

ABSTRACT

BISHOP, ABIGAIL CLAIRE. Involution Posets of Non-Crystallographic Coxeter Groups.
(Under the direction of Aloysius Helminck.)

In this thesis, we will examine the involution posets for non-crystallographic Coxeter groups. Involution posets occur in the study of symmetric spaces and their representations. These posets we consider are based on both the Richardson-Springer Order and an induced Bruhat order. We prove a number of properties for these posets.

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Involution Posets of Non-Crystallographic Coxeter Groups

by
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DEDICATION

To my family.

BIOGRAPHY

Abigail Claire Bishop was born in Raleigh, North Carolina to Tony and Louise Bishop. She has always loved school, so much in fact that after finishing second grade, her one complaint was the amount of homework; she wanted more. She attended Appalachian State University in Boone, NC where she received a Bachelor of Sciences in Mathematics, Secondary Education in Spring 2009. She then taught high school for a year before deciding to return to school in pursuit of a PhD. She began her graduate school career at North Carolina State University in Fall 2010 and completed her journey by graduating in Winter 2015 with a PhD in Mathematics.

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CHAPTER

1

INTRODUCTION

Coxeter groups play a fundamental role in representation theory. They can be used to describe the orbits of parabolic subgroups and Borel subgroups. For example if $G = GL(n)$, then any Borel subgroup B is isomorphic to an upper or lower triangular matrix. We have the $PA = LU$ decomposition where P is a permutation matrix, A is a matrix in G , L is a lower triangular matrix, and U is an upper triangular matrix. The B orbits in G/B can be characterized by subgroups of permutations, or Weyl groups, W . So we can write G as a disjoint union of these B orbits. Further, this induces an order on a poset given by the closure of these B orbits.

Similarly, orbits of Borel subgroups and parabolic subgroups acting on generalized symmetric spaces play a role in the representation theory of symmetric spaces. They can also be characterized as combinatorial objects, in this case as twisted involutions, \mathcal{I} , which are all w in W such that w is mapped to w^{-1} . Similar to the group case, the closure of orbits induce an order, but in this case we are looking at B orbits on our symmetric space Q . Using this order we can create the involution poset.

We have geometric definitions for both of these orders, but they can also be defined combinatorially. We use these combinatorial definitions to build posets. We can then use tools from combinatorics and graph theory to analyze the posets that we have created. These posets can be used to obtain results about orbit decompositions in the group or symmetric space.

In this dissertation, we examine involution posets for the Coxeter groups $I_2(m)$, H_2 , H_3 , and H_4 . There are four types of involution posets that we consider: the Richardson-Springer poset (also called the weak involution poset), the strong Richardson-Springer poset (also called the strong involution poset), the weak Bruhat involution poset, and the strong Bruhat involution poset. Both the weak and the strong Bruhat involution posets are posets whose edges are induced from the relationships presented in the weak Bruhat poset and strong Bruhat poset respectively. The study of these involution posets, is motivated by the work done by R.W. Richardson and T.A. Springer in [RS90]. The Richardson-Springer poset for various Weyl groups has been studied by Ruth Haas and Aloysius Helminck (see [HH12] and [HH11]), while the weak Bruhat involution poset for the classical Weyl groups has been studied by Federico Incitti (see [Inc03], [Inc04], and [Inc06]).

1.1 Summary of Results

1.1.1 Summary of Results for Coxeter Groups of type $I_2(m)$

For the Coxeter group $I_2(m)$, we primarily determined results related to the weak involution poset. We were able to fully classify the poset diagram of the weak involution poset. In doing this, we determined the topmost element and its length as well as the number of maximal chains that exist in the poset. Further, we found the poset diagram to be cycle leading us to results about the diagram being understood as a bipartite graph and also showing that there was a Hamiltonian cycle in any poset diagram of the weak involution poset. Finally we found that the poset was graded and also symmetric.

We were also able to determine a couple of results related to the other three types of involution posets for $I_2(m)$. We determined that the topmost element and its length for all three involution posets. We were also able to show the existence of a Hamiltonian cycle in the strong involution poset.

1.1.2 Summary of Results for Coxeter Groups of type H

For the Coxeter groups H_2 , H_3 , and H_4 , we were able to determine a number of results for the four involution posets. Over all posets and all three Coxeter groups of type H , we were able to determine their vertex sets. And for all four involution posets for H_2 , we were able to determine that each contains a Hamiltonian cycle. We also show that both the diagrams for the weak involution poset and the strong involution poset of H_3 contain Hamiltonian cycles. For H_3 we determined that neither the weak involution poset nor the strong involution poset is symmetric.

For the weak involution poset in particular, we determine that all maximal chains have the same length and further discovered the length of these maximal chains for all three Coxeter groups of type H . For H_2 and H_3 , we found a rank function for the weak involution poset. Also, for H_2 we showed that the edge set of the weak involution poset is a subset of the edge set of the other three types of involution posets. We also determined that we could not shift copies of H_2 in H_3 (respectively H_3 in H_4) to form the weak involution poset of H_3 (respectively H_4). Further, the weak involution poset diagrams of all three Coxeter groups of type H can be realized as bipartite graphs. We discovered the number of elements at each rank for the weak involution poset of all three Coxeter groups of type H . We also showed that the weak involution posets for all three Coxeter groups of type H are graded posets.

Finally, we found results related to the strong involution posets of these Coxeter groups of type H . We found that the strong involution poset was not actually a poset, but rather a Cayley graph for H_2 and H_3 . Finally, we found that in the case of H_2 , that Cayley graph is isomorphic to the complete graph on six vertices.

CHAPTER

2

BACKGROUND

2.1 Coxeter Groups

In this section, we will follow the notation set in [Hum90]. Let V be a real Euclidean vector space with a positive definite symmetric bilinear form (\cdot, \cdot) and let the group of all orthogonal transformations of V be denoted by $O(V)$.

Definition 2.1. *A reflection is a linear operator s on V , that takes a nonzero vector α to its mirror image across a hyperplane, H_α .*

So for $v \in V$, $sv = v$ if $v \in H_\alpha$ and $sv = -v$ if $v \in H_\alpha^\perp$ where H_α^\perp is the orthogonal complement. We can write $s = s_\alpha$ and truly we can write $s_\alpha = s_{c\alpha}$ for any $0 \neq c \in \mathbb{R}$. Further, we define a transformation s_α acting on $\lambda \in V$ by $s_\alpha \lambda = \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \alpha$. The preceding formula holds for $\lambda \in V = \mathbb{R}\alpha \oplus H_\alpha$. Further, $s_\alpha \in O(V)$ and $s_\alpha^2 = \text{id}$.

We will be considering a *finite reflection group* which is, perhaps clearly, a finite group that is generated by reflections. A familiar example of a finite reflection group is the dihedral

group of order $2n$, D_n . When first learning about the dihedral group, we often learn that it can be generated by a reflection and a rotation. However, we can also generate D_n by 2 reflections since we can combine reflections to get rotations as seen in Example 1. We will discuss the Dihedral group in more detail in Subsection 2.1.1

Example 1. Consider the dihedral group on 4 elements, D_4 . Recall, this is the symmetry group of a square. We will show that a clockwise rotation of $\pi/2$ radians is equivalent to 2 reflections performed in succession.

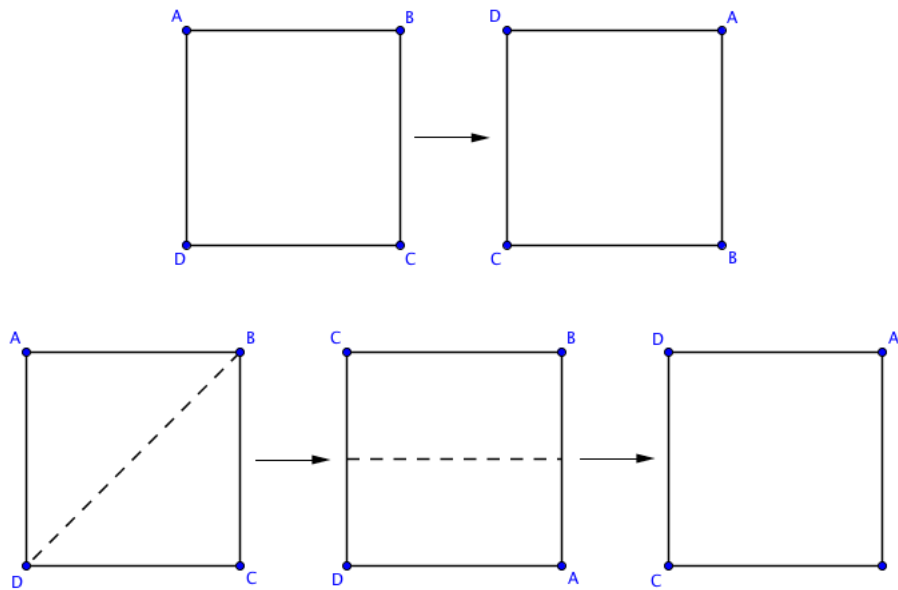


Figure 2.1 Rotations and reflections for D_4 .

Now, let W be a finite reflection group acting on the Euclidean space V . For each $s_\alpha \in W$ we get a reflecting hyperplane H_α and a line $L_\alpha = \mathbb{R}\alpha$ orthogonal to H_α . Proposition 2.2 shows us how W acts on lines, L_α .

Proposition 2.2. *If $t \in O(V)$ and α is any nonzero vector in V , then $t s_\alpha t^{-1} = s_{t\alpha}$. In particular, if $w \in W$, the $s_{w\alpha}$ belongs to W whenever s_α does.*

Therefore, W permutes lines of the form L_α using elements of W . We will axiomatize the situation as follows:

Let Φ be a finite set of vectors $0 \neq v \in V$ that satisfy the following two conditions

1. $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$
2. $s_\alpha \Phi = \Phi$ for all $\alpha \in \Phi$

We will further define W to not only be a finite reflection group, but to also be generated by all reflections s_α where $\alpha \in \Phi$.

Definition 2.3. *We will call Φ a root system with an associated reflection group W . The elements in Φ are called roots.*

Although the name is the same, this usage of root system differs from the same term used in the field of Lie Theory.

Fix a root system Φ in the Euclidean space V and let W be the finite reflection group generated by s_α where $\alpha \in \Phi$. Although we can use Φ to completely determine W , it may not always be convenient to do so based on the order of Φ compared to the dimension of V . Therefore, we may want to find a linearly independent subset of Φ , called a simple system. In a simple system, each α is a linear combination in \mathbb{R} of simple roots from our simple system with coefficients that are either all positive or all negative. Due to this, we are able to partition the roots into positive roots and negative roots, where all the positive roots form a subset of Φ , Π , called a positive system, and likewise all negative roots form a negative system or $-\Pi$.

Definition 2.4. *Δ , a subset of Φ , is called a simple system if Δ is a vector space basis for the \mathbb{R} -span of Φ in V . Further, each $\alpha \in \Phi$ is a linear combination of Δ with coefficients either all positive or all negative. We call the elements of Δ , simple roots.*

The following theorem address the existence of a simple system.

Theorem 2.5. (a) *If Δ is a simple system in Φ , then there is a unique positive system containing Δ .*

(b) Every positive system $\Pi \in \Phi$ contains a unique simple system.

Recall, simple systems are important because their order is smaller than that of Φ and therefore closer to the dimension of V .

Definition 2.6. *The rank of W is the cardinality of any simple system, or the dimension of the span of Φ in V*

Theorem 2.7. *Any two positive systems (respectively simple) in Φ are conjugate under W .*

Therefore, it does not matter which positive system we consider. So, we will fix a simple system Δ and a positive system Π in Φ . We have already addressed that W is a finite reflection group generated by reflections s_α for $\alpha \in \Phi$. However, we can take it one step further.

Theorem 2.8. *For a fixed simple system Δ , W is generated by the reflections s_α , where $\alpha \in \Delta$.*

We call the reflections described in the previous theorem *simple reflections*. We also have the following corollary which associates roots in Φ and simple roots.

Corollary 2.9. *Given Δ , for every $\beta \in \Phi$ there exists $w \in W$ such that $w\beta \in \Delta$.*

Now that we can see that W is generated by relatively few reflections, it is now convenient to determine a succinct way to write the presentation of W given the reflections $s_\alpha \in \Delta$ that generate it. We will use $(s_\alpha s_\beta)^{m(\alpha, \beta)} = 1$. Where $m(\alpha, \beta)$ gives the order of the product in W .

Since we know that W is generated by simple reflections, we also want to know how we can write an element $w \in W$ as a product of simple reflections. To do this, first let's solidify a bit of notation. Write s_{α_i} as s_i for all $s_{\alpha_i} \in \Delta$.

Definition 2.10. *If $w = s_1 s_2 \cdots s_r$, the length of w , $\ell(w)$, is the smallest r for which the product exists. We call this product of generators a reduced word for w .*

When describing the length of the identity element, we will use $\ell(\text{id}) = 0$. We can see that the length of any generator is 1 and that the length of w^{-1} is equal to the length of w . Further notice that $\det(w) = (-1)^{\ell(w)}$. This allows us to borrow from many properties of the determinant and apply them to the length function.

There is another function that is similar to the length function that we now want to introduce. Define $n(w)$ as the number of positive roots sent to negative roots by w .

Theorem 2.11. *Let $\alpha \in \Delta$, $w \in W$. Then,*

$$(a) \quad wa > 0 \rightarrow n(ws_\alpha) = n(w) + 1$$

$$(b) \quad wa < 0 \rightarrow n(ws_\alpha) = n(w) - 1$$

$$(c) \quad w^{-1}a > 0 \rightarrow n(ws_\alpha) = n(w) + 1$$

$$(d) \quad w^{-1}a < 0 \rightarrow n(ws_\alpha) = n(w) - 1$$

Corollary 2.12. *If $w \in W$ is written in any way as a product of simple reflections, say $w = s_1 s_2 \cdots s_r$, then $n(w) \leq r$. In particular $n(w) \leq \ell(w)$.*

Often, we want the shortest word possible when writing an element as a product of simple reflections. The following theorem shows how we can shorten words if necessary.

Theorem 2.13. *Fix a simple system Δ . Let $w = s_1 s_2 \cdots s_r$ be any expression of $w \in W$ as a product of simple reflections. Suppose $n(w) < r$, Then there exists indices $1 \leq i < j \leq r$ that satisfy*

$$1. \quad \alpha_i = (s_{i+1} \cdots s_{j-1})\alpha_j$$

$$2. \quad s_{i+1}s_{i+2} \cdots s_j = s_i s_{i+1} \cdots s_{j-1}$$

$$3. \quad w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_r \text{ where the hat denotes omission}$$

The third part of theorem gives us the *Deletion Condition*. This tells us that we can eventually reach a reduced word by removing pairs of reflections.

We also have the *Exchange Condition* which says:

Let $w = s_1 s_2 \cdots s_r$ where each s_i is a simple reflections and w is not necessarily reduced. If $\ell(ws) < \ell(w)$ for some simple reflection s , there there exists an index i for which $ws = s_1 \cdots \hat{s}_i \cdots s_r$ (and thus $w = s_1 \cdots \hat{s}_i \cdots s_r s$) In particular, w has a reduced expression ending in s if and only if $\ell(ws) < \ell(s)$.

We will now formalize the presentation of our reflection group that we previously discussed.

Theorem 2.14. Fix a simple system Δ in Φ . Then W is generated by the set $S = \{s_\alpha | \alpha \in \Delta\}$, subject only to the relations:

$$(s_\alpha s_\beta)^{m(\alpha, \beta)} = \text{id} \quad (\alpha, \beta \in \Delta)$$

We call any group W having a presentation such as this relative to its generating set S a *Coxeter group* and we call the pair (W, S) a *Coxeter System*.

Many may be familiar with some Coxeter groups by other names. For instance, the class of Coxeter groups A_n , is the symmetric group S_{n+1} and $I_2(m)$ is just dihedral group on m vertices. As we know from both the symmetric group and the dihedral group, Coxeter groups are built using generators and relations.

Definition 2.15. A *Coxeter matrix* is an $n \times n$ symmetric matrix with ones on the diagonal with all other entries are greater than 1. More formally, we will let S be a set. A matrix $m : S \times S \rightarrow \{1, 2, \dots, \infty\}$ is a *Coxeter matrix* if it satisfies the following conditions: $m(s_i, s_j) = m(s_j, s_i)$ and $m(s_i, s_j) = 1 \iff s_i = s_j$.

From the matrix we can get a *Coxeter graph* whose vertex set is S and whose edges are the unordered pairs $\{s_i, s_j\}$ where $m(s_i, s_j) \geq 3$. We label the edges with a number if $m(s_i, s_j) \geq 4$. Two generators commute if and only if they are not connected by an edge in a Coxeter graph. Through a Coxeter matrix (and therefore a Coxeter graph) we are able to further understand the presentation of a Coxeter group, W , using generators and relations. A Coxeter group has generators S and relations are determined by $(s_i s_j)^{m(s_i, s_j)} = \text{id}$, which notice is the same generator relation we gave in Theorem 2.14. From this presentation, we get that all of our generators in S are of order 2 since $m(s_i, s_i) = 1$.

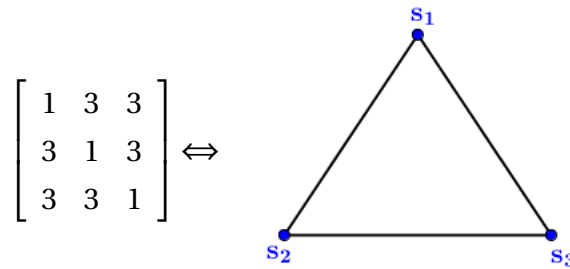
Example 2. Let $W = A_3 = S_4$. Our generators are $s_1 = (12)$, $s_2 = (23)$, and $s_3 = (34)$. The Coxeter matrix and corresponding Coxeter graph of A_3 are

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 3 \\ 2 & 3 & 1 \end{bmatrix} \iff \begin{array}{ccc} \bullet & \text{---} & \bullet \\ s_1 & & s_2 \end{array} \text{---} \begin{array}{c} \bullet \\ s_3 \end{array} .$$

Our relationships are $(s_1)^2 = (s_2)^2 = (s_3)^2 = (s_1 s_3)^2 = (s_1 s_2)^3 = (s_2 s_3)^3 = \text{id}$

Unless otherwise specified, the presentation from Example 2 will be the one used. Do note that by changing the set of generators, S , it is possible to obtain a new Coxeter matrix and therefore Coxeter graph.

