermore,

$$\bar{t}>1\Rightarrow z>-K. \tag{A.58}$$

$$\frac{\partial g(\bar{t},\bar{s})}{\partial s}=b-\frac{d}{\bar{t}}=b-\sqrt{-ad}\geq 0$$

implies that

$$y\geq -K$$

$$(y+K)\geq\sqrt{(-2K)(z-K)} \tag{A.59}$$

$$z-k\geq -\frac{1}{2K}(y+K)^2.$$

Finally,

$$\begin{aligned}
&g(t,s)\\
&\leq g(\bar{t},\bar{s})\\
&=a\sqrt{\frac{d}{-a}}-b+c-\frac{d}{\sqrt{\frac{d}{-a}}}\\
&=-2\sqrt{-ad}-b+c\leq 0
\end{aligned}$$

implies that

$$-2\sqrt{(-2K)(z-K)}\leq -x+z-2K. \tag{A.60}$$

$$x\leq z-2K+2\sqrt{(-2K)(z-K)}.$$

Case 4.3

Assume that the maximum value of $g(t,s)$ is achieved at $\bar{t}=1$, $\bar{s}<-1$.

From the optimality condition

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t}(t-\bar{t})+\frac{\partial g(\bar{t},\bar{s})}{\partial s}(s-\bar{s})\leq 0,$$

similar as the previous subsection, we have

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t}\leq 0,$$

$$\frac{\partial g(\bar{t},\bar{s})}{\partial s}=0.$$

$$\frac{\partial g\left(\bar{t},\bar{s}\right)}{\partial s} = b - \frac{d}{\bar{s}^2} = 0 \Rightarrow \bar{s} = -\sqrt{\frac{d}{b}}$$

Furthermore,

$$\bar{s} < -1 \Rightarrow y < K. \tag{A.61}$$

$$\frac{\partial g\left(\bar{t},\bar{s}\right)}{\partial t} = a - \frac{d}{\bar{s}} = a + \sqrt{bd} \leq 0$$

implies that

$$\begin{aligned} z &\leq K \\ z - K + \sqrt{2K\left(y+K\right)} &\leq 0 \\ y + K &\leq \frac{1}{2K}\left(z-K\right)^2. \end{aligned} \tag{A.62}$$

Finally,

$$\begin{aligned} & g\left(t,s\right) \\ & \leq g\left(\bar{t},\bar{s}\right) \\ & = a + c - 2\sqrt{bd} \leq 0 \end{aligned}$$

implies that

$$\begin{aligned} x + y + 2K &\leq 2\sqrt{2K\left(y+K\right)} \\ x &\leq -y - 2K + 2\sqrt{2K\left(y+K\right)}. \end{aligned} \tag{A.63}$$

Case 4.4

Assume that the minimum value of $g\left(t,s\right)$ is achieved at $\bar{t} = 1$, $\bar{s} = -1$.

From the optimality condition

$$\frac{\partial g\left(\bar{t},\bar{s}\right)}{\partial t}\left(t-\bar{t}\right) + \frac{\partial g\left(\bar{t},\bar{s}\right)}{\partial s}\left(s-\bar{s}\right) \leq 0,$$

similar as the previous subsection, we have

$$\begin{aligned} \frac{\partial g\left(\bar{t},\bar{s}\right)}{\partial t} &\leq 0, \\ \frac{\partial g\left(\bar{t},\bar{s}\right)}{\partial s} &\geq 0. \end{aligned}$$

$$\frac{\partial g\left(\bar{t},\bar{s}\right)}{\partial t} = a + d \leq 0$$

implies that

$$z \leq -K. \tag{A.64}$$

$$\frac{\partial g\left(\bar{t},\bar{s}\right)}{\partial s} = b - d \geq 0$$

implies that

$$y \geq K. \tag{A.65}$$

Furthermore,

$$\begin{aligned}&g(t,s)\\&\le g(\bar{t}=1,\bar{s}=-1)\\&=a-b+c-d\\&=x-K\le 0.\end{aligned}$$

implies that

$$x\le K. \tag{A.66}$$

However, (A.64) and (A.65) define a set that does not intersect the set defined by case 1 and case 2. Hence, we can discard case 4.4.

Discussion of case 4

The discussion is similar as case 3. Details are omitted.

Conclusion of case 4 (combine with case 1 and 2):

$$\begin{aligned}(x,y,z)\in\{&|x|\le K,|y|\le K,|z|\le K,\\&x\le -y+z-3K+3\sqrt[3]{(-2K)(y+K)(z-K)},\\&x\le z-2K+2\sqrt{(-2K)(z-K)},\\&x\le -y-2K+2\sqrt{2K(y+K)}\}\end{aligned}$$

A.3.5 Case 5

Case 5: $(c_{12},c_{03},c_{02})\in E_5: f(0,0)\ge 0,\ f\left(-\frac{1}{2},-\frac{1}{2}\right)\le 0,\ f\left(\frac{1}{2},-\frac{1}{2}\right)\ge 0$

Let

$$\begin{cases}A=h_i^x c_{12},\\B=h_i^x c_{12}+3h_j^y c_{03},\\C=2c_{02},\end{cases}$$

then

$$\begin{cases}c_{12}=\frac{1}{h_i^x}A,\\c_{03}=\frac{1}{3h_j^y}(B-A),\\c_{02}=\frac{1}{2}C,\end{cases}$$

and

$$\begin{aligned}&f(t,s)\\&=2c_{12}h_i^x t+6c_{03}h_j^y s+c_{12}h_i^x+3c_{03}h_j^y+2c_{02}\\&=2At+2(B-A)s+(B+C).\end{aligned}$$

Case 5 is equivalent to

$$\begin{aligned}f(0,0)=B+C\ge 0,\\f\left(-\frac{1}{2},-\frac{1}{2}\right)=C\le 0,\\f\left(\frac{1}{2},-\frac{1}{2}\right)=2A+C\ge 0.\end{aligned}$$

$$
\begin{aligned}
&F\left(c_{12}, c_{03}, c_{02}\right) \\
&= -h_i^x h_j^y \int_{-\frac{1}{2}}^{-\frac{B+C}{2B}} \int_{s}^{\frac{A-B}{A}s-\frac{B+C}{2A}} f\left(t,s\right) dtds \\
&+ h_i^x h_j^y \int_{-\frac{1}{2}}^{-\frac{B+C}{2B}} \int_{\frac{A-B}{A}s-\frac{B+C}{2A}}^{-s} f\left(t,s\right) dtds + h_i^x h_j^y \int_{-\frac{B+C}{2B}}^{0} \int_{s}^{-s} f\left(t,s\right) dtds \\
&= \frac{1}{12} h_i^x h_j^y \left\{ 2A + B + 3C - \frac{C^3}{AB} \right\}.
\end{aligned}
$$

Let

$$t = \frac{A}{C}, -\infty < t \leq -\frac{1}{2},$$

and

$$s = \frac{B}{C}, -\infty < s \leq -1,$$

then

$$
\begin{aligned}
&G_5\left(\xi, \eta, \gamma\right) \\
&= \sup_{\left(c_{12}, c_{03}, c_{02}\right) \in E_5} \left\{ \xi c_{12} + \eta c_{03} + \gamma c_{02} - F\left(c_{12}, c_{03}, c_{02}\right) \right\} \\
&= \sup_{\substack{C \leq 0, \\ 2A+C \geq 0, \\ B+C \geq 0}} \left\{ \xi \frac{1}{h_i^x} A + \eta \frac{1}{3h_j^y} \left(B - A\right) + \gamma \frac{1}{2} C - \frac{1}{12} h_i^x h_j^y \left[2A + B + 3C - \frac{C^3}{AB} \right] \right\} \\
&= \sup_{\substack{C \leq 0, \\ 2A+C \geq 0, \\ B+C \geq 0}} \left\{ \left(\frac{\xi}{h_i^x} - \frac{\eta}{3h_j^y} - \frac{h_i^x h_j^y}{6} \right) A + \left(\frac{\eta}{3h_j^y} - \frac{h_i^x h_j^y}{12} \right) B + \left(\frac{\gamma}{2} - \frac{h_i^x h_j^y}{4} \right) C + \frac{h_i^x h_j^y}{12} \frac{C^3}{AB} \right\} \\
&= \sup_{\substack{C \leq 0, \\ t \in \left(-\infty, -\frac{1}{2}\right], \\ s \in \left(-\infty, -1\right]}} \left\{ \left(\frac{\xi}{h_i^x} - \frac{\eta}{3h_j^y} - \frac{h_i^x h_j^y}{6} \right) t + \left(\frac{\eta}{3h_j^y} - \frac{h_i^x h_j^y}{12} \right) s + \left(\frac{\gamma}{2} - \frac{h_i^x h_j^y}{4} \right) + \frac{h_i^x h_j^y}{12} \frac{1}{ts} \right\} C \\
&= \sup_{\substack{C \leq 0, \\ t \in \left(-\infty, -\frac{1}{2}\right], \\ s \in \left(-\infty, -1\right]}} \left\{ g\left(t, s\right) C \right\}.
\end{aligned}
$$

Since $C \leq 0$, $G_5\left(\xi, \eta, \gamma\right)$ is finite if and only if $g\left(t, s\right) \geq 0$ for any .$t \in \left(-\infty, -\frac{1}{2}\right]$ and $s \in \left(-\infty, -1\right]$.

Consider a function

$$g\left(t, s\right) = at + bs + c + \frac{d}{ts}, t \in \left(-\infty, -\frac{1}{2}\right], s \in \left(-\infty, -1\right].$$

In this case, we have

$$
\begin{aligned}
a &= \frac{\xi}{h_i^x} - \frac{\eta}{3h_j^y} - \frac{h_i^x h_j^y}{6} = 2y - 2K, \\
b &= \frac{\eta}{3h_j^y} - \frac{h_i^x h_j^y}{12} = x - K, \\
c &= \frac{\gamma}{2} - \frac{h_i^x h_j^y}{4} = x + y - z - 3K, \\
d &= \frac{h_i^x h_j^y}{12} = K > 0,
\end{aligned}
$$

$$g(t, s) \geq 0 \text{ as } t \to -\infty$$

implies that

$$a \leq 0 \Longleftrightarrow y \leq K. \tag{A.67}$$

Similarly,

$$g(t, s) \geq 0 \text{ as } s \to -\infty$$

implies that

$$b \leq 0 \Longleftrightarrow x \leq K. \tag{A.68}$$

It is easy to show that $g(t, s)$ is strictly convex over its feasible set $\left(-\infty, -\frac{1}{2}\right] \times (-\infty, -1]$. Therefore, $(t^*, s^*) = \left(\sqrt[3]{\frac{bd}{a^2}}, \sqrt[3]{\frac{ad}{b^2}}\right)$ is the unique solution of $\nabla g = 0$.

Next we derive the conditions of $g(t, s) \geq 0$ in different cases.

Case 5.1

Case 5.1: (t^*, s^*) is an interior point of $\left(-\infty, -\frac{1}{2}\right] \times (-\infty, -1]$. Then $g(t, s)$ achieves its minimum value at the stationary point (t^*, s^*). We need

$$
\begin{aligned}
t^* &< -\frac{1}{2}, \\
s^* &< -1, \\
g(t^*, s^*) &\geq 0.
\end{aligned}
$$

where $t^* = \sqrt[3]{\frac{bd}{a^2}} < -\frac{1}{2}$ implies that

$$8bd < -a^2,$$

i.e.,

$$x - K < -\frac{1}{2K}(y - K)^2. \tag{A.69}$$

Similarly, $s^* = \sqrt[3]{\frac{ad}{b^2}} < -1$ implies that

$$ad < -b^2,$$

i.e.,

$$y - K < -\frac{1}{2K}(x - K)^2. \tag{A.70}$$

$g(t^*, s^*) \geq 0$ implies that

$$\begin{aligned} & g(t^*, s^*) \\ & = a\sqrt[3]{\frac{bd}{a^2}} + b\sqrt[3]{\frac{ad}{b^2}} + c + \frac{d}{\sqrt[3]{\frac{bd}{a^2}}\sqrt[3]{\frac{ad}{b^2}}} \\ & = 3\sqrt[3]{abd} + c \geq 0. \end{aligned}$$

i.e.,

$$z \leq x + y - 3K + 3\sqrt[3]{2K(x-K)(y-K)}. \tag{A.71}$$

Case 5.2

Assume that the minimum value of $g(t, s)$ is achieved at $\bar{t} < -\frac{1}{2}$, $\bar{s} = -1$.

