

ABSTRACT

TAYLOR, PADRAIC. On the Solvability of Nonlinear Discrete Multipoint Boundary Value Problems. (Under the direction of Dr. Jesús Rodríguez.)

In this manuscript we study nonlinear, discrete, multipoint boundary value problems. We investigate two types of problems.

We first consider scalar, nonlinear, multipoint boundary value problems of the form

$$y(t+n) + a_{n-1}(t)y(t+n-1) + \dots + a_0(t)y(t) = g(y(t+m-1))$$

subject to

$$\sum_{j=1}^n b_{ij}(0)y(j-1) + \sum_{j=1}^n b_{ij}(1)y(j) + \sum_{j=1}^n b_{ij}(2)y(j+1) + \dots + \sum_{j=1}^n b_{ij}(N)y(j+N-1) = 0,$$

for $i = 1, 2, \dots, n$.

We provide sufficient conditions for the existence of solutions. By allowing more general boundary conditions and by imposing less restrictions on the nonlinearities, we obtain results that extend previous work in the area of discrete boundary value problems [8, 9].

We also study weakly nonlinear, discrete systems of the form

$$x(t+1) = A(t)x(t) + \epsilon f(t, x(t))$$

subject to the multipoint boundary conditions

$$B_0x(0) + B_1x(1) + \dots + B_Nx(N) = 0.$$

We provide sufficient conditions for the existence of solutions and we present a qualitative analysis of the way the solutions depend on the parameter ϵ .

On The Solvability of Nonlinear Discrete Multipoint Boundary Value Problems

by

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BIOGRAPHY

Padraic “Paddy” Whittingham Taylor was born on April 5, 1977 in Silver Spring, Maryland. His parents are Stephen Taylor Sr. and Janice Taylor, and he has an older brother Stephen Jr. In 1995, Paddy graduated from Montgomery Blair High School in Silver Spring. In 1999, Paddy graduated Summa Cum Laude with honors in mathematics from McDaniel College (Formerly Western Maryland College). At McDaniel, he was a member of the men’s varsity basketball team, and met his future wife Kristine Knox during his sophomore year. Paddy and Kristy were married on October 26, 2003 in Raleigh, North Carolina. In 2003, he earned his Master of Science degree in Mathematics from North Carolina State University, and in March of 2005 he helped put the graduate/faculty intramural basketball championship in its rightful place - the math department!

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CHAPTER 1

INTRODUCTION

In this paper we study the solvability of a very general family of discrete, non-linear, boundary value problems. We consider multipoint boundary value problems of the form

$$x(t+1) = A(t)x(t) + f(x(t))$$

subject to

$$B_0x(0) + B_1x(1) + \dots + B_Nx(N) = 0.$$

Throughout this paper, we assume that for each t , $x(t)$ is a vector in \mathbb{R}^n and $A(t)$ is an invertible $n \times n$ matrix. We assume that $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is at least continuous, B_i is a constant $n \times n$ matrix for i , and N is a fixed integer larger than two. In most practical applications, N will be much larger than two.

Multipoint boundary value problems can occur in a variety of settings. The elasticity of an equally loaded, three-layered “sandwich beam” can be modeled by a three-point, third order ordinary differential equation [1, 13]. When studying physical phenomena where N states are observed at N times, the resulting model will include an N point boundary condition. To illustrate the scope of these types of boundary conditions, observe that periodic solutions to nonlinear discrete equations can be examined by letting $B_0 = I$, $B_N = -I$ and $B_i = 0$ for $i = 1, 2, \dots, N-1$.

In Chapter 2, we consider the solvability of n^{th} order, scalar, multipoint boundary value problems. We provide sufficient conditions for the existence of solutions when the dimension of the solution space of the associated linear homogeneous boundary value problem is less than two. Our approach depends on whether the solution space

of the linear boundary value is trivial or not, however in each case our arguments are based on fixed point theorems.

In Chapter 3, we study the solvability of weakly nonlinear systems of difference equations subject to multipoint boundary conditions. Our focus in this chapter is on the case where the associated linear homogeneous boundary value problem has non-trivial solutions, however, for completeness, we do consider the case when the linear problem has only the trivial solution. We provide sufficient conditions for the existence of solutions and show how these solutions are connected to the solutions of the linear problem. We use the Implicit Function Theorem to establish this connection.

The reader should note the parallel approach that we take in each of these two chapters, yet the reader is not required to read Chapter 2 in order to understand Chapter 3 and vice versa.

CHAPTER 2

SCALAR DISCRETE NONLINEAR MULTIPOINT BOUNDARY VALUE PROBLEMS

2.1. INTRODUCTION

This chapter is devoted to the study of nonlinear, discrete-time, multipoint boundary value problems. We consider equations of the form

$$(1) \quad y(t+n) + a_{n-1}(t)y(t+n-1) + \dots + a_0(t)y(t) = g(y(t+m-1))$$

subject to

$$(2) \quad \sum_{j=1}^n b_{ij}(0)y(j-1) + \sum_{j=1}^n b_{ij}(1)y(j) + \sum_{j=1}^n b_{ij}(2)y(j+1) + \dots + \sum_{j=1}^n b_{ij}(N)y(j+N-1) = 0$$

for $i = 1, 2, \dots, n$.

Throughout the chapter we will assume that g is a continuous real-valued function defined on \mathbb{R} , that m is fixed and belongs to $\{1, 2, \dots, n\}$, and that N is an integer larger than two. The coefficients b_{ij} and the functions a_0, a_1, \dots, a_{n-1} are all real-valued and $a_0(t) \neq 0$ for all t .

We establish sufficient conditions for the existence of solutions to the boundary value problem (1)-(2). By imposing less restrictions on the nonlinearity and by allowing more general boundary conditions, our results constitute a significant generalization of the work of Etheridge and Rodriguez [8, 9]. The present chapter

also complements other work in the area of discrete boundary value problems. The interested reader may consult [2, 3, 10, 12, 17, 18].

The structure of the solution space of the linear homogeneous boundary value problem

$$(3) \quad y(t+n) + a_{n-1}(t)y(t+n-1) + \dots + a_0(t)y(t) = 0$$

subject to

$$(4) \quad \sum_{j=1}^n b_{ij}(0)y(j-1) + \sum_{j=1}^n b_{ij}(1)y(j) + \dots + \sum_{j=1}^n b_{ij}(N)y(j+N-1) = 0, \text{ for } i = 1, 2, \dots, n,$$

plays a crucial role in our analysis. In this chapter we consider the case where the solution space of (3)-(4) has dimension less than two. If the only solution to the linear homogeneous boundary value problem is the trivial one, we are able to obtain existence results for (1)-(2) which are based solely on the growth rate of the nonlinear function g . A much more delicate situation arises when the solution space of (3)-(4) is nontrivial. In this case we approach the solvability of the boundary value problem (1)-(2) via a projection scheme. We obtain easily verifiable, sufficient conditions for the existence of solutions. These conditions depend of the limiting behavior of the nonlinear function g and on the structure of the solution space of the linear problem (3)-(4).

2.2. PRELIMINARIES

We are concerned with the existence of solutions of

$$y(t+n) + a_{n-1}(t)y(t+n-1) + \dots + a_0(t)y(t) = g(y(t+m-1))$$

subject to

$$\sum_{j=1}^n b_{ij}(0)y(j-1) + \sum_{j=1}^n b_{ij}(1)y(j) + \sum_{j=1}^n b_{ij}(2)y(j+1) + \dots + \sum_{j=1}^n b_{ij}(N)y(j+N-1) = 0$$

for $i = 1, 2, \dots, n$.

The multipoint boundary value problem (1)-(2) is analyzed as an $n \times n$ system of the form:

$$(5) \quad x(t+1) = A(t)x(t) + f(x(t))$$

subject to

$$(6) \quad B_0x(0) + B_1x(1) + \dots + B_Nx(N) = 0.$$

For each k in $\{0, 1, \dots, N\}$ we let B_k be the $n \times n$ matrix given by $B_k = (b_{ij}(k))$.

The $n \times n$ matrix $A(t)$ is given by

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0(t) & -a_1(t) & -a_2(t) & \dots & -a_{n-1}(t) \end{pmatrix}. \text{ We set}$$

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ y(t+1) \\ \vdots \\ y(t+n-1) \end{pmatrix}, \text{ and we define } f : \mathbb{R}^n \rightarrow \mathbb{R}^n \text{ by}$$

$$f \left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ g(x_m) \end{pmatrix}.$$

We study the boundary value problem (5)-(6) by using operators on finite dimensional sequence spaces. To facilitate this, we introduce the following spaces:

$$X = \{\phi : \{0, 1, \dots, N\} \rightarrow \mathbb{R}^n : B_0\phi(0) + B_1\phi(1) + \dots + B_N\phi(N) = 0\},$$

and

$$Y = \{\psi : \{0, 1, \dots, N-1\} \rightarrow \mathbb{R}^n\}.$$

In each of these finite dimensional spaces we will use the supremum norm, that is for x in X we define

$$\|x\| = \sup_{t=0,1,\dots,N} |x(t)|$$

and for y in Y we define

$$\|y\| = \sup_{t=0,1,\dots,N-1} |y(t)|,$$

where $|\cdot|$ denotes the Euclidean norm on \mathbb{R}^n . Clearly with these norms, X and Y are Banach Spaces. If we define the linear operator $L : X \rightarrow Y$ by

$$(Lx)(t) = x(t+1) - A(t)x(t)$$

and define $F : X \rightarrow Y$ by

$$(Fx)(t) = f(x(t)),$$

then x is a solution to (5)-(6) if and only if:

$$(7) \quad Lx = Fx.$$

Our approach will depend on the nature of the kernel of L . If L is invertible we establish conditions for the solvability of (7) by finding a fixed point of the map $L^{-1}F$. When L is not invertible, the situation is far more demanding. In this case, we use the Lyapunov Schmidt Procedure [4, 6, 7, 11, 16, 17], to gain a better understanding of the delicate interaction between the nonlinearity and the boundary conditions.

$$\text{Let } \Phi(t) = \begin{cases} I & \text{if } t = 0 \\ A(t-1)A(t-2)\dots A(0) & \text{for } t = 1, 2, \dots \end{cases}$$

It is well known [3, 12, 15] that Φ is a fundamental matrix solution of $x(t+1) = A(t)x(t)$.

PROPOSITION 2.2.1. *The solution space of the linear homogeneous boundary value problem (3)-(4) and $\ker(B_0 + B_1\Phi(1) + \dots B_N\Phi(N))$ have the same dimension.*

Proof.

Clearly the solution space of (3)-(4) has the same dimension as $\ker(L)$, and

$$\begin{aligned} x \in \ker(L) &\iff x(t+1) = A(t)x(t) \text{ and } B_0x(0) + B_1x(1) + \dots B_Nx(N) = 0 \\ &\iff \text{there exists a vector } c \text{ in } \mathbb{R}^n \text{ such that } x(t) = \Phi(t)c \text{ and} \\ &\quad B_0\Phi(0)c + B_1\Phi(1)c + \dots B_N\Phi(N)c = 0 \\ &\iff x(t) = \Phi(t)c \text{ where } c \in \ker(B_0 + B_1\Phi(1) + \dots B_N\Phi(N)) \quad \square \end{aligned}$$

When the solution space of (3)-(4) is nontrivial we choose a unit vector $d \in \ker(B_0 + B_1\Phi(1) + \dots B_N\Phi(N))$ and define $S : \{0, 1, \dots, N\} \rightarrow \mathbb{R}^n$ by

$$S(t) = \Phi(t)d.$$

COROLLARY 2.2.1. *If the solution space of (3)-(4) is nontrivial, then x is in the kernel of L if and only if $x(t) = S(t)\alpha$ for some real number α .*

2.3. THE CASE OF INVERTIBLE L

We will show that if $B_0 + B_1\Phi(1) + \dots + B_N\Phi(N)$ is invertible and $g : \mathbb{R} \rightarrow \mathbb{R}$ exhibits “sublinear” growth, then the boundary value problem:

$$y(t+n) + a_{n-1}(t)y(t+n-1) + \dots + a_0(t)y(t) = g(y(t+m-1))$$

subject to

$$\sum_{j=1}^n b_{ij}(0)y(j-1) + \sum_{j=1}^n b_{ij}(1)y(j) + \sum_{j=1}^n b_{ij}(2)y(j+1) + \dots + \sum_{j=1}^n b_{ij}(N)y(j+N-1) = 0$$

for $i = 1, 2, \dots, n$,

has a solution. This result extends previous work of Etheridge and Rodriguez [8, 9].