Example 3. *Once again, let $W = A_3 = S_4$, but choose generators $s_1 = (12)$, $s_2 = (13)$, $s_3 = (14)$. This time we'll get the following Coxeter matrix and Coxeter graph*



Thus our relationships are $(s_1)^2 = (s_2)^2 = (s_3)^2 = (s_1 s_3)^3 = (s_1 s_2)^3 = (s_2 s_3)^3 = \text{id}$

This further shows how important identifying a Coxeter group with its Coxeter system can be. Different Coxeter system can give us different presentations of the same Coxeter graph.

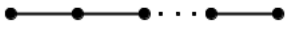
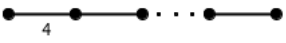
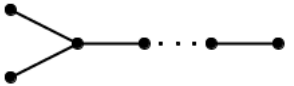


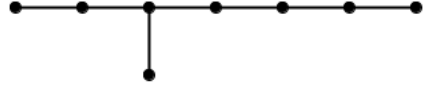
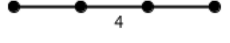


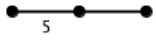


Theorem 2.16. *Up to isomorphism, there is a one-to-one correspondence between Coxeter systems and Coxeter matrices.*

All finite Coxeter groups can be realized as reflection groups of polytopes. Regular polytopes in the cases of $A_n, B_n, F_4, G_2, H_2, H_3, H_4$, and $I_2(m)$. We know that our Coxeter group is generated by simple reflections, but there are times when we want to consider the set of all reflections. We define the set of *reflections* as T , where $T = \{w s w^{-1} | s \in S, w \in W\}$. First, notice that all reflections are of order two. Also notice that if we let $w = \text{id}$, then we can see that $S \subset T$.

Example 4. *For $W = A_3 = S_4$, our set of reflections is $T = \{s_1, s_2, s_3, s_1 s_2 s_1, s_1 s_2 s_3 s_2 s_1\}$. Notice that $S = \{s_1, s_2, s_3\} \subset T$.*

The finite Coxeter Groups are listed in Table 2.1. It lists the order of the Coxeter group, $|W|$, the order of the reflections of the each group, $|T|$, the polytope or polytopes that W is the symmetry group of, as well as the Coxeter graph of each.

Table 2.1 Finite Coxeter Groups

Type	$ W $	$ T $	Symmetry group	Coxeter graph
A_n , $n \geq 1$	$(n+1)!$	$\binom{n+1}{2}$	n -simplex	
B_n , $n \geq 2$	$2^n n!$	n^2	n -hypercube	
D_n , $n \geq 4$	$2^{n-1} n!$	$n^2 - n$	n -demihypercube	
E_6	$2^7 \cdot 3^4 \cdot 5$	36	2_{21}	
E_7	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	63	2_{31}	
E_8	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$	120	2_{41}	
F_4	1152	24	24-cell	
G_2	12	6	hexagon	
H_2	10	4	pentagon	
H_3	120	15	icosahedron	
H_4	14,400	60	120-cell	
$I_2(m)$, $m \geq 3$	$2m$	m	m -gon	

Definition 2.17. A subgroup of G of $GL(V)$ is said to be crystallographic if it stabilizes a lattice L in V . So $gL \subset L$ for all $g \in G$

Further, the following proposition gives more information about which Coxeter groups are crystallographic.

Proposition 2.18. If W is crystallographic, then each integer $m(\alpha, \beta)$ must be 2, 3, 4, or 6 when $\alpha \neq \beta$.

Combining Proposition 2.18 and the results in Table 2.1, we can see that we get some cases where $m(\alpha, \beta) \neq 2, 3, 4, \text{ or } 6$. Those situations being H_2, H_3, H_4 and $I_2(m)$ when $m = 5$ or $m > 6$. Leading us to the fact that in those cases, H_2, H_3, H_4 and $I_2(m)$ are non-crystallographic.

We call a root system Φ crystallographic if it satisfies the condition $\frac{2(\alpha, \beta)}{(\beta, \beta)} \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$. We call these integers *Cartan integers*. The Coxeter groups that are generated by all reflections in a crystallographic root system are called *Weyl groups* of Φ . These Weyl groups are then crystallographic and will satisfy Proposition 2.18. Thus, we are further informed that of our finite Coxeter groups, $A_n, B_n, D_n, E_6, E_7, E_8, F_4,$ and G_2 are all Weyl groups.

A poset, $P = (P, \leq)$, or partially ordered set, is a set P and a partial order relation \leq . If $Q \subset P$, then Q is also a poset and further we call (Q, \leq) an induced subposet. We say two elements $x, y \in P$ are comparable if $x \leq y$ or $y \leq x$; otherwise they are called incomparable. Further (x_0, x_1, \dots, x_k) where $x_i \in P$ is called a chain if $x_0 \leq x_1 \leq \dots \leq x_k$. We call k the length of the chain. We call a chain maximal if its elements are not a proper subset of the elements in any other chain. Further if we have a maximal chain (x_0, \dots, x_n) then n is called the rank of the poset P .

There are many ways to get a partial ordering on W , but the way that we will focus on is the Bruhat ordering. Let T be the set of reflections in W as defined previously and let $u, w \in W$. Then, if $u = wt$ for some $t \in T$ where $\ell(u) > \ell(w)$, we will write $w \rightarrow u$. We will further say that $w < u$ if $w = u_0 \rightarrow u_1 \rightarrow \dots \rightarrow u_m = u$. From this we get a unique partial ordering that satisfies all necessary conditions: reflexivity, antisymmetry, and transitivity. We will let id be the unique minimal element. This ordering discussed above is called the *Bruhat order*. Although it seems typical to define the operation as multiplication of a reflection on the right, in practice, we will choose to use multiplication on the left. By multiplying on the left by any reflection as defined above, we will what is called, the *strong*

Bruhat order. If we choose instead to multiply on the left by only the generators, then we call our order the *weak Bruhat order*.

We build posets using the above stated partial orderings. For the weak Bruhat poset, the vertices are the elements of the Coxeter group and there is an edge from vertex u to vertex w if you can multiply on the left by a generator and get an element that does not yet exist in the poset.

Example 5. Let Coxeter group A_3 , which recall is isomorphic to S_4 , with the vertex set $s_1 = (12)$, $s_2 = (23)$, and $s_3 = (34)$. There is an edge between $u = s_1 = (12)$ and $v = s_2 s_1 = (132)$ since $s_2 u = v$.

The strong Bruhat poset has the same vertex set as the weak Bruhat poset. However, edges between two vertices are created through left multiplications of not only generators, but by all reflections.

In either case, when building the Bruhat poset, we must always begin at the identity element, id and build it up by multiplying on the left by generators or reflections. In Figure 2.2, we see the Weak and Strong Bruhat Posets of A_3 .

2.1.1 Coxeter Groups of type $I_2(m)$

$I_2(m)$, $m \geq 3$ is a Coxeter group of order $2m$ that is more familiarly known as the dihedral group. It consists of the orthogonal transformations that preserve a regular polygon with m sides that is centered at the origin on the Euclidean plane. $I_2(m)$ contains m rotations through the multiples of $2\pi/m$ and m reflections. This clearly leads us to the order of $I_2(m)$ as $2m$.

Let L_1 and L_2 be lines through the origin of the Euclidean plane. And let the angle between these two lines be equal to $\frac{\pi}{m}$ for an integer $m \geq 2$. Let r_1 be an orthogonal reflection through the line L_1 and let r_2 be an orthogonal reflection through L_2 . Notice that $r_1 r_2$ will give you a rotation in the plane through an angle of $\frac{2\pi}{m}$. Thus $(r_1 r_2)^m = \text{id}$. This allows us to generate the dihedral group using two reflections rather than a reflection and a rotation. We can define the Coxeter graph for the dihedral group to be as seen in Table 2.1. We can further understand the order of $I_2(m)$ by considering how many elements there are when written as products of generators. From here, it is clear that elements of $I_2(m)$ can be written

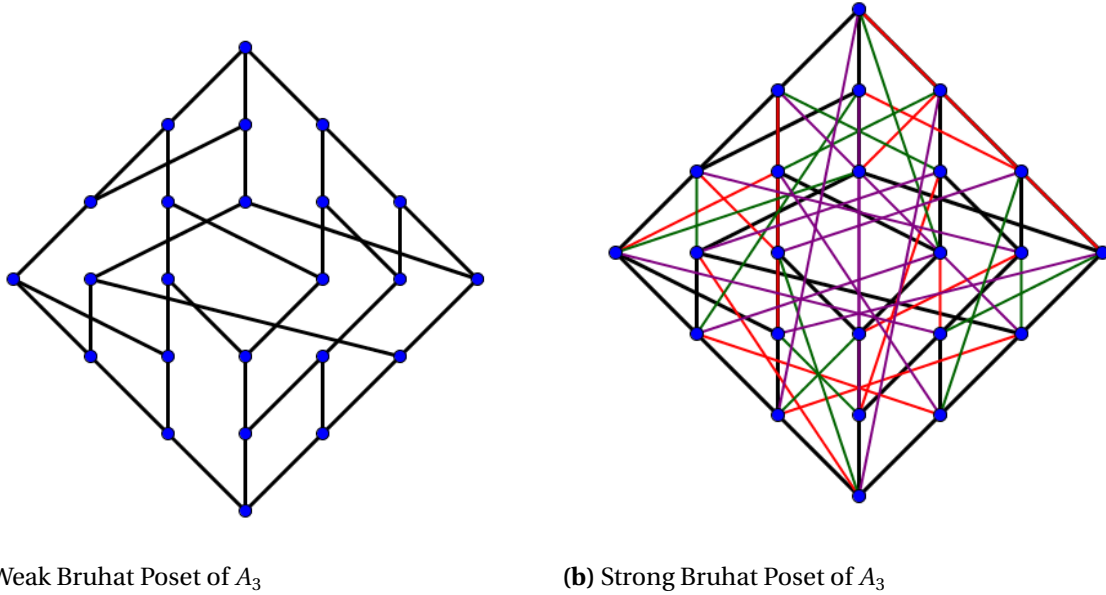


Figure 2.2 Bruhat Posets of A_3

as words by alternating s_1 and s_2 or vice versa. Since $(s_1 s_2)^m = \text{id}$, we know that our words can be no longer than m . We know that in $I_2(m)$ there are $2m - 2$ words, excluding the identity and the word of length m , since there are 2 words of each length: one starting with s_1 and another starting with s_2 . Once we add back in our shortest word and longest word, we can see that the order of $I_2(m)$ is $2m$.

2.1.2 Coxeter Groups of type H

The Coxeter groups of type H are H_2 , H_3 , and H_4 . H_2 is not always included in the Coxeter groups of type H since $H_2 = I_2(5)$. However for our purposes we will include it with those of type H . These groups are non-crystallographic. And they arise somewhat naturally as the symmetry groups of a regular pentagon in \mathbb{R}^2 , dodecahedron (a figure with 12 pentagonal faces) in \mathbb{R}^3 , and 120-sided solids with dodecahedral faces in \mathbb{R}^4 respectively. H_2 is of order 10 with 5 reflections, H_3 is of order 120 and contains 15 reflections, and H_4 is of order 14,400 and has 60 reflections.

H_3 can be thought of as isomorphic to the direct product of its center which is the set of

-1 and 1 and the simple group of order 60.

To verify the existence of the groups H_3 and H_4 we can find a root system in \mathbb{R}^3 and \mathbb{R}^4 respectively. If we can find a root system for H_4 then we can find one for H_3 since the root system for H_3 will be a subgroup of the root system for H_4 . By the alternating sum formula,

$$\sum_{T \subset S} (-1)^{|T|} \frac{|W|}{|W_T|} = 1,$$

we know that if the group H_3 does exist, then it has to have order 120. Thus we will be looking for a subgroup of order 120. To construct our root system for H_4 we will first connect vectors with elements of the ring of real quaternions \mathbf{H} . The vector $\lambda = (c_1, c_2, c_3, c_4)$ gives us $\lambda = c_1 + c_2i + c_3j + c_4k$. And our typical inner product $(\lambda, \mu) = \frac{1}{2}(\lambda\bar{\mu} + \mu\bar{\lambda})$ where $\bar{\lambda} = c_1 - c_2i - c_3j - c_4k$. We can compute inverses in the division ring \mathbf{H} as

$$\lambda^{-1} = \frac{\bar{\lambda}}{\|\lambda\|}$$

where $\|\lambda\| = \lambda\bar{\lambda}$. If we let $\alpha \in \mathbf{H}$ have norm 1 then the reflection transforms \mathbf{H} as follows: $\lambda \mapsto \alpha\bar{\lambda}\alpha$.

Lemma 2.19. *Any finite subgroup G of even order in \mathbf{H} is a root system (when regarded as a subset of \mathbb{R}^4).*

Due to Lemma 2.19, we will select a subgroup of \mathbf{H} that is of order 120. Thus that subgroup is the root system for H_4 and therefore we know there exists a finite reflection group of type H_4 . Which, in turn, means that there exists a subgroup of type H_3 .

We can construct the root system as follows. Since our Coxeter graph has a bond involving a 5, it may be clear that the angle $\pi/5$ should be a part of our root system construction. We will let $a = \cos(\pi/5) = \frac{1+\sqrt{5}}{2}$ and $b = \cos(2\pi/5) = \frac{-1+\sqrt{5}}{2}$. Then, we know that $2a = 2b + 1$, $4ab = 1$, $4a^2 = 2a + 1$, and $4b^2 = -2b + 1$. Let Φ be the unit vectors in \mathbf{H} obtained from 1 , $\frac{1}{2}(1 + i + j + k)$, and $a + \frac{1}{2}i + bj$ by even permutations of coordinates and arbitrary sign changes. We can then verify that Φ is indeed a group of order 120 and thus is a root system. Then we can let W be the resulting reflection group. W is irreducible and it must be of type H_4 . Further, we have the following simple system: $\alpha_1 = a - \frac{1}{2}i + bj$, $\alpha_2 = -a + \frac{1}{2}i + bj$, $\alpha_3 = \frac{1}{2} + bi - aj$, and $\alpha_4 = -\frac{1}{2} - ai + bk$. Note that the roots α_1, α_2 , and α_3 form a simple

system for the subgroup of type H_3 .

2.2 Orbits of Parabolic subgroups

In this section, we will follow the notation of [Hum75].

Let G be a reductive algebraic group and let $\theta \in \text{Aut}(G)$, the set of automorphisms of G where θ has order m .

Definition 2.20. We call $\theta \in \text{Aut}(G)$, where $\theta \neq \text{id}$ an involution if the order of θ is 2.

For our purposes, from this point on, we will let θ be an involution.

Definition 2.21. The fixed point group of θ is denoted as $H = G_\theta = \{g \in G \mid \theta(g) = g\}$

Using the preceding definitions, we can define a symmetric variety.

Definition 2.22. A symmetric variety is the homogeneous space G/H .

If we choose to define G and θ over a field k that is not algebraically closed then we have a symmetric k -variety rather than a symmetric variety

Definition 2.23. A symmetric k -variety is defined as $Q_k = G_k/H_k$ where G_k, H_k denote the sets of k -rational points of G, H respectively.

The orbits of a minimal parabolic k -subgroup P acting on the symmetric k -variety, $Q_k = G_k/H_k$ play an important part in the study of representations associated with these symmetric k -varieties. We can classify them in many different ways. First, we can describe them as P_k -orbits that are acting on the symmetric k -variety Q_k by θ twisted conjugation. We can define θ twisted conjugation for $x, g \in G_k$ as the operation $*$ where $g * x = g x \theta(g)^{-1}$. They can also be described as the number of H_k -orbits acting on the flag variety G_k/P_k by conjugation or as the set $P_k \backslash G_k/H_k$ of (P_k, H_k) -double cosets in G_k . The final description is the same as the set of $P_k \times H_k$ -orbits on G_k .

For k algebraically closed and $P=B$, a Borel subgroup, these orbits were characterized by Springer [Spr85]. He characterized the double cosets $B \backslash G/H$ in several different ways. For $k = \mathbb{R}$ and P a minimal parabolic k -subgroup, characterizations were given by Matsuki

[Mat79] and Rossmann [Ros79] and for general fields these orbits were characterized by Helminck and Wang [HW93].

For general fields one can first consider the set of (P, H) -double cosets in G . Then the (P_k, H_k) -double cosets in G_k can be characterized by the (P, H) -double cosets in G defined over k plus an additional invariant describing the decomposition of a (P, H) -double coset into (P_k, H_k) -double cosets.

Helminck and Wang classified the double coset $P_k \backslash G_k / H_k$ in many different ways. It can be classified as the H_k orbits on G_k / P_k . Let T be a torus of G_k ,

$$W = W_{G_k}(T) = N_{G_k}(T) / Z_{G_k}(T)$$

is the Weyl group of T in G_k . Further we can we can classify $W_{H_k}(T)$ as $\{w \in W_{G_k}(T) \mid w \text{ has a representative in } N_{H_k}(T)\}$. Also if T is θ -stable then so are all of the above.

The following result characterizes the (P, H) -double cosets in G . Let $T \subset P$ be a θ -stable maximal k -split torus of G , which we know exists by [HW93], $N = N_{G_k}(T)$ be the normalizer of T on G_k , $Z = Z_{G_k}(T)$ be the centralizer of T on G_k , and let $\mathcal{S}_\theta = \{w \in W \mid \theta(w) = w^{-1}\}$ be the set of θ -twisted involutions in W .

Theorem 2.24 ([HH11]). *The (P, H) -double cosets in G defined over k can be characterized by the pairs (\mathcal{O}, w) , where \mathcal{O} is a closed orbit and $w \in \mathcal{S}_\theta \subset W$.*

Using these \mathcal{O} orbits as described above we can induce an order called the *Bruhat Order*.

