Transient thermal stresses in a nonhomogeneous square cylinder with a square hole

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1 INTRODUCTION

The thermoelastic behavior of nonhomogeneous materials is important to estimate the strength of composite materials such as fiber-reinforced plastics and fiber-reinforced metals with local impurities at an elevated temperature. Originally, the interest in thermal stress in nonhomogeneous media arises from the fact that in general the mechanical and thermal properties of materials are highly temperature dependent. Thus, if such a dependence can be expressed in a certain mathematical model, the material properties become functions of temperature and hence of position within the body.

The thermal stresses in nonhomogeneous bodies under steady-state temperature fields have been analysed for an aeolotropic annular disc of variable thickness with thermal conductivity and elastic properties expressed in forms of the different power laws of distance from the center by Misra and Achari (1979) and for a thick plate with Kassir's nonhomogeneous elastic properties (1972) and a homogeneous thermal conductivity by Hata (1982). On the other hand Gosh and Mukherjee (1984) have presented a boundary element method formulation of thermoelastic problems in nonhomogeneous media. However, no transient thermal stress problems in nonhomogeneous media required particularly for the quantitative grasp on thermal shock effect at the starting in nuclear reactors, supersonic aircrafts and various types of turbines, and no thermal stress problems in multiply-connected regions made of various nonhomogeneous materials have been reported.

In this study we consider a plane-strain thermoelastic problem in a nonhomogeneous long square cylinder with a square hole as a formulation of plane thermoelastic problems in nonhomogeneous multiply-connected regions. The formulation is performed in terms of a stress function and new Michell's conditions are derived for an assurance of single-valuedness of the rotation and displacement components in multiply-connected regions. The effect of the nonhomogeneous material properties on temperature and thermal stresses is discussed from the results of numerical calculations which are carried out for exponential variations of the thermal and elastic properties with the rectangular Cartesian coordinate.

2 FORMULATION OF PLANE-STRAIN THERMOELASTIC PROBLEM IN A NONHOMOGENEOUS SQUARE HOLLOW CYLINDER BY STRESS FUNCTION METHOD

2.1 Temperature field

It is assumed that the nonhomogeneous square cylinder, initially at the same uniform temperature T_0 as the environments, is heated by the abrupt change T_1 in the temperature of the environment adjacent to the inner boundary and that the outer boundary surface dissipates heat by convection into the environment at constant temperature T_0 as illustrated in Fig. 1. We define the nonhomogeneous thermal conductivity, specific heat and density by $\lambda(x,y)$, $c(x,y)$ and $\rho(x,y)$, respectively. Then, the transient heat-conduction equation and the initial and boundary conditions in the nonhomogeneous square cylinder may be written as follows:

$$\frac{\partial}{\partial x}[\lambda(x,y)\frac{\partial T}{\partial x}] + \frac{\partial}{\partial y}[\lambda(x,y)\frac{\partial T}{\partial y}] = c(x,y)\rho(x,y)\frac{\partial T}{\partial t}, \tag{1}$$

$$T=T_0, \qquad \text{at } t=0 \tag{2}$$

$$\lambda(x,y)\frac{\partial T}{\partial x} + h_0(T-T_0)=0, \quad x=l_x \tag{3}$$

$$\lambda(x,y)\frac{\partial T}{\partial y} + h_0(T-T_0)=0, \quad y=l_y \tag{4}$$

$$\lambda(x,y)\frac{\partial T}{\partial x} - h_1(T-T_1)=0, \quad x=l'_x \tag{5}$$

$$\lambda(x,y)\frac{\partial T}{\partial y} - h_1(T-T_1)=0, \quad y=l'_y \tag{6}$$

where the heat transfer coefficients h_0 and h_1 on both boundary surfaces are taken to be constant.