In this paper we allow more general boundary conditions and we place less restriction on the nonlinearities.

PROPOSITION 2.3.1. *L is invertible if and only if $B_0 + B_1\Phi(1) + \dots + B_N\Phi(N)$ is invertible.*

Proof.

Let $h \in Y$. $Lx = h$ if and only if there exists an element x in X that satisfies

$$x(t+1) = A(t)x(t) + h(t) \text{ for } t = 0, 1, \dots, N-1.$$

By the well known Variation of Constants formula [3, 12, 15], $Lx = h$ if and only if

$$(8) \quad x(t) = \Phi(t)x(0) + \Phi(t) \sum_{i=0}^{t-1} \Phi^{-1}(i+1)h(i)$$

and

$$(9) \quad B_0x(0) + B_1x(1) + \dots + B_Nx(N) = 0.$$

Combining equations (8) and (9) we have $Lx = h$ if and only if

$$(10) \quad B_0x(0) + B_1(\Phi(1)x(0) + h(0)) + \dots + B_N \left(\Phi(N)x(0) + \Phi(N) \sum_{i=0}^{N-1} \Phi^{-1}(i+1)h(i) \right) = 0$$

or equivalently

$$(11) \quad (B_0 + B_1\Phi(1) + \dots + B_N\Phi(N))x(0) = - \left[B_1h(0) + \dots + B_N\Phi(N) \sum_{i=0}^{N-1} \Phi^{-1}(i+1)h(i) \right]$$

Notice that $Lx = h$ if and only if there exists $x(0) \in \mathbb{R}^n$ that satisfies equation (11). If we can find such an $x(0)$, then we define x as in equation (8) and we will have $Lx = h$. Observe that if $B_0 + B_1\Phi(1) + \dots + B_N\Phi(N)$ is invertible, then there is one and only one $x(0)$ that satisfies equation (11), which implies there is one and only one x that satisfies $Lx = h$, and thus L is invertible. Likewise if there is one and only one x that satisfies $Lx = h$, then there is one and only one $x(0)$ that satisfies equation (11), and thus $B_0 + B_1\Phi(1) + \dots + B_N\Phi(N)$ must be invertible. \square

PROPOSITION 2.3.2. *Assume $B_0 + B_1\Phi(1) + \dots + B_N\Phi(N)$ is invertible. If there exist nonnegative real numbers M_1, M_2 and a real number α with $0 < \alpha < 1$ such that $|g(x)| \leq M_1|x|^\alpha + M_2$ for all x , then there is at least one solution to equation (1)-(2).*

Proof.

Since $B_0 + B_1\Phi(1) + \dots + B_N\Phi(N)$ is invertible, then L is invertible and $Lx = Fx$ if and only if $x = L^{-1}Fx$. Define $T : X \rightarrow X$ by $Tx = L^{-1}Fx$. Then clearly T is continuous. Observe that

$$\|Fx\| = \sup_{t=0,1,\dots,N-1} |f(x(t))|,$$

and for each $t = 0, 1, \dots, N-1$:

$$|f(x(t))| = |g(x_m(t))| \leq M_1|x_m(t)|^\alpha + M_2 \leq M_1|x(t)|^\alpha + M_2 \leq M_1\|x\|^\alpha + M_2.$$

Now, if M is a positive real number and $\|x\| \leq M$ then

$$\|Tx\| = \|L^{-1}Fx\| \leq \|L^{-1}\| (M_1\|x\|^\alpha + M_2) \leq \|L^{-1}\| (M_1M^\alpha + M_2).$$

Therefore

$$\frac{\|Tx\|}{M} \leq \|L^{-1}\| \left(\frac{M_1}{M^{1-\alpha}} + \frac{M_2}{M} \right),$$

and thus for M sufficiently large, $\frac{\|Tx\|}{M} \leq 1$. Consequently, there exists a positive number M such that for all $x \in X$, if $\|x\| \leq M$ then $\|Tx\| \leq M$. If we let $B = \{x \in X : \|x\| \leq M\}$, then B is a closed, nonempty, convex set, and T maps B into itself. By the Brouwer Fixed Point Theorem, T has a fixed point in B . \square

2.4. THE CASE OF SINGULAR L

Now let us consider the case where the kernel of L is one dimensional. Recall from Corollary 2.1 that every element of $\ker(L)$ can be written as $S\alpha$ for some real number α .

In order to introduce our projection scheme, we now construct projections onto the kernel and image of L .

PROPOSITION 2.4.1. *If we define $P : X \rightarrow X$ by $(Px)(t) = S(t)d^T x(0)$, then P is a projection onto $\ker(L)$.*

Proof.

Clearly P is bounded and linear. We must show $P^2 = P$, and $\text{Im}(P) = \ker(L)$.

Claim 1: $P^2 = P$

proof of claim

$$\begin{aligned}(P(Px))(t) &= P(S(\cdot)d^T x(0))(t) \\ &= S(t)d^T S(0)d^T x(0) = S(t)(d^T d)d^T x(0) \\ &= S(t)d^T x(0) = (Px)(t).\end{aligned}$$

Claim 2: $\text{Im}(P) = \ker(L)$

proof of claim

Let $x \in X$. Then

$$(Px)(t) = S(t)d^T x(0) = S(t)\alpha, \text{ where } \alpha = d^T x(0).$$

Therefore $\text{Im}(P) \subset \ker(L)$.

Now let $x \in \ker(L)$. Then $x(t) = S(t)\beta$ for some $\beta \in \mathbb{R}$ and

$$(Px)(t) = S(t)d^T S(0)\beta = S(t)(d^T d)\beta = S(t)\beta = x(t).$$

Therefore $x \in \text{Im}(P)$ and thus $\ker(L) \subset \text{Im}(P)$, which implies $\text{Im}(P) = \ker(L)$.

Claims 1 and 2 verify that P is a projection onto $\ker(L)$. \square

The following definition and lemma are vital to the framework of our construction of a projection onto the image of L .

DEFINITION 2.4.1. *Assume $\ker((B_0 + B_1\Phi(1) + \dots + B_N\Phi(N))^T) = \text{span}\{c\}$ for some vector $c \in \mathbb{R}^n$. Define $\Psi : \{0, 1, \dots, N-1\} \rightarrow \mathbb{R}^n$ by*

$$\Psi(t) = \sum_{i=t+1}^N [B_i\Phi(i)\Phi^{-1}(t+1)]^T c$$

LEMMA 2.4.1. *Ψ is the zero map if and only if $\bigcap_{i=0}^N \ker(B_i^T) \neq \{0\}$.*

Proof.

Assume there is a nonzero vector v in $\bigcap_{i=0}^N \ker(B_i^T)$. Then $v \in \ker((B_0 + B_1\Phi(1) + \dots + B_N\Phi(N))^T)$. Therefore $v = \alpha c$ for some nonzero real number α , and thus $c = \frac{1}{\alpha}v$. As a result, $\Psi(t) = \sum_{i=t+1}^N [B_i\Phi(i)\Phi^{-1}(t+1)]^T \frac{1}{\alpha}v = 0$ for all $t = 0, 1, \dots, N-1$.

Now if Ψ is the zero map, then $\Psi(t) = 0$ for all $t = 0, 1, \dots, N-1$. Observe the following telescoping effect:

$$\begin{aligned} \Psi(N-1) = 0 &\implies B_N^T c = 0 \\ \Psi(N-2) = 0 &\implies B_{N-1}^T c + [B_N\Phi(N)\Phi(N-1)^{-1}]^T c = 0 \\ &\implies B_{N-1}^T c = 0 \\ &\vdots \\ \Psi(0) = 0 &\implies B_1^T c + \dots [B_N\Phi(N)\Phi(1)^{-1}]^T c = 0 \implies B_1^T c = 0 \end{aligned}$$

Therefore $c \in \bigcap_{i=1}^N \ker(B_i^T)$. However, since $c \in \ker((B_0 + B_1\Phi(1) + \dots + B_N\Phi(N))^T)$, then we must also have $c \in \ker(B_0^T)$ and thus $c \in \bigcap_{i=0}^N \ker(B_i^T)$. \square

PROPOSITION 2.4.2. h is in the image of L if and only if $\sum_{i=0}^{N-1} h^T(i)\Psi(i) = 0$.

Proof.

Let $h \in Y$. From Proposition (2.3.1) we know h is in the image of L if and only if there exists an element x in X such that

$$B_0x(0) + B_1(\Phi(1)x(0) + h(0)) + \dots + B_N\left(\Phi(N)x(0) + \Phi(N)\sum_{i=0}^{N-1}\Phi^{-1}(i+1)h(i)\right) = 0$$

This holds if and only if

$$B_1h(0) + B_2\sum_{i=0}^1\Phi(2)\Phi^{-1}(i+1)h(i) + \dots + B_N\sum_{i=0}^{N-1}\Phi(N)\Phi^{-1}(i+1)h(i)$$

is in the image of $B_0 + B_1\Phi(1) + \dots + B_N\Phi(N)$.

Therefore we must have

$$(12) \quad \left(B_1h(0) + B_2\sum_{i=0}^1\Phi(2)\Phi^{-1}(i+1)h(i) + \dots + B_N\sum_{i=0}^{N-1}\Phi(N)\Phi^{-1}(i+1)h(i)\right)^T \beta = 0$$

for all $\beta \in \ker((B_0 + B_1\Phi(1) + \dots + B_N\Phi(N))^T)$.

Since $\ker((B_0 + B_1\Phi(1) + \dots + B_N\Phi(N))^T)$ is a one dimensional space we see that equation (12) is equivalent to

$$\sum_{i=0}^{N-1} h^T(i) \left(\sum_{k=i+1}^N [B_k\Phi(k)\Phi^{-1}(i+1)]^T \right) c = 0,$$

where c spans $\ker((B_0 + B_1\Phi(1) + \dots + B_N\Phi(N))^T)$

Therefore, $h \in \text{Im}(L)$ if and only if $\sum_{i=0}^{N-1} h^T(i)\Psi(i) = 0$,

or equivalently $\sum_{i=0}^{N-1} \Psi^T(i)h(i) = 0$. \square

We have seen that if $\bigcap_{i=0}^N \ker(B_i^T) = \{0\}$, then Ψ is not identically zero. This allows us to define the map $W : Y \rightarrow Y$ by

$$(Wh)(t) = \Psi(t) \left(\sum_{i=0}^{N-1} |\Psi(i)|^2 \right)^{-1} \sum_{i=0}^{N-1} \Psi^T(i) h(i)$$

PROPOSITION 2.4.3. Assume $\bigcap_{i=0}^N \ker(B_i^T) = \{0\}$. Then $E = I - W$ is a projection onto the image of L .

Proof.

Since W is clearly bounded and linear, all we need to show is that W is a projection and $Im(E) = Im(L)$.

Claim 1: $W^2 = W$

proof of claim: Let $h \in \mathcal{Y}$

$$\begin{aligned} & (W(Wh))(t) \\ &= W \left(\Psi(\cdot) \left[\sum_{i=0}^{N-1} |\Psi(i)|^2 \right]^{-1} \sum_{k=0}^{N-1} \Psi(k)^T h(k) \right) \\ &= \Psi(t) \left[\sum_{\alpha=0}^{N-1} |\Psi(\alpha)|^2 \right]^{-1} \sum_{\beta=0}^{N-1} \Psi(\beta)^T \Psi(\beta) \left[\sum_{i=0}^{N-1} |\Psi(i)|^2 \right]^{-1} \sum_{k=0}^{N-1} \Psi(k)^T h(k) \\ &= \Psi(t) \left[\sum_{i=0}^{N-1} |\Psi(i)|^2 \right]^{-1} \sum_{k=0}^{N-1} \Psi(k)^T h(k) = (Wh)(t) \\ &\implies W^2 = W \end{aligned}$$

Since W is a projection, then E must also be a projection.

Claim 2: $Im(E) = Im(L)$

proof of claim: Let $h \in \mathcal{Y}$

$$\begin{aligned}
\sum_{i=0}^{N-1} \Psi(i)^T (Eh)(i) &= \sum_{i=0}^{N-1} \Psi(i)^T (h(i) - (Wh)(i)) \\
&= \sum_{i=0}^{N-1} \Psi(i)^T h(i) \\
&\quad - \sum_{i=0}^{N-1} \Psi(i)^T \left(\Psi(i) \left[\sum_{k=0}^{N-1} |\Psi(k)|^2 \right]^{-1} \sum_{k=0}^{N-1} \Psi(k)^T h(k) \right) \\
&= \sum_{i=0}^{N-1} \Psi(i)^T h(i) - \sum_{k=0}^{N-1} \Psi(k)^T h(k) = 0.
\end{aligned}$$

Therefore $Eh \in \text{Im}(L)$ and thus $\text{Im}(E) \subset \text{Im}(L)$.