Definition 2.25. *If $\mathcal{O}_1, \mathcal{O}_2$ are orbits as described in Theorem 2.24, then $\mathcal{O}_1 < \mathcal{O}_2$ if and only if \mathcal{O}_2 is contained in the orbit closure of \mathcal{O}_1 .*

The above definition gives us a geometric understanding of the Bruhat order, however it is not the only way that we can understand the Bruhat order geometrically. More directly, the Bruhat order arises from the Bruhat decomposition which can be understood as a generalization of the $PA = LU$ factorization of a nonsingular matrix A , where P is a permutation matrix, L is a lower triangular matrix, and U is an upper triangular matrix. Let B be a Borel subgroup of G which contains a maximal torus T . We know from [Hum75] that the $B \backslash G / B$ double cosets are in one-to-one correspondence with the Weyl group $W(T)$.

Definition 2.26. *The Bruhat Decomposition is characterized by*

$$G = \dot{\bigcup}_{w \in W(T)} B \dot{w} B$$

where $\dot{w} \in N_G(T)$ is a representative of $w \in W(T)$.

We can define the *Bruhat order* geometrically if we let $w_1, w_2 \in W(T)$ then $w_1 \leq w_2$ if and only if $B \dot{w}_1 B \subseteq \overline{B \dot{w}_2 B}$ where $\dot{w}_1, \dot{w}_2 \in N_G(T)$ are representatives of $w_1, w_2 \in W(T)$. Meaning w_1 is less than w_2 if and only if the orbit containing w_1 is contained in the closure of the orbit containing w_2 .

2.3 Richardson-Springer Order

In this section, we will examine some results about the orbit closures and the Richardson-Springer order induced by the closure relations. Let's begin with some notation.

Let θ be an involution such that $\theta(\Phi^+) = \Phi^+$, V be the set of orbits, B be a fixed Borel subgroup of G and T a θ -stable maximal torus of B . Let φ be the map $\varphi : V \rightarrow \mathcal{I}$ where \mathcal{I} is the set of twisted involutions. Let $\Phi = \Phi(G, T)$ denote the set of roots of T in G and $\Phi^+ = \Phi(B, T)$ with basis Δ . Let s_α denote the reflection defined by $\alpha \in \Phi$ and write $S = \{s_\alpha \mid \alpha \in \Delta\}$ for the set of simple reflections in W . Then (W, S) is a Coxeter group. For $\alpha \in \Phi$, let \mathfrak{g}_α be the root subspace of the Lie algebra $\mathfrak{g} = L(G)$ of G corresponding to α . We have the decomposition $\mathfrak{g} = L(G) = L(T) \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$. Given $\alpha \in \Delta$, let $P_\alpha = P_{s_\alpha}$ denote the standard parabolic subgroup of G containing B such that $\Phi(P_\alpha, T) = (\mathbb{Z}\alpha \cap \Phi) \cup \Phi^+$. Also, consider It is easy to see that

$$\dim(P_\alpha) = \dim(B) + \dim\left(\bigoplus_{\gamma \in \mathbb{Z}\alpha \cap \Phi^+} \mathfrak{g}_\gamma\right) = \dim(B) + 1. \quad (2.1)$$

Let $Q = \{g\theta(g)^{-1} \mid g \in G\}$ and let τ be a morphism from G to G where $\tau(x) = x\theta(x)^{-1}$ with $x \in G$. Notice $\tau(G) = Q$. The set Q is stable under the twisted action of G defined by $g * x = gx\theta(g)^{-1}$, $g \in G$, $x \in Q$. Set $\mathcal{V} = \{x \in G \mid \tau(x) \in N\}$. Let $\nu \in V$, $x = x(\nu)$ where $x(\nu) \in \mathcal{V}$ is a representative of the orbit ν in V , $\mathcal{O}_\nu = BxH$ and $n = x\theta(x)^{-1}$. Let $w_\Pi^0 \in W$ be the longest element with respect to Π where Π is a subset of Δ . In fact, this same definition will hold for any subset of Δ . In the following we will describe the closure of the orbit $B * n$

in Q . Since $\overline{\mathcal{O}_v} = \tau^{-1}(\overline{B * n})$ this will also give a description of the orbits $\mathcal{O}_v = BxH$ in G .

Proposition 2.27 ([HW93]). *If $w \in \mathcal{S}_\theta$, then there exist $s_1, \dots, s_h \in S$ and a θ -stable subset Π of Δ satisfying the following conditions.*

$$(i) \quad w = s_1 \dots s_h w_\Pi^0 \theta(s_h) \dots \theta(s_1) \text{ and } \ell(w) = 2h + \ell(w_\Pi^0).$$

$$(ii) \quad w_\Pi^0 \theta \alpha = -\alpha, \alpha \in \Phi_\Pi$$

Moreover if $w = t_1 \dots t_m w_\Lambda^0 \theta(t_m) \dots \theta(t_1)$, where $t_1, \dots, t_m \in S$ and Λ a θ -stable subset of Δ satisfying conditions (i) and (ii), then $m = h$, $s_1 \dots s_h \Pi = t_1 \dots t_h \Lambda$ and

$$s_1 \dots s_h \theta(s_h) \dots \theta(s_1) = t_1 \dots t_h \theta(t_h) \dots \theta(t_1).$$

Let w be the image of n in W . Where $w = s_1 \dots s_h w_\Pi^0 \theta(s_h) \dots \theta(s_1)$, with $\ell(w) = 2h + \ell(w_\Pi^0)$ as in Proposition 2.27. Choose $n_1, \dots, n_h \in N_G(T)$ with images s_1, \dots, s_h in W respectively. Set $s_i = s_{\alpha_i}$ and let $P_i = P_{\alpha_i}$, $1 \leq i \leq h$ and let P_Π be such that $\Phi(P_\Pi, T) = Z\Pi \cap \Phi^+$. If we write $u = n_h^{-1} \dots n_1^{-1} x$ and $m = u\theta(u)^{-1}$, then we have the following description of the orbit closures [HW93].

Proposition 2.28. *Let B, v, x, n, w, u, m, Π be as above. Then we have the following.*

$$(i) \quad \overline{B * n} = P_1 * \dots * P_h * P_\Pi * m.$$

$$(ii) \quad \overline{\mathcal{O}_v} = \overline{BxH} = \tau^{-1}(\overline{B * n}) = P_1 \dots P_h P_\Pi \mathcal{O}_u.$$

$$(iii) \quad \dim(B * n) = S_{i=1}^h (\dim(P_i) - \dim(B)) + \dim(P_\Pi * m) = h + \dim(P_\Pi * m).$$

This result gives a fairly detailed description of the orbit closures. It suggests that the closure of an orbit \mathcal{O}_v is to a large extent determined by the corresponding twisted involution $\varphi(v)$. To actually compute the orbits contained in the closure of an orbit one typically uses the Richardson-Springer order defined by the closure relations. We discuss this order in the next subsection.

2.3.1 The Richardson-Springer Order on V

Let $v \in V$, $x = x(v)$ and let $\mathcal{O}_v = BxH$ denote the corresponding double coset. This is a smooth subvariety and the closure $\overline{\mathcal{O}_v}$ is a union of double cosets. The Richardson-Springer order on V is the order \leq on V defined by the closure relations on the double cosets $\mathcal{O}_v = Bx(v)H$. Thus $v_1 \leq v$ if and only if $\mathcal{O}_{v_1} \subset \overline{\mathcal{O}_v}$.

We can represent the closure relations on the set of double cosets $Bx(v)H$ by a diagram, using the above Richardson-Springer order. This will be called the diagram of V .

In the case of the Bruhat decomposition of the group G (i.e. B -orbits on G/B), Chevalley gave a combinatorial description of this geometrically defined order. Here the Bruhat order on the $B \times B$ -orbits on G corresponds to the combinatorially defined Bruhat order on the Weyl group. In the case of B -orbits on the symmetric variety G/H , Richardson and Springer gave a similar combinatorial description of the Richardson-Springer Order on V (see [RS90]). They used the map $\varphi : V \rightarrow \mathcal{S}$ to map elements of V to twisted involutions in the Weyl group. An additional complication in this case is that this map is often not one to one. Note that Richardson-Springer order on V also induces an order on \mathcal{S} . A combinatorial description of this order on \mathcal{S} can be found in [RS90]. In this paper Richardson and Springer also prove that this combinatorial order on \mathcal{S} and the Richardson-Springer order are compatible (meaning $v_1 \leq v_2$ if and only if $\varphi(v_1) \leq \varphi(v_2)$).

In [RS90], the combinatorial description of the Richardson-Springer order on V and \mathcal{S} is given with respect to a standard pair (B, T) . Throughout this section, our \mathcal{S} will reflect a generalization of these results to an arbitrary pair (B, T) discussed in [Hel96].

When defining the combinatorial Richardson-Springer order on V , we need to analyze the description of the orbit closure in Proposition 2.28 in more detail. The set $P_{\Pi} \mathcal{O}_u$ as in Proposition 2.28 can also be described by a sequence $P_{s_1} \dots P_{s_k}$ acting on a closed orbit, where (s_1, \dots, s_k) is a sequence in S . To prove this, it is sufficient to analyze the rank one situation which gives us the following result: Let $v \in V$, $x = x(v)$ and $\mathcal{O}_v = BxH$ the corresponding double coset. If $s \in S$, then $P_s \mathcal{O}_v = P_s xH$ is a union of one, two or three $(B \times H)$ -orbits. For a detailed description of this, please refer to [RS90].

The rank one analysis leads to the following result, which enables us to define an action of the Coxeter group (W, S) on V .

Lemma 2.29. *Let $v \in V$ and $s \in S$. Then we have the following.*

- (i) $P_s \overline{\mathcal{O}}_v$ is closed.
- (ii) $P_s \mathcal{O}_v$ contains a unique dense $(B \times H)$ -orbit.

The first statement follows from a lemma given in [Ste74] and the second statement can be found in [RS90].

2.3.2 Admissible sequences in V

Related to a sequence $\mathbf{s} = (s_1, \dots, s_k)$ in S we can define now a sequence $\mathbf{v}(\mathbf{s}) = (v_0, v_1, \dots, v_k)$ in V as follows. Let \mathcal{O}_{v_0} be a closed orbit and for $i \in [1, k]$ let \mathcal{O}_{v_i} be the unique dense orbit in $P_{s_i} \mathcal{O}_{v_{i-1}}$. We will call \mathbf{s} an *admissible sequence for $v \in V$* if there exists a closed orbit \mathcal{O}_{v_0} , such that the sequence $\mathbf{v}(\mathbf{s}) = (v_0, v_1, \dots, v_k)$ in V satisfies

$$\dim P_{s_{i+1}} \dots P_{s_1} \mathcal{O}_{v_0} > \dim P_{s_i} \dots P_{s_1} \mathcal{O}_{v_0} \quad \text{for } i = 1, \dots, k-1, \quad (2.2)$$

and $v = v_k$. In this case we will also call the pair $(\mathcal{O}_{v_0}, \mathbf{s})$ an *admissible pair for v* .

The above sequences start at a closed orbit and build up from there.

It is possible to move up from any orbit to the open orbit by means of a sequence \mathbf{s} in S . The corresponding sequence in V is defined as follows. Let $v \in V$, $\mathbf{s} = (s_1, \dots, s_k)$ a sequence in S and $v_{\max} \in V$ the open orbit. Define the sequence $\mathbf{v}_o(\mathbf{s}) = (v_0, v_1, \dots, v_k)$ in V as follows. Let $v_0 = v$ and for $i \in [1, k]$ let \mathcal{O}_{v_i} be the unique dense orbit in $P_{s_i} \mathcal{O}_{v_{i-1}}$. If $v_k = v_{\max}$, we call this an *open-sequence for v in V* . We call an open-sequence for v in V an *admissible open-sequence for v in V* if it satisfies condition (2.2).

Through the following theorem, we know that both an admissible pair and an admissible open-sequence for each $v \in V$ exists.

Theorem 2.30. *Let $v \in V$. Then we have the following.*

- (i) *There exists a closed orbit \mathcal{O}_{v_0} and a sequence $\mathbf{s} = (s_1, \dots, s_k)$ in S such that $\overline{\mathcal{O}}_v = P_{s_k} \dots P_{s_1} \mathcal{O}_{v_0}$ and $\dim \mathcal{O}_v = k + \dim \mathcal{O}_{v_0}$.*
- (ii) *There exists an admissible open-sequence \mathbf{t} for v .*

This result is a refinement of Theorem 4.6 in [RS90] to arbitrary pairs (B, T) and follows easily from Proposition 2.28 and the rank one analysis.

If $\mathbf{s} = (s_1, \dots, s_k)$ is a sequence in S , then we write $\ell(\mathbf{s}) = k$ for the length of the sequence. The length $L(v)$ of an element $v \in V$ can be defined now as follows. Let $(\mathcal{O}_{v_0}, \mathbf{s})$ be an admissible pair for v . Define $L(v) = k = \dim \mathcal{O}_v - \dim \mathcal{O}_{v_0}$. Since all closed orbits have the same dimension the above definition of length does not depend on the admissible pair for v .

The relation between the length of an admissible open-sequence for v and an admissible sequence for v is now as follows [RS90].

Lemma 2.31. *Let $v \in V$, $v_{\max} \in V$ the open orbit, \mathbf{s} an admissible sequence for v and \mathbf{t} an admissible open-sequence for v . Then we have the following.*

$$(i) \quad L(v) = \ell(\mathbf{s}).$$

$$(ii) \quad \ell(\mathbf{t}) = L(v_{\max}) - L(v) = L(v_{\max}) - \ell(\mathbf{s}).$$

Remark 1. *The above discussion shows how a sequence $\mathbf{s} = (s_1, \dots, s_k)$ in S can be used to define a sequence $\mathbf{v}(\mathbf{s}) = (v_0, v_1, \dots, v_k)$ in V . Using the map $\varphi : V \rightarrow \mathcal{S}$ the sequence \mathbf{s} in S also defines sequences $\mathbf{a}(\mathbf{s}) = (a_0, a_1, \dots, a_k)$ in \mathcal{S} where $a_i = \varphi(v_i)$. In Definition 2.34 we will give a combinatorial description of these sequences in \mathcal{S} .*

The sets of admissible sequences and admissible open-sequences in V each have natural orders. These orders are defined as follows.

2.3.3 Combinatorial Richardson-Springer order on V

Let $x, y \in V$. Then we write $x \preceq y$ if there exist admissible pairs $(\mathcal{O}_{v_0}, \mathbf{s} = (s_1, \dots, s_k))$ for x and $(\mathcal{O}_{v_0}, \mathbf{t} = (t_1, \dots, t_r))$ for y with $k \leq r$ and $s_i = t_i$ for $i = 1, \dots, k$. We write $x \preceq_o y$ if there exist admissible open-sequences $\mathbf{s} = (s_1, \dots, s_k)$ for x and $\mathbf{t} = (t_1, \dots, t_r)$ for y with $k \geq r$ and $s_i = t_i$ for $i = 1, \dots, r$.

It is easy to see that \preceq and \preceq_o define partial orders on V . Moreover one easily shows that these orders are opposite:

Lemma 2.32. *Let $x, y \in V$. Then $x \preceq y$ if and only if $y \preceq_o x$.*

The order \preceq on V is the same as the standard order on V as defined in [RS90]. It follows now that the order \preceq on V and the geometric Richardson-Springer order on $V \leq$ are the same in [RS90].

Theorem 2.33. *Let \leq be the order on V as in Subsection 2.3.3, \leq the Richardson-Springer order on V and let $x, y \in V$. Then $x \leq y$ if and only if $x \leq y$.*

In the following we will define orders on the set \mathcal{I} of twisted involutions similar to the orders on V as in Subsection 2.3.3.

2.3.4 Admissible sequences in \mathcal{I}

As in V , a sequence $\mathbf{s} = (s_1, \dots, s_k)$ in S induces sequences in \mathcal{I} . These sequences, which we will call S -sequences in \mathcal{I} , are defined by induction as follows. The sequence in \mathcal{I} is $\mathbf{a}(\mathbf{s}) = (a_0, a_1, \dots, a_k)$, where $a_0 = \text{id}$ and $a_i = s_i \circ a_{i-1}$ for $i \in [1, k]$.

These S -sequences start at the identity in \mathcal{I} and build up from there. One could also start with the longest element in the Weyl group with respect to Δ and build the sequence in \mathcal{I} down from there. In that case we get the following. Let $w_\Delta^0 \in W$ be the longest element with respect to the basis Δ . If $\mathbf{s} = (s_1, \dots, s_k)$ is a sequence in S , then define a sequence $\mathbf{b}(\mathbf{s}) = (b_0, b_1, \dots, b_k)$ in \mathcal{I} by induction as follows. Let $b_0 = w_\Delta^0$ and for $i \in [1, k]$ let $b_i = s_i \circ b_{i-1}$. Such a sequence will be called a w_Δ^0 -sequence in \mathcal{I} .

Remark 2. *Since $\mathcal{I} w_\Delta^0 = \mathcal{I}_{\theta w_\Delta^0}$ it follows that the w_Δ^0 -sequences in \mathcal{I} correspond to S -sequences in $\mathcal{I} w_\Delta^0$.*

To define an order on \mathcal{I} compatible with this action we need first to define admissible sequences. These are defined as follows. For $w \in W$ let $\ell(w)$ denote the length of w with respect to the Bruhat order on W .

Definition 2.34. *Let $\mathbf{s} = (s_1, \dots, s_k)$ be a sequence in S and let $\mathbf{a}(\mathbf{s}) = (a_0, a_1, \dots, a_k)$ (resp. $\mathbf{b}(\mathbf{s}) = (b_0, b_1, \dots, b_k)$) be the S -sequence (resp. w_Δ^0 -sequence) in \mathcal{I} induced by \mathbf{s} . Then \mathbf{s} is called an admissible sequence (or an admissible S -sequence) if $0 = \ell(a_0) < \ell(a_1) < \dots < \ell(a_k)$. The sequence \mathbf{s} is called an admissible w_Δ^0 -sequence if $\ell(w_\Delta^0) = \ell(b_0) > \ell(b_1) > \dots > \ell(b_k)$. If $a \in \mathcal{I}$, then the sequence \mathbf{s} in S is called an admissible S -sequence for a (resp. an admissible w_Δ^0 -sequence for a) if \mathbf{s} is an admissible S -sequence (resp. admissible w_Δ^0 -sequence) and $a_k = a$ (resp. $b_k = a$).*

If $a \in \mathcal{I}$ and $\mathbf{s} = (s_1, \dots, s_k)$ an admissible S -sequence for a , then we define the length of a as $\ell(a) = \ell(\mathbf{s}) = k$.

