SMiRT-12 / K. Kussmaul (Editor)

B09/2

SHAKEDOWN ANALYSIS OF FRAME STRUCTURE

Liu-feng and Mei Zhan-xin

Department of Construction Engineering, Xi'an Institute of Metallurgy and Construction Engineering, Xi'an, 710055 Shaanxi, P.R. China

1. Introduction

In many fields of advanced technology, such as nuclear, space and aeronautic activities or off–shore exploration oil, very high requirements are imposed on structural safety and reliability. Because of the complexity of the practical loading, the response of structure to variable loading is quite different from that to the proportional load, and the failure forms are not the same. Generally there exist local failure due to plastic fatique (alternating plasticity) and a gradual divergence of the deformed configuration (incremental collapse of ratchetting). Prager once gave the word of shakedown to a safety structure and he pointed out that an elasto–plastic structure subjected to cyclic histories of loads and / or temperature distributions is said to shake down if its plastic work is bounded. This means that the structure attains at a finite time or approaches asymptotically a situation of purely elastic cycling. When this is not the case, nonadaptation will occur.

Since Melan and Koiter produced the upper and lower shakedown theorems respectively, a great achievements have been obtained in the area of theortical and practical shakedown analysis, especially since Maier pointed out the linear program method, a lot of unsolved problems have been analyzed on the basis of shakedown theorems. In this paper, we studied the linear program formula of upper and lower bound theorem, on the basis of comparison of these two formula, a conclusion was obtained that the shakedown parameter solved by lower bound theorem is a complete solution when surplus variables are not basic ones. Therefore it is unnecessary to solve a problem using upper bound theorem if surplus variables are zeros when it is solved by lower bound theorem, and this easilizes somewhat what is considered.

2. Basic Theorems and its Simplying

First of all we express the classical shakedown theorems, for they are the basis of our discussion. The basic assumption of the analysis of classic shakedown is: the material of structure is ideal elasto–plastic, calculation is geometric linearity, the yield surface is convex, the plastic strains are in conformity with flow condition and the material characteristics has nothing to do with the temperature, creep is not considered yet.

Melan's theorem of shakedown is:

A structure will shakedown to a loading if and only if there exists a time–independent residual stress distribution σ^r_{ij} in equilibrium with zero loads, so that the total stresses $(\sigma^e_{ij}+\sigma^r_{ij})$ will not violate

the yield criterion, where σ^e_{ij} is the time–dependent elastic stress distribution (assuming unlimited elastic response) under the loading.
That is:

$$f\left(\sigma^e_{ij}(t)+\sigma'_{ij}\right)\leq 0 \tag{1.1}$$

where f (•) is yield function. Melan's theorem can also be called lower bound shakedown theorem and static shakedown theorem.
Koiter's theorem of shakedown is:

Shakedown is impossible if there exists a kinematically admissible plastic strain rate cycle $\dot{\varepsilon}^p_{ij}$ of time interval T such that under any loading path,

$$\int_T(\int_v f_i\dot{U}_i dv+\int_{S_\sigma} T_i\dot{U}_i dS)dt>\int_T\left(\int_v\sigma_{ij}\dot{\varepsilon}^p_{ij}dv\right)dt \tag{1.2}$$

where f_i, T_i is body and surface force of the structure respectively, $\dot{U}_i$ is a kinematically admissible field corresponding to $\dot{\varepsilon}^p_{ij}$, σ_{ij} is on the boundary of yield loci with $\dot{\varepsilon}^p_{ij}$ arising according to the normality flow role. Koiter's theorem is also called upper bound shakedown theorem and kinematic shakedown theorem.

(1.2) shows that the rate of work of external loads on the strain rate cycle is bigger than the rate of energy dissipation associated with $\dot{\varepsilon}^p_{ij}$, so that the plastic strains continuously happen and make the structure nonadaptation.

Shakedown will occur if there exists a constant $\xi>1$ and for all kinematically admissible plastic strain rate cycles under all loading pathes

$$\xi\int_T(\int_v f_i\dot{U}_i dv+\int_{S_\sigma} T_i\dot{U}_i)dt\leq\int_T\int_v\sigma_{ij}\dot{\varepsilon}^p_{ij}dvdt \tag{1.3}$$

It's a common situation in many practical problems that the exact time–dependent history of a multiparameter loading is not known. Instead, a load domain (a convex hyperpolyhedron in the space of load parameters) is given. This is one of the reasons making shakedown analysis difficult. Therefore it is necessary to simply the loading domain, and do some simplifications. There cases can be obtained.