Now let $h \in \text{Im}(L)$. Then

$$(Eh)(t) = h(t) - \Psi(t) \left[\sum_{i=0}^{N-1} |\Psi(i)|^2 \right]^{-1} \sum_{k=0}^{N-1} \Psi(k)^T h(k) = h(t)$$

Therefore $Eh = h$, and thus $h \in \text{Im}(E)$ which implies $\text{Im}(E) = \text{Im}(L)$ \square

The construction of the projection W is analagous to a projection constructed for the special case of a two-point boundary value problem in [9]. The projections P and E enable us to study the existence of solutions to the boundary value problem (1)-(2) using the classic Lyapunov Schmidt Reduction. Our approach to this procedure is entirely self-contained. We do provide references, [4, 5, 6, 7, 11, 16, 17, 19, 21, 22] for readers interested in a more general formulation and for those interested in applications to differential or difference equations.

Since P is a continuous projection, we may now write $X = X_P \oplus X_{I-P}$ where $X_P = \text{Im}(P)$ and $X_{I-P} = \text{Im}(I - P)$. Note that $L : X_{I-P} \rightarrow \text{Im}(L)$ is a bijection and thus there exists a bounded linear map $M : \text{Im}(L) \rightarrow X_{I-P}$ such that:

$$i) \quad LMh = h \quad \text{for all } h \in \text{Im}(L)$$

$$ii) \quad MLx = x_{I-P} \quad \text{for all } x \in X.$$

PROPOSITION 2.4.4. $Lx = Fx$ if and only if there is a real number α such that $x = S\alpha + MEFx$ and $\sum_{i=0}^{N-1} g([S(i)\alpha + MEFx(i)]_m) [\Psi(i)]_n = 0$, where $[v]_k$ denotes the k^{th} row of the vector v .

Proof.

$$Lx = Fx \iff E(Lx - Fx) = 0 \text{ and } (I - E)(Lx - Fx) = 0$$

$$\iff Lx = EFx \text{ and } (I - E)Fx = 0$$

$$\iff x_{I-P} = MEFx \text{ and } Fx \in \text{Im}(L)$$

$$\iff x = x_P + MEFx \text{ and } \sum_{i=0}^{N-1} [Fx(i)]^T \Psi(i) = 0$$

$$\iff x = S\alpha + MEFx \text{ and } \sum_{i=0}^{N-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ g(x_m(i)) \end{bmatrix}^T \Psi(i) = 0$$

for some real number α

$$\iff x = S\alpha + MEFx \text{ and } \sum_{i=0}^{N-1} g([S(i)\alpha + MEFx(i)]_m) [\Psi(i)]_n = 0. \quad \square$$

Let $\lim_{x \rightarrow \infty} g(x) = g(\infty)$ and $\lim_{x \rightarrow -\infty} g(x) = g(-\infty)$. We introduce the following notation:

$$\mathcal{O}_1 = \{i \in \{0, 1, \dots, N-1\} : [S(i)]_m > 0\}$$

$$\mathcal{O}_2 = \{i \in \{0, 1, \dots, N-1\} : [S(i)]_m < 0\}$$

$$\mathcal{O}_3 = \{i \in \{0, 1, \dots, N-1\} : [S(i)]_m = 0\}$$

DEFINITION 2.4.2. For \mathcal{O}_1 and \mathcal{O}_2 defined above, define the real numbers J_1 and J_2 by

$$(13) \quad J_1 = g(\infty) \sum_{i \in \mathcal{O}_1} [\Psi(i)]_n + g(-\infty) \sum_{i \in \mathcal{O}_2} [\Psi(i)]_n$$

$$(14) \quad J_2 = g(-\infty) \sum_{i \in \mathcal{O}_1} [\Psi(i)]_n + g(\infty) \sum_{i \in \mathcal{O}_2} [\Psi(i)]_n.$$

PROPOSITION 2.4.5. Assume g is continuous, $g(\infty)$ and $g(-\infty)$ exist, \mathcal{O}_3 is empty and J_1 and J_2 are both different from 0. Then there exists a real number $\alpha_0 > 0$ such that for $\alpha \geq \alpha_0$, J_1 has the same sign as K_+ and J_2 has the same sign as K_- , where

$$K_{\pm} = \sum_{i=0}^{N-1} g([S(i)(\pm\alpha) + MEFx(i)]_m) [\Psi(i)]_n.$$

Proof.

Let $\epsilon > 0$. Here we will show that for α large, J_1 has the same sign as K_+ . The proof for J_2 and K_- will follow the same method. Since S and MEF are bounded, we can find $\alpha_0 > 0$ such that for $i \in \mathcal{O}_1$

$$g(\infty) - \epsilon < g([S(i)\alpha + MEFx(i)]_m) < g(\infty) + \epsilon$$

and for $i \in \mathcal{O}_2$

$$g(-\infty) - \epsilon < g([S(i)\alpha + MEFx(i)]_m) < g(-\infty) + \epsilon$$

for all $\alpha \geq \alpha_0$. Let

$$\mathcal{U}_1 = \{i \in \mathcal{O}_1 : [\Psi(i)]_n > 0\}$$

$$\mathcal{U}_2 = \{i \in \mathcal{O}_1 : [\Psi(i)]_n < 0\}$$

$$\mathcal{U}_3 = \{i \in \mathcal{O}_1 : [\Psi(i)]_n = 0\}$$

Now for $\alpha \geq \alpha_0$,

$$\sum_{\mathcal{O}_1} g([S(i)(\alpha) + MEFx(i)]_m) [\Psi(i)]_n =$$

$$\sum_{\mathcal{U}_1} g([S(i)(\alpha) + MEFx(i)]_m) [\Psi(i)]_n + \sum_{\mathcal{U}_2} g([S(i)(\alpha) + MEFx(i)]_m) [\Psi(i)]_n,$$

which yields

$$(15) \quad (g(\infty) - \epsilon) \sum_{\mathcal{U}_1} [\Psi(i)]_n + (g(\infty) + \epsilon) \sum_{\mathcal{U}_2} [\Psi(i)]_n <$$

$$(16) \quad \sum_{\mathcal{O}_1} g([S(i)(\alpha) + MEFx(i)]_m) [\Psi(i)]_n <$$

$$(17) \quad (g(\infty) + \epsilon) \sum_{\mathcal{U}_1} [\Psi(i)]_n + (g(\infty) - \epsilon) \sum_{\mathcal{U}_2} [\Psi(i)]_n.$$

In a similar fashion, define

$$\mathcal{W}_1 = \{i \in \mathcal{O}_2 : [\Psi(i)]_n > 0\}$$

$$\mathcal{W}_2 = \{i \in \mathcal{O}_2 : [\Psi(i)]_n < 0\}$$

$$\mathcal{W}_3 = \{i \in \mathcal{O}_2 : [\Psi(i)]_n = 0\}$$

Then for $\alpha \geq \alpha_0$,

$$\sum_{\mathcal{O}_2} g([S(i)(\alpha) + MEFx(i)]_m) [\Psi(i)]_n =$$

$$\sum_{\mathcal{W}_1} g([S(i)(\alpha) + MEFx(i)]_m) [\Psi(i)]_n + \sum_{\mathcal{W}_2} g([S(i)(\alpha) + MEFx(i)]_m) [\Psi(i)]_n,$$

and we have

$$(18) \quad (g(\infty) - \epsilon) \sum_{\mathcal{W}_1} [\Psi(i)]_n + (g(\infty) + \epsilon) \sum_{\mathcal{W}_2} [\Psi(i)]_n <$$

$$(19) \quad \sum_{\mathcal{O}_2} g([S(i)(\alpha) + MEFx(i)]_m) [\Psi(i)]_n <$$

$$(20) \quad (g(\infty) + \epsilon) \sum_{\mathcal{W}_1} [\Psi(i)]_n + (g(\infty) - \epsilon) \sum_{\mathcal{W}_2} [\Psi(i)]_n.$$

If we assume $\alpha \geq \alpha_0$ and add the inequalities given by (15)-(20) we obtain

$$J_1 - \epsilon \left(\sum_{\mathcal{U}_1} [\Psi(i)]_n - \sum_{\mathcal{U}_2} [\Psi(i)]_n + \sum_{\mathcal{W}_1} [\Psi(i)]_n - \sum_{\mathcal{W}_2} [\Psi(i)]_n \right) < K_+$$

and

$$K_+ < J_1 + \epsilon \left(\sum_{\mathcal{U}_1} [\Psi(i)]_n - \sum_{\mathcal{U}_2} [\Psi(i)]_n + \sum_{\mathcal{W}_1} [\Psi(i)]_n - \sum_{\mathcal{W}_2} [\Psi(i)]_n \right).$$

Therefore K_+ lies in an open interval centered at J_1 with radius $\epsilon \left(\sum_{i=0}^{N-1} |[\Psi(i)]_n| \right)$.

Observe that the radius of this interval is non trivial since $J_1 \neq 0$ implies the existence of at least one i where $[\Psi(i)]_n \neq 0$. Since we can make ϵ arbitrarily small and $J_1 \neq 0$, then there exists $\alpha_0 > 0$ such that K_+ and J_1 have the same sign for $\alpha \geq \alpha_0$. \square

THEOREM 2.4.1. *Suppose that $\ker(B_0 + B_1\Phi(1) + \dots + B_N\Phi(N))$ is one dimensional and $\bigcap_{i=0}^N \ker(B_i^T) = \{0\}$.*

If

i) $g : \mathbb{R} \rightarrow \mathbb{R}$ is continuous

ii) $g(\infty)$ and $g(-\infty)$ exist

iii) \mathcal{O}_3 is empty

iv) $J_1 J_2 < 0$

then there is at least one solution to equation (1)-(2).

Proof.

Suppose without loss of generality that $J_1 > J_2$. Take α_0 large enough so that for $\alpha \geq \alpha_0$, J_1 and K_+ have the same sign, and J_2 and K_- have the same sign. Also make sure $\alpha_0 > rN\|\Psi\|$ where $r = \sup_{t \in \mathbb{R}} |g(t)|$. Following Etheridge and Rodriguez, [9], define

$$H_1 : X \times \mathbb{R} \rightarrow X \quad \text{by} \quad H_1(x, \alpha) = S\alpha + MEFx,$$

$$H_2 : X \times \mathbb{R} \rightarrow \mathbb{R} \quad \text{by} \quad H_2(x, \alpha) = \alpha - \sum_{i=0}^{N-1} g([S(i)\alpha + MEFx(i)]_m) [\Psi(i)]_n,$$

$$\text{and } H : X \times \mathbb{R} \rightarrow X \times \mathbb{R} \quad \text{by} \quad H(x, \alpha) = (H_1(x, \alpha), H_2(x, \alpha)).$$

We will construct a nonempty, closed, convex set $\mathcal{B} \subset X \times \mathbb{R}$ such that $H(\mathcal{B}) \subset \mathcal{B}$. Since H is clearly continuous, H will have at least one fixed point in \mathcal{B} by the Brouwer Fixed Point Theorem.

Let

$$\mathcal{B} = \{(x, \alpha) : \|x\| \leq \|S\|\delta + \|ME\|r, \text{ and } |\alpha| \leq \delta \text{ where } \delta = \alpha_0 + rN\|\Psi\|\}.$$

Clearly \mathcal{B} is nonempty, closed and convex. Since $J_1 J_2 < 0$ and $J_1 > J_2$, for $\alpha \geq \alpha_0$ we have

$$\sum_{i=0}^{N-1} g([S(i)\alpha + MEFx(i)]_m) [\Psi(i)]_n > 0$$

and

$$\sum_{i=0}^{N-1} g([S(i)(-\alpha) + MEFx(i)]_m) [\Psi(i)]_n < 0$$

for all $x \in X$.

As a result, for $\alpha \in [\alpha_0, \delta]$,

$$H_2(x, \alpha) \leq \alpha \text{ and } H_2(x, -\alpha) \geq -\alpha.$$

Also, since $\left| \sum_{i=0}^{N-1} g([S(i)\alpha + MEFx(i)]_m) [\Psi(i)]_n \right| \leq rN\|\Psi\|$, if $\alpha \in [\alpha_0, \delta]$,

$$H_2(x, \alpha) \geq 0 \text{ and } H_2(x, -\alpha) \leq 0.$$

Therefore, if $\alpha \in [\alpha_0, \delta]$, $H_2(x, \alpha) \in [-\delta, \delta]$.