Using a similar argument to one used in [RS90], it follows that every element of \mathcal{S} has an admissible S -sequence and an admissible w_Δ^0 -sequence:

Lemma 2.35. *For every $a \in \mathcal{S}$, there exist sequences \mathbf{s} and \mathbf{t} in S , such that \mathbf{s} is an admissible S -sequence for a and \mathbf{t} is an admissible w_Δ^0 -sequence for a .*

From this result we get the following relation between the length of an admissible S -sequence for a and an admissible w_Δ^0 -sequence for a [RS90].

Lemma 2.36. *Let $a \in \mathcal{S}$, \mathbf{s} be an admissible S -sequence for a and \mathbf{t} an admissible w_Δ^0 -sequence for a . Then $\ell(\mathbf{t}) = \ell(w_\Delta^0) - \ell(a) = \ell(w_\Delta^0) - \ell(\mathbf{s})$.*

As in V the admissible sequences in \mathcal{S} induce partial orders on \mathcal{S} . In fact the S -sequences and w_Δ^0 -sequences in \mathcal{S} lead to opposite orders in \mathcal{S} . These orders are defined as follows.

2.3.5 Richardson-Springer Order on \mathcal{S}

Let $a, b \in \mathcal{S}$. Define the order \preceq_1 on \mathcal{S} with respect to the admissible S -sequences as follows. Write $a \preceq_1 b$ if there exist admissible S -sequences $\mathbf{s} = (s_1, \dots, s_k)$ for a and $\mathbf{t} = (t_1, \dots, t_r)$ for b with $k \leq r$ and $s_i = t_i$ for $i = 1, \dots, k$.

Similarly, using admissible w_Δ^0 -sequences we get an order \preceq_2 on \mathcal{S} as follows. If $a, b \in \mathcal{S}$, then write $a \preceq_2 b$ if there exists admissible w_Δ^0 -sequences $\mathbf{s} = (s_1, \dots, s_k)$ for a and $\mathbf{t} = (t_1, \dots, t_r)$ for b with $k \leq r$ and $s_i = t_i$ for $i = 1, \dots, k$.

The orders \preceq_1 and \preceq_2 define partial orders on \mathcal{S} . The order \preceq_1 will also be called the *Richardson-Springer order on \mathcal{S}* . Using a combinatorial argument one can show that this order on \mathcal{S} is compatible the partial order on \mathcal{S} induced by the Bruhat order on the Weyl group W . The orders \preceq_1 and \preceq_2 define opposite orders on \mathcal{S} . We summarize this in the following result [RS90].

Proposition 2.37. *Let $a, b \in \mathcal{S}$. Then we have the following.*

- (i) *An admissible sequence for a can be extended to an admissible sequence for w_Δ^0 .*
- (ii) *w_Δ^0 is the longest element of \mathcal{S} with respect to \preceq_1 .*

(iii) $a \preceq_1 b$ if and only if $b \preceq_2 a$.

From the above results it easily follows now that the admissible sequences in V lead to admissible sequences in \mathcal{S} .

Lemma 2.38. *Let $v \in V$. If $(\mathcal{O}_{v_0}, \mathbf{s} = (s_1, \dots, s_k))$ is an admissible pair for v and $\mathbf{v}(\mathbf{s}) = (v_0, v_1, \dots, v_r)$ is the corresponding sequence in V , then $\mathbf{a}(\mathbf{s}) = (a_0, a_1, \dots, a_k)$, with $a_i = \varphi(v_i)$ ($i = 1, \dots, k$), is an admissible sequence for $a = \varphi(v)$.*

Combining the above results we obtain now the following relation between the Richardson-Springer order on V and the Richardson-Springer order on \mathcal{S} .

Theorem 2.39. *Let $v \in V$, $a = \varphi(v) \in \mathcal{S}$ and $\mathbf{s} = (s_1, \dots, s_k)$ a sequence in S . Then \mathbf{s} is an admissible sequence for a if and only if there exists a closed orbit \mathcal{O}_{v_0} such that $(\mathcal{O}_{v_0}, \mathbf{s})$ is an admissible pair for v .*

Using the opposite sequences we also get the following result:

Corollary 2.40. *Let $v \in V$, $v_{\max} \in V$ be the open orbit, $a = \varphi(v)$, $a_{\max} = \varphi(v_{\max}) \in \mathcal{S}$, and $\mathbf{t} = (t_1, \dots, t_r)$ an admissible w_{Δ}^0 -sequence for a_{\max} . A sequence $\mathbf{s} = (s_1, \dots, s_k)$ in S is an admissible open-sequence for v if and only if $\mathbf{w} = (t_1, \dots, t_r, s_k, \dots, s_1)$ is an admissible w_{Δ}^0 -sequence for a .*

The question which remains now is how to compute the closure of an orbit x in V . As we have seen the closure of this orbit consists of $\{y \in V \mid y \leq x\}$ and therefore it becomes a question of the Richardson-Springer orders on V and \mathcal{S} . In the case of the Bruhat order on the Weyl group one uses Bruhat descendants to calculate all y in W which are less than or equal to a given element x with respect to the Bruhat order. Using the above characterizations of the orders on V and \mathcal{S} in terms of admissible sequences, we can use a similar procedure.

2.3.6 Bruhat descendants

Let $v \in V$ and let $w = \varphi(v) \in \mathcal{S}$. The *Bruhat descendants of v* are those elements $x \in V$ for which $x \leq v$ and $L(x) = L(v) - 1$. Similarly the *Bruhat descendants of w* are those $y \in \mathcal{S}$ for which $y \preceq_1 w$ and $\ell(y) = \ell(w) - 1$.

A description of the Bruhat descendants of ν or w follows easily from the following property of the Richardson-Springer order on V [RS90]: Recall, we defined the Exchange condition in Section 2.1. We can apply that condition to this situation as follows:

(Exchange condition). Let $x \in V$ and $(\mathcal{O}_{\nu_0}, \mathbf{s} = (s_1, \dots, s_k))$ be an admissible pair for x . Let $s \in S$ and $y \in V$ be such that $x = s \circ y$, $L(y) = L(x) - 1$. Then there exist $i \in [1, k]$ and an admissible pair $(\mathcal{O}_{\nu_0}, \mathbf{s} = (s_1, \dots, \hat{s}_i, \dots, s_k))$ for y , such that the pair $(\mathcal{O}_{\nu_0}, \mathbf{t} = (s_1, \dots, \hat{s}_i, \dots, s_k, s))$ is an admissible pair for x .

The Richardson-Springer order on \mathcal{S} satisfies a similar condition.

It follows from this that in order to compute the Bruhat descendants of $\nu \in V$ one needs to find two things.

- (1) The closed orbits contained in the closure of ν .
- (2) The Bruhat descendants of w in \mathcal{S} .

From the exchange condition it follows that the second problem is a matter of computing admissible subsequences. These are defined as follows. Let $\mathbf{s} = (s_1, \dots, s_k)$ be a sequence in S . A sequence $\mathbf{t} = (t_1, \dots, t_r)$ in S is called an *subsequence of \mathbf{s}* if $r < k$ and for all $i \in [1, r]$ we have $t_i = s_{j_i}$, where $j_i \in [1, k]$ and $j_i < j_{i+1}$. If \mathbf{s} is an admissible sequence for $w \in \mathcal{S}$, then a subsequence \mathbf{t} induces a sequence $\mathbf{b}(\mathbf{t}) = (b_0, b_1, \dots, b_r)$ as in Subsection 2.3.4. The subsequence \mathbf{t} in S is called an *admissible subsequence for w* if $\mathbf{b}(\mathbf{t})$ is an admissible S -sequence in \mathcal{S} .

Now let $w \in \mathcal{S}$. The Bruhat descendants of w can be obtained as follows. Let \mathbf{s} be an admissible sequence for w . Then the set of subsequences obtained from \mathbf{s} by removing a single reflection such that the resulting subsequence remains admissible, contains exactly one admissible sequence for each Bruhat descendent of w . From the exchange condition it follows that this is independent of the chosen admissible sequence \mathbf{s} for w . This notion of Bruhat descendants is similar to the one in the Weyl group.

Now any $y \in \mathcal{S}$ with $y \preceq_1 w$ can be obtained by starting from w and repeatedly moving from an element to one of its Bruhat descendants. This all leads to the following characterization of the elements contained in the closure of an orbit in V :

Proposition 2.41. *Let $x, y \in V$. Then $y \preceq x$ if and only if there exist an admissible pair $(\mathcal{O}_{\nu_0}, \mathbf{s})$ for x and a subsequence $\mathbf{t} = (t_1, \dots, t_r)$ of \mathbf{s} such that $(\mathcal{O}_{\nu_0}, \mathbf{t})$ is an admissible pair for y .*

The above Bruhat descendants give an inductive procedure to compute the closure of an orbit $v \in V$. This computation basically reduces to computing admissible subsequences in S .

2.4 Involution Posets

As we've examined before, the Coxeter group W acts on the set \mathcal{I}_θ by twisted conjugation $w * a = wa\theta(w)^{-1}$ for $w \in W$ and $a \in \mathcal{I}_\theta$. Now, we will define two operations. Let S be the generating set of reflections for our Coxeter group. Let $s \in S$ and $a \in \mathcal{I}_\theta$ then we will define $s \circ a = sa$ (left multiplication) if $s * a = a$ and $s * a$ otherwise. As shown in the following two lemmas, both operations are closed in \mathcal{I}_θ .

Lemma 2.42. *If $a \in \mathcal{I}_\theta$, then $sa\theta(s)^{-1} \in \mathcal{I}_\theta$*

Lemma 2.43. *If $a \in \mathcal{I}_\theta$, $s = s^{-1}$, and $sa\theta(s)^{-1} = a$, then $sw \in \mathcal{I}_\theta$*

It is convenient to have a visual tool to help us understand the Richardson-Springer order. Thus we will use an involution poset of a Coxeter group to help us see the order as explained in Section 2.3. To help us keep track of when we are using each operation, we will use a $\bar{}$ notation to signify that we are using twisted conjugation. Thus for $s \in S$, $a \in \mathcal{I}_\theta$ we will use the notation \bar{s}_i if $s \circ a_{i-1} = \bar{s}_i a_{i-1}$. Define $\bar{S} := \{\bar{s} \mid s \in S\}$ and let $\mathbf{r}_s = (r_1, r_2, \dots, r_k)$ be the sequence in $S \cup \bar{S}$ defined by $r_i = \bar{s}_i$ if $s_i \circ a_{i-1} = \bar{s}_i a_{i-1}$ in $\mathbf{a}(s)$ and $r_i = s_i$ otherwise.

In [HH12], Haas and Helminck came up with an algorithm for building the weak involution poset of Weyl groups. This algorithm allows for the input of a Weyl group and it outputs the involution poset. The algorithm for the weak involution poset when $\theta = \text{id}$ is as follows:

Algorithm 1. *Input Weyl group. Output poset for \mathcal{I}_{id} .*

0. *Let the stack consist of the identity element of rank 0, and set $k = 0$.*

I. *For each x of rank k from the stack, do*

A. *For each s_i such that there is no pointer into x labeled s_i or \bar{s}_i (no ascending admissible sequence for x begins with s_i or \bar{s}_i) do:*

- (i) Calculate $y' = s_i x s_i$. If $y' \neq x$ then let $y = y'$ which is of rank $k + 1$. If $y' = x$ then let $y = s_i x$, which is of rank $k + 1$.
- (ii) For each element z of rank $k + 1$, check if $y = z$ if so put a pointer from x to y labeled $s_i, (\overline{s_i})$ (add sequence of y to list of ascending admissible sequences for z). If no such z add y to stack as new element of rank $k + 1$, with a pointer from x labeled $s_i, (\overline{s_i})$.

II. If $k < k_0$, let $k = k + 1$ and go to step [I].

This algorithm can be generalized for any θ as follows:

Algorithm 2. I. Run Algorithm 1

II. For each element w of rank k found in 1 associate an element w' of rank $k_0 - k$ in \mathcal{I}_θ .

III. For each pointer $x \rightarrow w$ with label $s \in S \cup \overline{S}$ found in 1 of associate a pointer $w' \rightarrow x'$ in \mathcal{I}_θ with label $s \in S \cup \overline{S}$.

Example 6. Let $\theta = \text{id}$. We will begin by conjugating s_1 , a generator for $A_3 = S_4$, with s_3 , another generator of A_3

$$s_3 s_1 s_3 = s_1 s_3 s_3 = s_1$$

So we ended up with the same element that we started with. So instead we will left multiply.

$$s_3 s_1 = s_1 s_3$$

Now we've ended up with a new element. Therefore we will put it in our poset.

In Figure 2.3, we have the weak involution poset for $A_3 = S_4$. Where there are dotted lines, left multiplication was the operation used and where there are solid lines, conjugation was our operation. Each edge is labeled with the generator that we left multiplied or conjugated with. Notice that if $\theta \neq \text{id}$, then we are using twisted conjugation rather than conjugation.

Along with the weak involution poset, we can also introduce the *strong involution poset*. This poset is built the same way as the involution poset. The difference is related to the elements we twist-conjugate or left multiply by. In the involution poset, we only use the generators, $s \in S$. However, in the strong involution poset, we operate with all of the

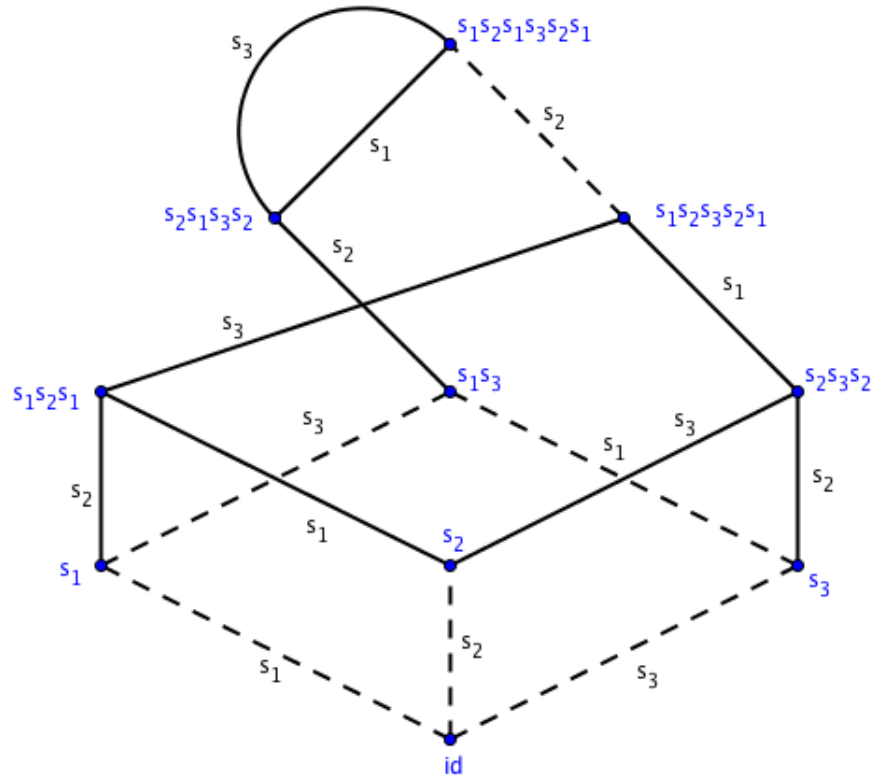


Figure 2.3 Weak Involution Poset of $A_3 = S_4$ for $\theta = \text{id}$

reflections. Thus, we share the same vertex set in both types of involution posets, but the order of the edge set increases in the strong involution poset.

In [BB05] and [Inc04], we are introduced to another type of involution poset which is denoted $\text{Invol}(n)$ and we will call the *Bruhat involution poset*. We can also extend this to the strong Bruhat order getting a fourth type of Involution Poset that we will call the *strong Bruhat involution poset*.

CHAPTER

3

PAST RESULTS FOR WEYL GROUPS

Recall that the Weyl groups are a subset of Coxeter groups which include A_n , B_n/C_n , D_n , E_6 , E_7 , E_8 , F_4 , and G_2 . Some involution posets of these Coxeter groups have been studied at some length. Incitti studied the Bruhat involution posets for B_n in [Inc03], A_n in [Inc04], and then extended those results to classical Weyl groups in [Inc06]. Haas and Helminck considered the weak involution posets (which they called the Richardson-Springer poset) as well as properties of these posets and the Richardson-Springer order itself on a variety of Weyl groups in [HH11].

3.1 Bruhat Involution Poset

Incitti approached the problem through combinatorics. This, before we begin, we will start with a little bit of combinatorial background on posets following from [Sta12] and [BB05].

Definition 3.1. *A sequence of elements, (x_0, \dots, x_h) in a poset P is called a chain if $x_0 \leq \dots \leq x_h$.*

The length of the chain is h .

Definition 3.2. A chain is maximal if the elements are not a proper subset of any other chain.

Definition 3.3. Let \triangleleft denotes a covering relation where $x \triangleleft y$ means that $x \leq y$ and there is no z such that $x \leq z \leq y$. A chain is called saturated if all relations are covering relations. Meaning (x_0, \dots, x_h) is saturated if $x_0 \triangleleft \dots \triangleleft x_h$

Definition 3.4. If all maximal chains are the same finite length, then we say P is pure.

Definition 3.5. A poset is bounded if it has a minimum and a maximum, which we denote by $\hat{0}$ and $\hat{1}$ respectively.

Definition 3.6. A poset P is said to be graded of rank n if it is finite, bounded, and if all maximal chains of P have the same length n .

Definition 3.7. Let P be a poset and let Q be a totally ordered set. An EL-Labeling of P is a function $\lambda : \{(x, y) \in P^2 | x \triangleleft y\} \rightarrow Q$ such that every $x, y \in P$, with $x \leq y$, satisfy the following properties:

1. there is exactly one saturated chain from x to y with non decreasing labels:

$$x = x_0 \triangleleft_{\lambda_1} \dots \triangleleft_{\lambda_h} x_h = y$$

with $\lambda_1 \leq \dots \leq \lambda_h$

2. this chain has the lexicographically minimal labeling: if

$$x = x_0 \triangleleft_{\mu_1} \dots \triangleleft_{\mu_h} x_h = y$$

is a saturated chain from x to y different from the one described above, then

$$(\lambda_1 \dots \lambda_h) < (\mu_1 \dots \mu_h)$$

Definition 3.8. A graded poset with an EL-labeling is called EL-shellable.