From the optimality condition

$$\frac{\partial g(\bar{t}, \bar{s})}{\partial t}(t - \bar{t}) + \frac{\partial g(\bar{t}, \bar{s})}{\partial s}(s - \bar{s}) \geq 0,$$

let $s = -1$, since $t - \bar{t}$ can be either positive or negative, we have

$$\frac{\partial g(\bar{t}, \bar{s})}{\partial t} = 0.$$

Similarly, let $t = \bar{t}$, since $s - \bar{s} \leq 0$, we have

$$\frac{\partial g(\bar{t}, \bar{s})}{\partial s} \leq 0.$$

$$\frac{\partial g(\bar{t}, \bar{s})}{\partial t} = a + \frac{d}{\bar{t}^2} = 0 \Rightarrow \bar{t} = -\sqrt{\frac{d}{-a}}.$$

Furthermore,

$$\bar{t} < -\frac{1}{2} \Rightarrow y > -K. \tag{A.72}$$

$$\frac{\partial g(\bar{t}, \bar{s})}{\partial s} = b - \frac{d}{\bar{t}} = b + \sqrt{-ad} \leq 0$$

implies that

$$\begin{gathered} x \leq K \\ (x-K) + \sqrt{(-2K)(y-K)} \leq 0 \\ y - k \geq -\frac{1}{2K}(x-K)^2. \end{gathered} \tag{A.73}$$

Finally,

$$\begin{aligned} & g(t, s) \\ & \geq g(\bar{t}, \bar{s}) \\ & = a\sqrt{\frac{d}{-a}} - b + c - \frac{d}{\sqrt{\frac{d}{-a}}} \\ & = 2\sqrt{-ad} - b + c \geq 0 \end{aligned}$$

implies that

$$z \leq y - 2K + 2\sqrt{(-2K)(y-K)}. \tag{A.74}$$

Case 5.3

Assume that the minimum value of $g(t,s)$ is achieved at $\bar{t} = -\frac{1}{2}$, $\bar{s} < -1$.

From the optimality condition

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t}(t-\bar{t}) + \frac{\partial g(\bar{t},\bar{s})}{\partial s}(s-\bar{s}) \geq 0,$$

similar as the previous subsection, we have

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t} \leq 0,$$
$$\frac{\partial g(\bar{t},\bar{s})}{\partial s} = 0.$$

$$\frac{\partial g(\bar{t},\bar{s})}{\partial s} = b + \frac{2d}{\bar{s}^2} = 0 \Rightarrow \bar{s} = -\sqrt{\frac{2d}{-b}}$$

Furthermore,

$$\bar{s} < -1 \Rightarrow x > -K. \tag{A.75}$$

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t} = a - \frac{4d}{\bar{s}} = a + \sqrt{-8bd} \leq 0$$

implies that

$$y \leq K$$
$$2(y-K) + \sqrt{(-8K)(x-K)} \leq 0 \tag{A.76}$$
$$x - K \geq -\frac{1}{2K}(y-K)^2.$$

Finally,

$$\begin{aligned} & g(t,s) \\ & \geq g(\bar{t},\bar{s}) \\ & = 2\sqrt{-2bd} - \frac{a}{2} + c \geq 0 \end{aligned}$$

implies that

$$z \leq x - 2K + 2\sqrt{(-2K)(x-K)} \tag{A.77}$$

Case 5.4

Assume that the minimum value of $g(t,s)$ is achieved at $\bar{t} = -\frac{1}{2}$, $\bar{s} = -1$.

From the optimality condition

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t}(t-\bar{t}) + \frac{\partial g(\bar{t},\bar{s})}{\partial s}(s-\bar{s}) \geq 0,$$

similar as the previous subsection, we have

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t} \leq 0,$$
$$\frac{\partial g(\bar{t},\bar{s})}{\partial s} \leq 0.$$

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t} = a + 4d \le 0$$

implies that

$$y \le -K. \tag{A.78}$$

$$\frac{\partial g(\bar{t},\bar{s})}{\partial s} = b + 2d \le 0$$

implies that

$$x \le -K. \tag{A.79}$$

Furthermore,

$$\begin{aligned} &g(t,s)\\ &\ge g\left(\bar{t} = -\frac{1}{2}, \bar{s} = -1\right)\\ &= -\frac{a}{2} - b + c + 2d\\ &= -z + K \ge 0. \end{aligned}$$

implies that

$$z \le K. \tag{A.80}$$

However, (A.78) and (A.79) define a set that does not intersect the set defined by case 1 and case 2. Hence, we can discard case 5.4.

Discussion of case 5

The discussion is similar as case 3. Details are omitted.

Conclusion of case 5 (combine with case 1 and 2):

$$\begin{aligned} (x,y,z) \in \{&|x| \le K, |y| \le K, |z| \le K,\\ &z \le x + y - 3K + 3\sqrt[3]{2K(x-K)(y-K)},\\ &z \le x - 2K + 2\sqrt{(-2K)(x-K)},\\ &z \le y - 2K + 2\sqrt{(-2K)(y-K)}\} \end{aligned}$$

A.3.6 Case 6

Case 6: $(c_{12}, c_{03}, c_{02}) \in E_6 : f(0,0) \le 0,\ f\left(-\frac{1}{2}, -\frac{1}{2}\right) \ge 0,\ f\left(\frac{1}{2}, -\frac{1}{2}\right) \le 0$

Let A, B, C be the same as defined for case 5

$$\begin{cases} A = h_i^x c_{12},\\ B = h_i^x c_{12} + 3h_j^y c_{03},\\ C = 2c_{02}, \end{cases}$$

then

$$\begin{cases} c_{12} = \frac{1}{h_i^x} A,\\ c_{03} = \frac{1}{3h_j^y}(B - A),\\ c_{02} = \frac{1}{2} C, \end{cases}$$

and

$$\begin{aligned}
&f(t,s)\\
&=2c_{12}h_i^x t+6c_{03}h_j^y s+c_{12}h_i^x+3c_{03}h_j^y+2c_{02}\\
&=2At+2(B-A)s+(B+C).
\end{aligned}$$

Case 6 is equivalent to

$$\begin{aligned}
f(0,0)&=B+C\le 0,\\
f\left(-\frac{1}{2},-\frac{1}{2}\right)&=C\ge 0,\\
f\left(\frac{1}{2},-\frac{1}{2}\right)&=2A+C\le 0.
\end{aligned}$$

$$\begin{aligned}
&F(c_{12},c_{03},c_{02})\\
&=h_i^x h_j^y\int_{-\frac{1}{2}}^{-\frac{B+C}{2B}}\int_{s}^{\frac{A-B}{A}s-\frac{B+C}{2A}} f(t,s)\,dtds\\
&-h_i^x h_j^y\int_{-\frac{1}{2}}^{-\frac{B+C}{2B}}\int_{\frac{A-B}{A}s-\frac{B+C}{2A}}^{-s} f(t,s)\,dtds-h_i^x h_j^y\int_{-\frac{B+C}{2B}}^{0}\int_{s}^{-s} f(t,s)\,dtds\\
&=\frac{1}{12}h_i^x h_j^y\left\{-2A-B-3C+\frac{C^3}{AB}\right\}.
\end{aligned}$$

Let

$$t=\frac{A}{C},-\infty<t\le-\frac{1}{2},$$

and

$$s=\frac{B}{C},-\infty<s\le-1,$$

then

$$\begin{aligned}
&G_6(\xi,\eta,\gamma)\\
&=\sup_{(c_{12},c_{03},c_{02})\in E_6}\left\{\xi c_{12}+\eta c_{03}+\gamma c_{02}-F(c_{12},c_{03},c_{02})\right\}\\
&=\sup_{\substack{C\ge 0,\\ 2A+C\le 0,\\ B+C\le 0}}\left\{\xi\frac{1}{h_i^x}A+\eta\frac{1}{3h_j^y}(B-A)+\gamma\frac{1}{2}C-\frac{1}{12}h_i^x h_j^y\left[-2A-B-3C+\frac{C^3}{AB}\right]\right\}\\
&=\sup_{\substack{C\le 0,\\ 2A+C\ge 0,\\ B+C\ge 0}}\left\{\left(\frac{\xi}{h_i^x}-\frac{\eta}{3h_j^y}+\frac{h_i^x h_j^y}{6}\right)A+\left(\frac{\eta}{3h_j^y}+\frac{h_i^x h_j^y}{12}\right)B+\left(\frac{\gamma}{2}+\frac{h_i^x h_j^y}{4}\right)C-\frac{h_i^x h_j^y}{12}\frac{C^3}{AB}\right\}\\
&=\sup_{\substack{C\ge 0,\\ t\in\left(-\infty,-\frac{1}{2}\right],\\ s\in(-\infty,-1]}}\left\{\left(\frac{\xi}{h_i^x}-\frac{\eta}{3h_j^y}+\frac{h_i^x h_j^y}{6}\right)t+\left(\frac{\eta}{3h_j^y}+\frac{h_i^x h_j^y}{12}\right)s+\left(\frac{\gamma}{2}+\frac{h_i^x h_j^y}{4}\right)-\frac{h_i^x h_j^y}{12}\frac{1}{ts}\right\}C\\
&=\sup_{\substack{C\ge 0,\\ t\in\left(-\infty,-\frac{1}{2}\right],\\ s\in(-\infty,-1]}}\left\{g(t,s)\,C\right\}.
\end{aligned}$$

Since $C \geq 0$, $G_6(\xi, \eta, \gamma)$ is finite if and only if $g(t,s) \leq 0$ for any $.t \in \left(-\infty, -\frac{1}{2}\right]$ and $s \in (-\infty, -1]$.

Consider a function

$$g(t,s) = at + bs + c + \frac{d}{ts}, t \in \left(-\infty, -\frac{1}{2}\right], s \in (-\infty, -1].$$

In this case, we have

$$\begin{aligned} a &= \frac{\xi}{h_i^x} - \frac{\eta}{3h_j^y} + \frac{h_i^x h_j^y}{6} = 2y + 2K, \\ b &= \frac{\eta}{3h_j^y} + \frac{h_i^x h_j^y}{12} = x + K, \\ c &= \frac{\gamma}{2} + \frac{h_i^x h_j^y}{4} = x + y - z + 3K, \\ d &= -\frac{h_i^x h_j^y}{12} = -K < 0, \end{aligned}$$

$$g(t,s) \leq 0 \text{ as } t \to -\infty$$

implies that

$$a \geq 0 \iff y \geq -K. \tag{A.81}$$

Similarly,

$$g(t,s) \leq 0 \text{ as } s \to -\infty$$

implies that

$$b \geq 0 \iff x \geq -K. \tag{A.82}$$

It is easy to show that $g(t,s)$ is strictly concave over its feasible set $\left(-\infty, -\frac{1}{2}\right] \times (-\infty, -1]$. Therefore, $(t^*, s^*) = \left(\sqrt[3]{\frac{bd}{a^2}}, \sqrt[3]{\frac{ad}{b^2}}\right)$ is the unique solution of $\nabla g = 0$.

Next we derive the conditions of $g(t,s) \leq 0$ in different cases.

Case 6.1

Case 6.1: (t^*, s^*) is an interior point of $\left(-\infty, -\frac{1}{2}\right] \times (-\infty, -1]$. Then $g(t,s)$ achieves its maximum value at the stationary point (t^*, s^*). We need

$$\begin{aligned} t^* &< -\frac{1}{2}, \\ s^* &< -1, \\ g(t^*, s^*) &\leq 0. \end{aligned}$$

where $t^* = \sqrt[3]{\frac{bd}{a^2}} < -\frac{1}{2}$ implies that

$$8bd < -a^2,$$

i.e.,

$$x + K > \frac{1}{2K}(y + K)^2. \tag{A.83}$$

Similarly, $s^* = \sqrt[3]{\frac{ad}{b^2}} < -1$ implies that

$$ad < -b^2,$$

i.e.,

$$y + K > \frac{1}{2K}(x+K)^2. \tag{A.84}$$

$g(t^*, s^*) \leq 0$ implies that

$$\begin{aligned} & g(t^*, s^*) \\ &= a\sqrt[3]{\frac{bd}{a^2}} + b\sqrt[3]{\frac{ad}{b^2}} + c + \frac{d}{\sqrt[3]{\frac{bd}{a^2}}\sqrt[3]{\frac{ad}{b^2}}} \\ &= 3\sqrt[3]{abd} + c \leq 0. \end{aligned}$$

i.e.,

$$z \geq x + y + 3K - 3\sqrt[3]{2K(x+K)(y+K)}. \tag{A.85}$$

Case 6.2

Assume that the maximum value of $g(t,s)$ is achieved at $\bar{t} < -\frac{1}{2}$, $\bar{s} = -1$.
From the optimality condition

$$\frac{\partial g(\bar{t}, \bar{s})}{\partial t}(t - \bar{t}) + \frac{\partial g(\bar{t}, \bar{s})}{\partial s}(s - \bar{s}) \leq 0,$$

let $s = -1$, since $t - \bar{t}$ can be either positive or negative, we have

$$\frac{\partial g(\bar{t}, \bar{s})}{\partial t} = 0.$$

Similarly, let $t = \bar{t}$, since $s - \bar{s} \leq 0$, we have

$$\frac{\partial g(\bar{t}, \bar{s})}{\partial s} \geq 0.$$

$$\frac{\partial g(\bar{t}, \bar{s})}{\partial t} = a + \frac{d}{\bar{t}^2} = 0 \Rightarrow \bar{t} = -\sqrt{\frac{-d}{a}}.$$

Furthermore,

$$\bar{t} < -\frac{1}{2} \Rightarrow y < K. \tag{A.86}$$

$$\frac{\partial g(\bar{t}, \bar{s})}{\partial s} = b - \frac{d}{\bar{t}} = b - \sqrt{-ad} \geq 0$$

implies that

$$\begin{aligned} x &\geq -K \\ x + K &\geq \sqrt{2K(y+K)} \\ y + k &\leq \frac{1}{2K}(x+K)^2. \end{aligned} \tag{A.87}$$

Finally,

$$
\begin{aligned}
& g(t,s) \\
& \leq g(\bar{t},\bar{s}) \\
& = a\sqrt{\frac{-d}{a}} - b + c - \frac{d}{\sqrt{\frac{-d}{a}}} \\
& = -2\sqrt{-ad} - b + c \leq 0
\end{aligned}
$$

implies that

$$z \geq y + 2K - 2\sqrt{2K(y+K)}. \tag{A.88}$$

Case 6.3

Assume that the maximum value of $g(t,s)$ is achieved at $\bar{t} = -\frac{1}{2}$, $\bar{s} < -1$.

