# SOME TOPICS IN SYSTEM THEORY

by

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ABSTRACT

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The foundations of measurement theory were investigated by model-theoretic methods. The purpose was to establish a firm basis for general system theory. One major result was the formulation of the concepts of scale, scale transformation and the proof of the existence, for an arbitrary scale, of the group of scale transformations which "leave the scale form invariant". As an illustration of the applicability of these concepts and because of its intrinsic interest an exposition was given of the theory of measurement for extensive quantities. A novel formulation of the usual axioms was developed which made them elementary formulae in the first order predicate calculus. Thus it was possible to show that this theory is model-complete. Then A. Robinson's theorems were applied to show that this theory is negation complete.

In the formulation of the existence theorem for scale transformation groups no restriction was placed on the empirical $\Sigma$-model or the numerical $\Sigma$-model and in the other illustrative examples it was shown how different choices of these $\Sigma$-models lead to different scale transformation groups. An example of a theory of a non-extensive quantity was presented but the methods used before would not yield model-completeness for this theory and thus an interesting unsolved problem remains.

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# BIOGRAPHY

Norman Francis Williamson, Jr., was born August 4, 1923, in Karuizawa, Japan, the only child of Baptist missionary parents who returned to the United States to live in 1934. His education was interrupted after two years at Mars Hill College, Mars Hill, North Carolina, by three years of military service. His educational goals having been changed by interests acquired during military service, he entered Armstrong College, Savannah, Georgia, as a freshman in the Electrical Engineering curriculum and subsequently obtained the B.S. degree from Emory University, Atlanta, Georgia, in 1950, presenting a double major in Physics and Mathematics.

After serving as a member of the technical staff of the Bell Telephone Laboratories and as Assistant Professor of Mathematics at Wesleyan College, Macon, Georgia, he returned to graduate school and received the M.S. degree from Tulane University in 1959. He then taught at Louisiana State University and the College of Charleston, Charleston, South Carolina. During the tenure of a PHS fellowship, 1964-1966, he satisfied most of the requirements for a Ph.D. degree at North Carolina State University. At the present he is a Research Associate in the Biomathematics program.

The author and Miss Elizabeth Cleopatra Tuten were married in 1944 and they have three children: Marcus Lee, 22 years old, Glen Lewis, 19 years old and Grover Francis, 17 years old.

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# 1. INTRODUCTION

The basic problems of System Theory may be stated as a list of questions:

1. What is a system?
2. How do we describe a system?
3. How do we predict the future behavior of a system?
4. How do we deduce the past behavior of a system?
5. How do we approximate a system?
6. How do we deduce the structure of a system?
7. How do we control a system?
8. How do we improve an operating system?

We will follow the advice of several eminent workers in the field and eschew a premature formalization of the definition of a system. Instead we will list a few examples of systems and give an informal description of some of the aspects that any definition would have to cope with. Examples of the system concept are:

1. a pond ecosystem
2. the General Motors Corporation
3. a Turing machine
4. an oil refinery
5. a differential equation with initial condition
6. a passive linear lumped parameter electrical circuit
7. the vascular bed of a mammal

8. a man on horseback

9. a Markov algorithm

10. the flora and fauna of the human axilla

Each of these systems possesses a collection of input functions, a collection of output functions, a collection of internal states and a state transition function. Perhaps the most fundamental problem, involved in some way in answering each of the questions listed above, is that of approximation: *e.g.*, how close is a predicted output function to an observed output function? It is immediately clear that the kinds of metrics definable on the space of output functions, or the space of input functions, are closely related to the kinds of metrics definable on the range space of any input function or output function. These in turn are determined by the measurement procedures by which these functions values are specified. S. S. Stevens, 1968, has stated a similar *caveat*, "the type of measurement achieved in an experiment may set bounds on the kinds of statistics that will prove appropriate....the relation between statistics and measurement is not a settled issue. Nor is it a simple issue, for it exhibits both theoretical and practical aspects." It is our contention that system theory will be able to build safely only on a foundation of measurement theory.

What then are the tools that will facilitate the development of a theory of measurement? The investigation of the foundations of various mathematical disciplines has produced a powerful methodology. In the case of algebra we can perceive three reasonably distinct periods. Prior to 1899 we may characterize the approach as taxonomic. Algebraists went forth to collect specimens from the profusion of algebraic structures, and when a new group or field was added to the fold there was great rejoicing. "The turning point was Hilbert's work on the foundations of geometry in 1899. Even though this did not concern algebra or arithmetic directly, it set a new and high standard of definiteness and completeness in the statement of all mathematical definitions or, what is equivalent, in the construction of postulate systems...The change for the better after 1900 was most marked..." (E.T. Bell, 1945). Between 1900 and 1930, the axiomatic period, practically every mathematical structure known to man was clothed in a set of postulates which included most of the assumptions used in making proofs of theorems about that structure. However, the Principia Mathematica and Godel's work pointed the way to a deeper level of formalization, and since 1930 the role of the symbolic language used has become more and more important. In the present model-theoretic period, a concrete set of postulates for a particular structure is not the central focus of interest. Workers on the foundations have turned to what is called Universal Algebra or Metamathematics. Writing in 1955, Mostowski says, "At present we do not attach so much importance to the actual axiomatization of various fragments of mathematics. Our interest is now concentrated on the general theory of models for structures characterized by sets of axioms." There is more in this approach than the mere unification and generalization of facts already known. The application of metamathematical results, notably Godel's theorems, has produced new and interesting theorems, in field theory for example. A typical theorem (G. Birkhoff, 1935) states that every class of structures closed with respect to the operations of forming homomorphic images, direct products and substructures, is always characterized by a set of elementary axioms. Such axioms contain no variables running over sets, classes of sets, relations, etc. They include, besides logical constants, variables of the lowest type and symbols for a certain number of constant operations and relations. Quantifiers appearing in these axioms are restricted to a fixed set called the carrier which is the union of the fields of the relations as well as of the domains and ranges of the functions whose symbols appear in the axioms. The rules of inference of an elementary system are in general the rules of the first order predicate calculus. Examples of non-elementary axioms, quite frequently used by mathematicians, are the induction axiom of natural number theory, the Archimedean axiom and the Dedekind completeness axiom for the real number system.

