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Limit Design of Reinforced Concrete Plates and Shells Under Combined Membrane Forces and Bending Moments

M. Beyer

Rechen- und Entwicklungsinstitut für EDV im Bauwesen — RIB e.V., Schulze-Delitzsch-Str. 28, D-7000 Stuttgart 80, Germany

W. Haas

RIB — Software und Systemberatung im Bauwesen GmbH, Schulze-Delitzsch-Str. 28, D-7000 Stuttgart 80, Germany

1. Abstract

A new method for the design of reinforced concrete plates and shells subjected to arbitrary combinations of membrane forces and bending moments is presented. It is based on Baumann's method [1] . The forces in the reinforcement and in the compression zone of the concrete are in exact equilibrium with the membrane forces and bending moments. Results gained with the proposed method and results using a method proposed by Bazant [5] , [6] are compared. The project was sponsored by the German ministry for research and technology in the project for offshore structures.

2. Introduction

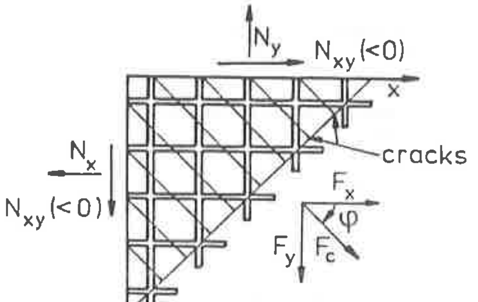
In the analysis of concrete plates or shells the computation of the reinforcement for the combined action of membrane forces and bending moments is a basic problem. Baumann presented in [1] a design method for concrete plates subjected to membrane forces. His main assumptions are the following:

- the reinforced concrete plate is represented by a truss model
- tension forces are carried by reinforcement and compression forces are carried by concrete
- concrete cannot carry any tension forces
- cracks are parallel to the direction of the compression force in concrete. This direction is calculated using the principle of minimum deformation energy.

In [1] Baumann applies this method also to the action of pure bending moments. The current paper describes the extension of Baumann's method to the combined action of membrane forces and bending moments. For the sake of simplicity, we will restrict the equations to rectangular web reinforcements. The equations for skew reinforcement are given in [2] .

3. Design of the reinforcement for membrane forces

We consider a reinforced plate subjected to the normal forces N_x, N_y and the shear force N_{xy}. Without any loss of generality, we can assume, that these forces have already been transformed to the directions of the reinforcement (Fig. 1).



The equilibrium condition between the internal forces N_x, N_y, N_{xy}, the reinforcement forces F_x, F_y and the compression force F_c is given by eqs. (1) to (4).

Fig. 1 Normal forces N_x, N_y, shear force N_{xy}, reinforcement forces F_x, F_y and compression force in concrete F_c

$$\mathbf{f} = \mathbf{T}\,\mathbf{n} \tag{1}$$

$$\mathbf{f} = \begin{bmatrix} F_x \\ F_y \\ F_c \end{bmatrix} \quad (2) \qquad \mathbf{n} = \begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} \quad (3) \qquad \mathbf{T}_m = \begin{bmatrix} 1 & 0 & -1/\tan\varphi \\ 0 & 1 & -\tan\varphi \\ 0 & 0 & 2/\sin 2\varphi \end{bmatrix} \quad (4)$$

For the calculation of φ , we further assume elastic behaviour of reinforcement and concrete. We obtain eq. (5) with eqs. (6) and (7).

$$\mathbf{e} = \mathbf{E}\,\mathbf{f} \tag{5}$$

$$\mathbf{e} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_c \end{bmatrix} \quad (6) \qquad \mathbf{E} = \begin{bmatrix} 1/(E_s A_x) & 0 & 0 \\ 0 & 1/(E_s A_y) & 0 \\ 0 & 0 & 1/(E_c d) \end{bmatrix} \quad (7)$$

where A_x, A_y are the areas of the reinforcement, d is the thickness of the plate and E_s, E_c are the elastic moduli of the reinforcement and the concrete.

Using eqs. (5), (6), (7), the deformation energy Φ can be calculated with eq. (8).

$$\Phi = \frac{1}{2}\int_V \sigma\varepsilon\, dV = \frac{1}{2}\int_V \frac{1}{d}\left(\frac{F_x^2}{E_s A_x} + \frac{F_y^2}{E_s A_y} + \frac{F_c^2}{E_c d}\right) dV \tag{8}$$

The principle of minimum deformation energy yields

$$\frac{d\Phi}{d\varphi} = \frac{1}{d}\left(\frac{F_x}{E_s A_x}\frac{dF_x}{d\varphi} + \frac{F_y}{E_s A_y}\frac{dF_y}{d\varphi} + \frac{F_c}{E_c d}\frac{dF_c}{d\varphi}\right) \tag{9}$$

Substituting eqs. (1), (5) into eq. (9) we obtain

$$\varepsilon_x \sin^2\left(\frac{\pi}{2}-\varphi\right) - \varepsilon_y \sin^2\varphi - \varepsilon_c \sin\left(\frac{\pi}{2}-2\varphi\right) = 0 \tag{10}$$

As shown in [2] eq. (10) physically means, that the cracks are opening vertically to the direction of the cracks.