From the optimality condition

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t}(t-\bar{t}) + \frac{\partial g(\bar{t},\bar{s})}{\partial s}(s-\bar{s}) \leq 0,$$

similar as the previous subsection, we have

$$
\begin{aligned}
\frac{\partial g(\bar{t},\bar{s})}{\partial t} &\geq 0, \\
\frac{\partial g(\bar{t},\bar{s})}{\partial s} &= 0.
\end{aligned}
$$

$$\frac{\partial g(\bar{t},\bar{s})}{\partial s} = b + \frac{2d}{\bar{s}^2} = 0 \Rightarrow \bar{s} = -\sqrt{\frac{-2d}{b}}$$

Furthermore,

$$\bar{s} < -1 \Rightarrow x < K. \tag{A.89}$$

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t} = a - \frac{4d}{\bar{s}} = a - \sqrt{-8bd} \geq 0$$

implies that

$$
\begin{aligned}
y &\geq -K \\
y + K &\geq \sqrt{2K(x+K)} \qquad \text{(A.90)} \\
x + K &\leq \frac{1}{2K}(y+K)^2.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
& g(t,s) \\
& \leq g(\bar{t},\bar{s}) \\
& = -2\sqrt{-2bd} - \frac{a}{2} + c \leq 0
\end{aligned}
$$

implies that

$$z \geq x + 2K - 2\sqrt{2K(x+K)} \tag{A.91}$$

Case 6.4

Assume that the maximum value of $g(t,s)$ is achieved at $\bar{t} = -\frac{1}{2}$, $\bar{s} = -1$.

From the optimality condition

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t}(t-\bar{t}) + \frac{\partial g(\bar{t},\bar{s})}{\partial s}(s-\bar{s}) \le 0,$$

similar as the previous subsection, we have

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t} \ge 0,$$
$$\frac{\partial g(\bar{t},\bar{s})}{\partial s} \ge 0.$$

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t} = a + 4d \ge 0$$

implies that

$$y \ge K. \tag{A.92}$$

$$\frac{\partial g(\bar{t},\bar{s})}{\partial s} = b + 2d \ge 0$$

implies that

$$x \ge K. \tag{A.93}$$

Furthermore,

$$\begin{aligned} & g(t,s) \\ & \le g\left(\bar{t} = -\frac{1}{2}, \bar{s} = -1\right) \\ & = -\frac{a}{2} - b + c + 2d \\ & = -z - K \le 0. \end{aligned}$$

implies that

$$z \ge -K. \tag{A.94}$$

However, (A.92) and (A.93) define a set that does not intersect the set defined by case 1 and case 2. Hence, we can discard case 6.4.

Discussion of case 6

The discussion is similar as case 3. Details are omitted.

Conclusion of case 6 (combine with case 1 and 2):

$$\begin{aligned} (x,y,z) \in \{ & |x| \le K, |y| \le K, |z| \le K, \\ & z \ge x + y + 3K - 3\sqrt[3]{2K(x+K)(y+K)}, \\ & z \ge x + 2K - 2\sqrt{2K(x+K)}, \\ & z \ge y + 2K - 2\sqrt{2K(y+K)} \} \end{aligned}$$

A.3.7 Case 7

Case 7: $(c_{12}, c_{03}, c_{02}) \in E_7 : f(0,0) \geq 0,\ f\left(-\frac{1}{2}, -\frac{1}{2}\right) \geq 0,\ f\left(\frac{1}{2}, -\frac{1}{2}\right) \leq 0$

Let

$$\begin{cases} A = h_i^x c_{12}, \\ B = h_i^x c_{12} - 3h_j^y c_{03}, \\ C = 2h_i^x c_{12} + 2c_{02}, \end{cases}$$

then

$$\begin{cases} c_{12} = \frac{1}{h_i^x} A, \\ c_{03} = \frac{1}{3h_j^y}(A - B), \\ c_{02} = -A + \frac{1}{2}C, \end{cases}$$

and

$$\begin{aligned} & f(t,s) \\ & = 2c_{12}h_i^x t + 6c_{03}h_j^y s + c_{12}h_i^x + 3c_{03}h_j^y + 2c_{02} \\ & = 2At + 2(A-B)s + (C-B). \end{aligned}$$

Case 7 is equivalent to

$$\begin{aligned} f(0,0) = C - B \geq 0, \\ f\left(-\frac{1}{2}, -\frac{1}{2}\right) = -2A + C \geq 0, \\ f\left(\frac{1}{2}, -\frac{1}{2}\right) = C \leq 0. \end{aligned}$$

$$\begin{aligned} & F(c_{12}, c_{03}, c_{02}) \\ & = h_i^x h_j^y \int_{-\frac{1}{2}}^{-\frac{B-C}{2B}} \int_{s}^{-\frac{A-B}{A}s + \frac{B-C}{2A}} f(t,s)\, dtds + h_i^x h_j^y \int_{-\frac{B-C}{2B}}^{0} \int_{s}^{-s} f(t,s)\, dtds \\ & - h_i^x h_j^y \int_{-\frac{1}{2}}^{-\frac{B-C}{2B}} \int_{-\frac{A-B}{A}s + \frac{B-C}{2A}}^{-s} f(t,s)\, dtds \\ & = \frac{1}{12} h_i^x h_j^y \left\{ -2A - B + 3C - \frac{C^3}{AB} \right\}. \end{aligned}$$

Let

$$t = \frac{A}{C}, \frac{1}{2} \leq t < \infty,$$

and

$$s = \frac{B}{C}, 1 \leq s < \infty,$$

then

$$
\begin{aligned}
&G_7(\xi,\eta,\gamma)\\
&= \sup_{(c_{12},c_{03},c_{02})\in E_7} \{\xi c_{12} + \eta c_{03} + \gamma c_{02} - F(c_{12},c_{03},c_{02})\}\\
&= \sup_{\substack{C\le 0,\\ -2A+C\ge 0,\\ C-B\ge 0}} \left\{ \xi \frac{1}{h_i^x} A + \eta \frac{1}{3h_j^y}(A-B) + \gamma\left(-A+\frac{1}{2}C\right) - \frac{1}{12} h_i^x h_j^y \left[-2A - B + 3C - \frac{C^3}{AB}\right]\right\}\\
&= \sup_{\substack{C\le 0,\\ -2A+C\ge 0,\\ C-B\ge 0}} \left\{ \left(\frac{\xi}{h_i^x} + \frac{\eta}{3h_j^y} - \gamma + \frac{h_i^x h_j^y}{6}\right) A + \left(-\frac{\eta}{3h_j^y} + \frac{h_i^x h_j^y}{12}\right) B + \left(\frac{\gamma}{2} - \frac{h_i^x h_j^y}{4}\right) C + \frac{h_i^x h_j^y}{12}\frac{C^3}{AB}\right\}\\
&= \sup_{\substack{C\le 0,\\ t\in\left[\frac{1}{2},+\infty\right),\\ s\in[1,+\infty)}} \left\{ \left(\frac{\xi}{h_i^x} + \frac{\eta}{3h_j^y} - \gamma + \frac{h_i^x h_j^y}{6}\right) t + \left(-\frac{\eta}{3h_j^y} + \frac{h_i^x h_j^y}{12}\right) s + \left(\frac{\gamma}{2} - \frac{h_i^x h_j^y}{4}\right) + \frac{h_i^x h_j^y}{12}\frac{1}{ts}\right\} C\\
&= \sup_{\substack{C\le 0,\\ t\in\left[\frac{1}{2},+\infty\right),\\ s\in[1,+\infty)}} \{g(t,s)\, C\}.
\end{aligned}
$$

Since $C \le 0$, $G_7(\xi,\eta,\gamma)$ is finite if and only if $g(t,s) \ge 0$ for any $.t \in \left[\frac{1}{2},+\infty\right)$ and $s \in [1,+\infty)$.

Consider a function

$$
g(t,s) = at + bs + c + \frac{d}{ts}, t \in \left[\frac{1}{2},+\infty\right), s \in [1,+\infty).
$$

In this case, we have

$$
\begin{aligned}
a &= \frac{\xi}{h_i^x} + \frac{\eta}{3h_j^y} - \gamma + \frac{h_i^x h_j^y}{6} = 2z + 2K,\\
b &= -\frac{\eta}{3h_j^y} + \frac{h_i^x h_j^y}{12} = -x + K,\\
c &= \frac{\gamma}{2} - \frac{h_i^x h_j^y}{4} = x + y - z - 3K,\\
d &= \frac{h_i^x h_j^y}{12} = K > 0,
\end{aligned}
$$

$$
g(t,s) \ge 0 \text{ as } t \to +\infty
$$

implies that

$$
a \ge 0 \Longleftrightarrow z \ge -K. \tag{A.95}
$$

Similarly,

$$
g(t,s) \ge 0 \text{ as } s \to +\infty
$$

implies that

$$
b \ge 0 \Longleftrightarrow x \le K. \tag{A.96}
$$

It is easy to show that $g(t,s)$ is strictly convex over its feasible set $\left[\frac{1}{2},+\infty\right)\times[1,+\infty)$. Therefore, $(t^*,s^*)=\left(\sqrt[3]{\frac{bd}{a^2}},\sqrt[3]{\frac{ad}{b^2}}\right)$ is the unique solution of $\nabla g=0$.

Next we derive the conditions of $g(t,s)\geq 0$ in different cases.

Case 7.1

Case 7.1: (t^*,s^*) is an interior point of $\left[\frac{1}{2},+\infty\right)\times[1,+\infty)$. Then $g(t,s)$ achieves its minimum value at the stationary point (t^*,s^*). We need

$$\begin{aligned} t^* &> \frac{1}{2}, \\ s^* &> 1, \\ g(t^*,s^*) &\geq 0. \end{aligned}$$

where $t^*=\sqrt[3]{\frac{bd}{a^2}}>\frac{1}{2}$ implies that

$$8bd>a^2,$$

i.e.,

$$x-K<-\frac{1}{2K}(z+K)^2. \tag{A.97}$$

Similarly, $s^*=\sqrt[3]{\frac{ad}{b^2}}>1$ implies that

$$ad>b^2,$$

i.e.,

$$z+K>\frac{1}{2K}(x-K)^2. \tag{A.98}$$

$g(t^*,s^*)\geq 0$ implies that

$$\begin{aligned} &g(t^*,s^*) \\ &=a\sqrt[3]{\frac{bd}{a^2}}+b\sqrt[3]{\frac{ad}{b^2}}+c+\frac{d}{\sqrt[3]{\frac{bd}{a^2}}\sqrt[3]{\frac{ad}{b^2}}} \\ &=3\sqrt[3]{abd}+c\geq 0. \end{aligned}$$

i.e.,

$$y\geq -x+z+3K-3\sqrt[3]{(-2K)(x-K)(z+K)}. \tag{A.99}$$

Case 7.2

Assume that the minimum value of $g(t,s)$ is achieved at $\bar{t}>\frac{1}{2}$, $\bar{s}=1$.

From the optimality condition

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t}(t-\bar{t})+\frac{\partial g(\bar{t},\bar{s})}{\partial s}(s-\bar{s})\geq 0,$$

let $s=1$, since $t-\bar{t}$ can be either positive or negative, we have

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t}=0.$$

Similarly, let $t = \bar{t}$, since $s - \bar{s} \geq 0$, we have

$$\frac{\partial g\left(\bar{t}, \bar{s}\right)}{\partial s} \geq 0.$$

$$\frac{\partial g\left(\bar{t}, \bar{s}\right)}{\partial t} = a - \frac{d}{\bar{t}^2} = 0 \Rightarrow \bar{t} = \sqrt{\frac{d}{a}}.$$

Furthermore,

$$\bar{t} > \frac{1}{2} \Rightarrow z < K. \tag{A.100}$$

$$\frac{\partial g\left(\bar{t}, \bar{s}\right)}{\partial s} = b - \frac{d}{\bar{t}} = b - \sqrt{ad} \geq 0$$

implies that

$$x \leq K$$

$$x - K + \sqrt{2K\left(z + K\right)} \leq 0 \tag{A.101}$$

$$z + k \leq \frac{1}{2K}\left(x - K\right)^2.$$

Finally,

$$\begin{aligned}
&g\left(t, s\right) \\
&\geq g\left(\bar{t}, \bar{s}\right) \\
&= a\sqrt{\frac{d}{a}} + b + c + \frac{d}{\sqrt{\frac{d}{a}}} \\
&= 2\sqrt{ad} + b + c \geq 0
\end{aligned}$$

implies that

$$y \geq z + 2K - 2\sqrt{2K\left(z + K\right)}. \tag{A.102}$$

Case 7.3

Assume that the minimum value of $g\left(t, s\right)$ is achieved at $\bar{t} = \frac{1}{2}$, $\bar{s} > 1$.