For the investigation of the foundations of measurement theory it will be seen that this sort of approach is quite essential even to state a definition of a scale. Also if the axioms for extensive measurement are formulated in this way the criterion of model-completeness may be applied to show that this theory is complete. Since there are several different definitions of completeness in use, it is necessary to state them and specify the one we shall be using. A theory is <u>deductively complete</u> if and only if every statement which is true in every model is provable. A theory T is <u>formally complete</u> provided that any theory T', which is formed by adjoining to the axioms of T a statement of T which is not already a theorem of T, is inconsistent. A theory is <u>negation complete</u> if, for any statement A of the theory, either A or $\sim$A is provable. It is easy to see, cf. (Stoll, 1963), that negation completeness and formal completeness are equivalent concepts for elementary first order theories. Negation completeness can also be described in terms of models. A theory T is negation complete if and only if every statement of T which is true in one model of T is true in every model of T. Godel's completeness theorem has been extended (Hasenjaeger,

1953) to the assertion that every elementary first order theory is deductively complete. For negation completeness, however, special methods must be used for each theory and that is what will be done in the sequel for the elementary theory of extensive measurement.

## 2. REVIEW OF LITERATURE

The purpose of this section is to contribute to the formulation of the concept of scale and scale transformation. We must first be clear about the nature of those quantities which can be measured by a scale before we can isolate the concept of a scale. With a definition of scale at hand we will then explicate the properties of a scale which should be preserved by those scale transformations which, as Stevens says, leave its "scale form" invariant. A brief overview of the process called measurement will help us allocate to its various phases the concepts and properties that scientists have recognized. The following figure exhibits the main constituents in a sort of flow diagram in which the boxes represent sets and the arrows represent mappings. Each pair of boxes joined by an arrow represents a conceptually distinguishable phase of the measurement process. In the legend which follows Figure 1 the word, object, denotes a physical object which may be measured; the word, point, denotes a mark on a scale or dial of a measuring instrument; and the word, calibration, denotes a correspondence between a set of numbers and a set of points, not a process of instrument adjustment.

$$
\begin{tikzcd}
\boxed{O} \arrow[r, "f"] \arrow[d, "\alpha"'] & \boxed{O^*} \arrow[r, "g"] & \boxed{R} \\
\boxed{D} \arrow[r, "f^*"] & \boxed{D^*} \arrow[r, "g^*"] & \boxed{Q} \arrow[u, "\beta"']
\end{tikzcd}
$$

Figure 1. The measurement procedure

$O$—the set of objects which possess quantity $p$; $O^*$—the set of equivalence classes of objects indistinguishable in respect of $p$; $R$—the set of real numbers; $D$—the set of points on a dial or scale of a measuring instrument; $D^*$—the set of equivalence classes of points indiscriminable by the observer; $Q$—the set of rational numbers; $f, f^*$—the natural or canonical maps of $O$ and $D$ on to their respective quotient sets $O^*$ and $D^*$; $g$—the map which assigns numbers to members of $O^*$, called quantification; $g^*$—the map which assigns numbers to points on the scale, called calibration; $\beta$—the map which assigns (statistical) estimates of the "true" value of a quantity $p$ to runs of measurements; $\alpha$—the map which assigns scale points to objects.

Philosophers of science have long exercised themselves with the problem: what is denoted by the words quantity and magnitude. An investigation of this problem is not our concern. Rather we will choose a position and state it so as to clear the ground for an attack on the notions of scale and scale transformation. When we speak of a quantity we will assume that a set, $O$, of objects and two physical operations have been chosen and that by means of these operations two relations, E and L, have been defined in the set O so that the triple $(O, E, L)$ satisfies the following axioms (Hempel, 1952, p. 59) where a, b, and c denote any members of O:

$$
\begin{array}{lll}
(1) & \text{(transitive)} & aEb \ \& \ bEc \Rightarrow aEc \\
(2) & \text{(symmetric)} & aEb \Rightarrow bEa \\
(3) & \text{(reflexive)} & aEa \\
(4) & \text{(transitive)} & aLb \ \& \ bLc \Rightarrow aLc \\
(5) & \text{(irreflexive)} & aEb \Rightarrow \sim aLb \\
(6) & \text{(connected)} & \sim aEb \Rightarrow aLb \text{ or } bLa
\end{array}
\tag{2.1}
$$