(I) Variable loading can be described as

$$f(X,t)=\sum_{i=1}^{\alpha}\mu_i(t)\bar{f}_i(X)$$

where f (X, t) stands for a total instantaneous load. $\bar{f}_i(X)$ is a constant load in time for the ith loading mode, μ_i (t) denotes the ith load parameter, The simplest load domain Ω E^α is the α dimensional rectangular parallele piped.

$$\bar{\mu}_i^-\leq\mu_i(t)\leq\bar{\mu}_i^+\qquad i=1,2,\cdots,\ \alpha \tag{1.4}$$

where the bounds $\bar{\mu}_i^-$, $\bar{\mu}_i^+$ are given usually by design codes.

(II) More generally, Ω can be described by a set of linear inequalities.

$$\sum_{i=1}^{\alpha}a_{ij}\mu_i\leq b_j\qquad j=1,2,\cdots,\ \beta \tag{1.5}$$

(III) Ω can also be described by a set of radius vectors μ_k, k = 1, 2, ⋯, γ .

The shakedown analysis is boiled down to find out maximum ξ so that the structure is made to shake down under such loading.

$$\zeta\overline{\mu}_i^- \leqslant \mu_i(t) \leqslant \zeta\overline{\mu}_i^+ \quad or \quad \sum_{i=1}^{n} a_{ij}\mu_i \leqslant \zeta b_j \quad or \quad \zeta\overline{\mu}_k \tag{1.6}$$

It's very difficult to trace every loading history, and sometimes it's impossible to do so for the complexity of varying loading. But it's possible to simplify the load domain according to convex analysis.

THEOREM 1 A structure can and only can shake down under a given variable repeated load if it can shake down to convex envelope C_0 (Ω) of Ω.

THEOREM 2 If a given structure shakes down in a cyclic loading process which covers the entire boundary of C_0 (Ω), then it shakes down in any load path contained within Ω.

THEOREM 3 If a given structure shakes down in a cyclic loading process which contains all the vertices of C_0 (Ω), then it shakes down in any loading path contained within Ω.

According to these three theorems, it's not necessary to exame the whole loading domain when shakedown analysis is considered, and what we should do is to make a study of boundary points or vertices of C_0 (Ω).

In order to linearize the computation of shakedown analysis, (1.1) must be replaced with sub–section linearized yield function, and this is a linearization problem of yield condition, that is to use convex body consisting of J hyper–planes replacing the original yield space.

To the ith stress examination points studied, we have

$$f_i = N_i\sigma_i - R_i \leqslant 0 \tag{1.7}$$

where

$$N_i = \left[\underline{N}_1, \underline{N}_2, \cdots, \underline{N}_J\right]^T \qquad \underline{N}_j = \{l_{j1}, l_{j2}, \cdots, l_{jI}\}^T \tag{1.8a}$$

$$R_i = \{R_{i1}, R_{i2}, \cdots, R_{iJ}\}^T \qquad \sigma_i = \{\sigma_{i1}, \sigma_{i2}, \cdots, \sigma_{iI}\}^T \tag{1.8b}$$

where l_{jk} is a included angle cosine of jth plane outward normals to kth stress components. $\underline{N}_j$ makes up the matrix of outward normals to yield loci. R_{ij} is a normal distance of coordinate original point of ith stress examination point to the jth plane of yield loci. I is dimensions of stress space.

If there are m stress examination points in the structure, then

$$f = N\sigma - R \leqslant 0 \tag{1.9}$$

where

$$R_{(m\times J)\times 1} = \left[R_1^T, R_2^T, \cdots, R_m^T\right]^T \qquad \sigma_{(I\times m)\times 1} = \left[\sigma_1^T, \sigma_2^T, \cdots, \sigma_m^T\right]^T \tag{1.10a}$$

$$N_{(m\times J)\times(I\times m)} = diag[N_1, N_2, \cdots, N_m] \tag{1.10b}$$

2. Linear Program and Discussion

According to yield condition (1.9), Melan's theorem can be written as

$$N\sigma^e(t) + N\sigma^r \leqslant R \tag{2.1}$$

where

$$\sigma^r = \left[\sigma_1^{rT}, \sigma_2^{rT}, \cdots, \sigma_m^{rT}\right]^T \tag{2.2}$$

