On the Three Dimensionnal Stress Intensity Factor Computation Using the Quarter Point Isoparametric Finite Element

Nguyen Dang Hung
University of Liège, Liege, Belgium
G. De Saxce
University of Technology of Compiègne, France

1. INTRODUCTION

The popularity of BARSOUM's collapsed quarter-point isoparametric element [1] is due to its simplicity in execution which does not require new subroutines to available 3-D finite element codes and their extension to fracture mechanic analysis. In fact, INGRAFFEA and MANU [2] have pointed out from this singular element some remarkable formulae giving directly stress intensity factors K_I, K_{II}, K_{III} in terms of the nodal displacements. It appears that in general, these formulae do not satisfy the rigid modes where one must expect to obtain no stress field and consequently no stress intensity factors. On the other hand, these formulae may lead to some drawbacks when loadings are imposed by temperature. The present paper tries to point out improved formulae with some additional terms which allow to satisfy the rigid modes and to give good results for thermal loading. One may find in the following numerical tests the ability and the accuracy of the new and correct formulae.

2. ANALYTICAL FORMULATION

Let us consider a 20-node isoparametric element surrounding the crack front such that the plane face ACFG of coordinate ξ=-1 constitutes one of the crack faces. After some calculations [2] INGRAFFEA and MANU propose the following formula for symmetric geometry and loading :

$$K_I = \frac{E}{4(1-\nu^2)}\sqrt{\frac{2\pi}{L_1}}\,\tilde{F}(v_i),\quad K_{II} = \frac{E}{4(1-\nu^2)}\sqrt{\frac{2\pi}{L_1}}\,\tilde{F}(u_i)\ ,\quad K_{III} = \frac{E}{4(1-\nu^2)}\sqrt{\frac{2\pi}{L_1}}\,\tilde{F}(w_i)$$

where E is YOUNG's modulus, ν is POISSON's ratio, L_1 is the element length along the crack face and $\tilde{F}$ is a function in terms of the nodal displacements (see figure 1).

$$\tilde{F}(d_i) = [2\,d_B - d_C + 2d_E - d_F + d_D + \frac{1}{2}\,\eta\,(-4d_B+d_C+4d_E-d_F)$$

$$+ 1/2\,\eta^2\,(d_F + d_C - 2d_D - d_A - d_G + 2d_H)] \qquad (2)$$

with d = v for K_I, d = u for K_{II} and d = w for K_{III}.

Let us imagine now a rigid translation of the structure ($d_B = d_C = d_E = d_F = d_G = d_H$), the function $F\ (\tilde{d}_i)$ is different to zero so that the program will provide stress intensity factors in case of no stress field.

This paradoxical situation may cause incorrect results when loadings are imposed by temperature and mostly when symmetry is not enough to fix the rigid modes. It appears that the deduction of the formula (1) has to be reformulated carefully. Let r the structural coordinate (figure 1) in the plan ACFG. One has :

$$r_A = r_H = r_G = 0,\ r_C = r_D = r_F = L_1, \quad r_B = r_E = L_1/4 \tag{3}$$

The shape function of the usual 2D-node isoparametric element are given by the following relationships :

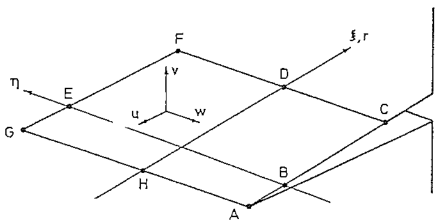

Figure 1.

For corner nodes :

$$N_i = \frac{1}{8} (1+\xi\xi_i)\ (1+\eta\eta_i)\ (1+\zeta\zeta_i)(\xi\xi_i+\eta\eta_i+\zeta\zeta_i - 2) \tag{4}$$

For mid-side nodes : B and E such that $\xi_i = 0$:

$$N_i = 1/4\ (1-\xi^2)(1+\eta\eta_i)(1+\zeta\zeta_i) \tag{5}$$

For mid-side nodes D and H such that $\eta_i = 0$:

$$N_i = 1/4\ (1-\eta^2)(1+\xi\xi_i)(1+\zeta\zeta_i) \tag{6}$$

Because the fact that r is zero for the nodes situated at the crack front, in the plan ACFG, one has for this isoparametric element :

$$\begin{aligned} r = {} & 1/2(1-\xi^2)(1+\eta)r_E + 1/2(1-\xi^2)(1-\eta)r_B + 1/4(1+\xi)(1+\eta)(\xi+\eta-1)r_F \\ & + 1/2(1-\eta^2)(1+\xi)\ r_D + 1/4\ (1+\xi)(1-\eta)(\xi - \eta - 1)\ r_C \end{aligned} \tag{7}$$

