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DISJOINING PERMUTATIONS IN FINITE BOOLEAN ALGEBRAS

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ABSTRACT

The existence of a map that permutes the members of a family of finite sets so that every set is mapped into a disjoint set is shown to be equivalent to a certain set of inequalities involving order-ideals of sets. Order-ideals themselves are shown to admit such permutations.

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1. INTRODUCTION

Let $\mathcal{B} = \mathcal{B}(X)$ be a finite Boolean algebra, which we will regard as the family of all subsets of a finite set X. We will reserve the name 'set' for subsets of X; 'family' will be used to refer to subsets of $\mathcal{B}$; i.e., sets of subsets of X. We will use Roman capitals for sets and script capitals for families. We will denote the operations of union, intersection, and containment, among subfamilies of $\mathcal{B}$ as well as among subsets of X, by $\cup$, $\cap$ and $\subseteq$, respectively. Complementation in $\mathcal{B}$ will be denoted by $A \mapsto A^*$; if $\mathcal{H}$ is a subfamily of $\mathcal{B}$, $\mathcal{H}^*$ is defined as the family of complements of members of $\mathcal{H}$: $\mathcal{H}^* = \{A^*: A \in \mathcal{H}\}$. The expression $A - B$, for sets or families, denotes the set of members of A which are not members of B. Note that for families, $\mathcal{H} - \mathcal{A}$ is not $\mathcal{H} \cap \mathcal{A}^*$, although $A-B = A\cap B^*$ for sets. The cardinality of a set A or of a family $\mathcal{A}$ is denoted by $|A|$ or $|\mathcal{A}|$. We note for later use that if $\mathcal{G}$ and $\mathcal{H}$ are subfamilies of $\mathcal{B}$, then

$$|\mathcal{H}| = |\mathcal{H}^*|,$$

$$(\mathcal{G}\cap\mathcal{H})^* = \mathcal{G}^* \cap \mathcal{H}^*, \qquad (\mathcal{G}\cup\mathcal{H})^* = \mathcal{G}^* \cup \mathcal{H}^*,$$

and thus

$$|\mathcal{G}\cap\mathcal{H}| = |\mathcal{G}^*\cap\mathcal{H}^*|, \qquad |\mathcal{G}\cup\mathcal{H}| = |\mathcal{G}^*\cup\mathcal{H}^*|.$$

Since we will not use the ring-theoretic notion of ideal, we define an *ideal* to be what is usually called an 'order-ideal': a subfamily $\mathcal{I}$ of $\mathcal{B}$

for which $A \subseteq B \in \mathcal{I}$ implies $A \in \mathcal{I}$. An ideal in a finite Boolean algebra is uniquely determined by its *generators*, i.e., its maximal elements. If $\mathcal{A}$ is any subfamily of $\mathcal{B}$, we denote by $\mathcal{I}(\mathcal{A})$ the smallest ideal containing $\mathcal{A}$, which is the family of all subsets of members of $\mathcal{A}$. A *principal* ideal is an ideal with one generator; we write $\mathcal{I}_A$ for $\mathcal{I}(\{A\})$. If $\mathcal{I}$ is an ideal, $\mathcal{I}^*$ is a *filter*; this may be taken as the definition.

F. J. Dyson has noted (unpublished communication) that if $\mathcal{H}$ and $\mathcal{I}$ are ideals in a finite Boolean algebra, then $|\mathcal{H} \cap \mathcal{I}| \geq |\mathcal{H} \cap \mathcal{I}^*|$. In other words, any ideal $\mathcal{H}$ has the property

(1) For any ideal $\mathcal{I}$, $|\mathcal{H} \cap \mathcal{I}| \geq |\mathcal{H} \cap \mathcal{I}^*|$.

Dyson's proof is by induction on $|X|$; we will give a proof below (Proposition 6 and Corollary 7) which, though by induction on $|X|$, is different from Dyson's.

It is easy to see (and is proved as part of Theorem 1 below) that property (1) is enjoyed by any subfamily $\mathcal{H}$ of $\mathcal{B}$ which has a *disjoining permutation*; that is, a bijection ϕ of $\mathcal{H}$ to $\mathcal{H}$ such that ϕA and A are disjoint for every A in $\mathcal{H}$. What is perhaps more surprising is the converse, that (1) implies the existence of a disjoining permutation of $\mathcal{H}$. This is easily obtained, however, by the application of P. Hall's 'marriage' theorem to the 'disjointness relation' of $\mathcal{H}$ (defined below), in the proof of Theorem 1. We will refer to (1) as 'Dyson's condition'.