Thus E is reflexive, symmetric and transitive (i.e., an equivalence relation) while L is irreflexive, transitive and connected. Since L and E are subsets of $O \times O$ we could define $LE = L \cup E$ and observe that LE is a full ordering. While we could define the concept of quantity using one relation, LE, and four axioms we have chosen the description in terms of two relations and six axioms since this facilitates the description of the quotient set $O/E$ or, as it was denoted in Figure 1, $O^*$. Also the problems involved in the empirical verification that transitivity holds for L and for E may be somewhat different. The set $O^*$ consists of the classes $[a] = \{x : aEx\}$ and a relation $L^*$ is defined in $O^*$ by: $[a] L^* [b]$ if and only if aLb. It can easily be shown that $O^*$ is fully ordered by $L^*$. We shall say that $[a]$ is a degree or intensity of the quantity.

A scale for the measurement of a quantity has been defined (Ellis, 1966, pp. 41-43) as a monotone function defined on $O^*$ with values in R. This quantification mapping assigns to each intensity of a quantity its "true" (Platonic ideal) numerical measure, but this conceptual scale is not the one used in empirical measurement. The conceptual scale need not be limited in either range or refinement (Ellis, 1966, p. 48), nor does $O^*$ have to be assumed finite; indeed, as we shall see later, the description of the concept of extensive quantity naturally includes the assumption that a system of standards is a subset of O. This assumption entails the theorems that $O^*$ is infinite and that it is dense in itself. On the other hand, any instrument used for measuring has a finite range and least count and can yield only a finite number of rational numbers as possible readings. Moreover a run of repeated measurements of "the same object under the same conditions" will show that usually one instrument reading provides only an unreliable estimate of the "true" numerical measure of the quantity. We are thus led to introduce a material scale (Bunge, 1967, vol. II, p. 221) as a monotone function defined on $D^* = D/I$ with values in Q. Here D denotes the set of points such as those on the dial of a meter or the surface of a meter stick, etc., and as before we assume that physical operations such as observing coincidence or displacement have defined relations I and M in D and that the triple (D,I,M) satisfies the axioms stated above for (O,E,L). The set of rational numbers is denoted by Q and the calibration mapping from $D^*$ into Q is monotone with respect to the order $M^*$ and the natural order in Q. A complete account of the measurement of a quantity must then introduce appropriate assumptions about the distribution function associated with the set of possible measured values by the measurement procedure and choose some appropriate statistical device such as estimation or hypothesis testing.

The necessity for such a complex conceptual framework becomes even more apparent when we consider various attempts to classify scales according to type. One of the most attractive proposals was made by S. S. Stevens (1946, pp. 677-680) who suggested that we should classify scales according to "their mathematical group structure." His exposition of this notion is suggestive but incomplete (Ellis, 1966, p. 58) and Ellis's proposed criterion (Ellis, 1966, p. 65) for invariance of scale form is also unsatisfactory. We shall propose below a definition of the group associated with a scale which may be the sort of thing Stevens had in mind and we will argue that this definition will provide a better criterion for classifying scales. The classical scheme proposed by Campbell (1920) depends on an analysis of measuring procedures which he classified as either fundamental or derived. Later workers also distinguished between extensive and intensive properties and described the categories of nominal, ordinal, ratio and interval scales and proposed various criteria for classifying scales. For example, Coombs (1952) proposed to use the kinds of applications of arithmetic, performed with numbers resulting from measurement on a particular scale which produce verifiably true statements, as a basis for classification. Stevens claims that if we know that a set of measurements was made on a particular scale then we can say what sorts of statistics are relevant to these measurements.

Before attempting to formulate a logically coherent set of definitions for these concepts it will be expedient to consider the related notions of fundamental measurement and extensive property. Postulates for the theory of extensive properties were given by Helmholtz (1887), Holder (1901), Campbell (1920), Suppes (1951), Hempel (1952), Menger (1959), Ellis (1966) and Whitney (1968).

In order to avoid confusion we must make a few definitions, first of the various order relations used by the authors of different definitions of quantity or extensive quantity, second of the concept of a semi-group and finally of some of the properties which may be possessed by a semi-group. We all know that the weight of two objects is equal to the sum of their separate weights. This means that the operation of combining two physical objects must have some similarities to the operation of adding numbers and these can best be examined using the language of semi-groups.

A relation R is a partial ordering of the set A if and only if R is reflexive, antisymmetric and transitive in A. A full ordering of A is a connected partial ordering of A. A relation R is a strict partial ordering of the set A if and only if R is asymmetric and transitive in A. A strict full ordering of A is a connected strict partial ordering of A. Since to each full ordering there corresponds a unique strict full ordering, and conversely, the distinction between full orderings and strict full orderings is substantively trivial (Suppes, 1957, p. 222). However, the alternative sets of axioms which may be used should be recognized as such. A transitive relation is irreflexive if and only if it is asymmetric. For a transitive relation R the following are equivalent: (1) R is irreflexive and connected, (2) R is asymmetric and connected, (3) R satisfies the trichotomy law. We can say then that Hempel's axioms (equations 2.1 above) define a quantity as a set together with an equivalence and a strict full order relation. We could have used any one of several sets of axioms provided only that $O^*$ has a full order or strict full order.