2.2 Thermal stresses

Let us formulate the plane-strain thermoelastic problem in a nonhomogeneous long square cylinder with a square hole by stress function method, where all the elastic material properties are expressible as functions of the rectangular Cartesian coordinate. We denote Young's modulus, Poisson's ratio and the coefficient of linear thermal expansion with $E(x,y)$, $\nu(x,y)$ and $\alpha(x,y)$, respectively, the stress-strain relations for the plane-strain thermoelastic problem are then:

$$\begin{aligned} \varepsilon_{xx} &= \frac{1+\nu(x,y)}{E(x,y)}\{[1-\nu(x,y)]\sigma_{xx} - \nu(x,y)\sigma_{yy} + E(x,y)\alpha(x,y)T\} \\ \varepsilon_{yy} &= \frac{1+\nu(x,y)}{E(x,y)}\{[1-\nu(x,y)]\sigma_{yy} - \nu(x,y)\sigma_{xx} + E(x,y)\alpha(x,y)T\} \\ \varepsilon_{xy} &= \frac{1+\nu(x,y)}{E(x,y)}\sigma_{xy} \end{aligned} \tag{7}$$

We introduce a stress function χ satisfying the equations of equilibrium for the rectangular Cartesian coordinate system as follows;

$$\sigma_{xx} = \frac{\partial^2\chi}{\partial y^2}, \qquad \sigma_{yy} = \frac{\partial^2\chi}{\partial x^2}, \qquad \sigma_{xy} = -\frac{\partial^2\chi}{\partial x\partial y} \tag{8}$$

The compatibility equation of strain may be expressed in terms of this stress function as follows;

$$\nabla^4\chi + [F_1(x,y)\frac{\partial}{\partial x} + F_2(x,y)\frac{\partial}{\partial y}]\nabla^2\chi + F_3(x,y)\frac{\partial^2\chi}{\partial x^2} + F_4(x,y)\frac{\partial^2\chi}{\partial y^2}$$

$$+ F_5(x,y)\frac{\partial^2\chi}{\partial x\partial y} = -E(x,y)a_1(x,y)\nabla^2[a_1(x,y)\alpha(x,y)T/a_2(x,y)] \tag{9}$$

where

$$
\left.
\begin{aligned}
& a_1(x,y)=1/[1-\nu^2(x,y)], \quad a_2(x,y)=1/[1-\nu(x,y)] \\
& F_1(x,y)=-2\{f_1(x,y)+2E(x,y)[a_2(x,y)-a_1(x,y)]f_6(x,y)\}/E(x,y) \\
& F_2(x,y)=-2\{f_2(x,y)+2E(x,y)[a_2(x,y)-a_1(x,y)]f_7(x,y)\}/E(x,y) \\
& F_3(x,y)=2f_1^2(x,y)/E^2(x,y)-2f_2^2(x,y)[a_2(x,y)-1]/E^2(x,y)+4[a_2(x,y) \\
& -a_1(x,y)]f_1(x,y)f_6(x,y)/E(x,y)+2[2a_2(x,y)-a_1(x,y)]f_2(x,y)f_7(x,y) \\
& /E(x,y)-f_3(x,y)/E(x,y)+f_4(x,y)[a_2(x,y)-1]/E(x,y)-2a_1(x,y)f_6^2(x,y) \\
& -2a_1(x,y)f_7^2(x,y)-2[a_2(x,y)-a_1(x,y)]f_8(x,y)-[2a_2(x,y)-a_1(x,y)] \\
& \times f_9(x,y), \quad F_4(x,y)=-2f_1^2(x,y)[a_2(x,y)-1]/E^2(x,y)+2f_2^2(x,y)/E^2(x,y) \\
& +2[2a_2(x,y)-a_1(x,y)]f_1(x,y)f_6(x,y)/E(x,y)+4[a_2(x,y)-a_1(x,y)] \\
& \times f_2(x,y)f_7(x,y)/E(x,y)+f_3(x,y)[a_2(x,y)-1]/E(x,y)-f_4(x,y)/E(x,y) \\
& -2a_1(x,y)f_6^2(x,y)-2a_1(x,y)f_7^2(x,y)-[2a_2(x,y)-a_1(x,y)]f_8(x,y) \\
& -2[a_2(x,y)-a_1(x,y)]f_9(x,y), \quad F_5(x,y)=2[2a_2(x,y)f_1(x,y)f_2(x,y) \\
& /E^2(x,y)-a_1(x,y)f_1(x,y)f_7(x,y)/E(x,y)-a_1(x,y)f_2(x,y)f_6(x,y) \\
& /E(x,y)-a_2(x,y)f_5(x,y)/E(x,y)+a_1(x,y)f_{10}(x,y)] \\
& f_1(x,y)=\frac{\partial E(x,y)}{\partial x}, \quad f_2(x,y)=\frac{\partial E(x,y)}{\partial y}, \quad f_3(x,y)=\frac{\partial^2 E(x,y)}{\partial x^2}, \quad f_4(x,y) \\
& =\frac{\partial^2 E(x,y)}{\partial y^2}, \quad f_5(x,y)=\frac{\partial^2 E(x,y)}{\partial x \partial y}, \quad f_6(x,y)=\frac{\partial \nu(x,y)}{\partial x}, \quad f_7(x,y)=\frac{\partial \nu(x,y)}{\partial y} \\
& f_8(x,y)=\frac{\partial^2 \nu(x,y)}{\partial x^2}, \quad f_9(x,y)=\frac{\partial^2 \nu(x,y)}{\partial y^2}, \quad f_{10}(x,y)=\frac{\partial^2 \nu(x,y)}{\partial x \partial y}.
\end{aligned}
\right\} \quad (10)
$$