We recall first some results from matching theory. The reader is referred to the books by C. L. Liu [4] and L. Mirsky [5] for elaboration and proofs. If $R \subseteq S \times T$ is any relation between sets, a *matching of* S *under* R is an injection of S to T which is contained (as a set of ordered pairs) in R. If $A \subseteq S$, $R(A)$ denotes $\{b \in T: (a,b) \in R$ for some a in $A\}$. If

S and T are finite, then according to P, Hall's theorem a matching of S under R exists if and only if $|A| \le |R(A)|$ for every subset A of S. (If S and T are not finite, then according to R, Rado's extension of Hall's theorem, which can be found in Mirsky's book, we need the additional hypothesis that $R(\{a\})$ is finite for each a in S; then a matching exists iff $|A| \le |R(A)|$ for each finite subset A of S.)

A corollary to Hall's theorem, found in Liu's book, provides in the finite case that if there is a positive integer k such that $R(\{a\}) \ge k \ge R^{-1}(\{b\})$ for all a in S and b in T, then a matching of S exists. In particular, if $R \subseteq S \times S$ is symmetric and $R(\{a\}) = k > 0$ for each a in S, then a matching of S to S under R exists.

We will be applying these results to the symmetric relation $D_{\mathcal{H}} \subseteq \mathcal{H} \times \mathcal{H}$ on a subfamily $\mathcal{H}$ of a Boolean algebra, defined by

$$(A,B) \in D_{\mathcal{H}} \text{ iff } A \text{ and } B \text{ are disjoint members of } \mathcal{H}.$$

2. DISJOINING PERMUTATIONS AND DYSON'S CONDITION

THEOREM 1: Let $\mathcal{H}$ be any subfamily of a finite Boolean algebra $\mathcal{B}$. Then $\mathcal{H}$ has a disjoining permutation if and only if $\mathcal{H}$ satisfies Dyson's condition (1).

Proof: Suppose first that ϕ is a disjoining permutation of $\mathcal{H}$, and let $\mathcal{I}$ be any ideal. If $\mathcal{H} \cap \mathcal{I}^* = \emptyset$, then the desired inequality is tribial. Otherwise, the image of $\mathcal{H} \cap \mathcal{I}^*$ under ϕ is contained in $\mathcal{H} \cap \mathcal{I}$ since ϕ is disjoining; since ϕ is injective, $|\mathcal{H} \cap \mathcal{I}| \ge |\mathcal{H} \cap \mathcal{I}^*|$

For the converse, we notice that a disjoining permutation is just a matching of $\mathcal{H}$ under $D_{\mathcal{H}}$; so by Hall's theorem, $\mathcal{H}$ has a disjoining permutation if and only if

(2) $|A| \leq |D_H(A)|$ for every subfamily A of H.

If a subfamily A of H is given, let $I = I(A^*)$. It is clear that $D(A) = H \cap I$ (since $A \cap B = \emptyset$ iff $A \subseteq B^*$), and that $A \subseteq H \cap I^*$. Property (2) follows. □

(A somewhat artificial transfinite version of this theorem runs as follows: If H is a set of members of a Boolean algebra (ring of sets) such that each member of H intersects all but finitely many members of H, then H has a disjoining permutation iff $|H \cap I| \geq |H \cap I^*|$ for every finitely-generated order-ideal I. The proof is accomplished by inserting the word 'finite' before each of the two occurrences of the word 'subfamily' in the above proof.)

The notion of a disjoining permutation has two obvious graph-theoretical formulations. (See the books by Harary [3] and Liu [4] for definitions of terms used in this and the next three paragraphs.) The use of Hall's theorem suggests the natural bipartite graph associated with the relation D_H, whose vertex set comprises two disjoint copies of H. On the other hand, we can consider the graph $G(H)$ whose vertices are the members of H, any pair being adjacent if and only if they are disjoint as sets. $G(H)$ is a graph without multiple edges; if $\emptyset$ is in H, then $G(H)$ has a single loop, and this vertex is adjacent to all others. Except for this possibility, $G(H)$ is just the complement of the intersection graph of H. Since every finite graph is the intersection graph of some family of nonempty subsets of a finite set, it follows that every finite graph without loops or multiple edges (or with a single loop whose vertex is adjacent to all others) is $G(H)$ for some H.