An ordered pair $(S,+)$ consisting of a set, $S$, and a binary operation, $+$, in $S$ is called a <u>semi-group</u> if $+$ is associative. Let a, b and c be any elements of a semi-group and for any integer n define nb recursively by the equations, $1b = b$ and $nb = ((n-1)b)+b$. A semi-group may in addition possess the following properties which will be important in defining extensive quantities.

$$\begin{array}{lll} (7) & \text{(commutative)} & a+b = b+a \\ (8) & \text{(divisible)} & (n)(\exists b)nb = a \end{array} \qquad (2.2)$$

If L is a partial ordering of S and $(S,+)$ is a semi-group, those properties (which establish a connection between these structures) that will be useful are:

$$\begin{array}{lll} (9) & \text{(monotony)} & aLb \Rightarrow (a+c)L(b+c) \\ (10) & \text{(positivity)} & aL(a+b) \\ (11) & \text{(natural order)} & aLb \Rightarrow (\exists c)b = a+c \\ (12) & \text{(Archimedean)} & aLb \Rightarrow (\exists n)bLna \end{array} \qquad (2.3)$$

A semi-group appears naturally in the course of defining the concept of extensive quantity. For a given quantity p possessed by the members of a set O of objects it may be that some members z of O are in fact composed of two other members x and y of O physically joined together in some specified manner. The result of joining x and y together will be denoted $\langle x,y\rangle$. If we make the reasonable assumption that if aEb and cEd then $\langle a,c\rangle E\langle b,d\rangle$, it will follow easily that the operation of physically joining objects in O induces an operation $+$ in $O^*$ by means of the definition,

$[a] + [b] = \{z : z = \langle x, y\rangle, x \epsilon [a], y \epsilon [b]\}$. That is, we are assuming that if equivalent objects are joined to equivalent objects then the resulting objects will also be equivalent in quantity p. The inference is that if each member of one equivalence class is paired with each member of another equivalence class then the resulting collection of composite objects will uniquely determine an equivalence class. If the operation + is to be associative we must assume that $\langle a, \langle b,c\rangle\rangle \, E \, \langle\langle a,b\rangle, c\rangle$ but all the authors mentioned above accept this and have proposed additional assumptions about the operation and its connection with the order relation.

Thus Helmholtz, Hempel, Menger and Ellis propose that $O^*$ is a commutative semi-group and in addition that also properties (9) through (11) of equations (2.3) hold except that Ellis omits (11) and proposes instead that (8) and (12) hold. Helmholtz and Menger also accept (8) and (12). The arguments proposed for acceptance or rejection of these various requirements are of two types: (i) those based on abstraction from actual measurement procedures and (ii) those based on the desire to make the function g of Figure 1 a homomorphism. To assume _ab initio_ that an extensive quantity has all of the structure of the real number system is too much (Suppes and Zinnes, 1963, p. 45). It would seem safer to rely on arguments of type (i) to formulate the concept of an extensive quantity axiomatically and then see if it can be mapped into or onto the real number system.

The following discussion is an example of an argument of type (i). A scale for the measurement of a quantity is often set up by the following procedure. First some object possessing the quantity in question is chosen as a standard. Another object, equivalent in respect of the quantity, is found and the composite system is formed. This or any object equivalent to it is then chosen as the second member of the set of standards. If the first object is denoted u, then the second could be denoted 2u and a third, constructed by repeating the process, 3u, and so on.

The scale is then extended by finding two equivalent objects whose composite is equivalent to u. Either of these might be denoted (1/2)u. Similarly (1/3)u, etc., could be found. Finally, by forming composites, standards are chosen so that any object may be approximated as closely as desired. This procedure depends on properties (8) and (12) and makes use of some properties of the set of positive integers.

The use of the positive integers here is an example of a procedure frequently used in mathematics. The best known instance is the use of positive integers to act on elements of a commutative group. To be more precise a sequence of functions is defined on and into an abelian group $(G,+)$ by the following formulae:

$$(x)\, f_1(x) = x$$
$$(x)\, f_n(x) = f_{n-1}(x) + x$$

Then $f_n$ is identified with $n$ and the notation $nx$ is introduced as an abbreviation for $f_n(x)$. Is this sequence of functions order-isomorphic to the natural numbers? A similar question and construction applies to semi-groups. It is recognized that the Peano axioms are not categorical (Henkin, 1950) and, furthermore, that the usual formulation of the induction axiom is non-elementary (Mostowski, 1955); hence some care must be exercised in answering these questions. Moreover, the Archimedean axiom itself is non-elementary (Robinson, 1965, p. 44) in the usual formulation. Hence while something like the properties (8) and (12) should be included in an axiomatization of the notion of extensive quantity, the usual formulations place serious difficulties in the way of an investigation of completeness.