We further assume that the strains $\varepsilon_x, \varepsilon_y$ in both directions are equal ε_s.

$$\varepsilon_x = \varepsilon_y = \varepsilon_s \tag{11}$$

By substituting eq. (11) into eq. (10), we obtain

$$\varphi = \pm \frac{\pi}{4} \tag{12}$$

wich physically means, that the direction of the cracks form angles φ = 45° to the directions of the reinforcement bars.

Eq. (12) is only valid if eq. (11) is satisfied. If we assume, that the reinforcement forces F_x, F_y are $\geqslant 0$, we must distinguish four different cases for the design of the reinforcement. These four cases are given in Tab. 1

Tab. 1 Reinforcement forces F_x, F_y for different membrane forces

case	condition		F_x	F_y	$\tan\varphi$
I	$N_2 \leq N_1 \leq 0$		0	0	–
II	$N_x > -\lvert N_{xy}\rvert$	$N_y > -\lvert N_{xy}\rvert$	$N_x + \lvert N_{xy}\rvert$	$N_y + \lvert N_{xy}\rvert$	$-\frac{N_{xy}}{\lvert N_{xy}\rvert}$
III	$N_1 > 0$	$N_y \leq -\lvert N_{xy}\rvert$	$N_x - \frac{N_{xy}^2}{N_y}$	0	$\frac{N_y}{N_{xy}}$
IV	$N_1 > 0$	$N_x \leq -\lvert N_{xy}\rvert$	0	$N_y - \frac{N_{xy}^2}{N_x}$	$\frac{N_{xy}}{N_x}$

In case I the concrete forces are identical with the principal membrane forces N_1, N_2. In case II-IV the concrete force F_c can be calculated using eq. (13)

$$N_1 + N_2 = N_x + N_y = F_x + F_y + F_c \tag{13}$$

The computation of the reinforcement forces according to Tab. 1 can be done graphically with Fig. 2. This Figure was introduced by Scharpf [3] .

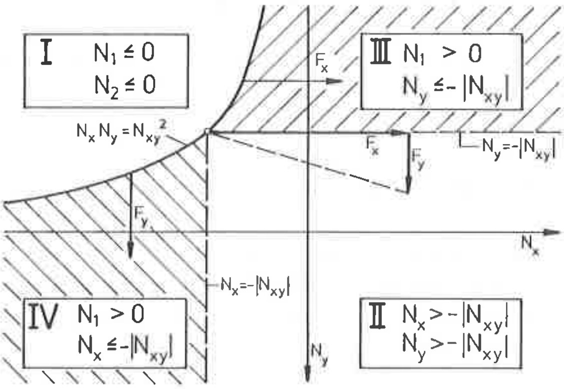


Fig. 2 Graphical construction of F_x, F_y

4. Design of the reinforcement for the combined action of membrane forces and bending moments

We first consider an excentric force F_i according to Fig. 3. Using equilibrium conditions we obtain the following eqs. for the contribution of F_i to the internal force vector

$$\mathbf{P}_i = \mathbf{t}_i F_i \qquad (14)$$

$$\mathbf{P}_i = \begin{bmatrix} N_{xi} \\ N_{yi} \\ N_{xyi} \\ M_{xi} \\ M_{yi} \\ M_{xyi} \end{bmatrix} \quad (15) \qquad \mathbf{t}_i = \begin{bmatrix} \cos^2\varphi_i \\ \sin^2\varphi_i \\ \cos\varphi_i \sin\varphi_i \\ z_i \cos^2\varphi_i \\ z_i \sin^2\varphi_i \\ z_i \cos\varphi_i \sin\varphi_i \end{bmatrix} \quad (16)$$

Eqs. (14), (15), (16) can be formulated for 4 reinforcement forces and the two compression forces of the lower and upper part of the concrete plate. The assumptions for the compression zone according to [4] are shown in Fig. 3

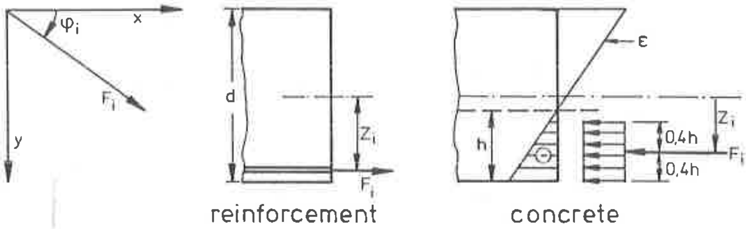


Fig. 3 Definition of force F_i with direction φ_i, and excentricity z_i

The six equations can be written in matrix form.