Definition 3.9. *A graded poset with a rank function ρ is Eulerian if*

$$|\{z \in [x, y] \mid \rho(z) \text{ is even}\}| = |\{z \in [x, y] \mid \rho(z) \text{ is odd}\}|$$

for every $x, y \in P$ such that $x \leq y$

We know that the Bruhat poset is a graded poset with a rank function that is given by the length and that it is EL-shellable. However, Incitti examined if the same properties carried over to the Bruhat involution poset.

In [Inc06], the main result is listed below.

Theorem 3.10. *Let W be a classical Weyl group. $\text{Invol}(W)$, the Bruhat involution poset, is*

1. *graded, with rank function given by*

$$\rho(w) = \frac{\ell(w) + a\ell(w)}{2}$$

for every $w \in \text{Invol}(W)$

2. *EL-shellable*
3. *Eulerian*

In the first part of theorem 3.10, Incitti lists a rank function that is given in terms of the length and the absolute length of an element. This is the same length function as described in Definition 2.10. The absolute length is the minimal k such that w can be written as a product of k reflections. Using this formula we can determine the rank of any element in W .

Remark 3. *A similar rank function can be considered for the weak involution poset. Rather than considering the absolute length of an element of w , we can instead focus on the number of unbarred operations, or left multiplications, that are applied to an element. We will consider this rank function further in Chapter 5.*

In [Inc04], the results are focused strictly on $\text{Invol}(A_n)$ rather than all Weyl groups. Thus more specific results are presented for this particular class of posets.

Theorem 3.11. *The poset $\text{Invol}(A_n)$ is graded, with rank function ρ given by*

$$\rho(w) = \frac{\text{inv}(w) + \text{exc}(w)}{2}$$

where $w \in \text{Invol}(A_n)$, $\text{inv}(w)$ is the number of inversions, and $\text{exc}(w)$ is the number of excedances. In particular $\text{Invol}(A_n)$ has rank

$$\rho(\text{Invol}(A_n)) = \left\lfloor \frac{(n+1)^2}{4} \right\rfloor$$

Example 7. *Consider the element $w = s_2 s_1 s_3 s_2$ or written in cycle notation (13)(24) in the weak Bruhat Involution poset for A_3 . We can determine the rank using the formula given in Theorem 3.11. We can see that $\text{inv}(w) = 4$ and $\text{exc}(w) = 2$. Thus $\rho(w) = \frac{4+2}{2} = 3$. Thus the rank of w is three and is on the third level of the poset, where zero-th level contains the identity and we count up from there. We can see this to be true on the poset in figure 3.1c.*

Example 8. *Using the formula for the rank of the poset given in Theorem 3.11, we can see that $\rho(A_3) = \left\lfloor \frac{(3+1)^2}{4} \right\rfloor = \left\lfloor \frac{16}{4} \right\rfloor = 4$. We can verify this with figure 3.1c.*

3.2 Weak Involution Poset

Haas and Helminck have also written about involution posets in [HH12] and [HH11]. However, rather than focusing on the Bruhat involution poset as Incitti did, they instead focused on the weak involution poset. As seen in Chapter 2, Haas and Helminck created Algorithms 1 and 2 to build the weak involution poset in both the situation where θ is the identity and when θ is any involution. They also determined many different properties of the weak involution poset.

Haas and Helminck found that in many cases for Weyl groups, it was only necessary to compute half of the weak involution poset due to the Corollary 3.12. Because [HH12] focused primarily on Algorithms that could then be written as code to run on a computer, the following result allows for the computation of half of a poset in certain cases which in turn makes it the result more quickly computed.

Corollary 3.12. *If Φ is of type $A_1, B_n, C_n, E_7, E_8, F_4$, or G_2 , then every ascending admissible sequence is also a descending admissible sequence and vice versa. That is, the poset is symmetric.*

We will examine whether involution posets of type H also have this property in Chapter 5.

In [HH12], Haas and Helminck created the following algorithm to identify the conjugacy classes of elements in \mathcal{S}_{id} .

Algorithm 3. *Input Weyl group W and element $w \in W$. This algorithm will return the set of elements conjugate to w .*

0. *Let the stack consist of the element w , where $d_w(w) = 0$, and set $k = 0$.*

I. *If there is no element x in the stack with $d_w(x) = k$ stop.*

For each x in the stack such that $d_w(x) = k$, do

A. *For each s_i such that there is no pointer into x labeled $\overline{s_i}$ (no admissible sequence for x begins with $\overline{s_i}$) do:*

(i) *Calculate $y' = s_i x s_i$. If $y' \neq x$ then let $y = y'$ and $d_w(y) = k + 1$.*

(ii) *For each element z such that $d_w(z) = k + 1$, check if $y = z$ if so put a pointer from x to z labeled $\overline{s_i}$ (add sequence of y to list of admissible sequences for z). If no such z add y to stack as new element of rank $k + 1$, with a pointer from x labeled $\overline{s_i}$.*

II. *Let $k = k + 1$ and go to step [I].*

However, that leads to the question to whether there is a similar way to find the conjugacy classes of elements in \mathcal{S}_θ . This could be completed in one of two ways:

1. Replace step (i) in Algorithm 3 with "(i) Calculate $y' = s_i x \theta(s_i^{-1})$. If $y' \neq x$ then let $y = y'$ and $d_w(y) = k + 1$ ".
2. Run the algorithm as intended and then convert results to \mathcal{S}_θ .

This second option require an understanding of how the elements of \mathcal{I}_{id} and \mathcal{I}_{θ} are related. If the involution θ is the restriction of an involution of a reductive algebraic group G to the root system of a maximal k -split torus as in [HH11], then we have the following result.

Corollary 3.13. *Let (W, S) be a Coxeter system, and θ an involution, such that $\theta(\Delta) = \Delta$ then for any $w \in \mathcal{I}_{\text{id}}$, $\mathbf{r} = (r_1, r_2, \dots, r_n)$ in $S \cup \bar{S}$ is an ascending admissible sequence for $w \in \mathcal{I}_{\text{id}}$ if and only if it is a descending w_0 -admissible sequence for $w w_0 \in \mathcal{I}_{\theta}$.*

However, we also have the following theorem which is more general.

Theorem 3.14. *Let (W, S) be a Coxeter system, and θ an involution, then there exists an element $w_0 \in W$ such that: $w \in \mathcal{I}_{\text{id}}$ if and only if $w w_0 \in \mathcal{I}_{\theta}$.*

In the same paper, Haas and Helminck gave us Algorithm 4 to compute the closure of an element in \mathcal{I}_{id} . This algorithm can then be extended to elements in \mathcal{I}_{θ} in a similar way that was done for finding the conjugacy classes above. The algorithm is as follows:

Algorithm 4. *Input an element w and specify Weyl group W . This algorithm will return the set of elements in the $+$ -closure of w .*

0. Let the stack consist of w , where $d_w(w) = 0$, and set $k = 0$.

I. For each x such that $d_w(x) = k$ from the stack, do

A. For each s_i such that there is no pointer into x labeled s_i or \bar{s}_i (no ascending w -admissible sequence for x begins with s_i or \bar{s}_i) do:

(i) Calculate $y' = s_i x s_i$. If $y' \neq x$ and $l(y') > l(x)$ then let $y = y'$ and $d_w(y) = k + 1$. If $y' = x$ and $l(s_i x) > l(x)$ then let $y = s_i x$, and $d_w(y) = k + 1$.

(ii) If $y = w_0$ stop.

(iii) For each element z with $d_w(z) = k + 1$, check if $y = z$ if so put a pointer from x to y labeled $s_i, (\bar{s}_i)$ (add sequence of y to list of ascending admissible sequences for z). If no such z add y to stack as new element of distance $k + 1$, with a pointer from x labeled $s_i, (\bar{s}_i)$.

II. Let $k = k + 1$ and go to step [I].

Further, they determined the order of \mathcal{I}_θ which in turn gave the number of elements in the weak involution poset.

Proposition 3.15. • For A_{n-1} there are

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{(n-2k)!2^k k!}$$

elements total in \mathcal{I}_{id} , and \mathcal{I}_θ .

• For B_n there are

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!2^{n-2k}}{(n-2k)!k!}$$

elements total in \mathcal{I}_{id} .

• For n odd, D_n has

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!2^{n-2k-1}}{(n-2k)!k!}$$

elements total in \mathcal{I}_{id} , and \mathcal{I}_θ .

• For n even, D_n has

$$\frac{n!}{(n/2)!} + \sum_{k=0}^{\lfloor n/2 \rfloor - 1} \frac{n!2^{n-2k-1}}{(n-2k)!k!}$$

elements total in \mathcal{I}_{id} and there are

$$\frac{-(n-1)!}{((n-2)/2)!} + \sum_{k=0}^{\lfloor (n-2)/2 \rfloor} \frac{(n-1)!2^{(n-2-2k)}(n-2k+1)}{k!(n-1-2k)!}$$

in \mathcal{I}_θ for the non-trivial automorphism.

They also determined the longest element w_0 for classical Weyl groups.

Theorem 3.16. • For Weyl group of type A_{n-1} , the longest element w_0 has rank $(n^2-1)/4$ if n is odd, $n^2/4$ if n is even.

- For Weyl group of type B_n , the longest element w_0 has rank $(n+1)n/2$.
- For Weyl group of type D_n , the longest element w_0 has rank $n^2/2$ if n is even and $(n+1)(n-1)/2$ if n is odd.

By combining the results of Theorem 3.16 and Theorem 3.11, we can see that the longest element of A_n has the same rank as A_n . Thus the longest element is at the top of the poset.

In [HH11], Haas and Helminck determined a number of results related to admissible sequences (recall Definition 2.34) for Weyl groups.

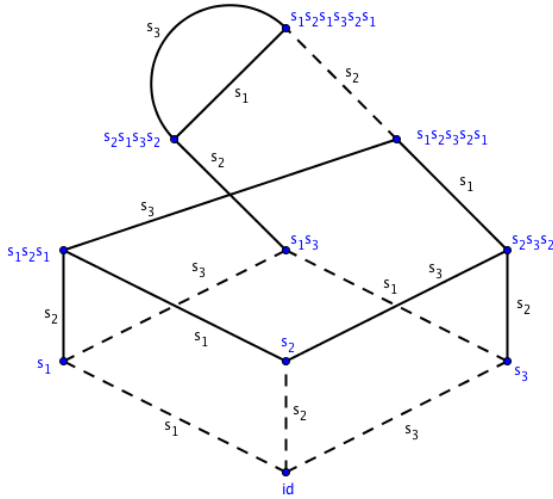
Theorem 3.17. *Let W be a Weyl group, S , Δ and Φ as in Chapter 2, and θ an involution such that $\theta(\Delta) = \Delta$. Denote by \cdot the action of a sequence $\mathbf{r} \in S \cup \bar{S}$ on \mathcal{I}_θ , and \cdot' the action of \mathbf{r} on \mathcal{I}_{id} .*

1. *A sequence $\mathbf{r} = (r_1, r_2, \dots, r_n)$ in $S \cup \bar{S}$ is an admissible ascending sequence for \mathcal{I}_θ if and only if it is an admissible descending sequence for \mathcal{I}_{id} .*
2. *A sequence $\mathbf{r} = (r_1, r_2, \dots, r_n)$ in $S \cup \bar{S}$ is an admissible descending sequence for \mathcal{I}_θ if and only if it is also an admissible ascending sequence for \mathcal{I}_{id} .*
3. *Let \mathbf{q} and \mathbf{r} be two sequences in $S \cup \bar{S}$. They are two admissible ascending sequences for \mathcal{I}_θ and $\mathbf{q} \cdot e = \mathbf{r} \cdot e$ in \mathcal{I}_θ if and only if they are two admissible descending sequences for \mathcal{I}_{id} and $\mathbf{q} \cdot' w_0 = \mathbf{r} \cdot w_0$ in \mathcal{I}_{id} .*

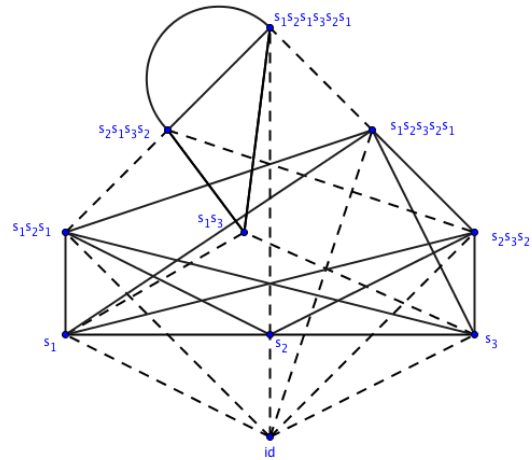
Lemma 3.18. *If $(r_1, \dots, r_k) \in S \cup \bar{S}$ is an admissible ascending sequence and $(r_1, \dots, r_k) \cdot e \neq w_0$ then there exists an element $r_{k+1} \in S \cup \bar{S}$ such that $(r_1, \dots, r_k, r_{k+1})$ is also an admissible ascending sequence.*

We have seen that Incitti focused on weak Bruhat involution posets (which he called $\text{Invol}(W)$), while Haas and Helminck instead chose to focus on weak involution posets (also called Richardson-Springer posets). Although these posets have the same vertex sets, their edge sets vary significantly. Below in Figure 3.1 all four types of posets including those considered by both Incitti and Haas and Helminck are shown below together for comparison. If we strictly compare the weak involution poset of A_3 and the weak Bruhat involution poset both for A_3 , we can see that their edge sets are not a subset of each other.

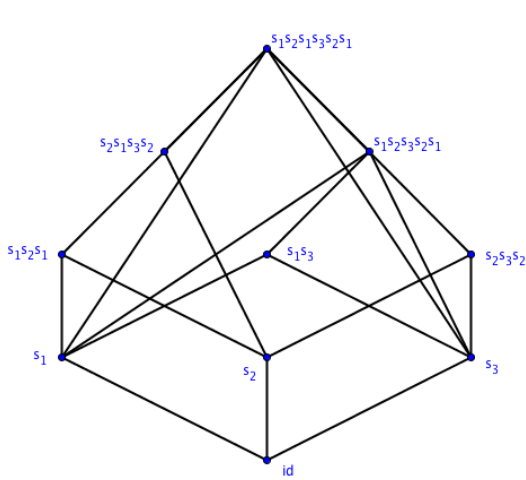
The weak involution poset has an edge between the vertices $s_1 s_3$ and $s_2 s_1 s_3 s_2$ and no edge between $s_1 s_2 s_1$ and $s_2 s_1 s_3 s_2$, while the weak Bruhat involution poset does not have an edge between the vertices $s_1 s_3$ and $s_2 s_1 s_3 s_2$ and yet has an edge between $s_1 s_2 s_1$ and $s_2 s_1 s_3 s_2$.



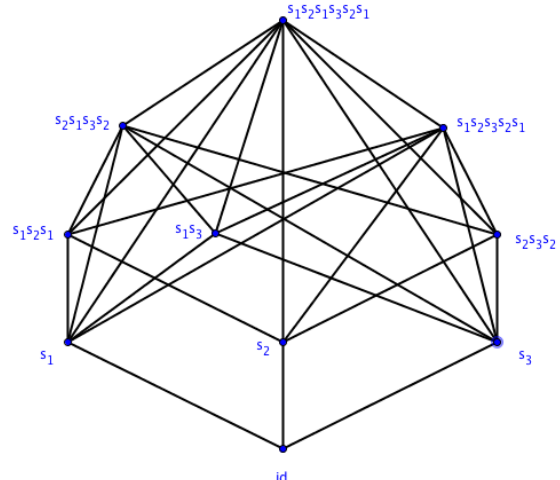
(a) Weak Involution Poset of A_3 , $\theta = \text{id}$



(b) Strong Involution Poset of A_3 , $\theta = \text{id}$



(c) Weak Bruhat Involution Poset of A_3



(d) Strong Bruhat Involution Poset of A_3

Figure 3.1 Involution Posets of A_3

CHAPTER

4

RESULTS FOR COXETER GROUPS OF TYPE $I_2(M)$

The Coxeter group of type $I_2(m)$, perhaps better known as the dihedral group, is the reflection group of regular polygons with m sides. As noted in Subsection 2.1.1, elements written as a product of 2 generators, both reflections, can be written by alternating the two reflections up to $(s_1 s_2)^m$ which gives us the identity. Note that in the following results, we let $\theta = \text{id}$.

Proposition 4.1. *The topmost element of any involution poset for $I_2(m)$ when m is even is $(s_1 s_2)^{m/2} = (s_2 s_1)^{m/2}$. If m is odd then it is $s_1 (s_2 s_1)^{\frac{m-1}{2}} = s_2 (s_1 s_2)^{\frac{m-1}{2}}$.*

Proof. To begin, we will focus on showing that the stated elements are the longest elements given the parity of m . In either case, recall $(s_1 s_2)^m = \text{id}$.

If m is even then we know $(s_1 s_2)^{m/2} = (s_2 s_1)^{m/2}$. On each side of the equality, we have equivalent elements of length m . If the power on $s_1 s_2$ decreases on a side, note that it will increase on the other. So $(s_1 s_2)^{m/2+1} = (s_2 s_1)^{m/2-1}$ and thus $\ell((s_1 s_2)^{m/2+1}) > \ell((s_2 s_1)^{m/2-1})$.

Therefore, if we ever have an element that includes a sequence longer than $(s_1 s_2)^{m/2}$ or $(s_2 s_1)^{m/2}$, we can reduce. To ensure that $(s_1 s_2)^{m/2} = (s_2 s_1)^{m/2}$ are the longest elements, we need to make sure they cannot be reduced. However, the only relationship that we have in $I_2(m)$ is $(s_1 s_2)^m = \text{id}$, meaning there is no relationship to allow us to reduce further. Now, we must ensure that $(s_1 s_2)^{m/2}$ is an involution. If that is the case then, We know $((s_1 s_2)^{m/2})^2 = \text{id}$. This is clear. Thus $(s_1 s_2)^{m/2}$ is not only the longest element of $I_2(m)$, when m is even, it is also the longest element of the weak involution poset, thus it sits at the top of the poset.