The traction-free boundary conditions on the inner and outer boundary surfaces may be expressed in terms of the stress function χ as follows:

$$
(\chi)_{p_1}=c_1 x_{p_1}+c_2 y_{p_1}+c_3, \quad (\partial\chi/\partial n')_{p_1}=c_1\cos(x,n')_{p_1}+c_2\cos(y,n')_{p_1} \quad \text{(on inner boundary } C_1\text{)} \quad (11)
$$

$$
(\chi)_{p_0}=(\partial\chi/\partial n')_{p_0}=0, \quad \text{(on outer boundary } C_0\text{)} \quad (12)
$$

where p_0 and p_1 are arbitrary points on the outer and inner boundary surfaces, respectively and n' is an arbitrary direction which does not coincide with the tangential direction to C_0 or C_1.

Michell(1899) have pointed out that three unknown constants appearing in the boundary conditions for the multiply-connecetd region should be determined from the conditions for single-valuedness of the rotation and displacements. But there seems to be no results reported on the conditions that are necessary for the assurance of single-valuedness of the rotation and displacements in nonhomogeneous multiply-connected regions. In this study, therefore, we derived the integral conditions taking into account arbitrary nonhomogeneous Young's modulus, Poisson's ratio and the coefficient of linear thermal expansion as follows:

$$
\begin{aligned}
& \oint_L \frac{\partial}{\partial n}[\nabla^2\chi/E(x,y)+a_2(x,y)\alpha(x,y)T]/a_1(x,y)ds + \oint_L \frac{\partial}{\partial y}[a_2(x,y) \\
& /a_1(x,y)E(x,y)]d\left(\frac{\partial\chi}{\partial x}\right) - \oint_L \frac{\partial}{\partial x}[a_2(x,y)/a_1(x,y)E(x,y)]d\left(\frac{\partial\chi}{\partial y}\right)=0 \qquad (13) \\
& \oint_L \left(x\frac{\partial}{\partial s} - y\frac{\partial}{\partial n}\right)[\nabla^2\chi/E(x,y) + a_2(x,y)\alpha(x,y)T]/a_1(x,y)ds + \oint_L [a_2(x,y) \\
& /a_1(x,y)E(x,y)]d\left(\frac{\partial\chi}{\partial x}\right) + \oint_L y\frac{\partial}{\partial x}[a_2(x,y)/a_1(x,y)E(x,y)]d\left(\frac{\partial\chi}{\partial y}\right)
\end{aligned}
$$