By taking account of (3), one obtains the following simplification :

$$r = \frac{L_1}{4} (\xi+1)^2, \text{ and inversely :} \quad \xi = -1 + 2\sqrt{\frac{r}{L_1}} \tag{8}$$

Now, with the same shape function (3-5), the in-plane displacement field is given by :

$$\begin{aligned} u = {} & -1/4\xi\ (1-\xi)[(1+\eta)\ u_G + (1-\eta)\ u_A] + 1/2\ (1-\xi^2)[(1+\eta)\ u_E + (1-\eta)\ U_B] \\ & - 1/4\ (1-\xi)(1-\eta^2)(u_A+u_G-2K_H) + 1/4\xi(1+\xi)[(1+\eta)\ u_F + (1-\eta)\ u_C] \\ & -1/4\ (1+\xi)(1-\eta^2)(u_C + u_F - 2\ u_D) \end{aligned}$$

Eliminate ξ with help of (8) one deduces :

$$\begin{aligned} u = {} & 1/2\ [(1+\eta)\ u_G + (1-\eta)\ u_A - (1-\eta^2)\ (u_A + u_G - 2u_H)] \\ & + \frac{r}{2L_1} [(1+\eta)(2u_G+2u_F-4u_E)+(1-\eta)(2u_A+2u_C-4u_B)] \end{aligned} \tag{9}$$

$$+ 1/2 \sqrt{\frac{r}{L_1}} \{(1+\eta)(4u_E-u_F-3u_G)+(1-\eta)(4u_B-u_C-3u_A)$$

$$+ (1-\eta^2) [(u_A+u_G-2u_H)-(u_C+u_F-2u_D)]\}$$

One may recognize in (9) that the first bracket contains the translation rigid modes, the second, the constant strain and the rotation modes and the third, the singular displacement field. Finally the identification of the field (9) with the theoretical displacement given by the fracture mechanics [2] on the crack face (θ = 180°) :

$$-v = \frac{2(1-\nu^2)}{E} \sqrt{\frac{r}{2\pi}} K_I, \quad u = \frac{2(1-\nu^2)}{E} \sqrt{\frac{r}{2\pi}} K_{II}, \quad w = \frac{2(1+\nu)}{E} \sqrt{\frac{r}{2\pi}} K_{III} \tag{10}$$

will lead to the formula :

$$K_I = \frac{E}{4(1-\nu^2)} \sqrt{\frac{2\pi}{L_1}} F(v_i), K_{II} = \frac{E}{4(1-\nu^2)} \sqrt{\frac{2\pi}{L_1}} F(u_i), K_{III} = \frac{E}{4(1+\nu)} \sqrt{\frac{2\pi}{L_1}} F(w_i) \tag{11}$$

with the new form of F (d_i)

$$F (d_i) = [(2\ d_B - d_C + 2\ d_E - d_F + d_D - 3/2\ d_A - 3/2\ d_G)$$

$$+ 1/2\ \eta\ (-4\ d_B + d_C + 4\ d_E - d_F + 3\ d_A - 3\ d_G)$$

$$+ 1/2\ \eta^2\ (d_F + d_C - 2\ d_D - d_A - d_G + 2d_H)] \tag{12}$$

where d = u,v,w corresponding respectively to K_{II}, K_I, K_{III}.

Comparing to (2), one may see that (12) contains certain additional terms which allow to satisfy completely the rigid modes.
It is important to mention that INGRAFFEA and MANU have presented in [2] some other correct formulae valuable for general cases of unsymmetrical geometry and loading and these formulae contain all rigid modes. However these formulae need nodal displacements belonged to both unrestrained crack surfaces and may not be employed in symmetric case where generally only a crack face is discretized using singular elements. The following example will point out some drawbacks provided from the INGRAFFEA-MANU's formulae of symmetric case.

3. NUMERICAL EXAMPLES

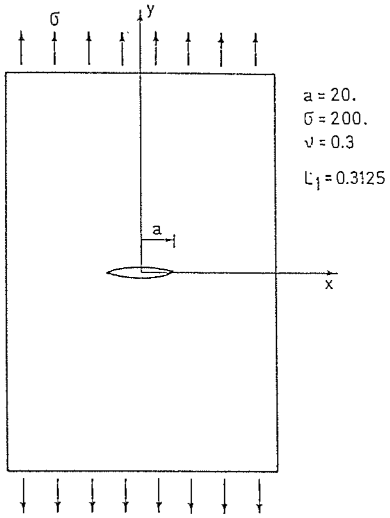

Figure 2.