In order to expose the concept of a ratio scale (Suppes and Zinnes, 1963, p. 9) we begin by explaining the action of $F^+$, the set of positive elements of some fully ordered field $(F, \#, \cdot, \leq)$, on a fully ordered semi-group $(S, +, L)$. Given the function, $\pi: F^+ \times S \to S$, let $m$ and $n$ denote members of $F^+$, let $a$ and $b$ denote arbitrary members and $u$ a fixed member of $S$, and let

$\pi(m,a)$ be abbreviated ma; then we will say that u generates S if $(b)(\exists m)\, mu = b$, and if the element m is determined uniquely for each b then we will say that u generates S simply. We will say that $F^+$ acts on S if S is simply generated by u and

$$\begin{aligned} m(na) &= (m \cdot n)a \\ mu + nu &= (m \# n)u \\ aLb &\Rightarrow ma\,L\,mb \end{aligned} \qquad (2.4)$$

The significance of this conceptual formulation is that if $F^+$ acts on $(O^*, +, L^*)$ then by choosing a member of $O^*$, say $[u]$, which generates $O^*$ simply we may make the statement that for any other element $[a]$ there is a unique element r of $F^+$ such that $r[u] = [a]$ or that the ratio of $[a]$ to $[u]$ is r. Moreover by assigning the number r to $[a]$ we have one way of constructing a map $g: O^* \rightarrow F^+$ which then justifies calling $(O^*, F^+, g)$ a ratio scale. While the material scale is a map into $Q^+$ or Q, as mentioned above, the conceptual scale should be a map into $R^+$ or R in order that laws of geometry and physics, which require algebraic and transcendental numbers, may hold in their usual formulations. In the reformulation proposed here the problem of defining extensive quantities, investigated by the authors mentioned above, was to define the structure of $(O^*, +, L^*)$ so that $Q^+$ or $R^+$ could act upon it. A similar approach was developed by Whitney (1968, p. 115), who showed a way in which one could construct a set of mathematical objects which act on a certain semi-group and showed that this set of objects was isomorphic to $R^+$. The number and type of relations and operations which are introduced as primitives and the assumptions about their properties which make up the definition of an extensive quantity may be selected in various ways so long as they hold for actual measurement procedures and make possible the construction of a ratio scale.

To the discussion of a quantity and a scale given at first we have now added the definitions of an extensive quantity and a ratio scale. All of the general features possessed by any quantity or scale have now appeared so that we are ready to attack the main problem of this section which is to contribute to the formulation of the concept of scale and scale transformation. We will begin by offering a new and more general definition of a conceptual scale which has been motivated by the preceding discussion. We will not consider material scales any further.

## 3. SCALES AND SCALE TRANSFORMATIONS

### 3.1. Structures, Models and Formal Language

First some notational devices will be introduced to facilitate the description of two important concepts; that of T-structure and that of $\Sigma$-model, which will be described in detail and then applied to measurement theory. If $f: A \to C$ and $B \subseteq A$ then $(B \times C) \cap f$ is a function which will be denoted by $f \mid B$ and is defined on $B$ with values in $C$. With the same notation $f(B)$ will denote $\{y \in C : (\exists x) x \in B \wedge f(x) = y\}$. We will denote an arbitrary member, $(y_1, y_2, \ldots, y_r)$, of $Y^r$ by $\vec{y}$ and if $f: Y \to Z$ we will define $f^r: Y^r \to Z^r$ by

$$f^r(\vec{y}) = (f(y_1), f(y_2), \ldots, f(y_r)).$$

A <u>structure</u> is a finite sequence

$$\mathfrak{A} = (A; S_1, S_2; \omega_{11}, \ldots, \omega_{1h}, \omega_{21}, \ldots, \omega_{2i}; \rho_{11}, \ldots, \rho_{1j}, \rho_{21}, \ldots, \rho_{2k}; \pi)$$

where $A$ is an arbitrary set called the <u>carrier</u> of $\mathfrak{A}$, $S_n$ are unary relations (<u>i.e.</u>, subsets in $A$) for $n = 1, 2$;

$\omega_{1m}$ is an operation in $\bar{S}_1 = \{x : S_1(x)\}$ $(1 \le m \le h)$,
$\omega_{2m}$ is an operation in $\bar{S}_2 = \{x : S_2(x)\}$ $(1 \le m \le i)$,
$\rho_{1m}$ is a relation in $\bar{S}_1$ $(1 \le m \le j)$,
$\rho_{2m}$ is a relation in $\bar{S}_2$ $(1 \le m \le k)$,

and $\pi: \bar{S}_2 \times \bar{S}_1 \to \bar{S}_1$ .

For a definition of operation and relation see Suppes (1957, pp.

211, 234). Moreover we require that for any $x \in A$ we have

$$[S_1(x) \vee S_2(x)] \wedge [\sim S_1(x) \vee \sim S_2(x)],$$

*i.e.*, the subsets $\bar{S}_1$ and $\bar{S}_2$ partition A. If $\omega_{nm}: \bar{S}_n^r \to \bar{S}_n$ we will call $r = a(\omega_{nm})$ the arity of $\omega_{nm}$ and if $\rho_{nm} \subseteq \bar{S}_n^r$ we will call $r = a(\rho_{nm})$ the arity of $\rho_{nm}$. We will say that a structure has type T or is a T-structure if

$$T = (a(\omega_{11}), \ldots, a(\omega_{2i}); a(\rho_{11}), \ldots, a(\rho_{2k})),$$

and when discussing a class of T-structures the same symbols, *i.e.*, $\omega_{nm}$ and $\rho_{nm}$, will usually be used to denote the operations and relations of any T-structure; this is an extension of the usual abuse of notation whereby + and $\times$ are used to denote different operations in various rings.