If m is odd, then we can know $s_1(s_2 s_1)^{\frac{m-1}{2}} = s_2(s_1 s_2)^{\frac{m-1}{2}}$. Again, we have elements on each side of length m . If we attempt to move elements, then note we will create a reducing relationship. So it's maximal. Also note, as before, we cannot reduce since the only reducing relationships are derived from $(s_1 s_2)^m$. Now, we want to show that we have an involution. Notice, $(s_1(s_2 s_1)^{\frac{m-1}{2}})^2 = \text{id}$. Thus, it's an involution. □

Corollary 4.2. *The length of the topmost element of any involution poset for $I_2(m)$ is m .*

Proof. By Proposition 4.1, we know the topmost element is $(s_1 s_2)^{m/2}$ when m is even and $s_1(s_2 s_1)^{\frac{m-1}{2}}$ when m is odd. When m is even, to get the length we multiply 2 and $m/2$ getting m . When m is odd, we multiply 2 with $\frac{m-1}{2}$ and add 1 for the leading s_1 , once again getting m . □

Proposition 4.3. *The weak involution poset of $I_2(m)$ has exactly two maximal chains. If m is even, the chains are of length $\frac{m}{2} + 1$ and if m is odd, the length of the chains are $\frac{m-1}{2} + 1$.*

Proof. From Proposition 4.1 we know the topmost element of the weak involution poset in both the cases when m is even and m is odd. Further, we know that to build the poset we start at the identity, and as usual begin with left multiplication of both generators. We then proceed by conjugating. Note as long as conjugation leads to an element of length less than or equal to m , we know that the element is not reducible since our only Coxeter relation, beyond those of the generators begin order 2, is that $(s_1 s_2)^m = \text{id}$. Thus, it is only when we reach an element with length longer than m that we can reduce using our relation. Thus when m is even, we can conjugate until we reach the element $s_2(s_1 s_2)^{\frac{m}{2}-1}$ or $s_1(s_2 s_1)^{\frac{m}{2}-1}$, depending on which side we're progressing up. We will focus on $s_2(s_1 s_2)^{\frac{m}{2}-1}$, noting that the process is similar for $s_1(s_2 s_1)^{\frac{m}{2}-1}$. If we conjugate $s_1(s_2 s_1)^{\frac{m}{2}-1}$, we will reach an element of

length $m + 1$. Therefore, we know we can simplify since our relation tells us that $(s_1 s_2)^{\frac{m}{2}} = (s_1 s_2)^{\frac{m}{2}}$. By conjugating the element $s_2(s_1 s_2)^{\frac{m}{2}-1}$ by s_1 , we get $s_1 s_2(s_1 s_2)^{\frac{m}{2}-1} s_1 = (s_1 s_2)^{\frac{m}{2}} s_1 = (s_2 s_1)^{\frac{m}{2}} s_1 = s_2(s_2 s_1)^{\frac{m}{2}-1} s_1 s_1 = s_2(s_2 s_1)^{\frac{m}{2}-1}$. Thus we get back to the element that we started with, so we must proceed to left multiplication, which in turn gives us the maximal element discussed in Proposition 4.1.

A similar process can be followed when m is odd. However, note that we can conjugate until we reach our maximal element and then our poset is finished.

In all cases, we can build each maximal chain from the identity until we reach a single element of maximum length.

Now, we must approach the issue of the lengths of these chains. First off, note that since both chains from the identity to the topmost element are maximal, they are of equal length. Thus we will focus on just one of the maximal chains in each case.

If m is even, then the length of an element increases by 1 when going from rank 0 to rank 1, then the length increases by 2 (since you are conjugating) until you reach the element $s_1(s_2 s_1)^{\frac{m}{2}-1}$ and then finally increases by length 1 to reach our maximal element. Thus we just need to determine how many times we conjugate to get from the element s_1 to the element $s_1(s_2 s_1)^{\frac{m}{2}-1}$ because each time we conjugate we are traveling up an edge. We know we're going from an element of length 1 to an element of length $m-1$. Thus, we must conjugate $\frac{m}{2} - 1$ times to go from s_1 to $s_1(s_2 s_1)^{\frac{m}{2}-1}$. Taking into account our $\frac{m}{2} - 1$ conjugations and our 2 left multiplications, once at the bottom and once at the top, we get that our chain is of length $\frac{m}{2} + 1$.

If m is odd, then the length of the element increase by 1 when going from rank 0 to rank 1 and then by two up to the maximal element, since we are conjugating from there. Thus, we can determine the number of times that we are conjugating to be $\frac{m-1}{2}$ and then we add one for the number of times that we left multiply. This gives us the length of the maximal chain to be $\frac{m-1}{2} + 1$. \square

Corollary 4.4. *The weak involution poset of $I_2(m)$ is a cycle. If m is even, then the cycle is of length $m + 2$. If m is odd, the cycle is of length $m + 1$.*

Proof. By Proposition 4.3, we know that we have two maximal chains and a unique top element as well as a unique bottom element. Thus, our poset is a cycle. Further, since we know the length of the maximal chains, we know the length of the cycle is double. Therefore,

if m is even, then the cycle has length $2(\frac{m}{2} + 1) = m + 2$ and if m is odd, then the cycle is of length $2(\frac{m-1}{2} + 1) = m + 1$. \square

Recall, a Hamiltonian cycle is spanning cycle meaning a cycle that passes through each vertex of the graph, or poset in this case.

Corollary 4.5. *The weak and strong involution poset of $I_2(m)$ contains a Hamiltonian cycle.*

Proof. By Corollary 4.4, we know that weak involution poset of $I_2(m)$ is a cycle made of the two maximal chains discussed in Proposition 4.3. Thus the cycle runs through every vertex in the poset, which means that it is a Hamiltonian cycle.

Since the strong involution poset and the weak involution poset of $I_2(m)$ contain the same vertices and the edge set of the weak involution poset is a subset of the edge set of the strong involution poset, the Hamiltonian cycle found in the weak involution poset also exists in the strong involution poset. \square

Recall, that a graph is bipartite if the vertex set is the union of two disjoint independent sets. Further, recall by a theorem credited to König, we know that a graph is bipartite if and only if it has no odd cycle.

Corollary 4.6. *The weak involution poset of $I_2(m)$ is a bipartite graph.*

Proof. By Corollary 4.4, we know that the weak involution poset of $I_2(m)$ is a cycle. By the theorem by König we mentioned above, we know that if this cycle is not odd, then our poset can be understood as a bipartite graph. When m is even, we know that our cycle is of length $m + 2$, thus giving us an even cycle. Likewise, when m is odd, our cycle is of length $m + 1$, once again giving us a cycle of even length. Therefore, the weak involution poset of $I_2(m)$ is a bipartite graph. \square

Using the preceding results, we get a good idea of what the weak involution poset for $I_2(m)$ will look like. Weak involution posets for $I_2(m)$ will look very similar no matter the m we are considering. In fact, they are so similar that we can give the following generalizations of the weak involution posets of $I_2(m)$ based on the parity of m and the parity of both $\frac{m}{2}$ and $\frac{m-1}{2}$ in Figures 4.1, 4.2, 4.3, and 4.4.

Proposition 4.7. *The weak involution poset of $I_2(m)$ is a graded poset.*

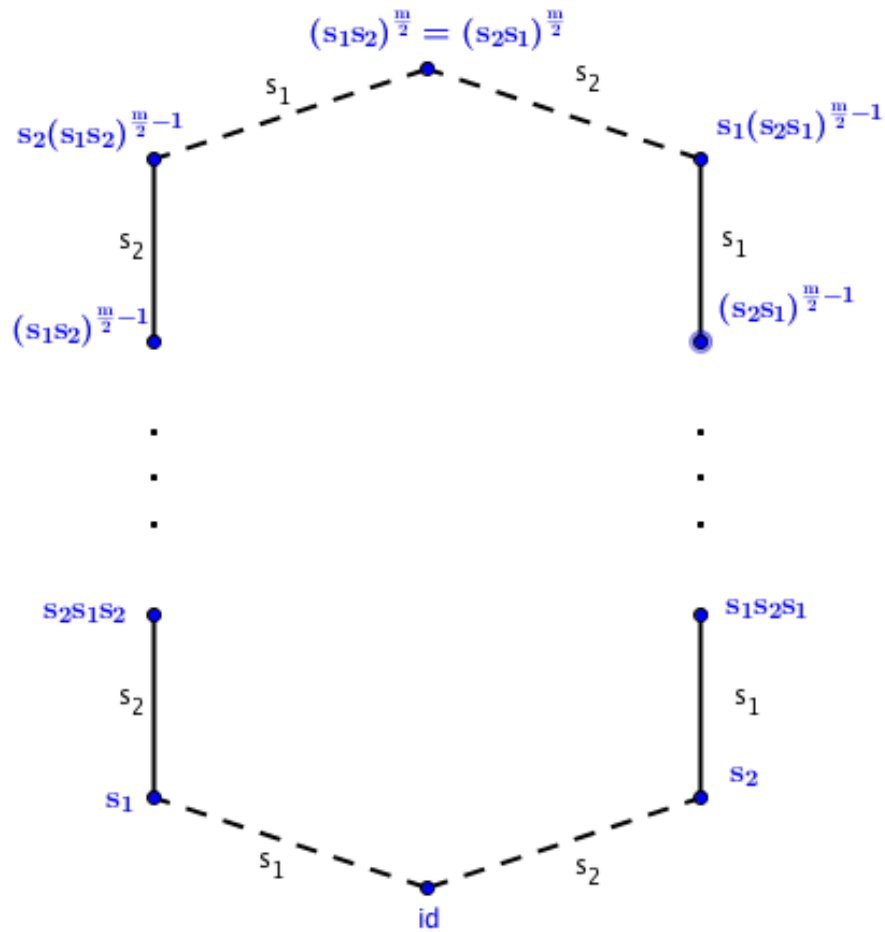


Figure 4.1 The Weak Involution Poset of $I_2(m)$ if both m and $\frac{m}{2}$ are even.

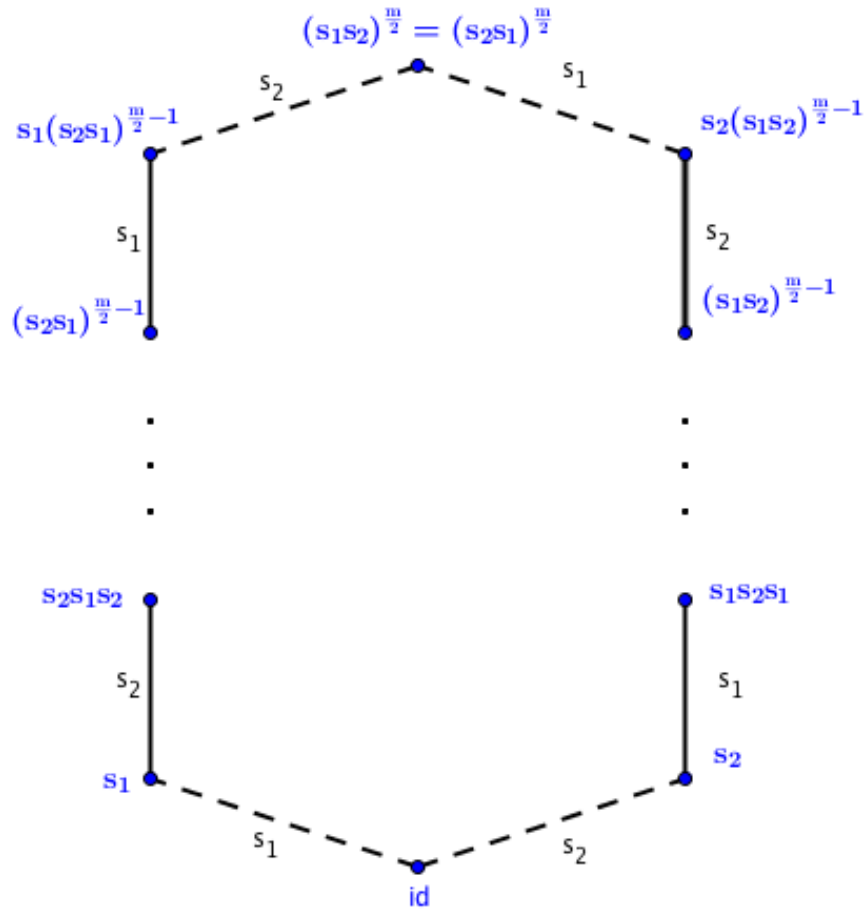


Figure 4.2 The Weak Involution Poset of $I_2(m)$ if m is even and $\frac{m}{2}$ is odd.

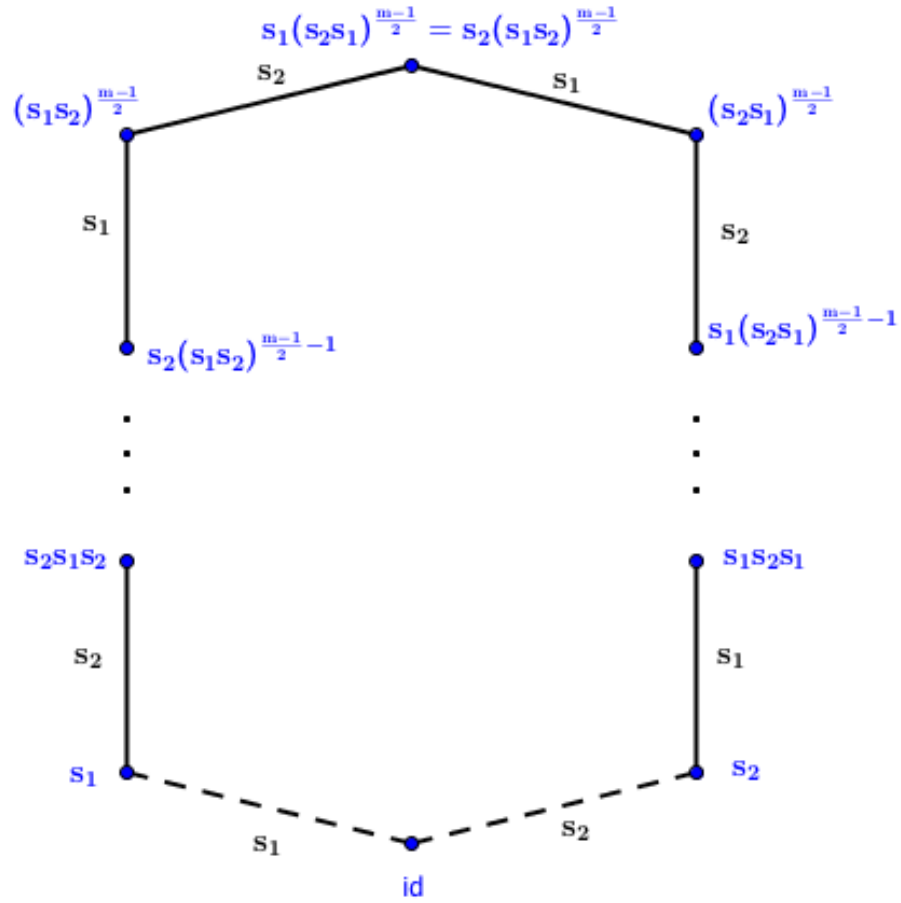


Figure 4.3 The Weak Involution Poset of $I_2(m)$ if m is odd and $\frac{m-1}{2}$ is even.

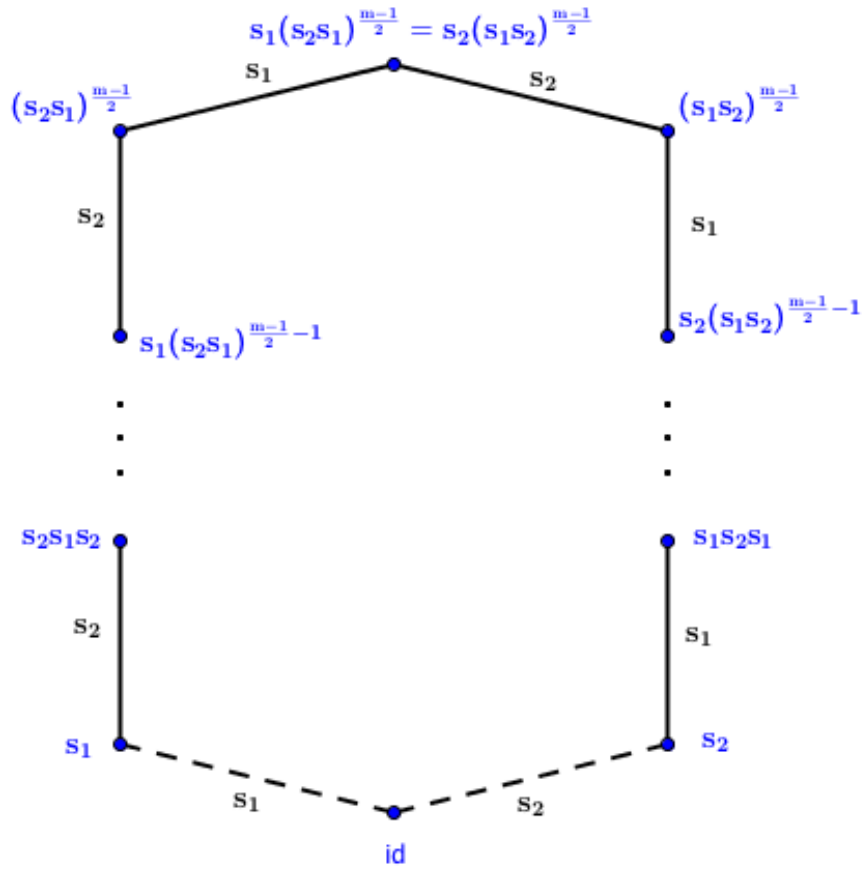


Figure 4.4 The Weak Involution Poset of $I_2(m)$ if m is odd and $\frac{m-1}{2}$ is odd.