3.1. Example 1

The first example is simply a centre cracked plate (figure 2) where because of symmetry, only a quarter of the structure is discretized into finite elements. The table 1 illustrates the comparison of results given respectively by an exact solution, INGRAFFEA-MANU's formulae (2) improved formulae (12).

The value of K_{II} given by the improved formula is good because near zero but that given by the INGRAFFEA-MANU's formula is strongly large ! This circumstance may be explained by the fact that the symmetry applied to the quarter of the plate cannot stop the transversal displacement on the crack tip :

$$u_A = a\,\epsilon_x = -\,a\,\nu\epsilon_y = -\,a\,\nu\,\frac{\sigma}{E}$$

By imposing the plane condition ($u_A = u_H = u_G, u_B = u_E, u_C = u_D = u_F$) and the rigid mode ($u_B = u_C = u_A$) in the formulae (1), one finds :

$$|K_{III}| = \frac{3E}{4(1-\nu^2)}\sqrt{\frac{2\pi}{L_1}}\,u_A = \frac{3\sigma a\nu}{4(1-\nu^2)}\sqrt{\frac{2\pi}{L_1}} = 13300$$

This value is very near the value obtained by the INGRAFFEA-MANU's formulae and this explains clearly that these formulae do not respect the rigid modes.

	K_I	K_{II}	K_{III}
Analytical solution	1625	0.	0.
INGRAFFEA-MANU's formula	1676.	13790.	0.
Improved formula	1676.	2.5	0.

Table 1. Comparison of results

3.2. Example 2

Let us consider a plane strain plate (fig. 3) with a centre crack subjected to a thermal gradient :

$$T(x) = T_o \left(\frac{x}{c}\right)^2$$

Along thickness	K_I/K_I^{th}	
	Present results (SAMCEF)	HELLEN (BERSAFE)
z = 0	1.046	1.003
z = t/2	1.017	0.913
z = t	1.062	0.954

Table 2. 2-D plate with crack subjected to temperature. Comparison of results.

The geometric and mechanic data of the problem are :

half the length of the crack $a = 20$
half the length of the plate $c = 100$
half the height of the plate $L = 250$
thickness : $t = 1$
YOUNG's modulus : $E = 10^6$
POISSON's ratio : $\nu = 0.3$
Thermal expansion coefficient : $\alpha = 10^{-5}$
Maximum temperature : $T_o = 100$

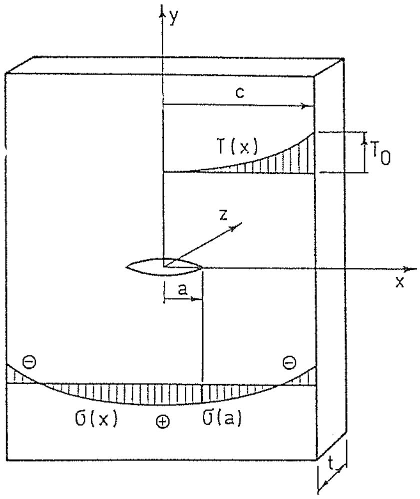

Figure 3.

A theoretical solution is given in [3] by HELLEN and CESARI basing on GREEN theory :

$$K_I^{th} = \frac{E\alpha T_o}{c^2}\sqrt{\pi a}\left(\frac{c^2}{3} - \frac{a^2}{2}\right) f\left(\frac{a}{c}\right), f(0.2) = 1.02 \qquad (13)$$

The table 2 presents the comparison of the results obtained from the formula (12) implemented in the SAMCEF package and those obtained by the V.C.E. method (Virtual Crack Extension) implemented by HELLEN [4] in the BERSAFE package. It is useful to mention that HELLEN has utilized 110 elements and 2544 degrees of freedom and the present analysis employed only 44 elements and 992 degrees of freedom. The length of the singular quarter-point elements along the face of the crack is $L_1 = 0.3125$.

The errors of both calculations are 3.2 % so that the present formulations seem very good.

9.3. Example 3

The same plate is considered but in the tridimensional conditions because the thickness t=10 is more important and the temperature varies both along the direction of the crack and the thickness (figure 4).