$$-\oint_L y\frac{\partial}{\partial y}[a_2(x,y)/a_1(x,y)E(x,y)]d(\frac{\partial\chi}{\partial x})=0 \quad (14)$$

$$\oint_L (y\frac{\partial}{\partial s} + x\frac{\partial}{\partial n})[\nabla^2\chi/E(x,y)+a_2(x,y)\alpha(x,y)T]/a_1(x,y)ds + \oint_L [a_2(x,y)/a_1(x,y)E(x,y)]d(\frac{\partial\chi}{\partial y}) - \oint_L x\frac{\partial}{\partial x}[a_2(x,y)/a_1(x,y)E(x,y)]d(\frac{\partial\chi}{\partial y})$$
$$+\oint_L x\frac{\partial}{\partial y}[a_2(x,y)/a_1(x,y)E(x,y)]d(\frac{\partial\chi}{\partial x})=0 \quad (15)$$

where L is an arbitrary closed integral path including the inner boundary curve C_1 in the interior of the nonhomogeneous square hollow cylinder, n is the normal direction to L and s denotes the arc length along L.

With equations (8)-(15) the formulation by stress function method is completed for the plane-strain thermoelastic problem in the nonhomogeneous square long cylinder with a square hole.

3 FINITE DIFFERENCE REPRESENTATION

We take finite diffrence grids with spatial intervals Δx and Δy and Δt as a time interval, and use the subscript i,j and the superscript k to denote some quantities on the discrete points. The fundamental differential equation (9) for the plane-strain thermoelastic problem in the nonhomogeneous square hollow cylinder may be then written in the following finite difference form:

$$A_1\bar{\chi}_{i,j}+A_2\bar{\chi}_{i+1,j}+A_3\bar{\chi}_{i-1,j}+A_4\bar{\chi}_{i,j+1}+A_5\bar{\chi}_{i,j-1}+A_6\bar{\chi}_{i+2,j}+A_7\bar{\chi}_{i-2,j}+A_8\bar{\chi}_{i,j+2}$$
$$+A_9\bar{\chi}_{i,j-2}+A_{10}\bar{\chi}_{i+1,j+1}+A_{11}\bar{\chi}_{i+1,j-1}+A_{12}\bar{\chi}_{i-1,j+1}+A_{13}\bar{\chi}_{i-1,j-1}=A_{14} \quad (16)$$

where $\bar{\chi}_{i,j}=\chi_{i,j}/[l_x E_0\alpha_0(T_1-T_0)]$, the nonhomogeneous Young's modulus, Poisson's ratio and coefficient of linear thermal expansion are disjointed into two factors, the first one dimensional and invariant, denoted by the subscript nought, and the second one dimensionless and a function of nondimensional coordinates (ξ,η), denoted by an asterisk:

$$E=E_0E^*(\xi,\eta), \quad \alpha=\alpha_0\alpha^*(\xi,\eta), \quad \nu=\nu_0\nu^*(\xi,\eta), \quad (17)$$

and the constants A_1-A_{14} can be expressed with the values of Young's modulus, Poisson's ratio, linear thermal expansion coefficient and temperature on the grid points.