A disjoining permutation then corresponds to a spanning subgraph of $G(H)$ which is the vertex-disjoint union of circuits and paths of length 1. The paths of length 1 correspond to cycles of length 2 in the permutation. Using this formulation we can easily prove

PROPOSITION 2: If H has a disjoining permutation, then it has a disjoining permutation in which all cycles are odd, and such that in any cycle, each set is disjoint only from its neighbors.

Proof: Let G' be the spanning subgraph of $G(H)$ corresponding to the given disjoining permutation. If G' has an even circuit, its vertices can be covered by a set of paths of length 1, viz. either of the two sets of alternate edges in the circuit. If G' has an odd circuit with a chord, then one of the two smaller circuits formed by the chord is odd; the remaining vertices are even in number and hence can be covered by a set of paths of length 1. Repeating this process will produce a spanning subgraph consisting of chordless odd circuits and paths of length 1. This subgraph corresponds to a disjoining permutation of the desired description. □

The following three propositions are easy consequences of Theorem 1.

COROLLARY 3: If H has a subfamily A with $A^* = A$, then H has a disjoining permutation if and only if $H - A$ does.

Proof: If I is any ideal, then

$$|H \cap I| - |H \cap I^*| = |(H-A) \cap I| - |(H-A) \cap I^*| + |A \cap I| - |A \cap I^*|.$$

But $|A \cap I^*| = |A^* \cap I| = |A \cap I|$. So $|H \cap I| \geq |H \cap I^*|$ iff $|(H-A) \cap I| \geq |(H-A) \cap I^*|$. □

COROLLARY 4: H and H^* both have disjoining permutations if and only if $H = H^*$.

Proof: If $H = H^*$, then complementation is a disjoining permutation of H (in fact, a disjoining *involution*, all cycles being of length 2).

Conversely, if both H and H^* have disjoining permutationa, then for any ideal I we have

$$|H \cap I| = |H^* \cap I^*| \leq |H^* \cap I| = |H \cap I^*| \leq |H \cap I|,$$

so that $|H \cap I| = |H \cap I^*|$. Now for any A in B, I_A is the principal ideal generated by A; let $I'_A = I_A - A$. Then A is in H iff $|H \cap I'_A| = |H \cap I_A|-1$ and A is in H^* iff $|H \cap I'_A{}^*| = |H \cap I_A{}^*|-1$. But $|H \cap I_A| = |H \cap I_A{}^*|$ and $|H \cap I'_A| = |H \cap I'_A{}^*|$, so A is in H iff A is in H^*. □

COROLLARY 5: If H satisfies Dyson's condition and H' is a family with an injection $u: H' \to H$ such that $A \subseteq uA$ for each A in H', then H' also satisfies Dyson's condition.

Proof: H has a disjoining permutation, say, ϕ. But then $u^{-1}\phi u$ is a disjoining permutation of H'; for $A \cap u^{-1}\phi uA \subseteq uA \cap \phi uA = \emptyset$. □

We conclude this section by noting that for Dyson's condition to hold it is not sufficient that $|H \cap I_A| \geq |H \cap I_A{}^*|$ for every principal ideal I_A. A counterexample is furnished for $|X| = 3$ by the family H consisting of $\emptyset$ and the three 2-subsets of X, which satisfies Dyson's condition for every principal ideal, but has no disjoining permutation.

Moreover, if we let B_j denote the family of j-subsets of X ($j = 1,2,\ldots,n = |X|$), it is not sufficient for Dyson's condition that $|H \cap B_j| \geq |H \cap B_{n-j}|$ when $j \leq \frac{1}{2}n$. Any saturated chain of sets (maximal family of sets which is totally ordered by inclusion) satisfies this condition but not Dyson's.

3. EXAMPLES

Here we give a few examples of types of families that have disjoining permutations. The only example in which the proof presents any difficulty is the first one: all ideals have disjoining permutations.

PROPOSITION 6: If $\mathcal{A}$ is any subfamily of $\mathcal{B}$, then $|\mathcal{A}| \le |D_{I(\mathcal{A})}(\mathcal{A})|$.