From the optimality condition

$$\frac{\partial g\left(\bar{t}, \bar{s}\right)}{\partial t}\left(t - \bar{t}\right) + \frac{\partial g\left(\bar{t}, \bar{s}\right)}{\partial s}\left(s - \bar{s}\right) \geq 0,$$

similar as the previous subsection, we have

$$\frac{\partial g\left(\bar{t}, \bar{s}\right)}{\partial t} \geq 0,$$
$$\frac{\partial g\left(\bar{t}, \bar{s}\right)}{\partial s} = 0.$$

$$\frac{\partial g\left(\bar{t}, \bar{s}\right)}{\partial s} = b - \frac{2d}{\bar{s}^2} = 0 \Rightarrow \bar{s} = \sqrt{\frac{2d}{b}}$$

Furthermore,

$$\bar{s} > 1 \Rightarrow x > -K. \tag{A.103}$$

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t} = a - \frac{4d}{\bar{s}} = a - \sqrt{8bd} \geq 0$$

implies that

$$\begin{aligned} z &\geq -K \\ z + K &\geq \sqrt{(-2K)(x-K)} \\ x - K &\geq -\frac{1}{2K}(z+K)^2 . \end{aligned} \tag{A.104}$$

Finally,

$$\begin{aligned} & g(t,s) \\ & \geq g(\bar{t},\bar{s}) \\ & = \frac{a}{2} + c + 2\sqrt{2bd} \geq 0 \end{aligned}$$

implies that

$$y \geq -x + 2K - 2\sqrt{(-2K)(x-K)} \tag{A.105}$$

Case 7.4

Assume that the minimum value of $g(t,s)$ is achieved at $\bar{t} = \frac{1}{2}$, $\bar{s} = 1$.

From the optimality condition

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t}(t-\bar{t}) + \frac{\partial g(\bar{t},\bar{s})}{\partial s}(s-\bar{s}) \geq 0,$$

similar as the previous subsection, we have

$$\begin{aligned} \frac{\partial g(\bar{t},\bar{s})}{\partial t} &\geq 0, \\ \frac{\partial g(\bar{t},\bar{s})}{\partial s} &\geq 0. \end{aligned}$$

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t} = a - 4d \geq 0$$

implies that

$$z \geq K. \tag{A.106}$$

$$\frac{\partial g(\bar{t},\bar{s})}{\partial s} = b - 2d \geq 0$$

implies that

$$x \leq -K. \tag{A.107}$$

Furthermore,

$$\begin{aligned} & g(t,s) \\ & \geq g\left(\bar{t} = \frac{1}{2}, \bar{s} = 1\right) \\ & = \frac{a}{2} + b + c + 2d \\ & = y + K \geq 0. \end{aligned}$$

implies that

$$y \geq -K. \tag{A.108}$$

However, (A.106) and (A.107) define a set that does not intersect the set defined by case 1 and case 2. Hence, we can discard case 7.4.

Discussion of case 7

The discussion is similar as case 3. Details are omitted.

Conclusion of case 7 (combine with case 1 and 2):

$$\begin{aligned}(x, y, z) \in \{&|x| \leq K, |y| \leq K, |z| \leq K,\\ &y \geq -x + z + 3K - 3\sqrt[3]{(-2K)(x-K)(z+K)},\\ &y \geq -x + 2K - 2\sqrt{(-2K)(x-K)},\\ &y \geq z + 2K - 2\sqrt{2K(z+K)}\}\end{aligned}$$

A.3.8 Case 8

Case 8: $(c_{12}, c_{03}, c_{02}) \in E_8 : f(0,0) \leq 0,\ f\left(-\frac{1}{2}, -\frac{1}{2}\right) \leq 0,\ f\left(\frac{1}{2}, -\frac{1}{2}\right) \geq 0$

Let A, B, C be defined the same as in case 7

$$\begin{cases} A = h_i^x c_{12}, \\ B = h_i^x c_{12} - 3h_j^y c_{03}, \\ C = 2h_i^x c_{12} + 2c_{02}, \end{cases}$$

then

$$\begin{cases} c_{12} = \frac{1}{h_i^x} A, \\ c_{03} = \frac{1}{3h_j^y}(A - B), \\ c_{02} = -A + \frac{1}{2}C, \end{cases}$$

and

$$\begin{aligned}&f(t, s)\\ &= 2c_{12}h_i^x t + 6c_{03}h_j^y s + c_{12}h_i^x + 3c_{03}h_j^y + 2c_{02}\\ &= 2At + 2(A - B)s + (C - B).\end{aligned}$$

Case 8 is equivalent to

$$\begin{aligned} f(0,0) = C - B &\leq 0,\\ f\left(-\frac{1}{2}, -\frac{1}{2}\right) = -2A + C &\leq 0,\\ f\left(\frac{1}{2}, -\frac{1}{2}\right) = C &\geq 0.\end{aligned}$$

$$
\begin{aligned}
&F\left(c_{12}, c_{03}, c_{02}\right) \\
&= -h_i^x h_j^y \int_{-\frac{1}{2}}^{-\frac{B-C}{2B}} \int_{s}^{-\frac{A-B}{A}s+\frac{B-C}{2A}} f\left(t,s\right) dtds - h_i^x h_j^y \int_{-\frac{B-C}{2B}}^{0} \int_{s}^{-s} f\left(t,s\right) dtds \\
&+ h_i^x h_j^y \int_{-\frac{1}{2}}^{-\frac{B-C}{2B}} \int_{-\frac{A-B}{A}s+\frac{B-C}{2A}}^{-s} f\left(t,s\right) dtds \\
&= \frac{1}{12} h_i^x h_j^y \left\{ 2A + B - 3C + \frac{C^3}{AB} \right\}.
\end{aligned}
$$

Let

$$
t = \frac{A}{C}, \frac{1}{2} \le t < \infty,
$$

and

$$
s = \frac{B}{C}, 1 \le s < \infty,
$$

then

$$
\begin{aligned}
&G_8\left(\xi, \eta, \gamma\right) \\
&= \sup_{\left(c_{12}, c_{03}, c_{02}\right) \in E_7} \left\{ \xi c_{12} + \eta c_{03} + \gamma c_{02} - F\left(c_{12}, c_{03}, c_{02}\right) \right\} \\
&= \sup_{\substack{C \ge 0, \\ -2A+C \le 0, \\ C-B \le 0}} \left\{ \xi \frac{1}{h_i^x} A + \eta \frac{1}{3h_j^y} \left(A - B\right) + \gamma \left(-A + \frac{1}{2}C\right) - \frac{1}{12} h_i^x h_j^y \left[2A + B - 3C + \frac{C^3}{AB} \right] \right\} \\
&= \sup_{\substack{C \ge 0, \\ -2A+C \le 0, \\ C-B \le 0}} \left\{ \left(\frac{\xi}{h_i^x} + \frac{\eta}{3h_j^y} - \gamma - \frac{h_i^x h_j^y}{6} \right) A + \left(-\frac{\eta}{3h_j^y} - \frac{h_i^x h_j^y}{12} \right) B + \left(\frac{\gamma}{2} + \frac{h_i^x h_j^y}{4} \right) C - \frac{h_i^x h_j^y}{12} \frac{C^3}{AB} \right\} \\
&= \sup_{\substack{C \ge 0, \\ t \in \left[\frac{1}{2}, +\infty\right), \\ s \in [1, +\infty)}} \left\{ \left(\frac{\xi}{h_i^x} + \frac{\eta}{3h_j^y} - \gamma - \frac{h_i^x h_j^y}{6} \right) t + \left(-\frac{\eta}{3h_j^y} - \frac{h_i^x h_j^y}{12} \right) s + \left(\frac{\gamma}{2} + \frac{h_i^x h_j^y}{4} \right) - \frac{h_i^x h_j^y}{12} \frac{1}{ts} \right\} C \\
&= \sup_{\substack{C \ge 0, \\ t \in \left[\frac{1}{2}, +\infty\right), \\ s \in [1, +\infty)}} \left\{ g\left(t,s\right) C \right\}.
\end{aligned}
$$

Since $C \ge 0$, $G_8\left(\xi, \eta, \gamma\right)$ is finite if and only if $g\left(t,s\right) \le 0$ for any $.t \in \left[\frac{1}{2}, +\infty\right)$ and $s \in [1, +\infty)$.

Consider a function

$$
g\left(t,s\right) = at + bs + c + \frac{d}{ts}, t \in \left[\frac{1}{2}, +\infty\right), s \in [1, +\infty).
$$

In this case, we have

$$\begin{aligned} a &= \frac{\xi}{h_i^x} + \frac{\eta}{3h_j^y} - \gamma - \frac{h_i^x h_j^y}{6} = 2z - 2K, \\ b &= -\frac{\eta}{3h_j^y} - \frac{h_i^x h_j^y}{12} = -x - K, \\ c &= \frac{\gamma}{2} + \frac{h_i^x h_j^y}{4} = x + y - z + 3K, \\ d &= -\frac{h_i^x h_j^y}{12} = -K < 0, \end{aligned}$$

$$g\left(t, s\right) \leq 0 \text{ as } t \to +\infty$$

implies that

$$a \leq 0 \Longleftrightarrow z \leq K. \tag{A.109}$$

Similarly,

$$g\left(t, s\right) \leq 0 \text{ as } s \to +\infty$$

implies that

$$b \leq 0 \Longleftrightarrow x \geq -K. \tag{A.110}$$

It is easy to show that $g\left(t, s\right)$ is strictly concave over its feasible set $\left[\frac{1}{2}, +\infty\right) \times [1, +\infty)$. Therefore, $(t^*, s^*) = \left(\sqrt[3]{\frac{bd}{a^2}}, \sqrt[3]{\frac{ad}{b^2}}\right)$ is the unique solution of $\nabla g = 0$.

Next we derive the conditions of $g\left(t, s\right) \leq 0$ in different cases.

Case 8.1

Case 8.1: (t^*, s^*) is an interior point of $\left[\frac{1}{2}, +\infty\right) \times [1, +\infty)$. Then $g\left(t, s\right)$ achieves its maximum value at the stationary point (t^*, s^*). We need

$$\begin{aligned} t^* &> \frac{1}{2}, \\ s^* &> 1, \\ g\left(t^*, s^*\right) &\leq 0. \end{aligned}$$

where $t^* = \sqrt[3]{\frac{bd}{a^2}} > \frac{1}{2}$ implies that

$$8bd > a^2,$$

i.e.,

$$x + K > \frac{1}{2K}\left(z - K\right)^2. \tag{A.111}$$

Similarly, $s^* = \sqrt[3]{\frac{ad}{b^2}} > 1$ implies that

$$ad > b^2,$$

i.e.,

$$z - K < -\frac{1}{2K}\left(x + K\right)^2. \tag{A.112}$$

$g(t^*, s^*) \leq 0$ implies that

$$\begin{aligned} & g(t^*, s^*) \\ &= a\sqrt[3]{\frac{bd}{a^2}} + b\sqrt[3]{\frac{ad}{b^2}} + c + \frac{d}{\sqrt[3]{\frac{bd}{a^2}}\sqrt[3]{\frac{ad}{b^2}}} \\ &= 3\sqrt[3]{abd} + c \leq 0. \end{aligned}$$

i.e.,

$$y \leq -x + z - 3K + 3\sqrt[3]{(-2K)(x+K)(z-K)}. \tag{A.113}$$

Case 8.2

Assume that the maximum value of $g(t, s)$ is achieved at $\bar{t} > \frac{1}{2}$, $\bar{s} = 1$.

From the optimality condition

$$\frac{\partial g(\bar{t}, \bar{s})}{\partial t}(t - \bar{t}) + \frac{\partial g(\bar{t}, \bar{s})}{\partial s}(s - \bar{s}) \leq 0,$$

let $s = 1$, since $t - \bar{t}$ can be either positive or negative, we have

$$\frac{\partial g(\bar{t}, \bar{s})}{\partial t} = 0.$$

Similarly, let $t = \bar{t}$, since $s - \bar{s} \geq 0$, we have

$$\frac{\partial g(\bar{t}, \bar{s})}{\partial s} \leq 0.$$

$$\frac{\partial g(\bar{t}, \bar{s})}{\partial t} = a - \frac{d}{\bar{t}^2} = 0 \Rightarrow \bar{t} = \sqrt{\frac{-d}{-a}}.$$

Furthermore,

$$\bar{t} > \frac{1}{2} \Rightarrow z > -K. \tag{A.114}$$

$$\frac{\partial g(\bar{t}, \bar{s})}{\partial s} = b - \frac{d}{\bar{t}} = b + \sqrt{ad} \leq 0$$

implies that

$$\begin{gathered} x \geq -K \\ x + K \geq \sqrt{(-2K)(z-K)} \\ z - k \geq -\frac{1}{2K}(x+K)^2. \end{gathered} \tag{A.115}$$

Finally,

$$\begin{aligned} & g(t, s) \\ &\leq g(\bar{t}, \bar{s}) \\ &= a\sqrt{\frac{-d}{-a}} + b + c + \frac{d}{\sqrt{\frac{-d}{-a}}} \\ &= -2\sqrt{ad} + b + c \leq 0 \end{aligned}$$

implies that

$$y \leq z - 2K + 2\sqrt{(-2K)(z-K)}. \tag{A.116}$$

Case 8.3

Assume that the maximum value of $g(t,s)$ is achieved at $\bar{t} = \frac{1}{2}$, $\bar{s} > 1$.