New Michell's conditions (13)-(15) for the assurance of single-valuedness of the rotation and displacements may be written in finite difference forms as follows:

$$\oint_L (B_{1m}\bar{\chi}_{i,j}+B_{2m}\bar{\chi}_{i+1,j}+B_{3m}\bar{\chi}_{i-1,j}+B_{4m}\bar{\chi}_{i,j+1}+B_{5m}\bar{\chi}_{i,j-1}+B_{6m}\bar{\chi}_{i+2,j}+B_{7m}\bar{\chi}_{i-2,j}$$
$$+B_{8m}\bar{\chi}_{i,j+2}+B_{9m}\bar{\chi}_{i,j-2}+B_{10m}\bar{\chi}_{i+1,j+1}+B_{11m}\bar{\chi}_{i+1,j-1}+B_{12m}\bar{\chi}_{i-1,j+1}+B_{13m}$$
$$\times\bar{\chi}_{i-1,j-1}+B_{14m})d\bar{s}=0 \quad (18)$$

where $d\bar{s}=ds/l_x$, the subscripts m=1,2,3 denote the finite difference representations of equations (13)-(15), respectively. Since the problem considered in this study is symmetric with respect to the x and y axes, the conditions (14) and (15) for the single-valuedness of the displacements in the x and y directions are identically satisfied and the unknown constants c_1 and c_2 in equations (11) become zero because of the symmetry of stress function. The integrations (18) can be estimated by the use of numerical integration methods.

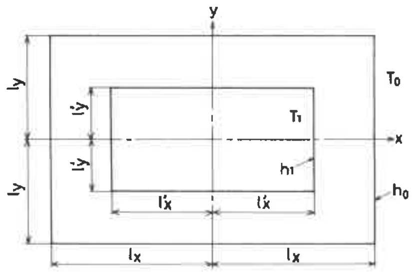

Fig.1 Nonhomogeneous square cylinder with square hole.

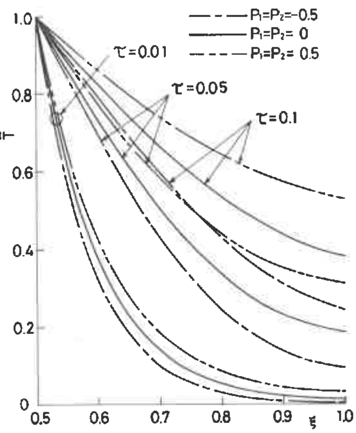

Fig.2 Temperature distribution.

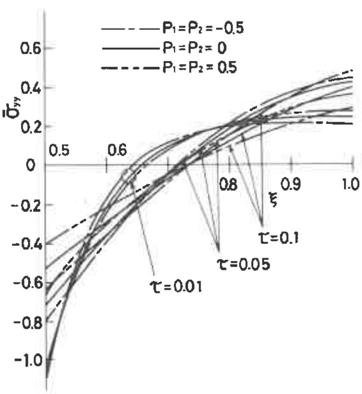

Fig.3 $\bar{\sigma}_{yy}$ on x axis.

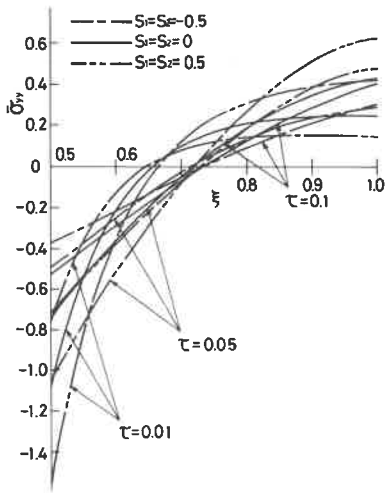

Fig.4 $\bar{\sigma}_{yy}$ on x axis.

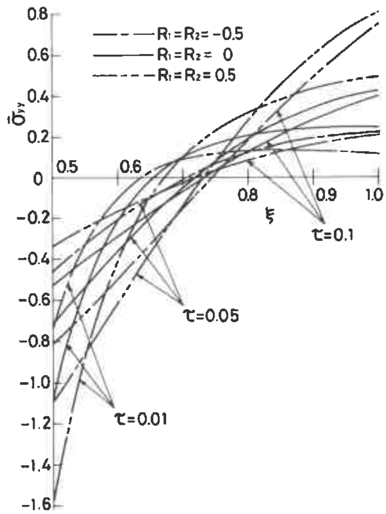

Fig.5 $\bar{\sigma}_{yy}$ on x axis.