Now consider $\alpha \in [0, \alpha_0)$.

$$\begin{aligned} |H_2(x, \pm\alpha)| &= \left| (\pm\alpha) - \sum_{i=0}^{N-1} g([S(i)(\pm\alpha) + MEFx(i)]_m) [\Psi(i)]_n \right| \\ &\leq |\alpha| + rN\|\Psi\| \\ &< \delta. \end{aligned}$$

Therefore $H_2(\mathcal{B}) \subset [-\delta, \delta]$.

Let $(x, \alpha) \in \mathcal{B}$. Then

$$\|H_1(x, \alpha)\| = \|S\alpha + MEFx\| \leq \|S\|\delta + \|ME\|r.$$

Since $H(\mathcal{B}) \subset \mathcal{B}$, by the Brouwer Fixed Point Theorem, there is at least one fixed point of H in \mathcal{B} . If $(\hat{x}, \hat{\alpha})$ is this fixed point, then $\hat{x} = S\hat{\alpha} + MEF\hat{x}$ and $\sum_{i=0}^{N-1} g([S(i)\hat{\alpha} + MEF\hat{x}(i)]_m) [\Psi(i)]_n = 0$. By Proposition 2.4.4, $L\hat{x} = F\hat{x}$ and equation (1)-(2) has at least one solution. \square

It is significant to observe that we do not require that $g(\infty)$ and $g(-\infty)$ have different signs. Previous papers, [8, 9] impose this restriction and they only consider two-point boundary conditions.

2.5. EXAMPLE

Consider

$$(21) \quad y(t+2) + 3y(t+1) + 2y(t) = g(y(t))$$

subject to

$$(22) \quad y(0) + \left(2 - 2^{\frac{3-N}{2}}\right) y\left(\frac{N-1}{2}\right) + \left(1 - 2^{\frac{3-N}{2}}\right) y\left(\frac{N+1}{2}\right) = 0$$

$$(23) \quad y(N) + y(N+1) = 0,$$

where N is an odd integer larger than 1, satisfying $\frac{N-1}{2}$ is an even integer.

Comparing this equation to equation (1)-(2), we note that $n = 2$ and $m = 1$.

We can write this in system form as

$$(24) \quad x(t+1) = A(t)x(t) + f(x(t))$$

subject to

$$(25) \quad B_0x(0) + B_{\frac{N-1}{2}}x\left(\frac{N-1}{2}\right) + B_Nx(N) = 0$$

where

$$A(t) = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \text{ for all } t,$$

$$B_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B_{\frac{N-1}{2}} = \begin{pmatrix} 2 - 2^{\frac{3-N}{2}} & 1 - 2^{\frac{3-N}{2}} \\ 0 & 0 \end{pmatrix}, B_N = \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix},$$

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ y(t+1) \end{pmatrix}, \text{ and}$$

$$f \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ g(x_1) \end{pmatrix}.$$

Since A is constant we can write $A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$ and thus $\Phi(t) = A^t$ for $t = 0, 1, 2, \dots, N$. Since N is odd, $\Phi(N) = A^N = \begin{pmatrix} 2^N - 2 & 2^N - 1 \\ 2 - 2^{N+1} & 1 - 2^{N+1} \end{pmatrix}$.

$$B_0 + B_{\frac{N-1}{2}} A^{\frac{N-1}{2}} + B_N A^N = \begin{pmatrix} 1 & 1 \\ 2^N & 2^N \end{pmatrix} \text{ and thus}$$

$$\ker \left(B_0 + B_{\frac{N-1}{2}} A^{\frac{N-1}{2}} + B_N A^N \right) = \text{span} \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

$$\text{Therefore } S(t) = A^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} (-1)^t \\ (-1)^{t+1} \end{pmatrix} \text{ for } t = 0, 1, \dots, N, \text{ and thus}$$

$\mathcal{O}_1 = \{0, 2, 4, \dots, N-1\}$, $\mathcal{O}_2 = \{1, 3, 5, \dots, N-2\}$, and \mathcal{O}_3 is empty.

Also, observe that $\bigcap_{i=0}^N \ker(B_i^T) = \{0\}$.

$$\text{Ker} \left((B_0 + B_{\frac{N-1}{2}} A^{\frac{N-1}{2}} + B_N A^N)^T \right) = \text{span} \left\{ \begin{pmatrix} 2^N \\ -1 \end{pmatrix} \right\} \text{ which gives us}$$

$$\Psi(t) = \begin{cases} \left(\left[B_{\frac{N-1}{2}} A^{\frac{N-1}{2} - (t+1)} \right]^T + \left[B_N A^{N - (t+1)} \right]^T \right) \begin{pmatrix} 2^N \\ -1 \end{pmatrix} & \text{for } t = 0, \dots, \frac{N-3}{2} \\ \left[B_N A^{N - (t+1)} \right]^T \begin{pmatrix} 2^N \\ -1 \end{pmatrix} & \text{for } t = \frac{N-1}{2}, \dots, N-1 \end{cases}$$

$$= \begin{cases} \begin{pmatrix} (-1)^t 2^{N+1} (1 - 2^{-t}) \\ (-1)^t 2^N (1 - 2^{-(t+1)}) \end{pmatrix} & \text{for } t = 0, 1, \dots, \frac{N-3}{2} \\ \begin{pmatrix} (-1)^t 2^{N - (t+1)} \\ (-1)^t 2^{N - (t+1)} \end{pmatrix} & \text{for } t = \frac{N-1}{2}, \dots, N-1. \end{cases}$$

We still need to make sure that $J_1 J_2 < 0$.

Observe that

$$\begin{aligned}
\sum_{i \in \mathcal{O}_1} [\Psi(i)]_2 &= 2^N \sum_{i \in \mathcal{O}_1, i=0}^{\frac{N-5}{2}} (-1)^i (1 - 2^{-(i+1)}) + 2^N \sum_{i \in \mathcal{O}_1, i=\frac{N-1}{2}}^{N-1} (-1)^i 2^{-(i+1)} \\
&= 2^N \left(\sum_{k=0}^{\frac{N-5}{4}} 1 - \sum_{k=0}^{\frac{N-5}{4}} \frac{1}{2} \left(\frac{1}{4}\right)^k + \sum_{k=\frac{N-1}{4}}^{\frac{N-1}{2}} \frac{1}{2} \left(\frac{1}{4}\right)^k \right) \\
&= 2^N \left(\frac{N-1}{4} + \sum_{k=0}^{\frac{N-1}{2}} \frac{1}{2} \left(\frac{1}{4}\right)^k - \sum_{k=0}^{\frac{N-5}{4}} \left(\frac{1}{4}\right)^k \right) \\
&= 2^{N-2} (N-1) + \left(\frac{-2^{N+1} + 2^{\frac{N+5}{2}} - 1}{3} \right).
\end{aligned}$$

Likewise

$$\begin{aligned}
\sum_{i \in \mathcal{O}_2} [\Psi(i)]_2 &= 2^N \sum_{i \in \mathcal{O}_2, i=1}^{\frac{N-3}{2}} (-1)^i (1 - 2^{-(i+1)}) + 2^N \sum_{i \in \mathcal{O}_2, i=\frac{N+1}{2}}^{N-2} (-1)^i 2^{-(i+1)} \\
&= 2^N \left(\sum_{k=0}^{\frac{N-5}{4}} (-1) - \sum_{k=0}^{\frac{N-5}{4}} (-1) \left(\frac{1}{2}\right) \left(\frac{1}{2}\right)^{2k+1} + \sum_{k=\frac{N-1}{4}}^{\frac{N-3}{2}} \left(\frac{1}{2}\right) \left(\frac{1}{2}\right)^{2k+1} \right) \\
&= 2^N \left(-\frac{N-1}{4} + \sum_{k=0}^{\frac{N-5}{4}} \left(\frac{1}{4}\right) \left(\frac{1}{4}\right)^k - \sum_{k=\frac{N-1}{4}}^{\frac{N-3}{2}} \left(\frac{1}{4}\right) \left(\frac{1}{4}\right)^k \right) \\
&= -2^{N-2} (N-1) + 2^N \left(\sum_{k=0}^{\frac{N-5}{4}} \left(\frac{1}{2}\right) \left(\frac{1}{4}\right)^k - \sum_{k=0}^{\frac{N-3}{2}} \left(\frac{1}{4}\right) \left(\frac{1}{4}\right)^k \right) \\
&= - \left(2^{N-2} (N-1) + \left(\frac{2^{\frac{N+3}{2}} - 2^N - 2}{3} \right) \right).
\end{aligned}$$

Therefore

$$\begin{aligned}
J_1 &= g(\infty) \left(2^{N-2}(N-1) + \left(\frac{-2^{N+1} + 2^{\frac{N+5}{2}} - 1}{3} \right) \right) \\
&\quad - g(-\infty) \left(2^{N-2}(N-1) + \left(\frac{2^{\frac{N+3}{2}} - 2^N - 2}{3} \right) \right)
\end{aligned}$$

and

$$\begin{aligned}
J_2 &= g(-\infty) \left(2^{N-2}(N-1) + \left(\frac{-2^{N+1} + 2^{\frac{N+5}{2}} - 1}{3} \right) \right) \\
&\quad - g(\infty) \left(2^{N-2}(N-1) + \left(\frac{2^{\frac{N+3}{2}} - 2^N - 2}{3} \right) \right).
\end{aligned}$$

Clearly there are infinitely many continuous functions g where the end behavior given by $g(\infty)$ and $g(-\infty)$ will make $J_1 J_2 < 0$, and thus establishing the existence of at least one solution to equation (21)-(23). Of notable interest is the fact that this can be achieved with a function g where $g(\infty)g(-\infty) > 0$. One simple example is given by

$$g(t) = \begin{cases} \frac{18}{\pi} \tan^{-1}(t) & \text{for } t \geq 0 \\ \frac{-96}{\pi} \tan^{-1}(t) & \text{for } t < 0 \end{cases}$$

Observe that $g(\infty) = 9$, $g(-\infty) = 48$ and $J_1 J_2 < 0$ for all $N \geq 5$.

CHAPTER 3

WEAKLY NONLINEAR DISCRETE MULTIPOINT BOUNDARY VALUE PROBLEMS

3.1. INTRODUCTION

In this chapter we consider weakly nonlinear, discrete-time systems

$$(26) \quad x(t+1) = A(t)x(t) + \epsilon f(t, x(t))$$

subject to multipoint boundary conditions of the form

$$(27) \quad B_0x(0) + B_1x(1) + \dots + B_Nx(N) = 0.$$

Throughout our discussion we assume that for each t , $A(t)$ is a nonsingular $n \times n$ matrix, f is a continuously differentiable map from \mathbb{R}^{n+1} into \mathbb{R}^n , ϵ is a “small” real parameter, each matrix B_k is $n \times n$, and N is an integer larger than two.

We are primarily interested in systems at resonance; that is, problems where the corresponding linear homogeneous boundary value problem

$$(28) \quad x(t+1) = A(t)x(t)$$

subject to

$$(29) \quad B_0x(0) + B_1x(1) + \dots + B_Nx(N) = 0,$$

has nontrivial solutions.

We establish sufficient conditions for the solvability of (1)-(2) and we provide a qualitative analysis of the dependence of the solutions on the parameter ϵ . This analysis allows us to establish a connection between the solution sets of the nonlinear system (1)-(2) and the linear homogeneous boundary value problem (3)-(4).

We analyze (1)-(2) using the Lyapunov-Schmidt Procedure. Our analysis is completely self contained; however we include references, [4, 5, 6, 7, 11, 16, 17, 19, 21, 22] for those who wish to see a more abstract approach as well as for readers interested in other applications.

The results presented here extend previous work of Rodriguez [20] where the boundary value problems are nonlinear perturbations of Sturm-Liouville scalar equations and of Rodriguez [17] who considered only endpoint boundary conditions. The present chapter also complements other results in the area of discrete boundary value problems. The interested reader may consult [2, 3, 8, 9, 10, 12, 18].