Let $B \subseteq A$ such that $\bar{R}_n = B \cap \bar{S}_n$ and $r = a(\omega_{nm})$; then if

$$\omega_{nm}(t, \vec{q}) \wedge (\vec{q} \in \bar{R}_n^r) \Rightarrow t \in \bar{R}_n$$

we will say that B admits the operation $\omega_{nm}$ and denote the operations in $\bar{R}_n$ obtained by restriction as $\omega_{nm}|B$ $(n = 1, 2)$. Similarly if

$$R_2(x) \wedge R_1(y) \wedge \pi(z, x, y) \Rightarrow R_1(z)$$

we will say that B admits the operation $\pi$ and denote the restriction by $\pi|B$. Also we denote $\rho_{nm} \cap B^{a(\rho_{nm})}$ by $\rho_{nm}|B$. Occasionally it will be more appropriate to use the more conventional $t = \omega_{nm}(\vec{q})$ in place of $\omega_{nm}(t, \vec{q})$.

The concept of $\Sigma$-model requires a syntactical specification of an admissible axiom. The axioms that are likely to be useful for the description of quantities will contain a finite set of variables; for example, a common set of axioms for an abelian group requires only three variables in the associative law, two in the commutative law, etc., so in fact three variables and five axioms will suffice. In general, depending on the arities of the operations and the complexity of the axioms, perhaps six or eight variables might be required for some quantity not yet envisioned. The set $\wp(T, I)$ introduced below has been defined so that it would contain any formula we could conceive of wanting to use as an axiom.

The set of formulae, $\wp(T, I)$, defined by the following recursive conditions forms a Boolean algebra for each type T and each set I of symbols or variables.

(1) $x \in I \wedge y \in I \Rightarrow (x = y) \in \wp(T, I)$

(2) $y \in I \wedge \vec{x} \in I^{a(\omega_{nm})} \Rightarrow \omega_{nm}(y, \vec{x}) \in \wp(T, I)$
for each operation $\omega_{nm}$

(3) $\vec{x} \in I^{a(\rho_{nm})} \Rightarrow \rho_{nm}(\vec{x}) \in \wp(T, I)$
for each relation $\rho_{nm}$ (3.1.1)

(4) $x \in I \Rightarrow S_n(x) \in \wp(T, I)$ for $n = 1, 2$

(5) $x \in I \wedge y \in I \wedge z \in I \Rightarrow \pi(z, x, y) \in \wp(T, I)$

(6) $P, Q \in \wp(T, I) \Rightarrow (P \vee Q) \in \wp(T, I)$

(7) $P \in \mathcal{P}(T, I) \Rightarrow \sim P \in \mathcal{P}(T, I)$

(8) $P \in \mathcal{P}(T, I) \Rightarrow (\exists x) P \in \mathcal{P}(T, I)$

It is understood that I will contain at most finitely many symbols and furthermore each member of $\mathcal{P}(T, I)$ will contain finitely many symbols, so clearly $\mathcal{P}(T, I)$ will contain at most countably many members. As usual in symbolic logic we define:

$$P \wedge Q \text{ for } \sim(\sim P \vee \sim Q)$$

and

$$(x)P \text{ for } \sim((\exists x) \sim P)$$

and observe that every member of $\mathcal{P}(T, I)$ in which all variables are bound by quantifiers may be written in the normal form,

$$(Qx_1), (Qx_2), \ldots, (Qx_n) P(x_1, x_2, \ldots, x_n)$$

where $P(x_1, x_2, \ldots, x_n)$ contains the variables $x_1, x_2, \ldots, x_n$ and no quantifiers and $(Qx_i)$ denotes either $(\exists x_i)$ or $(x_i)$.

## 3.2. Theories and Their Models

If a member of $\mathcal{P}(T, I)$ expressed in normal form has only universal quantifiers it is called a universal sentence, if it has only existential quantifiers it is called an existential sentence and otherwise a sentence. If it has no variables bound by quantifiers it is called an open formula. Any formula built up from formulae satisfying (1), (2), (3), (4) or (5) of equations 3.1.1 by using $\wedge$ and $\vee$ alone (without $\sim$) is called positive and any sentence that arises by quantification of a positive formula is called <u>positive</u>. A formula, satisfying one of (1), (2), (3), (4) or (5) of 3.1.1, which includes neither propositional connectives nor quantifiers will be called <u>atomic</u>. A sentence Y will be called <u>primitive</u> if it is of the form

$$Y = [(\exists y_1)(\exists y_2)\ldots(\exists y_n)\ Z(y_1, y_2, \ldots, y_n)] \qquad (3.2.1)$$

where Z is a conjunction of, and (or) negations of, atomic formulae. It is understood that a sentence will satisfy this definition if Z consists of a single atomic formula or of the negation of such a formula, or that Y does not include any quantifiers at all. Let $\mathfrak{A}$ be a T-structure, $P(\vec{x})$ a formula having n free variables, $\vec{a} \in A^n$ and $P(\vec{a})$ denote the formula which results when the components of $\vec{a}$ are substituted for the corresponding components of $\vec{x}$, then if $P(\vec{a})$ holds for each $\vec{a} \in A^n$ we will say that P is <u>valid</u> in $\mathfrak{A}$.