From the optimality condition

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t}(t-\bar{t}) + \frac{\partial g(\bar{t},\bar{s})}{\partial s}(s-\bar{s}) \le 0,$$

similar as the previous subsection, we have

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t} \le 0,$$
$$\frac{\partial g(\bar{t},\bar{s})}{\partial s} = 0.$$

$$\frac{\partial g(\bar{t},\bar{s})}{\partial s} = b - \frac{2d}{\bar{s}^2} = 0 \Rightarrow \bar{s} = \sqrt{\frac{-2d}{-b}}$$

Furthermore,

$$\bar{s} > 1 \Rightarrow x < K. \tag{A.117}$$

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t} = a - \frac{4d}{\bar{s}} = a + \sqrt{8bd} \le 0$$

implies that

$$z \le K$$
$$z - K + \sqrt{2K(x+K)} \le 0 \tag{A.118}$$
$$x + K \le \frac{1}{2K}(z-K)^2.$$

Finally,

$$\begin{aligned} & g(t,s) \\ & \le g(\bar{t},\bar{s}) \\ & = \frac{a}{2} + c - 2\sqrt{2bd} \le 0 \end{aligned}$$

implies that

$$y \le -x - 2K + 2\sqrt{2K(x+K)} \tag{A.119}$$

Case 8.4

Assume that the maximum value of $g(t,s)$ is achieved at $\bar{t} = \frac{1}{2}$, $\bar{s} = 1$.

From the optimality condition

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t}(t-\bar{t}) + \frac{\partial g(\bar{t},\bar{s})}{\partial s}(s-\bar{s}) \le 0,$$

similar as the previous subsection, we have

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t} \le 0,$$
$$\frac{\partial g(\bar{t},\bar{s})}{\partial s} \le 0.$$

$$\frac{\partial g(\bar{t},\bar{s})}{\partial t} = a - 4d \le 0$$

implies that

$$z \le -K. \tag{A.120}$$

$$\frac{\partial g(\bar{t},\bar{s})}{\partial s} = b - 2d \le 0$$

implies that

$$x \ge K. \tag{A.121}$$

Furthermore,

$$\begin{aligned} & g(t,s) \\ & \le g\left(\bar{t} = \frac{1}{2}, \bar{s} = 1\right) \\ & = \frac{a}{2} + b + c + 2d \\ & = y - K \le 0. \end{aligned}$$

implies that

$$y \le K. \tag{A.122}$$

However, (A.120) and (A.121) define a set that does not intersect the set defined by case 1 and case 2. Hence, we can discard case 8.4.

Discussion of case 8

The discussion is similar as case 3. Details are omitted.

Conclusion of case 8 (combine with case 1 and 2):

$$\begin{aligned} (x,y,z) \in \{ & |x| \le K, |y| \le K, |z| \le K, \\ & y \le -x + z - 3K + 3\sqrt[3]{(-2K)(x+K)(z-K)}, \\ & y \le -x - 2K + 2\sqrt{2K(x+K)}, \\ & y \le z - 2K + 2\sqrt{(-2K)(z-K)} \} \end{aligned}$$

A.3.9 Discussion of conjugate transform of $\left\|\frac{\partial^2 z}{\partial y^2}\right\|_1$

The coonjugate transform of $\left\|\frac{\partial^2 z}{\partial y^2}\right\|_1$ is defined in a convex set Ω, where the boundary of Ω is discussed in above 8 different cases. Case 1 and 2 define a cubic $|x| \le K, |y| \le K, |z| \le K$. Ω is a closed set inside this cubic. Hence, constraiints in case 1 and case 2 are redundant. Rest of constraints are in 6 groups. Each group consists 3 functions as discussed in case 3-8. In the next of this subsection, we show some of constraints are actually redundant.

Consider the constraints in case 6

$$\begin{aligned} & (1). z \ge x + y + 3K - 3\sqrt[3]{2K(x+K)(y+K)} \\ & (2). z \ge x + 2K - 2\sqrt{2K(x+K)} \\ & (3). z \ge y + 2K - 2\sqrt{2K(y+K)} \end{aligned}$$

Let Θ_i be the set in x-y space such that constraint i is active if consider case 6 only. For example,

$$\Theta_1 = \left\{ x + K \geq \frac{1}{2K}(y+K)^2, y + K \geq \frac{1}{2K}(x+K)^2 \right\}$$

$$\Theta_2 = \left\{ |x| \leq K, |y| \leq K, x + K \leq \frac{1}{2K}(y+K)^2 \right\}$$

$$\Theta_3 = \left\{ |x| \leq K, |y| \leq K, y + k \leq \frac{1}{2K}(x+K)^2 \right\}$$

Alsoconsider constraints in case 7

$$\begin{aligned}
&(4)\,.y \geq -x + z + 3K - 3\sqrt[3]{(-2K)(x-K)(z+K)} \\
&(5)\,.y \geq -x + 2K - 2\sqrt{(-2K)(x-K)} \\
&(6)\,.y \geq z + 2K - 2\sqrt{2K(z+K)}
\end{aligned}$$

and define Θ_4, Θ_5, Θ_6 by the same way. For example,

$$\Theta_6 = \left\{ |x| \leq K, |z| \leq K, z + k \leq \frac{1}{2K}(x-K)^2 \right\}$$

As discussed in case 6, in x-y space, Θ_1 and Θ_3 meet at the curve

$$y + K = \frac{1}{2K}(x+K)^2.$$

On this curve, function (1) and (3) reduce to

$$z = -2x + y$$

Hence, the boundary of (1) and (3) intersect at the curve

$$\begin{cases} x = x \\ y = \frac{1}{2K}(x+K)^2 - K \\ z = -2x + y \end{cases} \tag{A.123}$$

Similarly, the boundary of function (4) and (6) intersect at the curve

$$\begin{cases} x = x \\ z = \frac{1}{2K}(x-K)^2 - K \\ y = 2x + z \end{cases} \tag{A.124}$$

It is not difficult to show that (A.123) and (A.124) are equivalent, since from (A.123) we have

$$\begin{aligned}
z &= -2x + y \\
&= -2x + \frac{1}{2K}(x+K)^2 - K \\
&= \frac{1}{2K}\left[(x+K)^2 - 4Kx\right] - K \\
&= \frac{1}{2K}(x-K)^2 - K
\end{aligned}$$

i.e., the boundary of $(1),(3),(4),(6)$ intersect at the same curve.

If we consider case 6 only, then (3) is active on Θ_3, i.e.,

$$y+K\leq\frac{1}{2K}\left(x+K\right)^2.$$

On the boundary of case 6, z is decreasing about y. Hence, the projection of (3) over Θ_3 into x-z space becomes

$$z\geq\frac{1}{2K}\left(x-K\right)^2-K.$$

However, (6) is active when

$$z+K\leq\frac{1}{2K}\left(x-K\right)^2.$$

Therefore, (3) and (6) can not be active at the same time. Notice that as a function, (3) and (6) actually are the same. That means, the boundary of (3) over Θ_3 and (6) over Θ_6 are the two parts of the same curve partitioned by (A.123).

We have shown that (1) override (3) when (1) is active. So as (2). Therefore, (1) and (2) override (6). When (1) or (2) is active, (6) is satisfied automatically. When (3) is active, we just show that (6) can not be active at the same time. As a consequence, (6) is redundant.

The other thing need to check is that (1) and (4) are not the same function. It is easy to show, since when $x=y=0$, the value of z defined by (1) is $3\left(1-2^{\frac{1}{3}}\right)K$ and the value of z defined by (4) is $3\left(-2+3^{\frac{1}{2}}\right)K$.

Following the same logic, all quadratic constraints are redundant.

As a conclusion, the conjugate transform of $\left\|\frac{\partial^2 z}{\partial y^2}\right\|_1$ is defined in

$$\begin{aligned}
(x,y,z)\in\Big\{&x\leq -y+z-3K+3\sqrt[3]{(-2K)\left(y+K\right)\left(z-K\right)},\\
&x\geq -y+z+3K-3\sqrt[3]{(-2K)\left(y-K\right)\left(z+K\right)},\\
&y\leq -x+z-3K+3\sqrt[3]{(-2K)\left(x+K\right)\left(z-K\right)},\\
&y\geq -x+z+3K-3\sqrt[3]{(-2K)\left(x-K\right)\left(z+K\right)},\\
&z\leq x+y-3K+3\sqrt[3]{2K\left(x-K\right)\left(y-K\right)}\\
&z\geq x+y+3K-3\sqrt[3]{2K\left(x+K\right)\left(y+K\right)}\Big\}
\end{aligned}$$

where

$$\begin{cases}
x=\frac{\eta}{3h_j^y},\\
y=\frac{\xi}{2h_i^x}-\frac{\eta}{6h_j^y},\\
z=\frac{\xi}{2h_i^x}+\frac{\eta}{6h_j^y}-\frac{\gamma}{2},\\
K=\frac{1}{12}h_i^x h_j^y
\end{cases}$$

A.4 Conjugate transform of $\left\|\frac{\partial^2 z}{\partial x \partial y}\right\|_1$

$$\begin{aligned}\frac{\partial^2 z}{\partial x \partial y} &= 2c_{12}\left(y - y_j\right) + c_{11} \\ &= 2c_{12}h_j^y s + \left(c_{12}h_j^y + c_{11}\right), \quad s \in \left[-\frac{1}{2}, 0\right].\end{aligned}$$

Let

$$\begin{cases} A = c_{12}h_j^y, \\ B = c_{12}h_j^y + c_{11}, \end{cases}$$

then

$$\begin{cases} c_{12} = \frac{1}{h_j^y}A, \\ c_{11} = -A + B. \end{cases}$$

Let

$$\begin{aligned} f(s) &= 2c_{12}h_j^y s + \left(c_{12}h_j^y + c_{11}\right) \\ &= 2As + B, \quad s \in \left[-\frac{1}{2}, 0\right] \end{aligned}$$

which is a linear function of s. Let

$$\begin{aligned} &F\left(c_{12}, c_{11}\right) \\ &= \left\|\frac{\partial^2 z}{\partial x \partial y}\right\|_1 \\ &= \int_D \left|\frac{\partial^2 z}{\partial x \partial y}\right| dxdy \\ &= h_i^x h_j^y \int_{DI} \left|2c_{12}h_j^y s + \left(c_{12}h_j^y + c_{11}\right)\right| dtds \\ &= h_i^x h_j^y \int_{DI} \left|2As + B\right| dtds. \end{aligned}$$

A.4.1 Case 1

Case 1: $f(s) \geq 0, \forall s \in \left[-\frac{1}{2}, 0\right]$.