Proof: Write $\mathcal{A} = \{A_1,\ldots,A_n\}$; we need to show that among subsets of the sets $A_i - A_j$ $(i,j = 1,\ldots,n)$ are at least n distinct sets. We use induction on $|\bigcup_1^n A_i|$. If this number is 0 or 1, then $\mathcal{A}$ contains only one or two sets; in either case the assertion is easily seen to be true. Suppose then that the assertion is true when $|\bigcup_1^n A_i| = k-1 \ge 1$, and let $|\bigcup_1^n A_i| = k$. Let a be any member of $\bigcup_1^n A_i$. Denote $A_i - \{a\}$ by $A_i - a$.

If $A_1 - a,\ldots,A_n - a$ are distinct sets, then by the inductive hypothesis there are at least n distinct sets among subsets of the differences $(A_i - a) - (A_j - a)$. These are obviously subsets of the differences $A_i - A_j$ as well.

So we need to consider the case in which the sets $A_1 - a,\ldots,A_n - a$ are not all different. Notice that if $A - a = B - a$ while $A \ne B$, then one of A, B is the union of the other with $\{a\}$. Moreover, it is impossible that $A - a = B - a = C - a$ while A, B, C are all different. So we can renumber the sets:

$$\mathcal{A} = \{A_1,\ldots,A_m,A_{m+1},\ldots,A_{2m},A_{2m+1},\ldots,A_n\}$$

in such a way that $a \notin A_i$ $(i = 1,\ldots,m)$; $A_{m+i} = A_i \cup \{a\}$ $(i = 1,\ldots,m)$, and $A_{m+1} - a,\ldots,A_n - a$ are all distinct.

Thus, by the inductive hypothesis, there are at least $n-m$ distinct subsets of the sets $(A_i-a) - (A_j-a)$ for $i,j = m+1,\dots,n$. These are also subsets of $A_i - A_j$ for $i,j = m+1,\dots,n$, and none of them contains a.

Also, by the inductive hypothesis, there are at least m distinct subsets of the sets $A_i - A_j$ for $i,j = 1,\dots,m$; adjoining a to each of these produces at least m distinct subsets of the sets $A_i - A_j$ for $i = m+1,\dots,2m$, $j = 1,\dots,m$. And each of these sets contains a, so is distinct from each of the $n-m$ sets obtained earlier. The result is at least n distinct subsets of the sets $A_i - A_j$. □

COROLLARY 7: Every order ideal in a finite Boolean algebra admits a disjoining permutation.

Proof: For any subset A of an order ideal H, $I(A) \subseteq H$, and thus $|A| \le |D_{I(A)}(A)| \le |D_H(A)|$. □

An apparent strengthening of Proposition 6 is as follows. If $A_1,\dots,A_n$ are distinct sets, a *region of the Venn diagram* for $A_1,\dots,A_n$ is one of the 2^n (not necessarily distinct) sets $B_1 \cap \dots \cap B_n$, where each B_i is either A_i or $A_i{}^*$. If we apply Proposition 6 to the finite Boolean algebra whose points are the regions of the Venn diagram for $A_1,\dots,A_n$, we obtain

COROLLARY 8: If $A_1,\dots,A_n$ are distinct sets, then there are at least n distinct subsets of the differences $A_1 - A_j$ $(i,j = 1,\dots,n)$ which are unions of regions of the Venn diagram for $A_1,\dots,A_n$.

Recall now that $\mathcal{B}_j$ denotes the family of sets of size j in $\mathcal{B}$.

PROPOSITION 9: $\mathcal{B}_j \cup \mathcal{B}_{j+1} \cup \dots \cup \mathcal{B}_k$ has a disjoining permutation if and only if $j+k \le n$.

Proof: Let $H = \mathcal{B}_j \cup \mathcal{B}_{j+1} \cup \ldots \cup \mathcal{B}_k$. If $j+k > n$, then for $I = \mathcal{B}_0 \cup \mathcal{B}_1 \cup \ldots \cup \mathcal{B}_{n-j}$ we have $H \cap I^* = H$, but $H \cap I = \emptyset$ if $j > \frac{1}{2}n$ and $H \cap I = \mathcal{B}_j \cup \mathcal{B}_{j+1} \cup \ldots \cup \mathcal{B}_{n-j} \subsetneq H$ if $j \le \frac{1}{2}n$. So H has no disjoining permutation.

