On the First Passage Problem in Random Vibration for Simple Nonlinear Oscillators

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Abstract

A robust Petrov-Galerkin finite element method is used to determine the first passage survival probability for simple single degree-of-freedom nonlinear oscillators. No assumptions are made about the narrow-bandedness of the response, width of failure bounds, or magnitude of the nonlinearity. The versatility of the method is illustrated for the classical case of the VanderPol oscillator for various failure bound widths and nonlinearities. The accuracy of the solution is then demonstrated by comparison of the finite element results with Monte Carlo simulation.

1. Introduction

An important problem in the study of dynamical systems is the determination of the probability that a system, when subjected to random excitation, will not malfunction during a specified period of time. When failure occurs upon the first excursion of the response process out of the prescribed safe region, the failure process is generally referred to as a first passage problem. This class of problems has been the subject of considerable interest in recent years due to its application to the more general system reliability problem. Previous research into the solution of the first passage problem, particularly as applied to simple structures of engineering interest, encompasses a vast array of workers and methodologies. These efforts, though, have yet to lead to an exact solution the first passage problem, even for the case of the single degree-of-freedom linear oscillator. Thus, an extensive body of approximation theory has arisen over the years, with most of the work pertaining to the linear oscillator. All structures, however, display some degree of nonlinearity; yet the first passage problem for nonlinear systems has been investigated by only a limited number of authors. These approximations to the solution of the first passage problem for nonlinear systems include: digital simulation of the response process, (Goldberg, et al. [1] and Pi, et al. [2]); a discrete analog of of the continuous theory Fokker-Planck method, or random walk model, (Toland and Yang [3]); modeling of the response envelope by a one dimensional Markov process (Roberts [4-6], Seshadri, et al. [7], Spanos [8]) and experimental investigation (Roberts and Yosuri [9]).

In this paper, a method for determining the survival probability for simple memoryless nonlinear oscillators is developed. The versatility of the method is demonstrated for the classical case of the VanderPol oscillator. These results are then compared with Monte Carlo simulation and an approximate solution found in Ref. [8].

2. Problem Statement

Consider the single degree-of-freedom nonlinear oscillator depicted in Fig. 1 whose response is governed by the stochastic differential equation

$$\ddot{X}(t) + H(X(t), \dot{X}(t)) = -\ddot{N}(t), \quad t > 0 \tag{1}$$

with deterministic initial conditions

$$X(0) = x_0, \quad \dot{X}(0) = \dot{x}_0 . \tag{2}$$

Here, $X(t)$ is the random displacement process, $H(\cdot,\cdot)$ is an operator representing the restoring and dissipative forces, and $\ddot{N}(t)$ is a Gaussian white noise applied as base acceleration, with mean and covariance

$$E[\ddot{N}(t)] = 0, \quad E[\ddot{N}(t)\ \ddot{N}(t+t')] = 2\pi S_0\ \delta(t') \tag{3}$$

where S_0 is the magnitude of the constant two-sided spectral density function and $\delta(\cdot)$ is the Dirac delta function.

It is well known [10] that the response given by $\mathbf{X}(t) = [X(t), \dot{X}(t)]$ is a vectored Markov process if the correlation time of the excitation is much smaller than the characteristic time of the oscillator. Then, from Markov process theory, the first passage problem can be stated as a well-posed initial-boundary value problem related to the backward Kolmogorov equation, which is given by [11]

$$\frac{\partial F}{\partial t} = \pi S_0 \frac{\partial^2 F}{\partial \dot{x}_0^{\,2}} + \dot{x}_0 \frac{\partial F}{\partial x_0} - H(x_0,\dot{x}_0)\frac{\partial F}{\partial \dot{x}_0} \tag{4}$$

$$F(t|B,\dot{x}_0) = 0, \quad \dot{x}_0 > 0 \tag{5a}$$

$$F(t|-B,\dot{x}_0) = 0, \quad \dot{x}_0 < 0 \tag{5b}$$

$$F(t|x_0,\dot{x}_0) = 0, \quad |\dot{x}_0| \to \infty \tag{5c}$$

where $F = F(t|x_0,\dot{x}_0)$ is the survival probability of the oscillator, the initial condition is $F(0|x_0,\dot{x}_0) = 1$, $(x_0,\dot{x}_0 \in \Omega)$, and the safe domain Ω given by (5) is depicted in Fig. 2.

Solution to (4,5) is sought over the bounded symmetric domain

$$\Omega_Y = \{(x_0,\dot{x}_0)|x_0 < B,\ \dot{x}_0 > 0;\ x_0 > -B,\ \dot{x}_0 < 0;\ |\dot{x}_0| < Y\} \tag{6}$$

where Y is a large number (see Fig. 2). The equations are cast into the weak form and discretized using a standard weighted residual formulation with bilinear shape functions and biquadratic weighting functions. The resulting system of ordinary differential equations is further discretized in time using the Crank-Nicholson method. A discussion of the finite element formulation used herein and its applicability to the first passage problem has been given by Bergman and Spencer [12].