Since $f(s)$ is a linear function, this is the case that

$$\begin{aligned} f\left(-\frac{1}{2}\right) &= -A + B \geq 0, \\ f(0) &= B \geq 0, \end{aligned}$$

Therefore, this situation is equivalent to

$$B \geq A, B \geq 0.$$

i.e.,

$$\frac{A}{B} \leq 1, B \geq 0.$$

Hence,

$$
\begin{aligned}
&F\left(c_{12}, c_{11}\right) \\
&= h_i^x h_j^y \int_{DI} |2As + B|\, dtds \\
&= h_i^x h_j^y \int_{-\frac{1}{2}}^{0} \int_{-\frac{1}{2}}^{t} [2As + B]\, dsdt + h_i^x h_j^y \int_{0}^{-\frac{1}{2}} \int_{-\frac{1}{2}}^{-t} [2As + B]\, dsdt \\
&= h_i^x h_j^y \left(-\frac{A}{6} + \frac{B}{4}\right).
\end{aligned} \tag{A.125}
$$

In this situation, the conjugate transform of $F\left(c_{12}, c_{11}\right)$ is

$$
\begin{aligned}
&G\left(\xi, \eta\right) \\
&= \sup\left\{\xi c_{12} + \eta c_{11} - F\left(c_{12}, c_{11}\right)\right\} \\
&= \sup_{B \geq A, B \geq 0} \left\{\xi \frac{1}{h_j^y} A + \eta\left(-A + B\right) - h_i^x h_j^y \left(-\frac{A}{6} + \frac{B}{4}\right)\right\} \\
&= \sup_{B \geq A, B \geq 0} \left\{\left(\frac{\xi}{h_j^y} - \eta + \frac{h_i^x h_j^y}{6}\right) A + \left(\eta - \frac{h_i^x h_j^y}{4}\right) B\right\}.
\end{aligned}
$$

Let

$$
s = \frac{A}{B}, s \leq 1,
$$

then

$$
\begin{aligned}
&\left(\frac{\xi}{h_j^y} - \eta + \frac{h_i^x h_j^y}{6}\right) A + \left(\eta - \frac{h_i^x h_j^y}{4}\right) B \\
&= \left(\frac{\xi}{h_j^y} - \eta + \frac{h_i^x h_j^y}{6}\right) sB + \left(\eta - \frac{h_i^x h_j^y}{4}\right) B \\
&= \left[\left(\frac{\xi}{h_j^y} - \eta + \frac{h_i^x h_j^y}{6}\right) s + \left(\eta - \frac{h_i^x h_j^y}{4}\right)\right] B \\
&= h\left(s\right) B, s \leq 1
\end{aligned}
$$

where $h\left(s\right)$ is a linear function. Since $B \geq 0$, $G\left(\xi, \eta\right)$ is finite if and only if $h\left(s\right) \leq 0$, which is equivalent to $h\left(s\right)$ is nondecreasing and $h\left(1\right) \leq 0$. Hence,

$$
\left(\frac{\xi}{h_j^y} - \eta + \frac{h_i^x h_j^y}{6}\right) \geq 0
$$

implies

$$
\eta \leq \frac{1}{h_j^y}\left(\xi + \frac{h_i^x h_j^{y2}}{6}\right).
$$

And

$$\begin{aligned}
&h(1) \\
&= \left(\frac{\xi}{h_j^y} - \eta + \frac{h_i^x h_j^y}{6}\right) + \left(\eta - \frac{h_i^x h_j^y}{4}\right) \\
&= \frac{\xi}{h_j^y} - \frac{h_i^x h_j^y}{12} \leq 0
\end{aligned}$$

implies

$$\xi \leq \frac{h_i^x h_j^{y2}}{12}.$$

Therefore, in Case 1, the conjugate transform of $F(c_{12}, c_{11})$ is

$$G(\xi, \eta) = 0,$$

where

$$(\xi, \eta) \in \Omega_1 = \left\{ \xi \leq \frac{h_i^x h_j^{y2}}{12}, \eta \leq \frac{1}{h_j^y}\left(\xi + \frac{h_i^x h_j^{y2}}{6}\right) \right\}. \tag{A.126}$$

A.4.2 Case 2

Case 2: $f(s) \leq 0, \forall s \in \left[-\frac{1}{2}, 0\right]$.

This case is symmatric with case 1. Since $f(s)$ is a linear function, this is the case that

$$\begin{aligned}
f\left(-\frac{1}{2}\right) &= -A + B \leq 0, \\
f(0) &= B \leq 0,
\end{aligned}$$

Therefore, this situation is equivalent to

$$B \leq A, B \leq 0.$$

i.e.,

$$-1 \leq -\frac{A}{B}, B \leq 0.$$

Hence,

$$\begin{aligned}
&F(c_{12}, c_{11}) \\
&= h_i^x h_j^y \int_{DI} |2As + B| \, dtds \\
&= -h_i^x h_j^y \left(-\frac{A}{6} + \frac{B}{4}\right).
\end{aligned} \tag{A.127}$$

In this situation, the conjugate transform of $F\left(c_{12}, c_{11}\right)$ is

$$
\begin{aligned}
& G(\xi, \eta) \\
& =\sup \left\{\xi c_{12}+\eta c_{11}-F\left(c_{12}, c_{11}\right)\right\} \\
& =\sup_{B \leq A, B \leq 0}\left\{\xi \frac{1}{h_j^y} A+\eta(-A+B)+h_i^x h_j^y\left(-\frac{A}{6}+\frac{B}{4}\right)\right\} \\
& =\sup_{B \leq A, B \leq 0}\left\{\left(\frac{\xi}{h_j^y}-\eta-\frac{h_i^x h_j^y}{6}\right) A+\left(\eta+\frac{h_i^x h_j^y}{4}\right) B\right\} .
\end{aligned}
$$

Let

$$
s=-\frac{A}{B}, s \geq-1,
$$

then

$$
\begin{aligned}
& \left(\frac{\xi}{h_j^y}-\eta-\frac{h_i^x h_j^y}{6}\right) A+\left(\eta+\frac{h_i^x h_j^y}{4}\right) B \\
& =\left(-\frac{\xi}{h_j^y}+\eta+\frac{h_i^x h_j^y}{6}\right) s B+\left(\eta+\frac{h_i^x h_j^y}{4}\right) B \\
& =\left[\left(-\frac{\xi}{h_j^y}+\eta+\frac{h_i^x h_j^y}{6}\right) s+\left(\eta+\frac{h_i^x h_j^y}{4}\right)\right] B \\
& =h(s) B, s \geq-1
\end{aligned}
$$

where $h(s)$ is a linear function. Since $B \leq 0$, $G(\xi, \eta)$ is finite if and only if $h(s) \geq 0$, which is equivalent to $h(s)$ is nondecreasing and $h(-1) \geq 0$. Hence,

$$
\begin{aligned}
& h(-1) \\
& =\left(\frac{\xi}{h_j^y}-\eta-\frac{h_i^x h_j^y}{6}\right)+\left(\eta+\frac{h_i^x h_j^y}{4}\right) \\
& =\frac{\xi}{h_j^y}+\frac{h_i^x h_j^y}{12} \geq 0
\end{aligned}
$$

implies

$$
\xi \geq-\frac{1}{12} h_i^x h_j^{y 2} .
$$

And

$$
\left(-\frac{\xi}{h_j^y}+\eta+\frac{h_i^x h_j^y}{6}\right) \geq 0
$$

implies

$$
\eta \geq \frac{1}{h_j^y}\left(\xi-\frac{1}{6} h_i^x h_j^{y 2}\right) .
$$

Therefore, in Case 2, the conjugate transform of $F\left(c_{12}, c_{11}\right)$ is

$$
G(\xi, \eta)=0,
$$

where

$$(\xi, \eta) \in \Omega_2 = \left\{ \xi \geq -\frac{1}{12} h_i^x h_j^{y2}, \eta \geq \frac{1}{h_j^y} \left(\xi - \frac{1}{6} h_i^x h_j^{y2} \right) \right\}. \tag{A.128}$$

A.4.3 Case 3

Case 3: $f\left(-\frac{1}{2}\right) < 0$ and $f(0) > 0$.
this is the case that

$$f\left(-\frac{1}{2}\right) = -A + B < 0,$$
$$f(0) = B > 0,$$

whihch is equivalent to

$$A > B > 0.$$

Let $\hat{s}$ be the root of $f(s)$, i.e.,

$$f(\hat{s}) = 0, \hat{s} \in \left[-\frac{1}{2}, 0\right].$$

This is equivalent to

$$-\frac{1}{2} < \hat{s} = -\frac{B}{2A} < 0, A > B > 0.$$

$$\begin{aligned}
&F(c_{12}, c_{11}) = \left\| \frac{\partial^2 z}{\partial x \partial y} \right\|_1 \\
&= h_i^x h_j^y \left\{ \int_{-\frac{1}{2}}^{\hat{s}} \int_s^{-s} (-2As - B)\, dt ds + \int_{\hat{s}}^0 \int_s^{-s} (2As + B)\, dt ds \right\} \\
&= h_i^x h_j^y \left\{ \frac{8}{3} A\hat{s}^3 + 2B\hat{s}^2 + \frac{A}{6} - \frac{B}{4} \right\} \\
&= h_i^x h_j^y \left\{ \frac{1}{6} \frac{B^3}{A^2} + \frac{A}{6} - \frac{B}{4} \right\}.
\end{aligned}$$

Let

$$s = \frac{B}{A}, 0 < s < 1,$$

In this situation, the conjugate transform of $F(c_{12}, c_{11})$ is

$$\begin{aligned}
&G(\xi,\eta)\\
&=\sup\{\xi c_{12}+\eta c_{11}-F(c_{12},c_{11})\}\\
&=\sup_{A>B>0}\left\{\xi\frac{1}{h_j^y}A+\eta(-A+B)-h_i^x h_j^y\left[\frac{1}{6}\frac{B^3}{A^2}+\frac{A}{6}-\frac{B}{4}\right]\right\}\\
&=\sup_{0<s<1,A>0}\left\{\xi\frac{1}{h_j^y}A+\eta(-A+sA)-h_i^x h_j^y\left[\frac{1}{6}s^3A-\frac{sA}{4}+\frac{A}{6}\right]\right\}\\
&=\sup_{0<s<1,A>0}\left\{-\frac{h_i^x h_j^y}{6}s^3+\left(\eta+\frac{h_i^x h_j^y}{4}\right)s+\left(\frac{\xi}{h_j^y}-\frac{h_i^x h_j^y}{6}-\eta\right)\right\}A\\
&=\sup_{0<s<1,A>0}-\frac{h_i^x h_j^y}{6}\left\{s^3-\left(\frac{6\eta}{h_i^x h_j^y}+\frac{3}{2}\right)s+\left(-\frac{6\xi}{h_i^x h_j^{y2}}+1+\frac{6\eta}{h_i^x h_j^y}\right)\right\}A\\
&=\sup_{0<s<1,A>0}\left\{-\frac{h_i^x h_j^y}{6}g(s)A\right\}
\end{aligned}$$

Since $A>0$, $G(\xi,\eta)$ is finite if and only if

$$g(s)=s^3+\left(-\frac{6\eta}{h_i^x h_j^y}-\frac{3}{2}\right)s+\left(-\frac{6\xi}{h_i^x h_j^{y2}}+\frac{6\eta}{h_i^x h_j^y}+1\right)\geq 0, s\in(0,1).$$

$$\begin{aligned}
g'(s)&=3s^2+\left(-\frac{6\eta}{h_i^x h_j^y}-\frac{3}{2}\right)\\
&=3\left[s^2+\left(-\frac{2\eta}{h_i^x h_j^y}-\frac{1}{2}\right)\right]
\end{aligned}$$

Case 1 of $g(s)$: If $-\frac{2\eta}{h_i^x h_j^y}-\frac{1}{2}\geq 0$, i.e., $\eta\leq-\frac{h_i^x h_j^y}{4}$, then $g(s)$ is monotomically increasing, and $g(s)\geq 0, s\in(0,1)$, if and only if

$$g(0)=-\frac{6\xi}{h_i^x h_j^{y2}}+1+\frac{6\eta}{h_i^x h_j^y}\geq 0,$$

i.e.,

$$\eta\geq\frac{1}{h_j^y}\left(\xi-\frac{h_i^x h_j^{y2}}{6}\right).$$

Furthermore,

$$-\frac{h_i^x h_j^y}{4}\geq\eta\geq\frac{1}{h_j^y}\left(\xi-\frac{h_i^x h_j^{y2}}{6}\right)$$

implies that

$$\xi\leq-\frac{1}{12}h_i^x h_j^{y2}$$

Therefore, in this case, $G(\xi,\eta)$ is finite if and only if

$$\xi \leq -\frac{1}{12}h_i^x h_j^{y2}, \eta \leq -\frac{h_i^x h_j^y}{4}, \eta \geq \frac{1}{h_j^y}\left(\xi - \frac{h_i^x h_j^{y2}}{6}\right). \tag{A.129}$$

Case 2 of $g(s)$: If $-\frac{2\eta}{h_i^x h_j^y} - \frac{1}{2} < 0$, i.e., $\eta > -\frac{h_i^x h_j^y}{4}$, then there exists a s^* such that $g'(s^*) = 0$. We have

$$s^* = \sqrt{\frac{2\eta}{h_i^x h_j^y} + \frac{1}{2}} > 0.$$

Now we consider the situation that

$$0 \leq s^* < 1.$$

Hence, $g(s^*)$ is the mimnimum value of $g(s)$. $g(s) \geq 0$ if and only if $g(s^*) \geq 0$.

$$0 \leq s^* < 1 \Longrightarrow -\frac{h_i^x h_j^y}{4} \leq \eta < \frac{h_i^x h_j^y}{4}.$$

$$\begin{aligned}
&g(s^*) \\
&= \left(\frac{2\eta}{h_i^x h_j^y + \frac{1}{2}}\right)^{\frac{3}{2}} + \left(-\frac{6\eta}{h_i^x h_j^y} - \frac{3}{2}\right)\left(\frac{2\eta}{h_i^x h_j^y} + \frac{1}{2}\right)^{\frac{1}{2}} + \left(-\frac{6\xi}{h_i^x h_j^{y2}} + 1 + \frac{6\eta}{h_i^x h_j^y}\right) \\
&= -2\left(\frac{2\eta}{h_i^x h_j^y} + \frac{1}{2}\right)^{\frac{3}{2}} + \left(-\frac{6\xi}{h_i^x h_j^{y2}} + 1 + \frac{6\eta}{h_i^x h_j^y}\right)
\end{aligned}$$