3.2. PRELIMINARIES

We are concerned with the existence of solutions of

$$x(t+1) = A(t)x(t) + \epsilon f(t, x(t))$$

subject to

$$B_0x(0) + B_1x(1) + \dots + B_Nx(N) = 0.$$

We analyze equation (26)-(27) by using operators on finite dimensional sequence spaces. In order to do this, we introduce the following spaces:

$$X = \{\phi : \{0, 1, \dots, N\} \rightarrow \mathbb{R}^n : B_0\phi(0) + B_1\phi(1) + \dots + B_N\phi(N) = 0\},$$

and

$$Y = \{\psi : \{0, 1, \dots, N-1\} \rightarrow \mathbb{R}^n\}.$$

We will use the supremum norm on each of these finite dimensional spaces. For x in X we define

$$\|x\| = \sup_{t=0,1,\dots,N} |x(t)|$$

and for y in Y we define

$$\|y\| = \sup_{t=0,1,\dots,N-1} |y(t)|,$$

where $|\cdot|$ denotes the Euclidean norm on \mathbb{R}^n . It is evident that with these norms, X and Y are Banach Spaces. For $v = (v_1, v_2, \dots, v_m)$ in a product space of the form $V_1 \times V_2 \times \dots \times V_m$ we will let $\|v\| = \sum_{i=1}^m \|v_i\|_{V_i}$ where $\|\cdot\|_{V_i}$ denotes the norm on V_i .

We define the linear map $L : X \rightarrow Y$ by

$$(Lx)(t) = x(t+1) - A(t)x(t)$$

and $F : X \rightarrow Y$ is given by

$$(Fx)(t) = f(t, x(t)).$$

From this it follows that x is a solution of (26)-(27) if and only if:

$$(30) \quad Lx = \epsilon Fx.$$

$$\text{Let } \Phi(t) = \begin{cases} I & \text{if } t = 0 \\ A(t-1)A(t-2) \dots A(0) & \text{for } t = 1, 2, \dots \end{cases}$$

It is well known [3, 12, 15], that Φ is a fundamental matrix solution of $x(t+1) = A(t)x(t)$.

PROPOSITION 3.2.1. *The solution space of the linear homogeneous boundary value problem (28)-(29) has the same dimension as $\ker(B_0 + B_1\Phi(1) + \dots + B_N\Phi(N))$.*

Proof.

Clearly the solution space of (28)-(29) has the same dimension as $\ker(L)$, and

$$x \in \ker(L) \iff x(t+1) = A(t)x(t) \text{ and } B_0x(0) + B_1x(1) + \dots + B_Nx(N) = 0$$

$$\iff \text{there exists a vector } c \text{ in } \mathbb{R}^n \text{ such that } x(t) = \Phi(t)c \text{ and}$$

$$B_0\Phi(0)c + B_1\Phi(1)c + \dots + B_N\Phi(N)c = 0$$

$$\iff x(t) = \Phi(t)c \text{ where } c \in \ker(B_0 + B_1\Phi(1) + \dots + B_N\Phi(N)). \quad \square$$

Since $\ker(B_0 + B_1\Phi(1) + \dots + B_N\Phi(N))$ has dimension less than or equal to n , there exists vectors b_1, b_2, \dots, b_j , $0 \leq j \leq n$, in \mathbb{R}^n such that $\{b_1, b_2, \dots, b_j\}$ is a basis for $\ker(B_0 + B_1\Phi(1) + \dots + B_N\Phi(N))$.

DEFINITION 3.2.1. $S(t)$ is the $n \times j$ matrix whose i^{th} column is $\Phi(t)b_i$.

COROLLARY 3.2.1. x is in the kernel of L if and only if $x(t) = S(t)\alpha$ for some $\alpha \in \mathbb{R}^j$.

3.3. THE CASE OF INVERTIBLE L

We show that if $B_0 + B_1\Phi(1) + \dots + B_N\Phi(N)$ is invertible and $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuously differentiable, then the boundary value problem:

$$x(t+1) = A(t)x(t) + \epsilon f(t, x(t))$$

subject to

$$B_0x(0) + B_1x(1) + \dots + B_Nx(N) = 0,$$

has a solution for sufficiently small ϵ .

PROPOSITION 3.3.1. *L is invertible if and only if $B_0 + B_1\Phi(1) + \dots + B_N\Phi(N)$ is invertible. In this case, the bounded linear map $L^{-1} : Y \rightarrow X$ is defined by:*

$$\begin{aligned} (L^{-1}h)(t) &= -\Phi(t)D^{-1} \left[B_1\Phi(1)\Phi^{-1}(1)h(0) + \dots + B_N\Phi(N) \sum_{i=0}^{N-1} \Phi^{-1}(i+1)h(i) \right] \\ &\quad + \Phi(t) \sum_{i=0}^{t-1} \Phi^{-1}(i+1)h(i) \end{aligned}$$

where $D = B_0 + B_1\Phi(1) + \dots + B_N\Phi(N)$.

Proof.

Let $h \in Y$. $Lx = h$ if and only if there exists an element x in X that satisfies

$$x(t+1) = A(t)x(t) + h(t) \text{ for } t = 0, 1, \dots, N-1.$$

By the well known Variation of Constants formula [**3, 12, 15**], $Lx = h$ if and only if

$$(31) \quad x(t) = \Phi(t)x(0) + \Phi(t) \sum_{i=0}^{t-1} \Phi^{-1}(i+1)h(i)$$

and

$$(32) \quad B_0x(0) + B_1x(1) + \dots B_Nx(N) = 0.$$

Combining equations (31) and (32) we have $Lx = h$ if and only if

$$(33) \quad B_0x(0) + B_1(\Phi(1)x(0) + h(0)) + \dots B_N \left(\Phi(N)x(0) + \Phi(N) \sum_{i=0}^{N-1} \Phi^{-1}(i+1)h(i) \right) = 0$$

or equivalently

$$(34) \quad (B_0 + B_1\Phi(1) + \dots B_N\Phi(N))x(0) = - \left[B_1h(0) + \dots B_N\Phi(N) \sum_{i=0}^{N-1} \Phi^{-1}(i+1)h(i) \right]$$

Notice that $Lx = h$ if and only if there exists $x(0) \in \mathbb{R}^n$ that satisfies equation (34). If we can find such an $x(0)$, then we define x as in equation (31) and we will have $Lx = h$. Observe that if $B_0 + B_1\Phi(1) + \dots B_N\Phi(N)$ is invertible, then there is one and only one $x(0)$ that satisfies equation (34), which implies there is one and only one x that satisfies $Lx = h$, and thus L is invertible, with

$$\begin{aligned} (L^{-1}h)(t) &= -\Phi(t)D^{-1} \left[B_1\Phi(1)\Phi^{-1}(1)h(0) + \dots B_N\Phi(N) \sum_{i=0}^{N-1} \Phi^{-1}(i+1)h(i) \right] \\ &\quad + \Phi(t) \sum_{i=0}^{t-1} \Phi^{-1}(i+1)h(i) \end{aligned}$$

where $D = B_0 + B_1\Phi(1) + \dots B_N\Phi(N)$.

Likewise if there is one and only one x that satisfies $Lx = h$, then there is one and only one $x(0)$ that satisfies equation (34), and thus $B_0 + B_1\Phi(1) + \dots B_N\Phi(N)$ must be invertible. \square

In order to use arguments based on the Implicit Function Theorem, we now establish that F is continuously Frechet differentiable. Readers interested in Calculus on Banach Spaces may consult [11, 14].

PROPOSITION 3.3.2. *The map $F: X \rightarrow Y$ is continuously Frechet differentiable and $DF(\phi): X \rightarrow Y$ is given by:*

$$(DF(\phi))(h)(t) = \left(\frac{\partial f}{\partial x}(t, \phi(t)) \right) (h(t)).$$

Proof.

We will first show that F is differentiable.

Let $\phi \in X$, and let $A : X \rightarrow Y$ be defined by

$$(Ah)(t) = \frac{\partial f}{\partial x}(t, \phi(t))h(t).$$

Let $\epsilon > 0$. Since f is continuously differentiable, for each $t = 0, 1, \dots, N-1$ there exists $\delta_t > 0$ such that

$$|h(t)| < \delta_t \implies \left| f(t, \phi(t) + h(t)) - f(t, \phi(t)) - \frac{\partial f}{\partial x}(t, \phi(t))(h(t)) \right| < \epsilon|h(t)|.$$

Let $\|h\| < \delta$ where $\delta = \min_{t \in \{0, 1, \dots, N-1\}} \{\delta_t\}$.

Then $\|F(\phi + h) - F(\phi) - Ah\|$

$$\begin{aligned} &= \sup_{t \in \{0, 1, \dots, N-1\}} |F(\phi + h)(t) - F(\phi)(t) - (Ah)(t)| \\ &= \sup_{t \in \{0, 1, \dots, N-1\}} \left| f(t, \phi(t) + h(t)) - f(t, \phi(t)) - \frac{\partial f}{\partial x}(t, \phi(t))(h(t)) \right| < \epsilon\|h\| \end{aligned}$$

Therefore $\frac{\|F(\phi + h) - F(\phi) - Ah\|}{\|h\|} \leq \epsilon$, and thus F is differentiable, with

$$(DF(\phi))(h)(t) = \left(\frac{\partial f}{\partial x}(t, \phi(t)) \right) (h(t)).$$

We now must show that F is continuously differentiable. Let $\phi \in X$ and let $\epsilon > 0$.

Since f is continuously differentiable, for each $t \in \{0, 1, \dots, N-1\}$ there exists $\delta_t > 0$ such that

$$|\phi(t) - \psi(t)| < \delta_t \implies \left| \frac{\partial f}{\partial x}(t, \phi(t)) - \frac{\partial f}{\partial x}(t, \psi(t)) \right| < \epsilon.$$

Let $\delta = \min_{t \in \{0, 1, \dots, N-1\}} \{\delta_t\}$.

Then

$$\|\phi - \psi\| < \delta \implies \left| \frac{\partial f}{\partial x}(t, \phi(t)) - \frac{\partial f}{\partial x}(t, \psi(t)) \right| < \epsilon \text{ for all } t \in \{0, 1, \dots, N-1\}.$$

Note that for $\|h\| = 1$

$$\left| \frac{\partial f}{\partial x}(t, \phi(t))(h(t)) - \frac{\partial f}{\partial x}(t, \psi(t))(h(t)) \right| \leq \left| \frac{\partial f}{\partial x}(t, \phi(t)) - \frac{\partial f}{\partial x}(t, \psi(t)) \right|$$

for all $t \in \{0, 1, \dots, N-1\}$.

$$\text{Therefore } \|\phi - \psi\| < \delta \implies \|DF(\phi) - DF(\psi)\| = \sup_{\|h\|=1} \|DF(\phi)h - DF(\psi)h\|$$

$$= \sup_{\|h\|=1} \left(\sup_{t \in \{0, 1, \dots, N-1\}} \left| \frac{\partial f}{\partial x}(t, \phi(t))(h(t)) - \frac{\partial f}{\partial x}(t, \psi(t))(h(t)) \right| \right) < \epsilon$$

Therefore F is continuously differentiable. \square

THEOREM 3.3.1. *Suppose $B_0 + B_1\Phi(1) + \dots + B_N\Phi(N)$ is invertible. Then for each ϵ small enough, there is a solution to the boundary value problem (26)-(27).*

Proof.

Since $B_0 + B_1\Phi(1) + \dots + B_N\Phi(N)$ is invertible, then L^{-1} exists and x is a solution of (26)-(27) if and only if $x = \epsilon L^{-1}Fx$. If we define $T : \mathbb{R} \times X \rightarrow X$ by $T(\epsilon, x) = x - \epsilon L^{-1}Fx$, then x is a solution of (26)-(27) if and only if $T(\epsilon, x) = 0$. Clearly T is continuously Frechet differentiable, $T(0, 0) = 0$ and $\frac{\partial T}{\partial x}(0, 0)$ is the identity map which of course is a bijection. Therefore, by the Implicit Function Theorem, for each ϵ small enough, there exists x_ϵ such that $T(\epsilon, x_\epsilon) = 0$ and thus x_ϵ is a solution of (26)-(27). \square

3.4. THE CASE OF SINGULAR L

In order to analyze (26)-(27) using the Lyapunov Schmidt Procedure, we now construct projections onto the kernel and image of L .

DEFINITION 3.4.1. Define $P : X \rightarrow X$ by

$$(Px)(t) = S(t) (S(0)^T S(0))^{-1} S(0)^T x(0).$$

PROPOSITION 3.4.1. P is a projection onto $\ker(L)$.

Proof.

First we must show that $S(0)^T S(0)$ is invertible.

Claim 1: $S(0)^T S(0)$ is invertible.

proof of claim:

Let $c \in \mathbb{R}^j$ and assume $S(0)^T S(0)c = 0$. Then

$$\begin{aligned} c^T S(0)^T S(0)c &= 0 \\ \implies (S(0)c)^T (S(0)c) &= 0 \implies |S(0)c| = 0 \\ \implies S(0)c &= 0 \implies \Phi(0)b_1 c_1 + \cdots + \Phi(0)b_j c_j = 0 \quad (c = (c_1, \dots, c_j))^T \\ \implies c_1 b_1 + \cdots + c_j b_j &= 0 \implies c_i = 0 \text{ for all } i = 1, 2, \dots, j. \end{aligned}$$

Therefore $S(0)^T S(0)$ is invertible.