If $j+k \le n$, then $H = \mathcal{B}_j \cup \mathcal{B}_{j+1} \cup \ldots \cup \mathcal{B}_{n-k-1} \cup I_1$, where $I_1 = \mathcal{B}_{n-k} \cup \mathcal{B}_{n-k-1} \cup \ldots \cup \mathcal{B}_k$ (or $I_1 = \emptyset$ if $k < \frac{n}{2}$). $I_1^* = I_1$, so that complementation is a disjoining permutation for I_1. And if $i \le \frac{1}{2}n$, then $\mathcal{B}_i$ itself has a disjoining permutation, for each member of $\mathcal{B}_i$ is disjoint from exactly $\binom{n-i}{i}$ other members of $\mathcal{B}_i$. □

As a consequence of Proposition 9, we observe that for any $k \le n-1$, $G = \mathcal{B}_0 \cup \mathcal{B}_1 \cup \ldots \cup \mathcal{B}_k \cup \mathcal{B}_n$ has a disjoining permutation. G is a *combinatorial geometry* (see Crapo and Rota [1]) of a special type: the truncation of a free geometry (i.e. Boolean algebra). We conjecture that the family of flats of any finite combinatorial geometry has a disjoining permutation. We note that Greene [2] has proved what would seem to be a partial result in this direction: that in any combinatorial geometry there is an injection ϕ from the points to the copoints which is disjoining in that p is not a member of the copoint ϕp for any $p \in X$.

If the family of flats of a combinatorial geometry has a disjoining permutation, it follows (since $\mathcal{B}_0 \cup \ldots \cup \mathcal{B}_j$ is an ideal for any j) that there are at least as many flats of size $\le j$ as of size $\ge n-j$, for $j = 1,2,\ldots,n$. This is in contrast to (but certainly does not contradict) Rota's celebrated conjecture that there are no more flats of *rank* j (in the geometry) than of rank $n-j$, for $j \le \frac{1}{2}n$.

We conclude with a sufficient condition for the existence of a disjoining permutation for a lattice of sets which is not a boolean algebra. Let G be an arbitrary subfamily of $\mathcal{B}(X)$ which forms, under intersection, an infimum-sublattice. If $\emptyset \in \mathcal{G}$, then $\mathcal{G}$ is also a lattice. Suppose that $\mathcal{G}$ is complemented, and also that X is in $\mathcal{G}$. Denote complementation in $\mathcal{G}$ by $A \mapsto \bar{A}$, and for a subfamily $\mathcal{H}$ of $\mathcal{G}$ let $\bar{\mathcal{H}}$ be the family of all $\mathcal{G}$-complements of members of $\mathcal{H}$. In general, of course, $|\bar{\mathcal{H}}| \geq |\mathcal{H}|$, so the hypothesis of the following proposition is quite strong. A $\mathcal{G}$-*ideal* is the intersection of $\mathcal{G}$ with an ideal of $\mathcal{B}$.

PROPOSITION 10: If $\mathcal{H}$ is a subfamily of $\mathcal{G}$ with the property that $|\mathcal{H} \cap \mathcal{I}| \geq |\mathcal{H} \cap \bar{\mathcal{I}}|$ for every $\mathcal{G}$-ideal $\mathcal{I}$, then $\mathcal{H}$ has a disjoining permutation.

Proof: For any $\mathcal{A} \subseteq \mathcal{H}$, let $\mathcal{I}$ be the $\mathcal{G}$-ideal generated by $\bar{\mathcal{A}}$. Then $\bar{\mathcal{A}} \subseteq D_{\mathcal{H}}(\mathcal{A})$ and $D_{\mathcal{H}}(\mathcal{A})$ is a $\mathcal{G}$-ideal, so $\mathcal{I} \subseteq D_{\mathcal{H}}(\mathcal{A})$. Moreover, $\mathcal{A} \subseteq \bar{\mathcal{I}}$. So $\mathcal{A} \subseteq \mathcal{H} \cap \bar{\mathcal{I}}$ and $\mathcal{H} \cap \mathcal{I} \subseteq D_{\mathcal{H}}(\mathcal{A})$. Thus $|\mathcal{A}| \leq |D_{\mathcal{H}}(\mathcal{A})|$. □

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FOOTNOTES

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