Hence, $g(s^*) \geq 0$ implies

$$\begin{aligned}
-2\left(\frac{2\eta}{h_i^x h_j^y} + \frac{1}{2}\right)^{\frac{3}{2}} &\geq \frac{6\xi}{h_i^x h_j^{y2}} - 1 - \frac{6\eta}{h_i^x h_j^y} \\
\left(\frac{2\eta}{h_i^x h_j^y} + \frac{1}{2}\right)^{3} &\leq \left(\frac{3\xi}{h_i^x h_j^{y2}} - \frac{1}{2} - \frac{3\eta}{h_i^x h_j^y}\right)^2 \\
\xi &\leq h_j^y \eta - \frac{h_i^x h_j^{y2}}{3}\left(\frac{2\eta}{h_i^x h_j^y} + \frac{1}{2}\right)^{\frac{3}{2}} + \frac{h_i^x h_j^{y2}}{6}.
\end{aligned}$$

In summary, case 2 is the situation such that

$$\begin{aligned}
-\frac{2\eta}{h_i^x h_j^y} - \frac{1}{2} < 0 \\
0 \leq s^* < 1 \\
g(s^*) \geq 0
\end{aligned}$$

i.e.,

$$-\frac{h_i^x h_j^y}{4} \leq \eta < \frac{h_i^x h_j^y}{4}, \xi \leq h_j^y \eta - \frac{h_i^x h_j^{y2}}{3}\left(\frac{2\eta}{h_i^x h_j^y} + \frac{1}{2}\right)^{\frac{3}{2}} + \frac{h_i^x h_j^{y2}}{6}$$

Property: From function

$$\xi = h_j^y\eta - \frac{h_i^x h_j^{y2}}{3}\left(\frac{2\eta}{h_i^x h_j^y} + \frac{1}{2}\right)^{\frac{3}{2}} + \frac{h_i^x h_j^{y2}}{6}$$

we can derive that

$$\frac{d\xi}{d\eta} = h_j^y - h_j^y\left(\frac{2\eta}{h_i^x h_j^y} + \frac{1}{2}\right)^{\frac{1}{2}} > h_j^y - h_j^y = 0$$

$$\frac{d^2\xi}{d\eta^2} = -\frac{1}{h_i^x}\left(\frac{2\eta}{h_i^x h_j^y} + \frac{1}{2}\right)^{-\frac{1}{2}} < 0$$

Since this function is monotonically increaeing about η, so

$$\xi \le h_j^y\eta - \frac{h_i^x h_j^{y2}}{3}\left(\frac{2\eta}{h_i^x h_j^y} + \frac{1}{2}\right)^{\frac{3}{2}} + \frac{h_i^x h_j^{y2}}{6}\Bigg|_{\eta=\frac{h_i^x h_j^y}{4}} = \frac{1}{12}h_i^x h_j^{y2}.$$

So case 2 is the situation that

$$\xi \le \frac{1}{12}h_i^x h_j^{y2}, -\frac{h_i^x h_j^y}{4} \le \eta < \frac{h_i^x h_j^y}{4}, \xi \le h_j^y\eta - \frac{h_i^x h_j^{y2}}{3}\left(\frac{2\eta}{h_i^x h_j^y} + \frac{1}{2}\right)^{\frac{3}{2}} + \frac{h_i^x h_j^{y2}}{6}, \qquad \text{(A.130)}$$

which defines a convex set.

Case 3 of $g(s)$: If $-\frac{2\eta}{h_i^x h_j^y} - \frac{1}{2} < 0$ and $s^* \ge 1$, then $g(s)$ is monotonically decreasing in $(0,1)$. Hence, $g(s) \ge 0$ if and only if $g(1) \ge 0$.

$$\begin{aligned}
&g(1)\\
&= 1 + \left(-\frac{6\eta}{h_i^x h_j^y} - \frac{3}{2}\right) + \left(-\frac{6\xi}{h_i^x h_j^{y2}} + 1 + \frac{6\eta}{h_i^x h_j^y}\right)\\
&= -\frac{6\xi}{h_i^x h_j^{y2}} + \frac{1}{2}
\end{aligned}$$

$g(1) \ge 0$ implies that

$$\xi \le \frac{h_i^x h_j^{y2}}{12}.$$

$$s^* = \sqrt{\frac{2\eta}{h_i^x h_j^y} + \frac{1}{2}} \ge 1$$

implies

$$\eta \ge \frac{h_i^x h_j^y}{4}.$$

In summary, case 3 is the situation that

$$\begin{aligned}
-\frac{2\eta}{h_i^x h_j^y} - \frac{1}{2} &< 0\\
s^* &\ge 1\\
g(1) &\ge 0
\end{aligned}$$

i.e.,

$$\xi \leq \frac{1}{12} h_i^x h_j^{y2}, \eta \geq \frac{1}{4} h_i^x h_j^y, \tag{A.131}$$

Conclusion: in the situation of $A > B > 0$, the conojugate transform of $F(c_{12}, c_{11})$ is

$$G(\xi, \eta) = 0,$$

where (ξ, η) is defined over the union of sets (A.129), (A.130), and (A.131), i.e.,

$$\Omega_3 = \left\{ \begin{array}{c} \xi \leq -\frac{1}{12} h_i^x h_j^{y2}, \eta \geq \frac{1}{h_j^y}\left(\xi - \frac{1}{6} h_i^x h_j^{y2}\right) \text{ if } \eta \leq -\frac{h_i^x h_j^y}{4}, \\ \xi \leq \frac{1}{12} h_i^x h_j^{y2}, \xi \leq h_j^y \eta - \frac{1}{3} h_i^x h_j^{y2}\left(\frac{2\eta}{h_i^x h_j^y} + \frac{1}{2}\right)^{\frac{3}{2}} + \frac{h_i^x h_j^{y2}}{6} \text{ if } -\frac{h_i^x h_j^y}{4} \leq \eta < \frac{h_i^x h_j^y}{4}, \\ \xi \leq \frac{1}{12} h_i^x h_j^y \text{ if } \eta \geq \frac{h_i^x h_j^y}{4}. \end{array} \right\}. \tag{A.132}$$

This is a convex set. The boundary of this set is C^1-smooth.

A.4.4 Case 4

Case 4: $f\left(-\frac{1}{2}\right) > 0$ and $f(0) < 0$

This is the case that

$$f\left(-\frac{1}{2}\right) = -A + B > 0,$$
$$f(0) = B < 0,$$

which is equivalent to

$$A < B < 0.$$

Let $\hat{s}$ be the root of $f(s)$, i.e.,

$$f(\hat{s}) = 0, \hat{s} \in \left[-\frac{1}{2}, 0\right].$$

This is equivalent to

$$-\frac{1}{2} < \hat{s} = -\frac{B}{2A} < 0, A < 0, B < 0.$$

$$\begin{aligned} F(c_{12}, c_{11}) &= \left\| \frac{\partial^2 z}{\partial x \partial y} \right\|_1 \\ &= h_i^x h_j^y \left\{ \int_{-\frac{1}{2}}^{\hat{s}} \int_s^{-s} (2As + B)\, dt ds + \int_{\hat{s}}^0 \int_s^{-s} (-2As - B)\, dt ds \right\} \\ &= h_i^x h_j^y \left\{ -\frac{1}{6} \frac{B^3}{A^2} - \frac{A}{6} + \frac{B}{4} \right\}. \end{aligned}$$

Let

$$s = \frac{B}{A}, 0 < s < 1,$$

In this situation, the conjugate transform of $F(c_{12}, c_{11})$ is

$$\begin{aligned}
& G(\xi, \eta) \\
&= \sup \{\xi c_{12} + \eta c_{11} - F(c_{12}, c_{11})\} \\
&= \sup_{A<B<0} \left\{ \xi \frac{1}{h_j^y} A + \eta(-A+B) - h_i^x h_j^y \left[-\frac{1}{6}\frac{B^3}{A^2} - \frac{A}{6} + \frac{B}{4} \right] \right\} \\
&= \sup_{0<s<1, A<0} \left\{ \xi \frac{1}{h_j^y} A + \eta(-A+sA) - h_i^x h_j^y \left[-\frac{1}{6} s^3 A + \frac{A}{4} s - \frac{A}{6} \right] \right\} \\
&= \sup_{0<s<1, A<0} \left\{ \frac{h_i^x h_j^y}{6} s^3 + \left(\eta - \frac{h_i^x h_j^y}{4} \right) s + \left(\frac{\xi}{h_i^x} - \eta + \frac{h_i^x h_j^y}{6} \right) \right\} A \\
&= \sup_{0<s<1, A<0} \frac{h_i^x h_j^y}{6} \left\{ s^3 + \left(\frac{6\eta}{h_i^x h_j^y} - \frac{3}{2} \right) s + \left(\frac{6\xi}{h_i^x h_j^{y2}} - \frac{6\eta}{h_i^x h_j^y} + 1 \right) \right\} A \\
&= \sup_{0<s<1, A<0} \left\{ \frac{h_i^x h_j^y}{6} g(s) A \right\}
\end{aligned}$$

Since $A < 0$, $G(\xi, \eta)$ is finite if and only if

$$g(s) = s^3 + \left(\frac{6\eta}{h_i^x h_j^y} - \frac{3}{2} \right) s + \left(\frac{6\xi}{h_i^x h_j^{y2}} - \frac{6\eta}{h_i^x h_j^y} + 1 \right) \geq 0, s \in (0,1).$$

$$\begin{aligned}
g'(s) &= 3s^2 + \left(\frac{6\eta}{h_i^x h_j^y} - \frac{3}{2} \right) \\
&= 3 \left[s^2 + \left(\frac{2\eta}{h_i^x h_j^y} - \frac{1}{2} \right) \right]
\end{aligned}$$

Case 1 of $g(s)$: If $\left(\frac{2\eta}{h_i^x h_j^y} - \frac{1}{2} \right) \geq 0$, i.e., $\eta \geq \frac{h_i^x h_j^y}{4}$, then $g(s)$ is monotomically increasing, and $g(s) \geq 0, s \in (0,1)$, if and only if

$$g(0) = \frac{6\xi}{h_i^x h_j^{y2}} - \frac{6\eta}{h_i^x h_j^y} + 1 \geq 0,$$

i.e.,

$$\eta \leq \frac{1}{h_j^y} \left(\xi + \frac{h_i^x h_j^{y2}}{6} \right).$$

And

$$\frac{1}{4} h_i^x h_j^y \leq \eta \leq \frac{1}{h_j^y} \left(\xi + \frac{h_i^x h_j^{y2}}{6} \right)$$

implies

$$\xi \geq \frac{1}{12} h_i^x h_j^{y2}.$$

Therefore, in this case, $G(\xi,\eta)$ is finite if and only if

$$\xi \geq \frac{1}{12} h_i^x h_j^{y2}, \eta \geq \frac{h_i^x h_j^y}{4}, \eta \leq \frac{1}{h_j^y}\left(\xi + \frac{h_i^x h_j^{y2}}{6}\right). \tag{A.133}$$

Case 2 of $g(s)$: If $\left(\frac{2\eta}{h_i^x h_j^y} - \frac{1}{2}\right) < 0$, i.e., $\eta < \frac{h_i^x h_j^y}{4}$, then there exists a s^* such that $g'(s^*) = 0$. We have

$$s^* = \sqrt{\frac{1}{2} - \frac{2\eta}{h_i^x h_j^y}} > 0.$$

Now we consider the situation that

$$0 \leq s^* < 1.$$

Hence, $g(s^*)$ is the minimum value of $g(s)$. $g(s) \geq 0$ if and only if $g(s^*) \geq 0$.

$$0 \leq s^* < 1 \Longrightarrow -\frac{h_i^x h_j^y}{4} < \eta \leq \frac{h_i^x h_j^y}{4}.$$

$$\begin{aligned}
& g(s^*) \\
&= \left(\frac{1}{2} - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{3}{2}} + \left(\frac{6\eta}{h_i^x h_j^y} - \frac{3}{2}\right)\left(\frac{1}{2} - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{1}{2}} + \left(\frac{6\xi}{h_i^x h_j^{y2}} - \frac{6\eta}{h_i^x h_j^y} + 1\right) \\
&= -2\left(\frac{1}{2} - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{3}{2}} + \left(\frac{6\xi}{h_i^x h_j^{y2}} - \frac{6\eta}{h_i^x h_j^y} + 1\right)
\end{aligned}$$

Hence, $g(s^*) \geq 0$ implies

$$\begin{aligned}
\left(\frac{1}{2} - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{3}{2}} &\leq \frac{3\xi}{h_i^x h_j^{y2}} - \frac{3\eta}{h_i^x h_j^y} + \frac{1}{2} \\
\left(\frac{1}{2} - \frac{2\eta}{h_i^x h_j^y}\right)^{3} &\leq \left(-\frac{3\xi}{h_i^x h_j^{y2}} + \frac{3\eta}{h_i^x h_j^y} - \frac{1}{2}\right)^2 \\
\xi &\geq h_j^y \eta + \frac{1}{3} h_i^x h_j^{y2}\left(\frac{1}{2} - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{3}{2}} - \frac{1}{6} h_i^x h_j^{y2}.
\end{aligned}$$

Property: From function

$$\xi = h_j^y \eta + \frac{1}{3} h_i^x h_j^{y2}\left(\frac{1}{2} - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{2}{3}} - \frac{1}{6} h_i^x h_j^{y2}$$

we can derive that

$$\frac{d\xi}{d\eta} = h_j^y - h_j^y \left(\frac{1}{2} - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{1}{2}} \geq h_j^y - h_j^y = 0$$

$$\frac{d^2\xi}{d\eta^2} = \frac{1}{h_i^x}\left(\frac{1}{2} - \frac{2\eta}{h_i^x h_j^y}\right)^{-\frac{1}{2}} \geq 0$$

Since ξ is monotonically increasing about η,

$$\xi \geq h_j^y \eta + \frac{1}{3} h_i^x h_j^{y2}\left(\frac{1}{2} - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{3}{2}} - \frac{1}{6} h_i^x h_j^y \Bigg|_{\eta = -\frac{1}{4} h_i^x h_j^y} = -\frac{1}{12} h_i^x h_j^{y2}$$

In summary, case 2 is the situation such that

$$\frac{2\eta}{h_i^x h_j^y} - \frac{1}{2} < 0$$
$$0 \leq s^* < 1$$
$$g(s^*) \geq 0$$

i.e.,

$$-\frac{h_i^x h_j^y}{4} < \eta \leq \frac{h_i^x h_j^y}{4}, \xi \geq -\frac{1}{12} h_i^x h_j^{y2}, \xi \geq h_j^y \eta + \frac{1}{3} h_i^x h_j^{y2}\left(\frac{1}{2} - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{3}{2}} - \frac{1}{6} h_i^x h_j^{y2}, \quad \text{(A.134)}$$

which defines a convex set.