Claim 2: $P^2 = P$

proof of claim:

Let $x \in X$.

$$\begin{aligned}
(P(Px))(t) &= P(S(\cdot) (S(0)^T S(0))^{-1} S(0)^T x(0)) \\
&= S(t) (S(0)^T S(0))^{-1} S(0)^T \left[S(0) (S(0)^T S(0))^{-1} S(0)^T x(0) \right] \\
&= S(t) (S(0)^T S(0))^{-1} S(0)^T x(0) \\
&= (Px)(t).
\end{aligned}$$

Therefore $P^2 = P$.

Claim 3: $\text{Im}(P) = \text{Ker}(L)$

proof of claim:

Let $x \in X$, then

$$(Px)(t) = S(t) (S(0)^T S(0))^{-1} S(0)^T x(0) = S(t)\alpha,$$

where $\alpha = (S(0)^T S(0))^{-1} S(0)^T x(0) \in \mathbb{R}^j$.

Therefore $\text{Im}(P) \subset \text{ker}(L)$.

Now let $x \in \text{ker}(L)$. Then $x(t) = S(t)\beta$ for some $\beta \in \mathbb{R}^j$ and

$$(Px)(t) = S(t) (S(0)^T S(0))^{-1} S(0)^T S(0)\beta = S(t)\beta = x(t).$$

Therefore $x \in \text{Im}(P)$ and thus $\text{ker}(L) \subset \text{Im}(P)$, which implies $\text{Im}(P) = \text{ker}(L)$.

Since P is clearly bounded and linear, then Claims 1-3 verify that P is a projection onto $\text{ker}(L)$. \square

The following definition and lemmas are vital in the construction of a projection onto the image of L .

DEFINITION 3.4.2. Assume $\{c_1, c_2, \dots, c_j\}$ forms a basis for $\text{ker}((B_0 + B_1\Phi(1) + \dots + B_N\Phi(N))^T)$, for some vectors $c_1, c_2, \dots, c_j \in \mathbb{R}^n$. We define

$v_k : \{0, 1, \dots, N-1\} \rightarrow \mathbb{R}^n$ by

$$v_k(t) = \sum_{i=t+1}^N [B_i \Phi(i) \Phi^{-1}(t+1)]^T c_k$$

LEMMA 3.4.1. v_k is the zero map if and only if $c_k \in \bigcap_{i=0}^N \ker(B_i^T)$.

Proof.

If v_k is the zero map, then $v_k(t) = 0$ for all $t = 0, 1, \dots, N-1$. Observe the following telescoping effect:

$$\begin{aligned} v_k(N-1) = 0 &\implies B_N^T c_k = 0 \\ v_k(N-2) = 0 &\implies B_{N-1}^T c_k + [B_N \Phi(N) \Phi(N-1)^{-1}]^T c_k = 0 \\ &\implies B_{N-1}^T c_k = 0 \\ &\vdots \\ v_k(0) = 0 &\implies B_1^T c_k + \dots [B_N \Phi(N) \Phi(1)^{-1}]^T c_k = 0 \implies B_1^T c_k = 0 \end{aligned}$$

Therefore $c_k \in \bigcap_{i=1}^N \ker(B_i^T)$.

However, since $c_k \in \ker((B_0 + B_1 \Phi(1) + \dots + B_N \Phi(N))^T)$, then we must also have $c_k \in \bigcap_{i=0}^N \ker(B_i^T)$. The proof in the other direction is trivial. \square

LEMMA 3.4.2. If $c_k \notin \bigcap_{i=0}^N \ker(B_i^T)$ for each $k = 1, 2, \dots, j$, then $\{v_1, v_2, \dots, v_j\}$ is a linearly independent set.

Proof.

Assume $\sum_{k=1}^j \alpha_k v_k = 0$ for some real numbers $\alpha_1, \alpha_2, \dots, \alpha_j$. Then we must have $\sum_{k=1}^j \alpha_k v_k(t) = 0$ for all $t = 0, 1, \dots, N-1$.

Consider

$$\sum_{k=1}^j \alpha_k v_k(N-1) = \sum_{k=1}^j \alpha_k B_N^T c_k = 0.$$

Now, for all i such that $c_i \notin \ker(B_N^T)$, we must have $\alpha_i = 0$, since whenever $c_i \notin \ker(B_N^T)$, $B_N^T c_i$ is a basis element for the image of the map that sends each $x \in \ker(B_0 + B_1\Phi(1) + \dots + B_N\Phi(N)^T)$ into $B_N^T x$.

If there exists an integer k such that $c_k \in \ker(B_N^T)$, then we continue this process, letting $t = N - 2$.

Observe

$$\sum_{k=1}^j \alpha_k v_k(N-2) = \sum_{k \text{ such that } c_k \in \ker(B_N^T)} \alpha_k (B_{N-1})^T c_k = 0.$$

For all k such that $c_k \notin \ker(B_{N-1}^T)$, we must have $\alpha_k = 0$ since whenever $c_i \notin \ker(B_{N-1}^T)$, $B_{N-1}^T c_i$ is a basis element for the image of the map that sends each $x \in \ker(B_0 + B_1\Phi(1) + \dots + B_N\Phi(N)^T)$ into $B_{N-1}^T x$.

Now if there are any coefficients, α_k , which we have not yet shown to be 0, then $c_k \in \bigcap_{i=N-1}^N \ker(B_i^T)$ and we continue letting $t = N - 3$ and repeat this process if necessary until $t = 0$. If we indeed need to continue until $t = 0$, then there exists an integer k such that $c_k \in \bigcap_{i=2}^N \ker(B_i^T)$. However, for each of these c_k , $c_k \notin \ker(B_1^T)$, for if it were, since $c_k \in \ker((B_0 + B_1\Phi(1) + \dots + B_N\Phi(N))^T)$, we would also have to have $c_k \in \ker(B_0^T)$, which would contradict our hypothesis that $c_k \notin \bigcap_{i=0}^N \ker(B_i^T)$.

Finally, if we consider

$$\sum_{k=1}^j \alpha_k v_k(0) = \sum_{k \text{ such that } c_k \in \bigcap_{i=2}^N \ker(B_i^T)} \alpha_k (B_1)^T c_k = 0,$$

then all the remaining α_k must be 0, by the same argument presented for $t = N - 1$ and $t = N - 2$.

Since $\alpha_i = 0$ for all $i = 1 \dots j$, then $\{v_1, v_2, \dots, v_j\}$ is a linearly independent set.

□

REMARK 3.4.1. For the rest of the paper we will assume that for each k in $\{1, 2, \dots, j\}$, $c_k \notin \bigcap_{i=0}^N \ker(B_i^T)$.

DEFINITION 3.4.3. $\Psi(t)$ is the $n \times j$ matrix whose i^{th} column is $v_i(t)$.

PROPOSITION 3.4.2. If h is in Y , then h is in the image of L if and only if
$$\sum_{i=0}^{N-1} h^T(i)\Psi(i) = 0.$$

Proof.

Let $h \in Y$. From Proposition (3.3.1) we know h is in the image of L if and only if there exists an element x in X such that

$$B_0x(0) + B_1(\Phi(1)x(0) + h(0)) + \dots + B_N\left(\Phi(N)x(0) + \Phi(N)\sum_{i=0}^{N-1}\Phi^{-1}(i+1)h(i)\right) = 0$$

This holds if and only if

$$B_1h(0) + B_2\sum_{i=0}^1\Phi(2)\Phi^{-1}(i+1)h(i) + \dots + B_N\sum_{i=0}^{N-1}\Phi(N)\Phi^{-1}(i+1)h(i)$$

is in the image of $B_0 + B_1\Phi(1) + \dots + B_N\Phi(N)$.

Therefore we must have

$$(35) \quad \left(B_1h(0) + B_2\sum_{i=0}^1\Phi(2)\Phi^{-1}(i+1)h(i) + \dots + B_N\sum_{i=0}^{N-1}\Phi(N)\Phi^{-1}(i+1)h(i)\right)^T \beta = 0$$

for all $\beta \in \ker((B_0 + B_1\Phi(1) + \dots + B_N\Phi(N))^T)$.

Since $\{c_1, c_2, \dots, c_j\}$ spans $\ker((B_0 + B_1\Phi(1) + \dots + B_N\Phi(N))^T)$ we see that the above holds if and only if

$$(36) \quad \sum_{i=0}^{N-1} h^T(i) \left(\sum_{m=i+1}^N [B_m\Phi(m)\Phi^{-1}(i+1)]^T \right) c_k = 0, \text{ for each } k = 1, 2, \dots, j,$$

which is equivalent to

$$(37) \quad \sum_{i=0}^{N-1} h^T(i) v_k(i) = 0, \text{ for each } k = 1, 2, \dots, j.$$

Therefore, $h \in \text{Im}(L)$ if and only if $\sum_{i=0}^{N-1} h^T(i)\Psi(i) = 0$,

or equivalently $\sum_{i=0}^{N-1} \Psi^T(i)h(i) = 0$. \square

We have seen that if $c_k \notin \bigcap_{i=0}^N \ker(B_i^T)$ for each $k = 1, 2, \dots, j$, then $\{v_1, v_2, \dots, v_j\}$ is a linearly independent set. This allows us to make the following definition.

DEFINITION 3.4.4. *The map $W : Y \rightarrow Y$ is defined by*

$$(Wh)(t) = \Psi(t) \left(\sum_{i=0}^{N-1} \Psi(i)^T \Psi(i) \right)^{-1} \sum_{i=0}^{N-1} \Psi^T(i)h(i).$$

PROPOSITION 3.4.3. *The map E given by $E = I - W$ is a projection onto the image of L .*

Proof.

We must first show that W is well defined.

Claim 1: $\sum_{i=0}^{N-1} \Psi(i)^T \Psi(i)$ is invertible.

proof of claim: Let $\alpha \in \mathbb{R}^j$ and write $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_j)^T$.

Assume $\left(\sum_{i=0}^{N-1} \Psi(i)^T \Psi(i) \right) \alpha = 0$, then

$$\alpha^T \left(\sum_{i=0}^{N-1} \Psi(i)^T \Psi(i) \right) \alpha = 0 \implies \sum_{i=0}^{N-1} (\Psi(i)\alpha)^T \Psi(i)\alpha = 0 \implies \sum_{i=0}^{N-1} |\Psi(i)\alpha|^2 = 0.$$

Therefore $\Psi(i)\alpha = 0$ for all $i = 0, 1, \dots, N-1$, and thus

$$v_1(i)\alpha_1 + v_2(i)\alpha_2 + \dots + v_j(i)\alpha_j = 0 \text{ for each } i = 0, 1, \dots, N-1.$$

Since $\{v_1, v_2, \dots, v_j\}$ is a linearly independent set, then we must have $\alpha = 0$ and thus $\sum_{i=0}^{N-1} \Psi(i)^T \Psi(i)$ is invertible.

Since W is clearly bounded and linear, all we need to show is that W is a projection and $Im(E) = Im(L)$.

Claim 2: $W^2 = W$

proof of claim: Let $h \in \mathcal{Y}$

$$\begin{aligned} & (W(Wh))(t) \\ = & W \left(\Psi(\cdot) \left[\sum_{i=0}^{N-1} \Psi(i)^T \Psi(i) \right]^{-1} \sum_{k=0}^{N-1} \Psi(k)^T h(k) \right) \\ = & \Psi(t) \left[\sum_{\alpha=0}^{N-1} \Psi(\alpha)^T \Psi(\alpha) \right]^{-1} \sum_{\beta=0}^{N-1} \Psi(\beta)^T \Psi(\beta) \left[\sum_{i=0}^{N-1} \Psi(i)^T \Psi(i) \right]^{-1} \sum_{k=0}^{N-1} \Psi(k)^T h(k) \\ = & \Psi(t) \left[\sum_{i=0}^{N-1} \Psi(i)^T \Psi(i) \right]^{-1} \sum_{k=0}^{N-1} \Psi(k)^T h(k) = (Wh)(t) \\ \implies & W^2 = W \end{aligned}$$

Since W is a projection, then E must also be a projection.