is time–independent residual stress field in the structure, where

$$\sigma_i^r = \left\{\sigma_{i1}^r, \sigma_{i2}^r, \cdots, \sigma_{iI}^r\right\}^T \tag{2.3a}$$

and

$$\sigma^{e}(t)=\left[\sigma_1^{e\,T}(t),\sigma_2^{e\,T}(t),\cdots,\ \sigma_m^{e\,T}(t)\right]^T \tag{2.3b}$$

is elastic response of the structure, where

$$\sigma_i^{e}(t)=\{\sigma_{i1}(t),\sigma_{i2}(t),\cdots,\ \sigma_{il}(t)\}^T \tag{2.4}$$

Let ρ be generalized variables of the structure, W be applying matrix which is relevant to the structural materials and geometric shape. If $W\rho$ is in equilibrium with zero loads, then residual stress fields can be described as

$$\sigma^{r}=W\rho \tag{2.5}$$

(2.1) can be simplified when theorem 3 is used. Let elastic response of ith stress examination point be σ_{bi}^{el} due to the loading of the lth vertice of basic load domain, and

$$M_j^i=\max_{l=1,k}\left(N_j^T\sigma_{bi}^{el}\right) \tag{2.6a}$$

is a maxium elastic response of ith stress point to jth yield plane. So we have

$$M^i=\left\{M_1^i,M_2^i,\cdots,\ M_J^i\right\}^T \tag{2.6b}$$

If m stress examination points are studied we have

$$M=[M^{1\,T},M^{2\,T},\cdots,\ M^{m\,T}]^T \tag{2.6c}$$

If loading parameter is ξ, substitute (2.5), (2.6) into (1.9).

$$\xi M+NW\rho\leqslant R \tag{2.7}$$

Therefore to solve a shakedown parameter ξ is to solve a linear program LP_1

$$LP_1:\quad \max\quad s=\xi\,/\,s.t.\quad \xi M+NW\rho\leqslant R,\quad \xi\geqslant 0 \tag{2.8}$$

ρ is free variables, we can let $\rho=\rho_+-\rho_-$ and we get a standard linear program LP_1'

$$LP_1':\ \max\ s=\xi\,/\,s.t.\ \xi M+NW\rho_+-NW\rho_-\leqslant R,\ \xi\geqslant 0,\ \rho_+\geqslant 0,\ \rho_-\geqslant 0 \tag{2.9}$$

A dual linear program DLP_1 can be easily got

$$DLP_1:\quad \min\quad s=R^T\alpha\,/\,s.t.\quad M^T\alpha\geqslant 1,\quad (NW)^T\alpha=0,\quad \alpha\geqslant 0 \tag{2.10}$$

Now we study the linear program formula corresponding to Koiter's theorem. If the generalized plastic strain rate $\dot{\varepsilon}^p$ satisfies

$$\dot{\varepsilon}^p=\frac{\partial f}{\partial\sigma}\dot{\lambda}=N^T\dot{\lambda} \tag{2.11}$$

where $\dot{\lambda}$ is plastic flow parameter, and it is kinematically admissible, then Koiter's theorem (1.3) can be written as

$$\xi M^T\lambda>R^T\lambda \tag{2.12}$$

$\varepsilon^p=N^T\lambda$ is kinematically admissible for $\dot{\lambda}$ is kinematically admissible, so

$$\sigma^{r^T}\varepsilon^p=0 \tag{2.13}$$

Considering (2.5), we find if

$$W^TN^T\lambda=0 \tag{2.14}$$

then $\dot{\lambda}$ is kinematically admissible. Therefore we get linear program formula LP_2 corresponding to kinematic shakedown theorem

$$LP_2:\quad \min\quad s=R^T\lambda\,/\quad s.t.\quad M^T\lambda=1,\quad (NW)^T\lambda=0,\quad \lambda\geqslant 0 \tag{2.15}$$

Let's compare LP_2 with DLP_1 corresponding to upper and lower bound shakedown theorem. Let effect domain corresponding to LP_2 and DLP_1 be k_2 and k_1 respectively, it can be found out $k_2 \propto k_1$, so $s_2 > s_1$, where s_2 and s_1 is the most excellent value corresponding to LP_2 and DLP_1 respectively, and this shows that the shakedown parameter solved from upper bound theorem can not be less than the one solved from lower bound theorem. Especially if we introduce surplus variables η in DLP_1 we get DLP'

$$DLP' \quad min \quad s = R^T\alpha / \quad s.t. \quad M^T\alpha - \eta = 1, \quad (NW)^T\alpha = 0, \quad \alpha \geqslant 0, \quad \eta \geqslant 0 \qquad (2.16)$$

When DLP_1' is solved with simplex algorithm method, and if η can be nonbasic variables, then the feasible domains of DLP_1 and LP_2 are the same, their object functions are also the same, we get a complete solution, then the shakedown parameter obtained from DLP_1 is a complete one.