Case 3 of $g(s)$: If $\left(\frac{2\eta}{h_i^x h_j^y} - \frac{1}{2}\right) < 0$ and $s^* \geq 1$, then $g(s)$ is monotonically decreasing in $(0, 1)$. Hence, $g(s) \geq 0$ if and only if $g(1) \geq 0$.

$$\begin{aligned} & g(1) \\ & = 1 + \frac{6\eta}{h_i^x h_j^y} - \frac{3}{2} + \frac{6\xi}{h_i^x h_j^{y2}} - \frac{6\eta}{h_i^x h_j^y} + 1 \\ & = \frac{6\xi}{h_i^x h_j^{y2}} + \frac{1}{2} \end{aligned}$$

$g(1) \geq 0$ implies that

$$\xi \geq -\frac{1}{12} h_i^x h_j^y.$$

$$s^* = \sqrt{\frac{1}{2} - \frac{2\eta}{h_i^x h_j^y}} \geq 1$$

implies

$$\eta \leq -\frac{1}{4} h_i^x h_j^y.$$

In summary, case 3 is the situation that

$$\left(\frac{2\eta}{h_i^x h_j^y} - \frac{1}{2}\right) < 0$$
$$s^* \geq 1$$
$$g(1) \geq 0$$

i.e.,

$$\xi \geq -\frac{1}{12}h_i^x h_j^y, \eta \leq -\frac{1}{4}h_i^x h_j^y \tag{A.135}$$

Conclusion: in the situation of $A < B < 0$, the conojugate transform of $F\left(c_{12}, c_{11}\right)$ is

$$G\left(\xi, \eta\right) = 0,$$

where (ξ, η) is defined over the union of sets (A.133), (A.134), and (A.135), i.e.,

$$\Omega_4 = \left\{ \begin{array}{c} \xi \geq \frac{1}{12}h_i^x h_j^{y2}, \eta \leq \frac{1}{h_j^y}\left(\xi + \frac{h_i^x h_j^{y2}}{6}\right) \text{ if } \eta \geq \frac{h_i^x h_j^y}{4}, \\ \xi \geq -\frac{1}{12}h_i^x h_{j2}^y, \xi \geq h_j^y \eta + \frac{1}{3}h_i^x h_j^{y2}\left(\frac{1}{2} - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{3}{2}} - \frac{1}{6}h_i^x h_j^{y2} \text{ if } -\frac{h_i^x h_j^y}{4} < \eta \leq \frac{h_i^x h_j^y}{4}, \\ \xi \geq -\frac{1}{12}h_i^x h_j^{y2} \text{ if } \eta \leq -\frac{1}{4}h_i^x h_j^y. \end{array} \right\}. \tag{A.136}$$

This is a convex set. The boundary of this set is C^1-smooth.

A.4.5 Conclusion of conjugate transform of $\left\|\frac{\partial^2 z}{\partial x \partial y}\right\|_1$

The functions from Case 1 and Case 2

$$\begin{array}{l} \eta \leq \frac{1}{h_j^y}\left(\xi + \frac{1}{6}h_i^x h_j^{y2}\right) \\ \eta \geq \frac{1}{h_j^y}\left(\xi - \frac{1}{6}h_i^x h_j^{y2}\right) \end{array}$$

can be re-stated as

$$\begin{array}{l} \xi \geq h_j^y \eta - \frac{h_i^x h_j^{y2}}{6} \\ \xi \leq h_j^y \eta + \frac{h_i^x h_j^{y2}}{6} \end{array}$$

Consider four functions

$$f_1\left(\eta\right) = h_j^y \eta - \frac{1}{3}h_i^x h_j^{y2}\left(\frac{2\eta}{h_i^x h_j^y} + \frac{1}{2}\right)^{\frac{3}{2}} + \frac{h_i^x h_j^{y2}}{6} \tag{A.137}$$

$$f_2\left(\eta\right) = h_j^y \eta + \frac{1}{3}h_i^x h_j^{y2}\left(\frac{1}{2} - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{3}{2}} - \frac{h_i^x h_j^{y2}}{6} \tag{A.138}$$

$$g_1\left(\eta\right) = h_j^y \eta + \frac{h_i^x h_j^{y2}}{6} \tag{A.139}$$

and

$$g_2\left(\eta\right) = h_j^y \eta - \frac{h_i^x h_j^{y2}}{6} \tag{A.140}$$

defined over the set $\eta \in \left[-\frac{h_i^x h_j^y}{4}, \frac{h_i^x h_j^y}{4}\right]$.

Let

$$H_1(\eta) = f_1(\eta) - g_1(\eta)$$
$$= -\frac{1}{3}h_i^x h_j^{y2}\left(\frac{2\eta}{h_i^x h_j^y} + \frac{1}{2}\right)^{\frac{3}{2}}, \eta \in \left[-\frac{h_i^x h_j^y}{4}, \frac{h_i^x h_j^y}{4}\right].$$

$$H_1'(\eta) = -h_j^y\left(\frac{2\eta}{h_i^x h_j^y} + \frac{1}{2}\right)^{\frac{1}{2}} \leq 0, \eta \in \left[-\frac{h_i^x h_j^y}{4}, \frac{h_i^x h_j^y}{4}\right].$$

Since $H_1(\eta)$ is monotonically decreasing while $\eta \in \left[-\frac{h_i^x h_j^y}{4}, \frac{h_i^x h_j^y}{4}\right]$, the maximum value of $H_1(\eta)$ while $\eta \in \left[-\frac{h_i^x h_j^y}{4}, \frac{h_i^x h_j^y}{4}\right]$ is $H_1\left(-\frac{h_i^x h_j^y}{4}\right) = 0$. i.e.,

$$f_1(\eta) \leq g_1(\eta), -\frac{h_i^x h_j^y}{4} \leq \eta \leq -\frac{h_i^x h_j^y}{4}.$$

The constraint $\xi \leq g_1(\eta)$ is automatically satisfied when $\xi \leq f_1(\eta)$. In other words, If

$$\xi \leq h_j^y \eta - \frac{1}{3}h_i^x h_j^{y2}\left(\frac{2\eta}{h_i^x h_j^y} + \frac{1}{2}\right)^{\frac{3}{2}} + \frac{h_i^x h_j^{y2}}{6}$$

and

$$-\frac{h_i^x h_j^y}{4} \leq \eta \leq -\frac{h_i^x h_j^y}{4},$$

then automatically,

$$\eta \geq \frac{1}{h_j^y}\left(\xi - \frac{1}{6}h_i^x h_j^{y2}\right).$$

Let

$$H_2(\eta) = f_2(\eta) - g_2(\eta)$$
$$= \frac{1}{3}h_i^x h_j^{y2}\left(\frac{1}{2} - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{3}{2}}, \eta \in \left[-\frac{h_i^x h_j^y}{4}, \frac{h_i^x h_j^y}{4}\right].$$

$$H_2'(\eta) = -h_j^y\left(\frac{1}{2} - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{1}{2}} \leq 0, \eta \in \left[-\frac{h_i^x h_j^y}{4}, \frac{h_i^x h_j^y}{4}\right].$$

Since $H_2(\eta)$ is monotonically decreasing while $\eta \in \left[-\frac{h_i^x h_j^y}{4}, \frac{h_i^x h_j^y}{4}\right]$, the minimum value of $H_2(\eta)$ while $\eta \in \left[-\frac{h_i^x h_j^y}{4}, \frac{h_i^x h_j^y}{4}\right]$ is $H_2\left(\frac{h_i^x h_j^y}{4}\right) = 0$. i.e.,

$$f_2(\eta) \geq g_2(\eta), -\frac{h_i^x h_j^y}{4} \leq \eta \leq -\frac{h_i^x h_j^y}{4}.$$

The constraint $\xi \geq g_2(\eta)$ is automatically satisfied when $\xi \geq f_2(\eta)$.

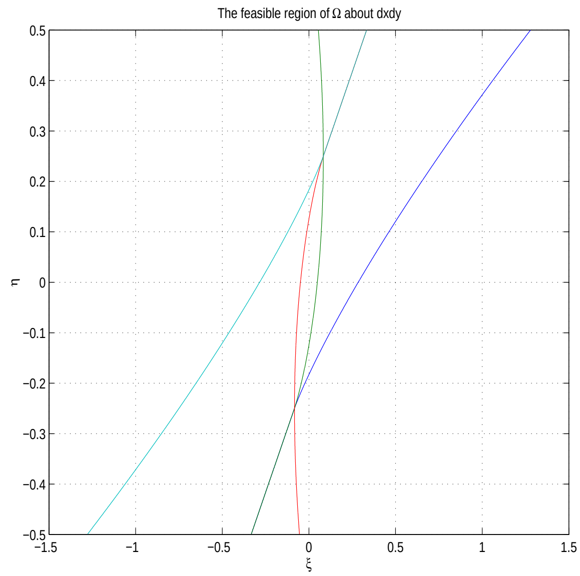


Figure A.4: The feasible region Ω of $\left\|\frac{\partial^2 z}{\partial x \partial y}\right\|_1$

In other words, If

$$\xi \geq h_j^y \eta + \frac{1}{3} h_i^x h_j^{y2} \left(\frac{1}{2} - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{3}{2}} - \frac{h_i^x h_j^{y2}}{6}$$

and

$$-\frac{h_i^x h_j^y}{4} \leq \eta \leq -\frac{h_i^x h_j^y}{4},$$

then automatically,

$$\eta \leq \frac{1}{h_j^y}\left(\xi + \frac{1}{6} h_i^x h_j^{y2}\right).$$

It's also easy to see that $f_1(s)$ and $g_2(s)$ only intersect at point $(\xi, \eta) = \left(\frac{1}{12} h_i^x h_j^{y2}, \frac{1}{4} h_i^x h_j^y\right)$. So $\eta \leq \frac{1}{h_j^y}\left(\xi + \frac{1}{6} h_i^x h_j^{y2}\right)$ is redundant to the constraint $\xi \leq h_j^y \eta - \frac{1}{3} h_i^x h_j^{y2}\left(\frac{2\eta}{h_i^x h_j^y} + \frac{1}{2}\right)^{\frac{3}{2}} + \frac{h_i^x h_j^{y2}}{6}$. Similarly, $f_2(s)$ and $g_1(s)$ only intersect at point $(\xi, \eta) = \left(-\frac{1}{12} h_i^x h_j^{y2}, -\frac{1}{4} h_i^x h_j^y\right)$. So $\eta \geq \frac{1}{h_j^y}\left(\xi - \frac{1}{6} h_i^x h_j^{y2}\right)$ is redundant to the constraint $\xi \geq h_j^y \eta + \frac{1}{3} h_i^x h_j^{y2}\left(\frac{1}{2} - \frac{2\eta}{h_i^x h_j^y}\right)^{\frac{3}{2}} - \frac{h_i^x h_j^{y2}}{6}$.

The conjugate transform of $\left\|\frac{\partial^2 z}{\partial x \partial y}\right\|_1$ is

$$G(\xi, \eta) = 0,$$

where (ξ, η) is defined over the set

$$
\begin{aligned}
&\Omega \\
&= \Omega_1 \cap \Omega_2 \cap \Omega_3 \cap \Omega_4 \\
&= \left\{ \begin{array}{c} \left(\frac{1}{2} + \frac{2\eta}{h_i^x h_j^y}\right)^3 \le \left(-\frac{3\xi}{h_i^x h_j^{y2}} + \frac{3\eta}{h_i^x h_j^y} + \frac{1}{2}\right)^2, \\ \left(\frac{1}{2} - \frac{2\eta}{h_i^x h_j^y}\right)^3 \le \left(\frac{3\xi}{h_i^x h_j^{y2}} - \frac{3\eta}{h_i^x h_j^y} + \frac{1}{2}\right)^2, \end{array} \right\},
\end{aligned}
$$

which is a convex set bounded by two cubic functions developed in Case 3 and Case 4. (Please refer to Figure A.4.)