Claim 3: $Im(E) = Im(L)$

proof of claim: Let $h \in \mathcal{Y}$

$$\begin{aligned}
\sum_{i=0}^{N-1} \Psi(i)^T (Eh)(i) &= \sum_{i=0}^{N-1} \Psi(i)^T (h(i) - (Wh)(i)) \\
&= \sum_{i=0}^{N-1} \Psi(i)^T h(i) \\
&\quad - \sum_{i=0}^{N-1} \Psi(i)^T \left(\Psi(i) \left[\sum_{k=0}^{N-1} \Psi(k)^T \Psi(k) \right]^{-1} \sum_{k=0}^{N-1} \Psi(k)^T h(k) \right) \\
&= \sum_{i=0}^{N-1} \Psi(i)^T h(i) - \sum_{k=0}^{N-1} \Psi(k)^T h(k) = 0.
\end{aligned}$$

Therefore $Eh \in Im(L)$ and thus $Im(E) \subset Im(L)$.

Now let $h \in Im(L)$. Then

$$(Eh)(t) = h(t) - \Psi(t) \left[\sum_{i=0}^{N-1} \Psi(i)^T \Psi(i) \right]^{-1} \sum_{k=0}^{N-1} \Psi(k)^T h(k) = h(t).$$

Therefore $Eh = h$, and thus $h \in Im(E)$ which implies $Im(E) = Im(L)$ \square

A very special case of the projection E appears in [17], where only two-point boundary value problems are considered.

Since P and E are continuous projections, we may now write $X = X_P \oplus X_{I-P}$ and $Y = Y_{I-E} \oplus Y_E$ where $X_P = Im(P)$, $X_{I-P} = Im(I-P)$, $Y_E = Im(E)$, and $Y_{I-E} = Im(I-E)$.

LEMMA 3.4.3. *The dimension of X_P is the same as the dimension of Y_{I-E} .*

Proof.

Since $c_k \notin \bigcap_{i=0}^N \ker(B_i^T)$ for each $k = 1, 2, \dots, j$, then by Lemma 3.4.2, the span of $\{v_1, v_2, \dots, v_j\}$ is a j dimensional space. Let $h \in Y$. Then

$$\begin{aligned} (Wh)(t) &= \Psi(t)(a_1, a_2, \dots, a_j)^T \quad \text{for some real numbers } a_1, a_2, \dots, a_j. \\ \implies (Wh)(t) &= a_1 v_1(t) + a_2 v_2(t) + \dots + a_j v_j(t) \quad \text{for each } t = 0, 1, \dots, N-1. \end{aligned}$$

Therefore $\text{Im}(W) \subset \text{span}\{v_1, v_2, \dots, v_j\}$.

Now let $h \in \text{span}\{v_1, v_2, \dots, v_j\}$. Then $h(t) = \Psi(t)(b_1, b_2, \dots, b_j)^T$ for some real numbers

b_1, b_2, \dots, b_j . Therefore

$$\begin{aligned} (Wh)(t) &= \Psi(t) \left[\sum_{i=0}^{N-1} \Psi(i)^T \Psi(i) \right]^{-1} \sum_{i=0}^{N-1} \Psi(i)^T \Psi(i) (b_1, b_2, \dots, b_j)^T \\ &= \Psi(t)(b_1, b_2, \dots, b_j)^T = h(t) \end{aligned}$$

Therefore $Wh = h$ and thus $h \in \text{Im}(W)$ which implies $\text{span}\{v_1, v_2, \dots, v_j\} = \text{Im}(W)$. Consequently both X_P and Y_{I-E} are j dimensional spaces. \square

Note that $L : X_{I-P} \rightarrow \text{Im}(L)$ is a bijection and thus there exists a bounded linear map $M : \text{Im}(L) \rightarrow X_{I-P}$ such that:

$$i) \quad LMh = h \quad \text{for all } h \in \text{Im}(L)$$

$$ii) \quad MLx = x_{I-P} \quad \text{for all } x \in X.$$

DEFINITION 3.4.5. Define $H : \mathbb{R} \times X_P \times X_{I-P} \rightarrow Y_{I-E} \times X_{I-P}$ by

$$H(\epsilon, x_P, x_{I-P}) = \begin{pmatrix} WF(x_P + x_{I-P}) \\ x_{I-P} - \epsilon MEF(x_P + x_{I-P}) \end{pmatrix}$$

PROPOSITION 3.4.4. For $\epsilon \neq 0$, $Lx = \epsilon Fx$ if and only if $H(\epsilon, x_P, x_{I-P}) = 0$.

Proof.

$$\begin{aligned}
Lx = \epsilon Fx &\iff E(Lx - \epsilon Fx) = 0 \text{ and } (I - E)(Lx - \epsilon Fx) = 0 \\
&\iff Lx = \epsilon E Fx \text{ and } \epsilon W Fx = 0 \\
&\iff x_{I-P} - \epsilon M E F(x_P + x_{I-P}) = 0 \text{ and } W F(x_P + x_{I-P}) = 0 \\
&\iff H(\epsilon, x_P, x_{I-P}) = 0 \quad \square
\end{aligned}$$

PROPOSITION 3.4.5. If F is differentiable, then H is differentiable for each $(\epsilon, x_P, x_{I-P}) \in \mathbb{R} \times X_P \times X_{I-P}$, with $DH(\epsilon, x_P, x_{I-P}) : \mathbb{R} \times X_P \times X_{I-P} \rightarrow Y_{I-E} \times X_{I-P}$ defined by

$$DH(\epsilon, x_P, x_{I-P})(\alpha, p, q) = \begin{pmatrix} WDF(x_P + x_{I-P})(p + q) \\ q - \alpha M E F(x_P + x_{I-P}) - \epsilon M E D F(x_P + x_{I-P})(p + q) \end{pmatrix}.$$

Proof.

Let $(\epsilon, x_P, x_{I-P}) \in \mathbb{R} \times X_P \times X_{I-P}$. Let $\omega > 0$. Let $\hat{\phi} = (p, q) \in X_P \times X_{I-P}$ and $\alpha \in \mathbb{R}$ with $\|(\alpha, \hat{\phi})\| < \min\{\delta_1, \delta_2, \delta_3\}$ where $\{\delta_1, \delta_2, \delta_3\}$ will be determined.

Now if we let $x = x_P + x_{I-P}$ and $\phi = p + q$, then

$$\begin{aligned}
& \|H(\epsilon + \alpha, x_P + p, x_{I-P} + q) - H(\epsilon, x_P, x_{I-P}) - DH(\epsilon, x_P, x_{I-P})(\alpha, p, q)\| \\
&= \left\| \begin{pmatrix} WF(x + \phi) - WF(x) - WDF(x)(\phi) \\ -\epsilon MEF(x + \phi) + \epsilon MEF(x) + \epsilon MEDF(x)(\phi) - \alpha MEF(x + \phi) + \alpha MEF(x) \end{pmatrix} \right\| \\
&= \|WF(x + \phi) - WF(x) - WDF(x)(\phi)\| + \|\epsilon MEF(x + \phi) - \epsilon MEF(x) - \epsilon MEDF(x)(\phi)\| + \\
&\quad \|\alpha MEF(x + \phi) - \alpha MEF(x)\| \\
&\leq \|W\| \|F(x + \phi) - F(x) - DF(x)(\phi)\| + |\epsilon| \|ME\| \|F(x + \phi) - F(x) - DF(x)(\phi)\| + \\
&\quad |\alpha| \|ME\| \|F(x + \phi) - F(x)\|.
\end{aligned}$$

Now a) Since F is differentiable, there exists $\delta_1 > 0$ such that

$$\|\phi\| < \delta_1 \implies \|F(x + \phi) - F(x) - DF(x)(\phi)\| \leq \frac{\omega}{2\|W\|} \|\phi\|.$$

b) Likewise, since F is differentiable, there exists $\delta_2 > 0$ such that

$$\|\phi\| < \delta_2 \implies \|F(x + \phi) - F(x) - DF(x)(\phi)\| \leq \frac{\omega}{2|\epsilon|\|ME\|} \|\phi\|.$$

c) Since F is continuous, there exists $\delta_3 > 0$ such that

$$\|\phi\| < \delta_3 \implies \|(x + \phi) - x\| < \delta_3 \implies \|F(x + \phi) - F(x)\| \leq \frac{\omega}{\|ME\|}.$$

d) $\|\phi\| = \|p + q\| \leq \|p\| + \|q\| \leq |\alpha| + \|p\| + \|q\| = \|(\alpha, \hat{\phi})\|.$

From statements a-d we can conclude that

$$\begin{aligned}
& \|H(\epsilon + \alpha, x_P + p, x_{I-P} + q) - H(\epsilon, x_P, x_{I-P}) - DH(\epsilon, x_P, x_{I-P})(\alpha, p, q)\| \\
& \leq \frac{\omega}{2}\|\phi\| + \frac{\omega}{2}\|\phi\| + |\alpha|\omega \\
& = \omega(\|\phi\| + |\alpha|).
\end{aligned}$$

Therefore

$$\begin{aligned}
& \frac{\|H(\epsilon + \alpha, x_P + p, x_{I-P} + q) - H(\epsilon, x_P, x_{I-P}) - DH(\epsilon, x_P, x_{I-P})(\alpha, p, q)\|}{\|(\alpha, p, q)\|} \\
& \leq \frac{\omega(\|\phi\| + |\alpha|)}{|\alpha| + \|p\| + \|q\|} \leq \omega.
\end{aligned}$$

and thus H is differentiable with

$$DH(\epsilon, x_P, x_{I-P})(\alpha, p, q) = \begin{pmatrix} WDF(x_P + x_{I-P})(p + q) \\ q - \alpha MEF(x_P + x_{I-P}) - \epsilon MEDF(x_P + x_{I-P})(p + q) \end{pmatrix}.$$

□

PROPOSITION 3.4.6. *If F is continuously differentiable then H is continuously differentiable.*

Proof.

Let $(\epsilon_1, x_P, x_{I-P}) \in \mathbb{R} \times X_P \times X_{I-P}$ and let $\omega > 0$.

Assume $\|(\epsilon_1, x_P, x_{I-P}) - (\epsilon_2, z_P, z_{I-P})\| < \delta$, where δ will be determined. If we let

$x = x_P + x_{I-P}$, and $z = z_P + z_{I-P}$, then

$$\begin{aligned}
& \|DH(\epsilon_1, x_P, x_{I-P}) - DH(\epsilon_2, z_P, z_{I-P})\| \\
&= \sup_{\|(\epsilon, h_P, h_{I-P})\|=1} \| (DH(\epsilon_1, x_P, x_{I-P}) - DH(\epsilon_2, z_P, z_{I-P})) (\epsilon, h_P, h_{I-P}) \| \\
&\leq \|W\| \|DF(x) - DF(z)\| + \|ME\| \|F(z) - F(x)\| \\
&+ \|\epsilon_2 MEDF(z) - \epsilon_1 MEDF(x)\|.
\end{aligned}$$

a) Since F is continuously differentiable, there exists $\delta_1 > 0$ such that

$$\|x - z\| < \delta_1 \implies \|DF(x) - DF(z)\| < \frac{\omega}{4\|W\|}.$$

b) Since F is continuous there exists $\delta_2 > 0$ such that

$$\|x - z\| < \delta_2 \implies \|F(z) - F(x)\| < \frac{\omega}{4\|ME\|}.$$

c) If we let $\alpha = \epsilon_2 - \epsilon_1$, then

$$\begin{aligned}
& \|\epsilon_2 MEDF(z) - \epsilon_1 MEDF(x)\| = \|(\epsilon_1 + \alpha)MEDF(z) - \epsilon_1 MEDF(x)\| \\
&\leq |\epsilon_1| \|ME\| \|DF(z) - DF(x)\| + |\epsilon_2 - \epsilon_1| \|ME\| \|DF(z)\|
\end{aligned}$$

d) Since F is continuously differentiable, there exists $\delta_3 > 0$ such that

$$\|x - z\| < \delta_3 \implies \|DF(z) - DF(x)\| < \frac{\omega}{4|\epsilon_1|\|ME\|} \text{ and } \|DF(z)\| < \|DF(x)\| + 1.$$

e) We can make $|\epsilon_2 - \epsilon_1| < \frac{\omega}{4\|ME\|(\|DF(x)\| + 1)}$

Therefore, if $\delta = \min\{\delta_1, \delta_2, \delta_3, \frac{\omega}{4\|ME\|(\|DF(x)\|+1)}\}$ then

$$\|DH(\epsilon_1, x_P, x_{I-P}) - DH(\epsilon_2, z_P, z_{I-P})\| \leq \|W\| \frac{\omega}{4\|W\|} + \frac{\omega}{4} + \frac{\omega}{4} + \frac{\omega}{4} \leq \omega,$$

and thus H is continuously differentiable \square

THEOREM 3.4.1. *Assume f is continuously differentiable and $c_k \notin \bigcap_{i=0}^N \ker(B_i^T)$ for each $k = 1, 2, \dots, j$. If there exists $\hat{\alpha} \in \mathbb{R}^j$ such that*

$$\sum_{i=0}^{N-1} f(i, S(i)\hat{\alpha})^T \Psi(i) = 0 \text{ and } \sum_{i=0}^{N-1} \Psi(i)^T \frac{\partial f}{\partial x}(i, S(i)\hat{\alpha}) S(i) \text{ is invertible,}$$

then for each ϵ small enough, there exists a solution, x_ϵ , to the boundary value problem (26)-(27). Furthermore, $\lim_{\epsilon \rightarrow 0} \|x_\epsilon - S(\cdot)\hat{\alpha}\| = 0$.