3. Survival Probability for the VanderPol Oscillator

In order to demonstrate the utility of the method, the classical case of the VanderPol oscillator was considered. The nonlinear operator from Eq. (1) is then given by

$$H(X(t), \dot{X}(t)) = 2\zeta\omega_n\dot{X}\left(\frac{\varepsilon}{\sigma^2}X^2 - 1\right) + \omega_n^2 X \tag{7}$$

where $\sigma^2 = \pi S_0/2\zeta\omega_n^3$ is the variance of the stationary displacement response and ω_n is the undamped natural frequency of the related linear system. The initial-boundary value problem for the reliability of the VanderPol oscillator given by Eqs. (4,5) was solved as described previously. Nonlinear parameters $\varepsilon = 0.1$ and $\varepsilon = 0.5$ were examined for symmetric barrier widths $B = 2$ and $B = 3$, and damping ratio $\zeta = 0.01$. While the solution provides the survival probability and probability density function over the entire phase plane of initial conditions, due to space limitations the graphical results are presented only for the case of quiescent initial conditions. The converged results for $\varepsilon = 0.1$ and $\varepsilon = 0.5$ are presented in Figs. 3 and 4, respectively.

It was found that the survival probability of the oscillator was much more sensitive to nonzero initial velocity than to nonzero initial displacement. Also, the VanderPol oscillator is unstable about the origin for all values of ε, and thus the survival probability of the VanderPol oscillator is much lower than that of the related linear oscillator. The finite element solutions exhibit the expected oscillatory behavior [13] in the probability density functions at a frequency of two per cycle [12], while many of the available solutions average out this characteristic behavior of the solution.

In order to demonstrate the accuracy of the finite element method, extensive Monte Carlo simulation has been conducted for the VanderPol oscillator previously discussed. A Gaussian white noise was approximated by rectangular pulses, and the equations of motion were integrated in time using a 4th order Runge-Kutta scheme to assure accuracy. Ten thousand records were simulated in order to obtain a good approximation for the survival probability. Without loss of generality, $\omega_n = 1$ and $\sigma = 1$. The survival probabilities obtained from the simulation are superimposed on Figs. 3 and 4 for comparison. As can be seen, the Monte Carlo simulation cannot be distinguished from the finite element results for most of the time scale. In addition, results from Ref. [8] are overlaid for comparison. Finally, the first two ordinary moments of first passage time computed in Ref. [12] were used in conjunction with a maximum entropy distribution given by Dowson and Wragg [14] to obtain an approximate survival probability for the VanderPol oscillator. These results are also depicted in Figs. 3 and 4.

4. Conclusions

The first passage problem for simple nonlinear oscillators subjected to Gaussian white noise excitation has been solved by a Petrov-Galerkin finite element method. The survival probability has been obtained for the classical case of the VanderPol oscillator, and in the absence of exact analytical solutions, these results are the most accurate available to date. In addition, it should be remembered that the solution is obtained over the entire phase plane of initial conditions, while the Monte Carlo simulation applies to only a single set of initial conditions. Thus, hundreds of simulations would have to be performed in order to obtain the information contained in single finite element solution.