Proof. By Proposition 4.3, we know that the involution poset for $I_2(m)$ has exactly two maximal chains and further, we know those two chains are the same length. By Definition 3.5, we know that our posets is bounded, with $\hat{0} = \text{id}$ and $\hat{1}$ equal to the element listed in Proposition 4.1. Further, by Definition 3.6, we know that since our poset is bounded, finite, with all maximal chains of equal length that our poset is graded. \square

Proposition 4.8. *The weak involution poset of $I_2(m)$ is symmetric when m is even.*

Proof. If a weak involution poset is symmetric, then each admissible ascending sequence is also an admissible descending sequence. Since we have generalizations of the weak involution posets of $I_2(m)$ when m is even, seen in Figures 4.1 and 4.2, we can easily see that our 2 admissible ascending sequences spanning from id to $(s_1 s_2)^{\frac{m}{2}} = (s_2 s_1)^{\frac{m}{2}}$ is also an admissible descending sequence. \square

CHAPTER

5

RESULTS FOR COXETER GROUPS OF TYPE H_2 , H_3 , AND H_4

The following results have been determined for Coxeter groups of type H , made up of H_2 , H_3 , and H_4 . For the weak involution poset and for the strong involution poset, we let $\theta = \text{id}$. Note that all four involution posets share the same vertex set, it is the edge set that varies between them. Specifically, the strong involution poset contains many more edges than the weak involution poset and similarly, the strong Bruhat involution poset will contain more edges than the weak Bruhat involution poset. In the case of the Bruhat Involution poset and the strong Bruhat involution poset, we are inducing edges from the weak Bruhat poset and the strong Bruhat poset and retaining the relative rank of the vertices. As noted in site Chapter 3, some refer to the weak Bruhat involution poset as $\text{Invol}(W)$, where W is a Coxeter group.

We will begin by proving what makes up the vertex set of our involution posets.

Proposition 5.1. *The vertex set for each of the involution posets of Coxeter groups of type H , excluding the identity, is the set of order 2 elements of the group (or involutions of the group)*

Proof. In the case of the weak and strong Bruhat Involution posets, we are specifically taking the set of involutions as our vertex set. Also, note that if the weak involution poset has a vertex set made up of the order 2 elements of the group, then the strong involution poset will have the same vertex set. Ultimately, we only need to show that it's true for the weak involution poset. By Lemmas 2.42 and 2.43, we know that based on our two operations, that our vertices will be involutions. However, we then have to ensure that we are not just getting a subset of the set of involutions. However, fortunately the order of the set of involutions of the Coxeter groups of type H is known. There are 5 elements of order 2 in H_2 , 31 elements of order 2 in H_3 and 571 elements of order 2 in H_4 . Thus the order of the set of involutions for these Coxeter groups is equal to the vertex set of our weak involution poset (excluding the identity). \square

Recall the definition of a covering relation and saturated given in Definition 3.3. All chains in the weak involution poset are saturated because all relations are covering relations. This clear based on how we build our weak involution poset as stated in Algorithm 1.

Theorem 5.2. *In the weak involution poset of H_2 , H_3 , and H_4 , all maximal chains have the same length. Further, the length of a maximal chain is equal to the rank of the poset.*

Proof. All chains are saturated in the weak involution poset of Coxeter groups of type H . Further, in the case of the weak involution posets of H_2 , H_3 , and H_4 , there is a unique element at the top of poset as seen in Figures 5.1, 5.2, and 5.3 and a unique element at the bottom of the poset, id. Any saturated chain that runs from id to the topmost element, must travel along a single edge when traveling from an element of rank i to one of rank $i + 1$. Further, each edge we travel on increases the length of the chain by one. Thus, if the rank of the poset is n , then the length of a maximal chain must also be n . \square

Proposition 5.3. *Neither the weak involution poset nor the strong involution poset for H_3 is symmetric.*

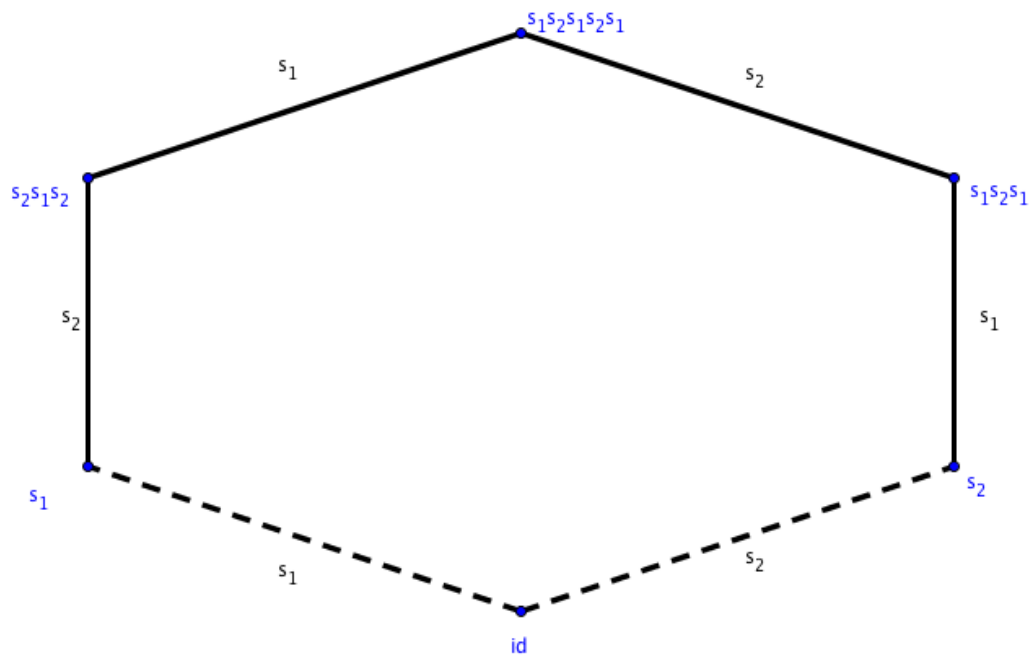


Figure 5.1 Weak Involution Poset of H_2

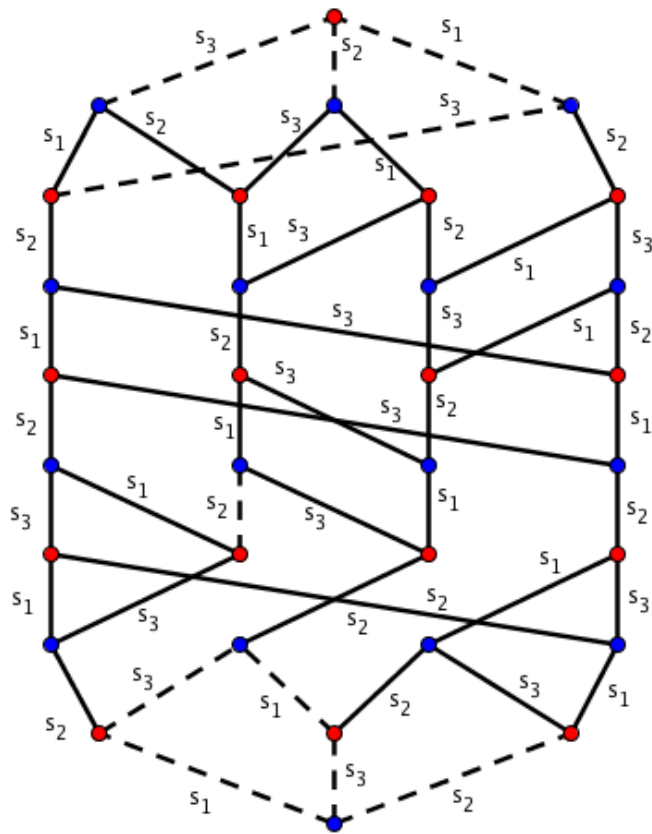


Figure 5.2 Weak Involution Poset of H_3

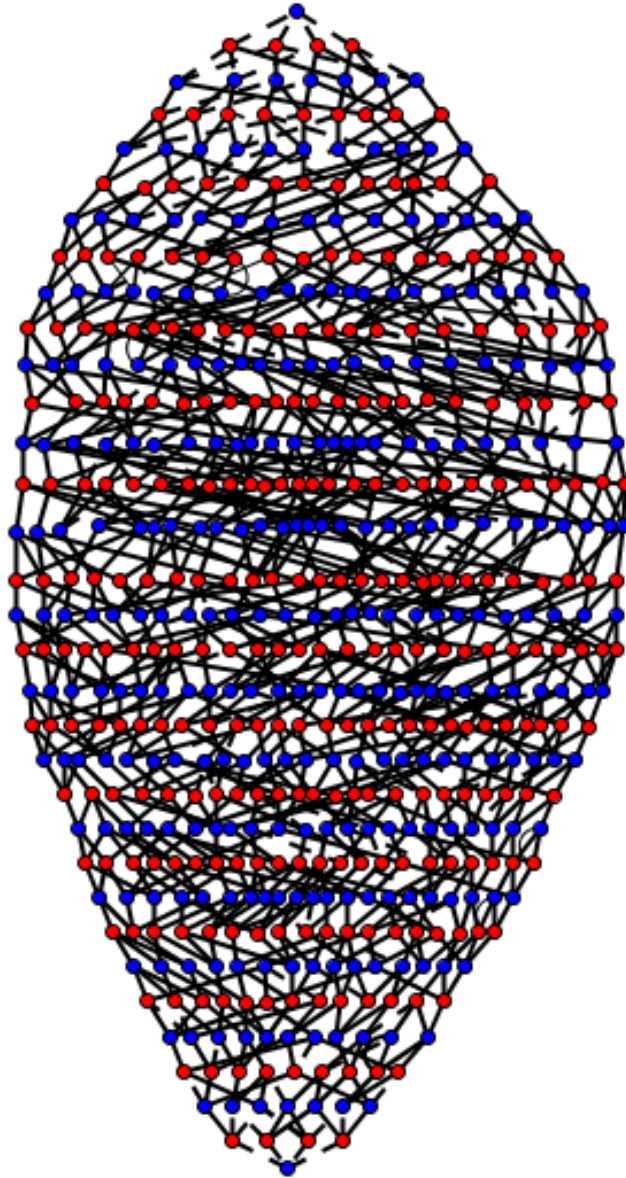


Figure 5.3 Weak Involution Poset of H_4

Proof. If the weak involution poset for H_3 is not symmetric then there exists an ascending admissible sequence in the poset that is not also a descending admissible sequence. The ascending admissible sequence $(s_2, \overline{s_1}, \overline{s_3}, \overline{s_2}, \overline{s_1}, \overline{s_2}, \overline{s_3}, \overline{s_2}, s_1)$ is not a descending admissible sequence. Meaning, if you apply that ascending admissible sequence, taking into account the correct bar and unbar operations as presented in Figure 5.2 to the topmost element, $s_1 s_2 s_1 s_2 s_1 s_3 s_2 s_1 s_2 s_1 s_3 s_2 s_1 s_2 s_3$, in the weak involution poset of H_3 , then you will not get the identity. Instead, we get the element s_3 . Note that if this ascending admissible sequence exists in weak involution poset, then it also exists in the strong involution poset. Thus, the ascending admissible sequence is not a descending admissible in that case as well. \square

Proposition 5.4. *The only copies of H_2 in H_3 are at the top or bottom of the weak involution poset. This also holds for H_3 in H_4 .*

Proof. It can be easily seen that there exist copies of H_n in H_{n+1} for $n = 2, 3$ at both the bottom and the top of the poset. In each case we just restrict the generators to the set from the smaller group and can easily be viewed in pink for H_3 in Figure 5.4. However, if we attempt copies of H_n in H_{n+1} , then we come up short. Notice, if you try to find a copy of H_2 in H_3 by shifting it, thus starting your H_2 at s_3 and then attempt to build the poset by alternately operating by s_1 and s_2 , we don't get the closed loop that we would need to get to have a copy of H_2 embedded in H_3 . In the case of H_3 in H_4 , it is perhaps even clearer that there exists no copy of H_3 in H_4 beyond the standard ones found at the bottom of the poset, created by restricting the generators. When attempting to find a copy of H_3 in H_4 , we begin by shifting up a level and using the vertex s_4 as our starting point. Here we progress to three vertices as is necessary since there are exactly three vertices adjacent to s_4 . However, when we progress to level 2 (noting s_4 is our level 0), we run in to trouble. Rather than those vertices being adjacent to four other vertices as is the case in the weak involution poset of H_3 , instead in the weak involution poset for H_4 , our new level 1 vertices are adjacent to 5 vertices in our new level 2. Thus we cannot achieve a copy of H_3 in H_4 by shifting. Please see Figure 5.5 for a visual counterexample. \square

Proposition 5.5. *The strong involution posets of H_2 , H_3 , and H_4 are not posets, but rather Cayley graphs.*

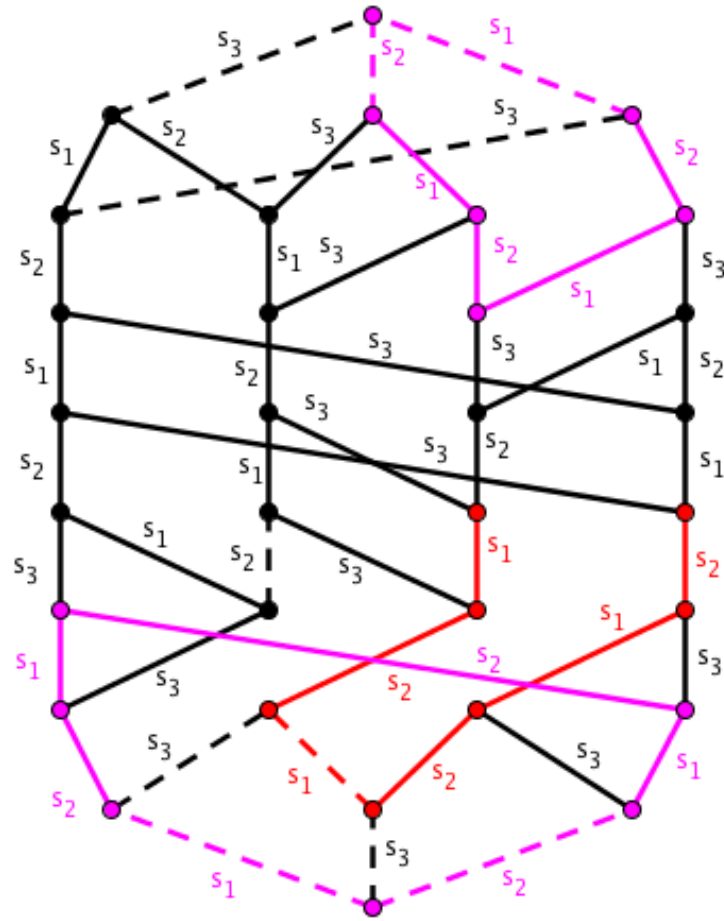


Figure 5.4 In pink, we have the copies of the weak involution poset for H_2 in the weak involution poset for H_3 . In red, we have an attempt at shifting the weak involution poset for H_2 in the weak involution poset of H_3 by shifting to start at s_3 .

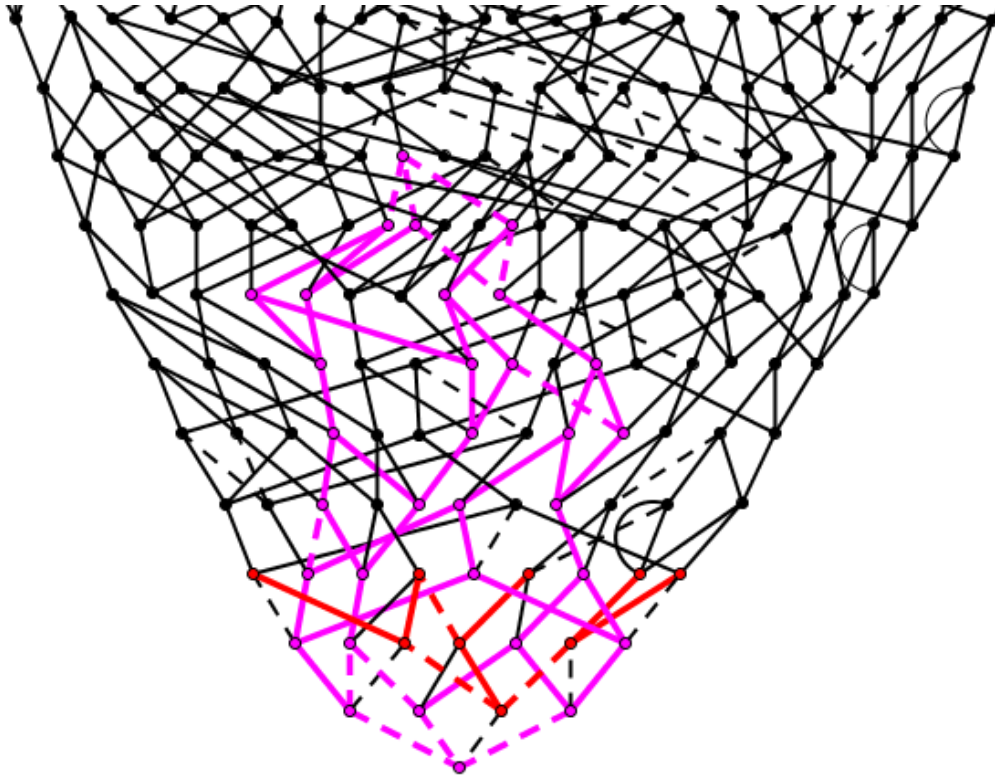


Figure 5.5 In pink, we have a copy of the weak involution poset for H_3 in the weak involution poset for H_4 . In red, we have an attempt to locate a copy of the weak involution poset for H_3 in the weak involution poset of H_4 by shifting it up a level and starting at s_4 .