3. Example

Consider a portal frame (fig.1), the frame is subjected to a horizontal load H (t) varying within the limits $-H_0 < H < H_0$. The load is applied to the top of the column. In the middle of the span a vertical load V (t) is applied such that its pulsation is $0 < V < V_0$. Let

$$h(t) = \frac{H(t)l}{M_p}, \quad v(t) = \frac{V(t)l}{M_p} \qquad (3.1)$$

be nondimensional values of loading. M_p is plastic limit moment of the frame. The reference load domain is shown in fig.2. Five nodes are considered, and let

$$m = \frac{M}{M_p}, \quad \rho_m = \frac{M_r}{M_p}, \quad \rho_h = \frac{H_r l}{M_p}, \quad \rho_v = \frac{V_r l}{M_p} \qquad (3.2)$$

where H_r, M_r, V_r is generalized residual stress when a static basis is selected as fig.3.

Five nodal moments consist of the generalized stress field of the structure, and (1.10) is

$$\sigma^{eT} = \{m_1, m_2, m_3, m_4, m_5\}$$

while (1.10a) is

$$K^T = \{1,1,1,1,1,1,1,1,1,1\}$$

and

$$\rho^T = \{\rho_m, \rho_h, \rho_v\}$$

$$W = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 & 0 \\ 2 & 2 & 1 & 0 & 0 \end{bmatrix}^T$$

The elastic envelope is

$$M^T = \{0.4125, 0.3125, 0.1875, 0.3875, 0.3, 0.0, 0.1875, 0.3875, 0.4125, 0.3125\}$$

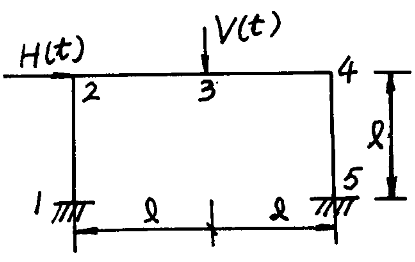

FIGURE 1 Portal frame

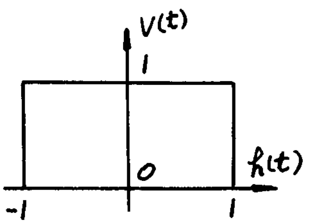

FIGURE 2 Reference load

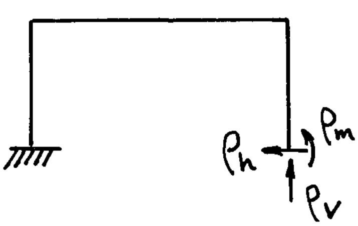

FIGURE 3 Static basis

Substitute above equations into (2.16) and solve it

$$S = 2.759 \quad \rho_m = -0.138 \quad \rho_h = -0.207 \quad \rho_v = 0.0 \quad \eta = 0$$

$$\lambda^T = \{1.379, 1.379, 0,0,0,0,0,0,0,0\}$$

$\eta = 0$ shows that we get a complete solution.

The shakedown limits of the structure is

$$0 \leqslant V(t) \leqslant 2.759\frac{M_p}{l}, \quad -2.759\frac{M_p}{l} \leqslant H(t) \leqslant 2.759\frac{M_p}{l} \qquad (3.3)$$

4. Reference

1. A. Sawczuk, Shakedown Analysis of Elasto–plastic Structures, Nucl. Eng. Des. 28(1974).
2. J. A. Konig, G. Maier, Shakedown Analysis of Elasto–plastic Structures, A Review of Recent Development, Nucl. Eng. Des. 66(1981)
3. W. T. Koiter, General Theorems for Elasto–plastic Solids, in Progress in Solid Mechanics, North–Holland, 1960.
4. A. Borkowski, M. Kleiber, On a Numerical Approach to Shakedown Analysis of Structures, Computer Methods in Applied Mechanics and Engineering, 22(1980).