Proof.

Let $\hat{x}_P \in X_P$ be given by $\hat{x}_P(t) = S(t)\hat{\alpha}$. Since f is continuously differentiable, then F and H are continuously Frechet differentiable. Also, since $\sum_{i=0}^{N-1} f(i, \hat{x}_P(i))^T \Psi(i) = 0$, then $F(\hat{x}_P)$ is in the image of L , and thus $H(0, \hat{x}_P, 0) = 0$.

We now show that partial Frechet derivative of H with respect to (x_P, x_{I-P}) at $(0, \hat{x}_P, 0)$, $\frac{\partial H}{\partial(x_P, x_{I-P})}(0, \hat{x}_P, 0)$ is a bijection from $X_P \times X_{I-P}$ onto $Y_{I-E} \times X_{I-P}$.

First we show that $\frac{\partial H}{\partial(x_P, x_{I-P})}(0, \hat{x}_P, 0) : X_P \times X_{I-P} \rightarrow Y_{I-E} \times X_{I-P}$ is given by

$$\frac{\partial H}{\partial(x_P, x_{I-P})}(0, \hat{x}_P, 0)(h_1, h_2) = \begin{pmatrix} WDF(\hat{x}_P)(h_1 + h_2) \\ h_2 \end{pmatrix}.$$

Let $\omega > 0$.

$$\begin{aligned}
& \left\| H(0, \hat{x}_P + h_1, h_2) - H(0, \hat{x}_P, 0) - \frac{\partial H}{\partial(x_P, x_{I-P})}(0, \hat{x}_P, 0)(h_1, h_2) \right\| \\
&= \left\| \begin{pmatrix} W(F(\hat{x}_P + h_1 + h_2) - F(\hat{x}_P) - DF(\hat{x}_P)(h_1 + h_2)) \\ 0 \end{pmatrix} \right\| \\
&\leq \|W\| \|F(\hat{x}_P + h) - F(\hat{x}_P) - DF(\hat{x}_P)(h)\| \text{ where } h = h_1 + h_2.
\end{aligned}$$

Since F is differentiable, there exists $\delta > 0$ such that

$$\|h\| < \delta \implies \|F(\hat{x}_P + h) - F(\hat{x}_P) - DF(\hat{x}_P)(h)\| < \frac{\omega \|h\|}{\|W\|}.$$

Now if $\|(h_1, h_2)\| = \|h_1\| + \|h_2\| < \delta$, then $\|h_1 + h_2\| < \delta$, and we have

$$\begin{aligned}
& \frac{\|H(0, \hat{x}_P + h_1, h_2) - H(0, \hat{x}_P, 0) - \frac{\partial H}{\partial(x_P, x_{I-P})}(0, \hat{x}_P, 0)(h_1, h_2)\|}{\|(h_1, h_2)\|} \\
&< \frac{\|W\| \frac{\omega \|h\|}{\|W\|}}{\|(h_1, h_2)\|} = \frac{\omega \|h_1 + h_2\|}{\|(h_1, h_2)\|} \leq \omega.
\end{aligned}$$

Therefore

$$\frac{\partial H}{\partial(x_P, x_{I-P})}(0, \hat{x}_P, 0)(h_1, h_2) = \begin{pmatrix} WDF(\hat{x}_P)(h_1 + h_2) \\ h_2 \end{pmatrix}.$$

We now show that since X_P and Y_{I-E} have the same dimension, then $WDF(\hat{x}_P) : X_P \rightarrow Y_{I-E}$ a bijection if and only if $\sum_{i=0}^{N-1} \Psi(i)^T \frac{\partial f}{\partial x}(i, \hat{x}_P(i)) S(i)$ is invertible.

Assume $\sum_{i=0}^{N-1} \Psi(i)^T \frac{\partial f}{\partial x}(i, \hat{x}_P(i)) S(i)$ is invertible and let $h \in \ker(WDF(\hat{x}_P))$. Since $h \in X_P$, we can write $h(t) = S(t)\alpha$ for some $\alpha \in \mathbb{R}^j$. Since $h \in \ker(WDF(\hat{x}_P))$, then

$$\Psi(t) \left(\sum_{i=0}^{N-1} \Psi(i)^T \Psi(i) \right)^{-1} \sum_{i=0}^{N-1} \Psi(i)^T \frac{\partial f}{\partial x}(i, \hat{x}_P(i)) h(i) = 0 \text{ for all } t = 0, 1, \dots, N-1.$$

Since the columns of Ψ are linearly independent then

$$\left(\sum_{i=0}^{N-1} \Psi(i)^T \Psi(i) \right)^{-1} \sum_{i=0}^{N-1} \Psi(i)^T \frac{\partial f}{\partial x}(i, \hat{x}_p(i)) S(i) \alpha = 0,$$

and thus

$$\sum_{i=0}^{N-1} \Psi(i)^T \frac{\partial f}{\partial x}(i, \hat{x}_p(i)) S(i) \alpha = 0.$$

Since $\sum_{i=0}^{N-1} \Psi(i)^T \frac{\partial f}{\partial x}(i, \hat{x}_p(i)) S(i)$ is invertible, then we must have $\alpha = 0$ and thus $h = 0$.

Now assume $WDF(\hat{x}_p)$ is a bijection and assume there exists a nonzero vector $\alpha \in \mathbb{R}^j$ such that

$$\sum_{i=0}^{N-1} \Psi(i)^T \frac{\partial f}{\partial x}(i, \hat{x}_p(i)) S(i) \alpha = 0.$$

Let $h(i) = S(i)\alpha$. Then $h \in X_P$, $h \neq 0$ and

$$\sum_{i=0}^{N-1} \Psi(i)^T \frac{\partial f}{\partial x}(i, \hat{x}_p(i)) h(i) = 0,$$

which implies $WDF(\hat{x}_p)h = 0$. This contradicts the invertibility of $WDF(\hat{x}_p)$, and

therefore $\sum_{i=0}^{N-1} \Psi(i)^T \frac{\partial f}{\partial x}(i, \hat{x}_p(i)) S(i)$ is invertible.

Since $\sum_{i=0}^{N-1} \Psi(i)^T \frac{\partial f}{\partial x}(i, \hat{x}_p(i)) S(i)$ is invertible, then we now have that $WDF(\hat{x}_p)$ is a bijection.

Let $(h_1, h_2) \in \ker \left(\frac{\partial H}{\partial(x_P, x_{I-P})}(0, \hat{x}_p, 0) \right)$. Since

$$\begin{pmatrix} WDF(\hat{x}_p)(h_1 + h_2) \\ h_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ then } h_2 = 0.$$

Also, since $h_1 + h_2 \in \ker(WDF(\hat{x}_p))$, then $h_1 + h_2 = 0$ and thus $h_1 = 0$.

Therefore $\frac{\partial H}{\partial(x_P, x_{I-P})}(0, \hat{x}_p, 0)$ is one to one. Since the dimension of $X_P \times X_{I-P}$ is the same as the dimension of $Y_{I-E} \times X_{I-P}$ then $\frac{\partial H}{\partial(x_P, x_{I-P})}(0, \hat{x}_p, 0)$ is a bijection.

Since H is continuously differentiable, $H(0, \hat{x}_P, 0) = 0$, and $\frac{\partial H}{\partial(x_P, x_{I-P})}(0, \hat{x}_P, 0)$ is a bijection, then by the Implicit Function Theorem, for each ϵ small enough, there exists $(x_P, x_{I-P})_\epsilon$ such that $H(\epsilon, (x_P, x_{I-P})_\epsilon) = 0$. If we let $x_\epsilon = (x_P)_\epsilon + (x_{I-P})_\epsilon$, then $Lx_\epsilon = \epsilon F(x_\epsilon)$ and thus x_ϵ is a solution of (26)-(27). Furthermore, x_ϵ is a continuously differentiable function of ϵ and $\lim_{\epsilon \rightarrow 0} \|x_\epsilon - \hat{x}_P\| = 0$. \square

3.5. EXAMPLE

Consider

$$(38) \quad x(t+1) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} x(t) + \epsilon f(t, x(t))$$

subject to

$$(39) \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x(0) + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(1) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} x(N) = 0$$

To put this boundary value problem into our framework we let

$$A(t) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \text{ for each } t = 0, 1, 2, \dots,$$

$$\text{and we let } B_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B_N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Observe that since $A(t)$ is a constant matrix, then $\Phi(t) = A^t$.

$$\ker(B_0 + B_1 A + B_N A^N) = \ker \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}, \text{ and thus}$$

$$S(t) = \Phi(t) \begin{pmatrix} -1 \\ 1 \end{pmatrix} = A^t \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1-t \end{pmatrix}.$$

$$\ker((B_0 + B_1 A + B_N A^N)^T) = \ker \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}.$$

Note that $\begin{pmatrix} 1 \\ -2 \end{pmatrix} \notin \ker(B_0^T)$, and thus X_P and Y_{I-E} are both one dimensional

spaces with $\Psi(t)$ given by

$$\Psi(t) = \begin{cases} \begin{pmatrix} -2 \\ 1 \end{pmatrix} & \text{for } t = 0 \\ \begin{pmatrix} -2 \\ 0 \end{pmatrix} & \text{for } t = 1, 2, \dots, N-1. \end{cases}$$

Now assume

$$f(t, x_1, x_2) = \begin{pmatrix} f_1(x_1) \\ f_2(t, x_1, x_2) \end{pmatrix}$$

where $f_1 : \mathbb{R} \rightarrow \mathbb{R}$ and $f_2 : \mathbb{R}^3 \rightarrow \mathbb{R}$ are both continuously differentiable, $f_1(-\alpha) = 0$ for some real number α , $f_2(0, x_1, x_2) = 0$ and

$$2Nf_1'(-\alpha) + \frac{\partial f_2}{\partial x_1}(0, -\alpha, \alpha) + \frac{\partial f_2}{\partial x_2}(0, -\alpha, \alpha) \neq 0.$$

If we let $\hat{x}_p = S(t)\alpha$, then

$$\begin{aligned} & \sum_{k=0}^{N-1} f(k, \hat{x}_p(k))^T \Psi(k) \\ &= \sum_{k=0}^{N-1} \begin{pmatrix} f_1(-\alpha) \\ f_2(k, -\alpha, \alpha(1-k)) \end{pmatrix}^T \Psi(k) = f_2(0, -\alpha, \alpha) - 2Nf_1(-\alpha) = 0. \end{aligned}$$

Furthermore the real number

$$\begin{aligned} & \sum_{i=0}^{N-1} \Psi(i)^T \frac{\partial f}{\partial x}(i, \hat{x}_p(i)) S(i) \\ &= \sum_{i=0}^{N-1} \Psi(i)^T \frac{\partial f}{\partial x}(i, -\alpha, \alpha(1-i)) \begin{pmatrix} -1 \\ 1-i \end{pmatrix} \\ &= 2Nf_1'(-\alpha) + \frac{\partial f_2}{\partial x_1}(0, -\alpha, \alpha) + \frac{\partial f_2}{\partial x_2}(0, -\alpha, \alpha) \neq 0. \end{aligned}$$

Therefore by Theorem 3.4.1, for each ϵ small enough there is a unique solution to the boundary value problem (38)-(39). A simple example of a function f satisfying our criteria is given by

$$f(t, x_1, x_2) = \begin{pmatrix} x_1^2 + x_1 - 6 \\ t(x_1^2 + x_2^2) \end{pmatrix}.$$

Observe that $f_1(-3) = 0$, $f_2(0, x_1, x_2) = 0$, and

$$2Nf_1'(-3) + \frac{\partial f_2}{\partial x_1}(0, -3, 3) + \frac{\partial f_2}{\partial x_2}(0, -3, 3) = -10N \neq 0.$$

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