By the finite difference method solving numerically the plane-strain thermoelastic problem in the nonhomogeneous square hollow cylinder formulated in terms of stress function is reduced to the determination of the values of the stress function at the grid points in the interior of the cylinder and of the unknown constant c_3 by solving the simultaneous equations obtained from the application of equation (16) to the same grid points and the numerical integration of equation (18).

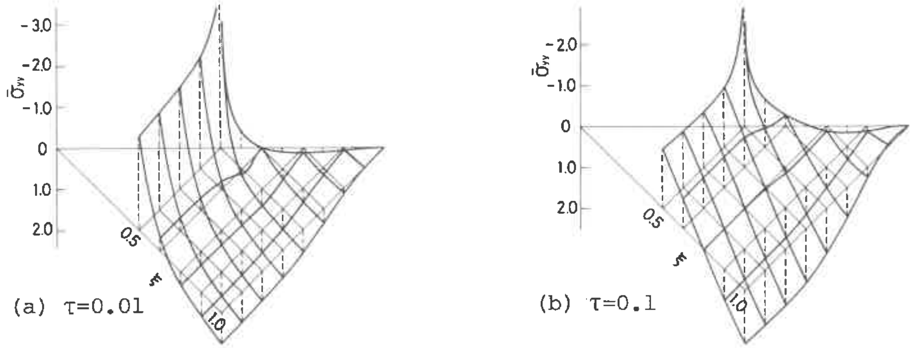

Fig.6 Distribution of $\bar{\sigma}_{yy}$.

4 NUMERICAL RESULTS AND DISCUSSION

Numerical calculations were carried out for the temperature and thermal stresses in the nonhomogeneous square hollow cylinder with the following nonhomogeneities of material properties:

$$\lambda = \lambda_0 \exp(P_1\xi + P_2\eta),\ \rho = \text{const.},\ c = \text{const.},\ E = E_0 \exp(R_1\xi + R_2\eta)$$
$$\alpha = \alpha_0 \exp(S_1\xi + S_2\eta),\ \nu = \text{const.},\ (P_1 = P_2, R_1 = R_2, S_1 = S_2) = -0.5, 0, 0.5,$$

and the following values were used

$$\bar{l}_y = l_y/l_x = 1,\ \bar{l}'_x = l'_x/l_x = 0.5,\ \bar{l}'_y = l'_y/l_x = 0.5,\ (m_0, m_1) = (h_0, h_1)l_x/\lambda_0$$
$$= (1, \infty),\ \bar{T}_1 = 1,\ \bar{T}_0 = 0,\ \Delta\tau = \kappa\Delta t/l_x^2 = 0.005,\ (\Delta\xi, \Delta\eta) = (\Delta x/l_x, \Delta y/l_x) = 0.0625,$$

where $\kappa = \lambda_0/\rho c$. The nondimensional temperature and thermal stress in Figs. 2-6 were given by

$$\bar{T} = (T - T_0)/(T_1 - T_0),\quad \bar{\sigma}_{ij} = \sigma_{ij}/[E_0\alpha_0(T_1 - T_0)].$$

Figs. 2 and 3 show the effect of the nonhomogeneity of thermal conductivity on the distributions of temeprature and thermal stress $\bar{\sigma}_{yy}$ in the y direction on the x axis. Figs. 4 and 5 show the effect of the nonhomogeneity of Young's modulus and linear thermal expansion coefficient on the distribution of $\bar{\sigma}_{yy}$. Figs. 6 (a) and (b) show the distribution of $\bar{\sigma}_{yy}$ for $P_1 = P_2 = -0.5$, $R_1 = R_2 = 0.5$ and $S_1 = S_2 = 0.5$, by the use of which the largest thermal stresses are obtained.

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