From now on we will assume that the carrier of any T-structure we mention will be a subset of some universe U, fixed but otherwise unspecified. We will denote by [T] the set of all T-structures and define two mappings:

(1) to any class $C \subseteq [T]$ there corresponds the set $C^*$ of all formulae which are valid in each T-structure of C, and

(3.2.2)

(2) to any set $\Sigma \subseteq \mathcal{P}(T, I)$ there corresponds the set $\Sigma^*$ of all those T-structures in which all the formulae of $\Sigma$ are valid.

This defines a bijection between $\mathcal{B}(\mathcal{P}(T,I))$, the collection of all subsets of $\mathcal{P}(T,I)$, and $\mathcal{B}([T])$, the collection of all subsets of $[T]$, with the following properties:

$$
\begin{aligned}
&(1)\quad C_1 \subseteq C_2 \subseteq [T] \Rightarrow C_2^* \subseteq C_1^* \\
&\qquad\ \Sigma_1 \subseteq \Sigma_2 \subseteq \mathcal{P}(T,I) \Rightarrow \Sigma_2^* \subseteq \Sigma_1^* \\
&(2)\quad C \subseteq C^{**},\ \Sigma \subseteq \Sigma^{**} \\
&(3)\quad C^{***} = C^*,\ \Sigma^{***} = \Sigma^*
\end{aligned}
\qquad (3.2.3)
$$

Such a correspondence is usually called a Galois connection and it is easily seen that the mappings $C \to C^{**}$ and $\Sigma \to \Sigma^{**}$ are closure operators and that $X = \{C : C \subseteq [T]$ and $C = \Sigma^*$ for some set $\Sigma$ of sentences$\}$ is a closure system as is $M = \{\Sigma : \Sigma \subseteq \mathcal{P}(T,I)$ and $\Sigma = C^*$ for some $C \subseteq [T]\}$. (The terminology introduced in the preceding sentence is that of Birkhoff, 1967, p. 124.) Any member of $X$ will be called an <u>axiomatic</u> class and any member of $Y$ will be called a <u>model-closed</u> set; moreover if $C$ is non-empty then $C^*$ will be called a <u>theory</u>; $\Sigma$ is called a <u>set of axioms</u> for $\Sigma^*$ and any member of $\Sigma^*$ is called a <u>$\Sigma$-model</u>. We could also introduce a set of operators having the form of rules of deduction (Hilbert and Ackerman, 1950, pp. 68-70) and for each set $\Sigma \subseteq \mathcal{P}(T,I)$ define $\Sigma^\circ$ to be the set of all formulae derivable from those in $\Sigma$ using the rules of deduction. It is interesting to observe (Cohn, 1965, p. 206) that "the model-closed sets are precisely the sets admitting the rules of deduction," i.e. $\Sigma^\circ = \Sigma^{**}$. Moreover X and Y are equinumerous as can easily be seen using the mappings defined in equations 3.2.2.

For any T-structure $\mathfrak{A}$ the set $D(\mathfrak{A})$ of atomic formulae and negations of atomic formulae which hold in $\mathfrak{A}$ will be called its diagram. If for any formula X both the formulae X and $\sim X$ belong to a set $\Sigma$ then $\Sigma$ will be called inconsistent and $\Sigma$ will be called consistent if it is not inconsistent. If $\Sigma$ is a non-empty consistent set of formulae in $\mathcal{P}(T, I)$ then $\Sigma$ will be called model-complete if for every T-structure $\mathfrak{A} \in \Sigma^*$ the set $\Sigma \cup D(\mathfrak{A})$ is complete. We recall that $\Sigma$ is said to be complete if for every X in $\mathcal{P}(T, I)$ either $X \in \Sigma^\circ$ or $\sim X \in \Sigma^\circ$. A. Robinson has given examples to show that neither of these concepts includes the other and also he has stated useful conditions under which model-completeness implies completeness in the ordinary sense.

We turn now to a consideration of isomorphism for $\Sigma$-models and related concepts. If $\mathfrak{B} = (B; R_1, R_2; \psi_{11}, \ldots, \psi_{2i}, \sigma_{11}, \ldots, \sigma_{2k}; \tau)$ we will call $\mathfrak{B}$ a substructure of $\mathfrak{A}$ if they have the same type and the following conditions are satisfied:

$$\begin{aligned} &(S1) \quad B \subseteq A, \\ &(S2) \quad R_1(x) \Rightarrow S_1(x), S_2(x) \Leftrightarrow R_2(x), \end{aligned} \tag{3.2.4}$$

(S3) $\quad B$ admits $\omega_{nm}$ and $\omega_{nm} | B = \psi_{nm}$

for each operation $\omega_{nm}$,

(S4) $\quad B$ admits $\pi$ and $\pi | B = \tau$, (3.2.4) (cont.)

(S5) $\quad \sigma_{nm} = \rho_{nm} | B$ for each relation $\rho_{nm}$.

If $\mathfrak{A} \in \Sigma^*$ and $\mathfrak{B} \in \Sigma^*$ and $\mathfrak{B}$ is a substructure of $\mathfrak{A}$ then $\mathfrak{B}$ will be called a sub-$\Sigma$-model.