5. References

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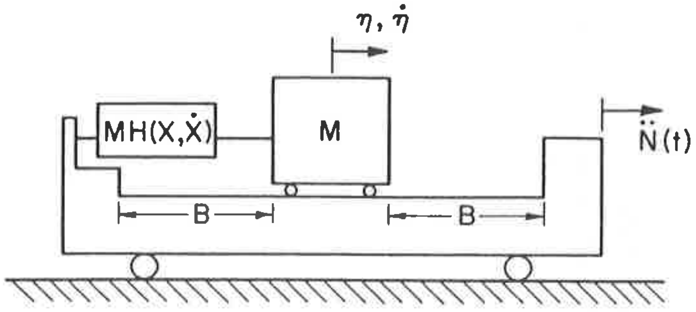
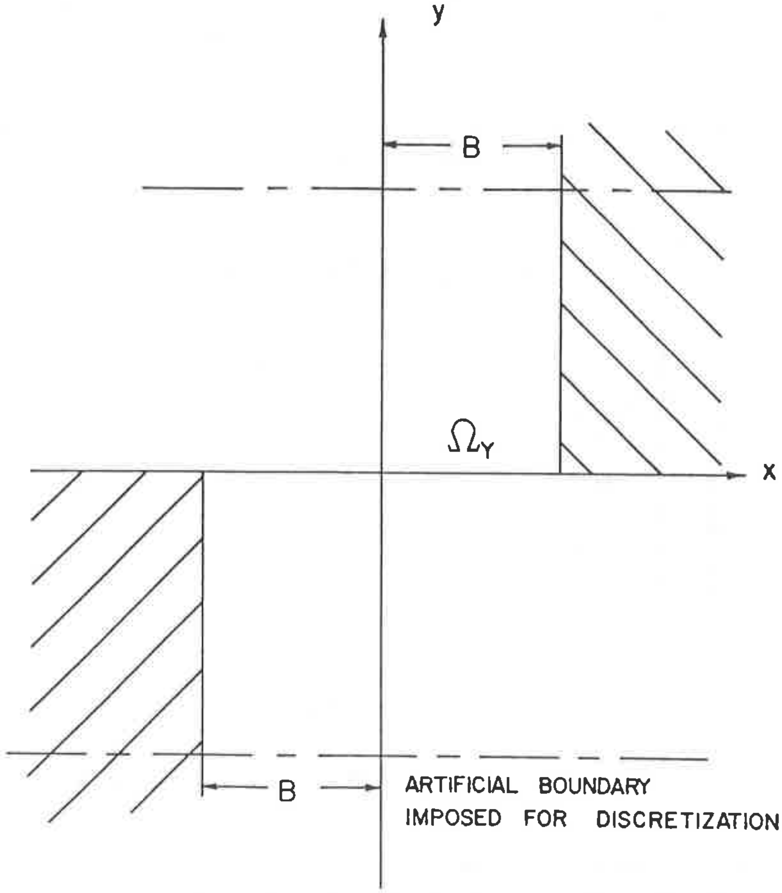
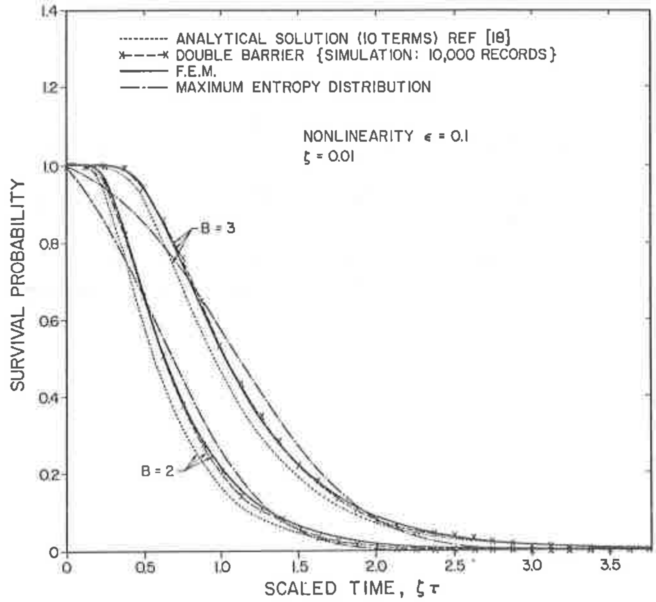


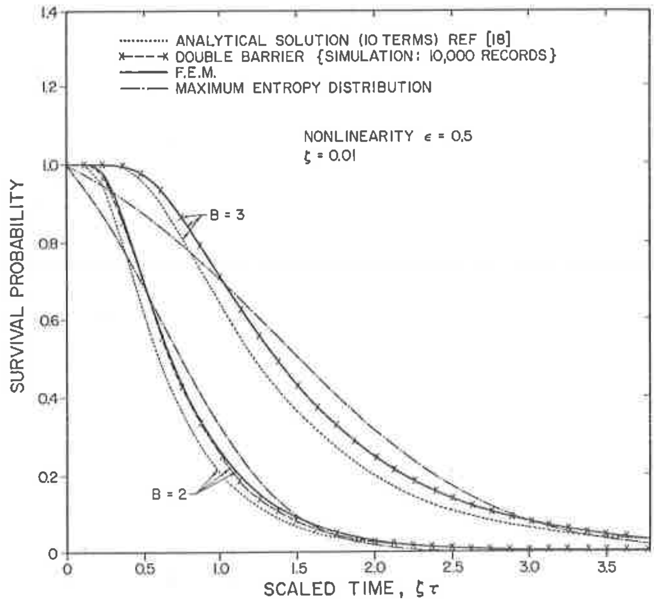
1) Simple nonlinear oscillator subjected to white noise excitation at the base.



2) The safe domain with artificial boundaries imposed for discretization.



3) Survival probability versus scaled time; comparison of F.E.M. results with maximum entropy distribution, simulation and approximate method of Ref. [8] for $\varepsilon = 0.1$.



4) Survival probability versus scaled time; comparison of F.E.M. results with maximum entropy distribution, simulation and approximate method of Ref. [8] for $\varepsilon = 0.5$.