Proof. The strong involution posets of H_2 , H_3 , and H_4 all contain horizontal edges where two elements with the same rank are connected with an edge. This horizontal edge prevents these diagrams from being called posets. In all cases, one example of a horizontal edges within the diagram is created by conjugating s_1 with the reflection $s_1 s_2 s_1 s_2 s_1$. Completing the calculation:

$$\begin{aligned} s_1 s_2 s_1 s_2 s_1 s_1 s_1 s_2 s_1 s_2 s_1 &= s_1 s_2 s_1 s_2 (s_1 s_1) s_1 s_2 s_1 s_2 s_1 \\ &= s_1 s_2 s_1 s_2 (s_1 s_2 s_1 s_2 s_1) \\ &= s_1 s_2 s_1 s_2 (s_2 s_1 s_2 s_1 s_2) \\ &= s_2 \end{aligned}$$

We see that we end up with s_2 which is the same rank as s_1 , thus giving us a horizontal edge. □

Proposition 5.6. *The weak involution poset of H_2 is a subset of the other three types of involution posets (or Cayley graph when appropriate) for H_2 .*

Proof. It can be easily seen that all involution posets of H_2 have the same vertex set, that of the involutions of H_2 . However, the edge sets of each type of involution posets vary. As you can see in Figures 5.1, 5.6, 5.7, and 5.8 the edge set of the weak involution poset is a subset of the edge set of the Bruhat involution poset which in turn is a subset of the edge set of the strong Bruhat involution poset which is a subset of the edge set of the strong involution Cayley graph. □

Corollary 5.7. *The strong involution Cayley graph of H_2 is isomorphic to K_6 , the complete graph on 6 vertices.*

Recall, a Hamiltonian cycle is spanning cycle meaning a cycle that passes through each vertex of the graph, or poset in this case.

Proposition 5.8. *All four involution posets of H_2 contain a Hamiltonian cycle.*

Proof. Since H_2 contains 2 generators, the weak involution poset is made up of a single cycle containing 6 vertices as you can see in Figure 5.1. Thus the poset itself is a Hamilton cycle.

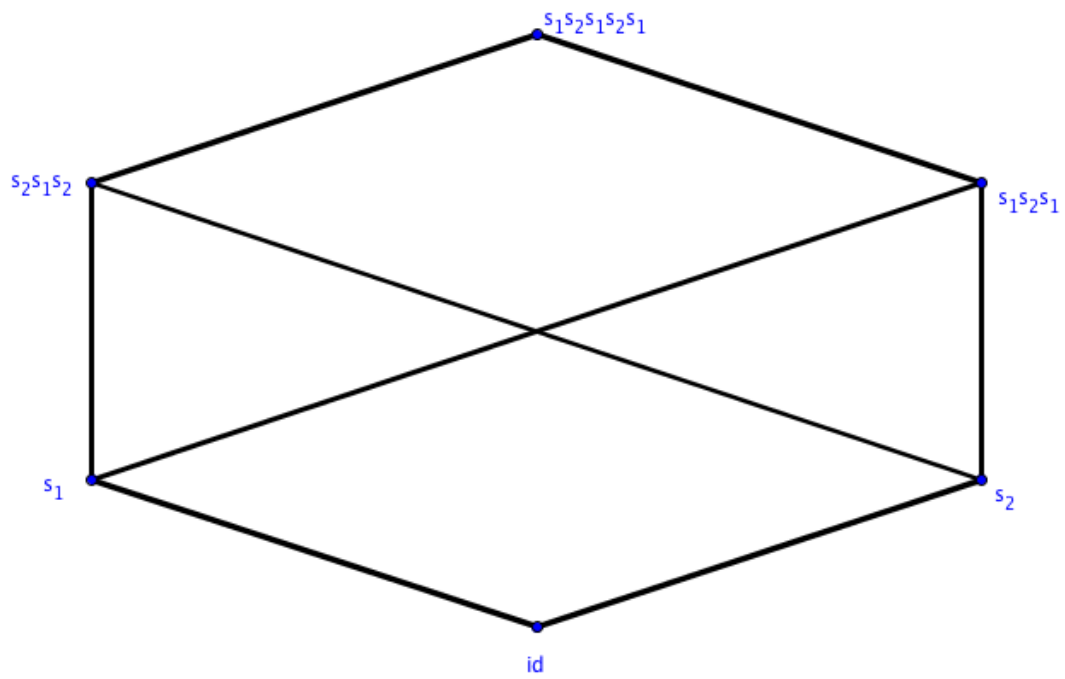


Figure 5.6 Weak Bruhat Involution Poset of H_2

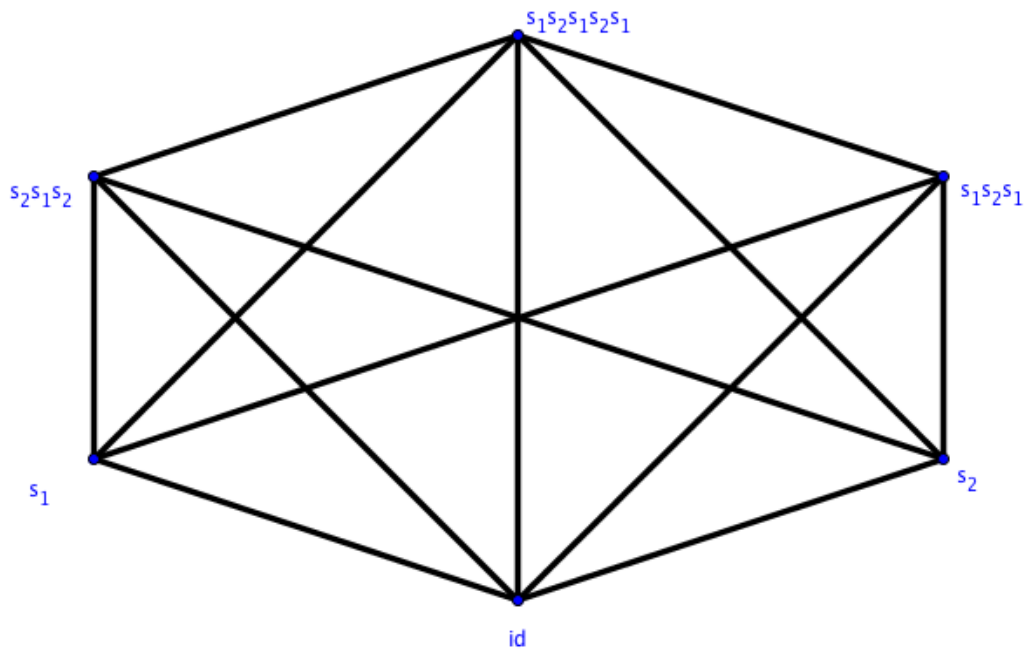


Figure 5.7 Strong Bruhat Involution Poset of H_2

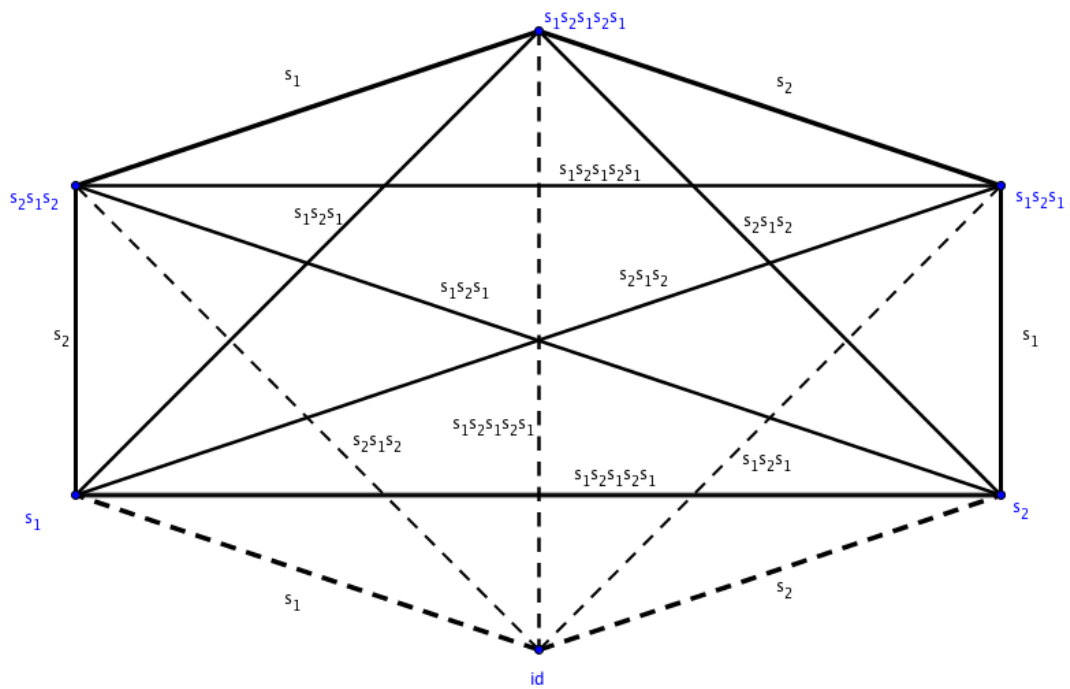


Figure 5.8 Strong Involution Poset of H_2

Because of Proposition 5.6, we know that the weak involution poset is a subset of the strong involution poset, the weak Bruhat involution poset, and the strong Bruhat involution poset. Further, we know that each involution poset has the same vertex set. Thus the Hamilton cycle that exists in H_2 also exists in the other three involution poset as seen in Figures 5.6, 5.7, and 5.8. \square

Proposition 5.9. *Both the weak involution poset and the strong involution poset of H_3 contain a Hamiltonian cycle.*

Proof. We know if the weak involution poset has a Hamiltonian cycle, then so does the strong involution poset since the edge set of the weak involution poset is a subset of the edge set of the strong involution poset. We can exhibit the Hamiltonian cycle written as a sequence of edges starting and ending at the identity element at the bottom of the poset as follows: $s_1, s_3, \overline{s_2}, \overline{s_3}, \overline{s_2}, \overline{s_3}, \overline{s_1}, \overline{s_3}, \overline{s_2}, \overline{s_1}, \overline{s_2}, \overline{s_1}, \overline{s_2}, \overline{s_3}, s_2, s_1, \overline{s_2}, \overline{s_1}, \overline{s_2}, \overline{s_3}, \overline{s_2}, \overline{s_3}, \overline{s_2}, \overline{s_1}, \overline{s_2}, \overline{s_1}, \overline{s_2}, \overline{s_3}, \overline{s_1}, \overline{s_3}, \overline{s_2}, s_3$. Further, the Hamiltonian cycle for the weak involution poset of H_3 is shown in Figure 5.9. \square

Proposition 5.10. *The weak involution posets for H_2 , H_3 , and H_4 are bipartite graphs.*

Proof. In the weak involution poset, edges travel from a vertex on the n -th level to a vertex on the $(n + 1)$ -th level. Thus, we can partition the edges into 2 sets, sending all vertices of even rank to one vertex set and all vertices of odd rank to the other vertex set. \square

Proposition 5.11. *The number of elements of rank n is equal to the number of elements of rank $3 - n$ for the weak involution poset of H_2 .*

Proof. Refer to Table 5.1a for an explicit list of the number of elements at each rank. \square

Proposition 5.12. *The number of elements of rank n is equal to the number of elements of rank $9 - n$ for the weak involution poset of H_3 .*

Proof. Refer to Table 5.1b for an explicit list of the number of elements at each rank. \square

Proposition 5.13. *The number of elements of rank n is equal to the number of elements of rank $32 - n$ for the weak involution poset of H_4 .*

Proof. Refer to Table 5.1c for an explicit list of the number of elements at each rank. \square

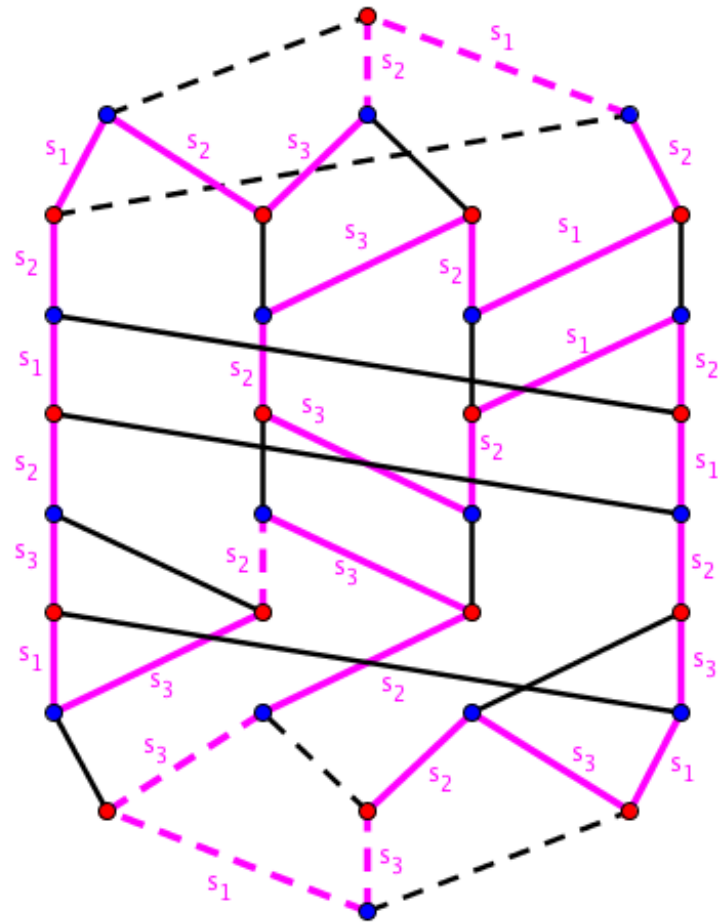


Figure 5.9 Hamiltonian cycle in weak involution poset of H_3 .

Table 5.1 Rank and the order of the rank for the weak involution poset for Coxeter groups of type H .

(a) Rank and the order of the rank for the weak involution poset of H_2 .

rank	number of elements at each rank
0	1
1	2
2	2
3	1

(b) Rank and the order of the rank for the weak involution poset of H_3 .

rank	number of elements at each rank
0	1
1	3
2	4
3	4
4	4
5	4
6	4
7	4
8	3
9	1

(c) Rank and the order of the rank for the weak involution poset of H_4 .

rank	number of elements at each rank	rank	number of elements at each rank
0	1	17	27
1	4	18	28
2	7	19	27
3	9	20	25
4	11	21	23
5	13	22	22
6	15	23	22
7	18	24	21
8	21	25	18
9	22	26	15
10	22	27	13
11	23	28	11
12	25	29	9
13	27	30	7
14	28	31	4
15	27	32	1
16	26		

Proposition 5.14. *The weak involution poset of H_4 contains double edges.*

Proof. The weak involution poset of H_4 contains multiple double edges. These occurs when an element can be conjugated and/or left multiplied by 2 different elements and end with the same result. We will calculate one of these double edges explicitly.

If we take the element $s_3 s_2 s_4 s_3$ and we conjugate with either s_2 or s_4 , we end up with the same result, $s_2 s_3 s_2 s_4 s_3 s_2$. It is clear that conjugating by s_2 will give the stated result, so we will explicitly look at the calculation when conjugating $s_3 s_2 s_4 s_3$ by s_4 .

$$\begin{aligned}
 s_4 s_3 s_2 s_4 s_3 s_4 &= s_4 s_3 s_2 s_3 s_4 s_3 \\
 &= s_4 s_2 s_3 s_2 s_4 s_3 \\
 &= s_2 s_4 s_3 s_2 s_4 s_3 \\
 &= s_2 s_4 s_3 s_4 s_2 s_3 \\
 &= s_2 s_3 s_4 s_3 s_2 s_3 \\
 &= s_2 s_3 s_4 s_2 s_3 s_2 \\
 &= s_2 s_3 s_2 s_4 s_3 s_2
 \end{aligned}$$

Thus we can see that by using the relations given by the Coxeter group structure, we can end up with the same result by conjugating $s_3 s_2 s_4 s_3$ with s_4 as we did by conjugating by s_2 . \square

Proposition 5.15. *The weak involution posets of Coxeter groups of type H are a graded posets.*

Proof. By Theorem 5.2, we know that in the involution poset for H_2 , H_3 , and H_4 all maximal chains have the same length. By Definition 3.5, we know that our posets is bounded, with $\hat{0} = \text{id}$ and $\hat{1}$ equal to $s_1 s_2 s_1 s_2 s_1$ in H_2 , $s_1 s_2 s_1 s_2 s_1 s_3 s_2 s_1 s_2 s_1 s_3 s_2 s_1 s_2 s_3$ in the case of H_3 , and $s_1 s_2 s_1 s_2 s_1 s_3 s_2 s_1 s_2 s_1 s_3 s_2 s_1 s_2 s_3 s_4 s_3 s_2 s_1 s_2 s_1 s_3 s_2 s_1 s_2 s_3 s_4 s_3 s_2 s_1 s_2 s_1 s_3 s_2 s_1 s_2 s_3 s_4 s_3 s_2 s_1 s_2 s_1 s_3 s_2 s_1 s_2 s_3 s_4$ in the case of H_4 . Further, by Definition 3.6, we know that since our poset is bounded, finite, with all maximal chains of equal length that our poset is graded. \square

Theorem 5.16. *The rank function for the weak involution poset of the Coxeter group W*

where $W = H_2$ or $W = H_3$ is

$$\rho(w) = \frac{\ell(w) + \text{unbar}(w)}{2}$$

for $w \in W$.

Proof. Recall, Remark 3 where we discussed the similarity between this rank function and the one presented by Incitti in Theorem 3.10. By $\text{unbar}(w)$, we mean the number of unbarred operations when listing the admissible ascending sequence, so number of left multiplications. For instance for the element $w = s_2 s_1 s_2$ in H_2 , $\ell(w) = 3$, $\text{unbar}(w) = 1$. So $\rho(w) = \frac{\ell(w) + \text{unbar}(w)}{2} = \frac{3+1}{2} = 2$. We can verify with Figure 5.1 that we are correct. Similarly for $w = s_1 s_2 s_3 s_2 s_1 s_2 s_1 s_3$ in H_3 , $\ell(w) = 8$ and $\text{unbar}(w) = 2$. So, $\rho(w) = \frac{\ell(w) + \text{unbar}(w)}{2} = \frac{8+2}{2} = 5$. Which can once again be verified using Figure 5.2. The element $w = s_1 s_2 s_3 s_2 s_1 s_2 s_1 s_3$ is the vertex on the fifth level, second to the left. The ranks of all other elements can be easily verified using the relevant weak involution poset. \square

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