$$\mathbf{P} = \mathbf{T}\,\mathbf{F} \qquad (17)$$

$$\mathbf{P} = \sum_{1}^{6} \mathbf{P}_i \quad (18), \qquad \mathbf{T} = \{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4, \mathbf{t}_5, \mathbf{t}_6\} \quad (19), \qquad \mathbf{F} = \begin{bmatrix} F_x \\ F_y \\ F_c \\ F_x \\ F_y \\ F_c \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} \text{upper part} \\ \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} \text{lower part} \end{matrix} \quad (20)$$

Inversion of eq. 17 yields

$$\mathbf{F} = \mathbf{T}^{-1}\mathbf{P} \qquad (21)$$

which corresponds to the equilibrium condition eq. (1) for the action of membrane forces alone.

Since the coefficients of $\mathbf{T}$ are not known, the force vector F_i cannot be calculated directly. An iterative procedure must be applied on the basis of eq. 21, Fig. 3 and Tab. 1. Usually the iteration yields accurate results after a few steps.

5. Comparison with existing methods

Bažant and Tsubaki proposed in 5_ a new method for the design of the reinforcement in concrete plates, subjected to membrane forces. The main purpose of the new method is to avoid slip in the cracks.

In this slip free limit design, the reinforcement must not only balance the applied load but must also produce enough compression in concrete to develop the friction force that must accompany the sliding. Their method was not proposed to replace the classical method but as an alternative concept for situations, where the reduction in cracking and deformation is desirable.

Fig. 4 shows a comparison of results gained using the slip free design method [5] with results gained on the basis of Tab. 1 and the additional assumption according to Baumann [1] , that the reinforcement in any direction is at least 20 % of the reinforcement of the other direction. This "20 % rule" is an integral part of Baumann's classical design method. It guarantees, that the behaviour of the reinforced concrete is not much influenced by the following effects:

- minor changes of the directions of the principal normal forces and bending moments
- deviation of real cracks of the supposed direction.

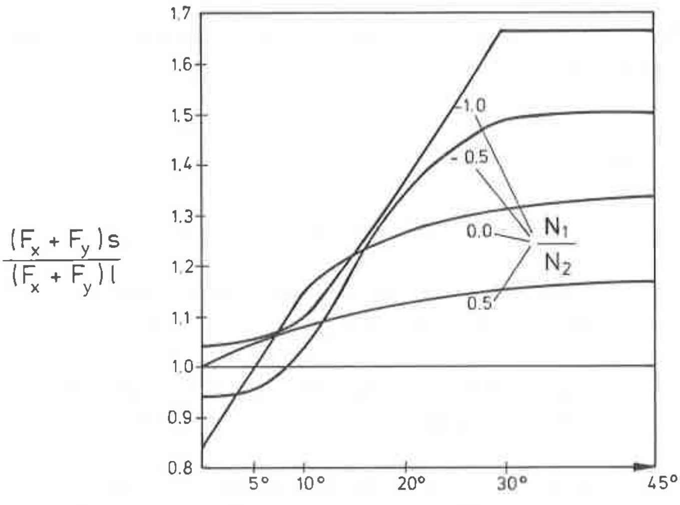


Fig. 4 $(F_x+F_y)s$=reinforcement forces according to [5]
$(F_x+F_y)_l$=reinforcement forces according to Tab. 1 and with "20 % rule"

In Fig. 4 we can see that only in cases, where the directions of the reinforcement deviate approximately less then 10° from the directions of the principal forces N_1, N_2, the slip free design method may yield less reinforcement than the classical method, which is due to the "20 % rule". In all other cases, the slip free design method yields more reinforcement than the classical method.

The slip-free design was extended in [6] to the combined action of membrane forces and bending moments. In this paper the equilibrium conditions eq. 4 in [6] are formulated for forces N_n, N_{nx} and moments M_n, M_{nx} in the plane t of the cracks. Therefore only these forces and moments are considered. The action of N_t and M_t parallel to t is neglected. These forces however give a contribution to the reinforcement forces and the compression force in the concrete. The neglection of N_t and M_t leads to an underdimensioning of the reinforcement and especially to an important underestimation of the compression force in concrete.

At the end, the authors would like to emphasize that their comments to the slip-free concept only consider the mathematical transposition of the method to the case of combined membrane forces and bending moments and not the basic ideas, since the experimental knowledge of the behaviour of reinforced plates under combined bending moments and membrane forces is far behind the theoretical knowledge [7] .

6. References

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