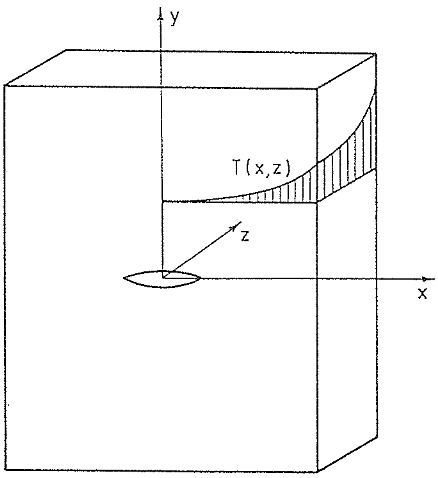

Figure 4.

$$T(x,z) = (z^2 + 50)\left(\frac{x}{c}\right)^2 \qquad (14)$$

All other geometric and mechanic data are the same as previous example.
Here there is no analytical solution but an approximated one may be performed using (13) with $T_o = T(z) = z^2 + 50$:

$$K_I^{app} = \frac{E\alpha}{c^2}(z^2+50)\sqrt{\pi a}\left(\frac{c^2}{3} - \frac{a^2}{2}\right) f\left(\frac{a}{c}\right) \qquad (15)$$

The present finite element calculation is compared with the HELLEN ones [4] and the approximate solution (15) in figure 5.

HELLEN's finite element mesh possesses 330 elements, 5466 degrees of freedom distributed on three layers along the thickness. The results concern the V.C.E. method.

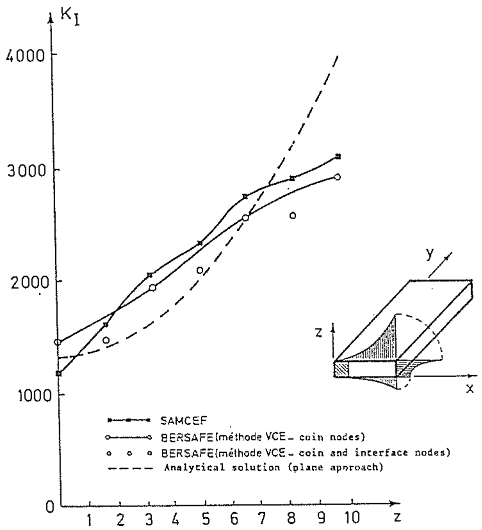

Figure 5.

The present results are obtained from a mesh of 132 elements and also three layers along the thickness. The plate is free on the face z = 0 and simply supported on the face y = L. To obtain the continuous curve, HELLEN has illuminated the results given by interface nodes because where the accuracy is somewhat poor for his V.C.E. method. One may see that the present results obtained from SAMCEF package are very near the best ones obtained by HELLEN from BERSAFE package. The detailed informations of the present paper may be found in [5].

4. CONCLUSIONS

The improved formulae presented in this paper give good results when the collapsed quarter-point isoparametric is used to obtain directly the intensity factors of cracked 3-D structures. These formulae are indispensable mostly for symmetric geometry and loading where only a crack face is discretized using singular elements.

ACKNOWLEDGMENT

The results obtained from SAMCEF package are realized under M.S.M. - S.E.P. - SAMTECH contract n° 87022. The authors are grateful to the responsibles of the University of Liège, Department M.S.M., the society SEP (Société Européenne des Propulsions) and the society SAMTECH (Bd Frère Orban, 25 4000 LIEGE) for the authorization of the publication of this paper.

REFERENCES

[1] BARSOUM, R.S. (1976). On the Use of Isoparametric Finite Elements in Linear Fracture Mechanics. Int. J. Numer. Meth. in Eng., vol. 10, pp. 25-37.

[2] INGRAFFEA, A.R., MANU, C. (1980). Stress-Intensity Factor Computation in Three-Dimensions with Quarter-Point Elements. Int. J. Num. Meth. in Eng., vol. 15, pp. 1427-1445.

[3] HELLEN, T.K., CESARI, K. (1979). On the Solution of the Center-Cracked Plate with a Quadratic Thermal Gradient. Eng. Fract. Mech., vol. 12, pp. 469-478.

[4] HELLEN, T.K. (1983). A substructuring application of the virtual crack extension method. Int. J. Numer. Meth. in Eng., vol. 19, pp. 1713-1717.

[5] NGUYEN DANG HUNG and DE SAXCE, G. (1988). Evaluation des facteurs d'intensité de contrainte de structures à 3 dimensions (in French). Internal Report n° 175, May 1988, Department M.S.M., University of Liège.