If $\mathfrak{A}$ and $\mathfrak{B}$ are any two T-structures and $\phi : A \to B$ we will call $\phi$ a homomorphism if

(H1) $\quad \phi(\bar{S}_1) \subseteq \bar{R}_1$

(H2) $\quad \phi(\bar{S}_2) \subseteq \bar{R}_2$

(H3) $\quad \phi^{r+1}(\omega_{nm}) \subseteq \psi_{nm}$ where $r = a(\omega_{nm})$ (3.2.5)

(H4) $\quad \phi^{r}(\rho_{nm}) \subseteq \sigma_{nm}$ where $r = a(\rho_{nm})$

(H5) $\quad \phi^{3}(\pi) \subseteq \tau$

If a homomorphism has an inverse which is also a homomorphism then it is called an isomorphism. If $a(\omega) = r$ and $(t, \vec{q}) \in \omega$ is denoted $t = \omega(\vec{q})$ then the following statements are equivalent,

(1) $\phi^{r+1}(\omega) \subseteq \psi$

(2) $(t, \vec{q}) \in \omega \Rightarrow (\phi(t), \phi^{r}(\vec{q})) \in \psi$

(3) $t = \omega(\vec{q}) \Rightarrow \psi(\phi^{r}(\vec{q})) = \phi(t)$ (3.2.6)

(4) $\phi(t) = \phi(\omega(\vec{q})) \Rightarrow \psi(\phi^{r}(\vec{q})) = \phi(t)$

(5) $\phi(\omega(\vec{q})) = \psi(\phi^{r}(\vec{q}))$.

However the last is easily seen to be the usual definition of a homorphism. By an application of the Schroder-Bernstein theorem (Kolodner, 1967, p. 995) we see that if $\phi$ is an automorphism we have $\phi(\bar{S}_1) = \bar{S}_1$.

We digress briefly to recall some syntactical notions and related properties of model classes. A class of models determined as in (2) of (3.2.2) above by a set $\Sigma$ of universal sentences is called a <u>universal class</u> (or also an <u>open class</u> since it may be defined by open formulae (in which no variable is bound)). A model class (of $\Sigma$-models with respect to some set $\Sigma$) will be called <u>abstract</u> if it contains with any model all its isomorphic copies. For any abstract class of models, C, we write SC for the class of submodels of members of C and observe that S is a closure operator. An S-closed class (<u>i.e.</u>, an abstract class C such that SC = C) is said to be <u>hereditary</u>. We can now state two important results (Cohn, 1965, p. 226, 238).

(I.) A class of models is hereditary and axiomatic if and only if it is a universal class.

(II.) A universal class of models admits homorphic images if and only if it can be defined by positive universal sentences.

The requirement that a model class be abstract does place some restriction on the form of the members of $\Sigma$ and we will assume throughout the following discussion that $\Sigma$ defines an abstract universal class of models which by the results quoted above is hereditary.

The relation of isomorphism between T-structures determines an equivalence relation in an axiomatic class $\Sigma^*$ and if there is only one equivalence class then $\Sigma$ is said to be categorical. If $\Sigma$ is not categorical, but a subset M of $\Sigma^*$ can be found with the property that every member of $\Sigma^*$ is isomorphic to a member of M, then we say that $\Sigma$-models may be represented by members of M.

3.3. Definition of Scales and Scale Transformations

We now apply these concepts to the theory of quantitative measurement. Let $\Sigma$ be a set of formulae which describe a quantity (relative to a class of T-structures) *e.g.*, (1) - (12) of section 2 referred to in describing an extensive quantity as a fully ordered semi-group satisfying certain additional properties. Further let $\mathfrak{n}$ denote a $\Sigma$-model whose carrier is N where $N = R \cup Z$, R is the set of real numbers, Z is the set of natural numbers and let M denote the set of sub-$\Sigma$-models of $\mathfrak{n}$. We propose that quantities may be represented by members of M as pointed out in the preceding paragraph, $\mathfrak{n}$ will be called a full numerical $\Sigma$-model, any other member of M will be called a numerical $\Sigma$-model and a member of $\Sigma^*$ not in M whose carrier is a set of equivalence classes of physical objects will be called an <u>empirical</u> <u>$\Sigma$-model</u>. As pointed out by Suppes and Zinnes (1963, p. 7) the first fundamental problem of measurement is that of showing that "any empirical relational system that purports to measure (by a simple number) a given property of the elements in the domain of the system is isomorphic (or possibly homomorphic) to an appropriately chosen numerical relational system." (What they call a relational system is similar to a T-structure and the domain of a relational system corresponds to the carrier of a T-structure.)

When we consider concrete examples of scales and scale transformation as instances of the general conceptual framework developed here it will become clear that this approach has definite advantages over that developed by Suppes and Zinnes. The fundamental problem, of finding a representation relative to M, can also be described as that of finding a scale. If $\mathfrak{A}$ is an empirical $\Sigma$-model with carrier A then we will let $\aleph(\mathfrak{A},\mathfrak{h})$ denote $\{f:f$ is an isomorphism, $f:A \rightarrow B$ and B is the carrier of some member of M$\}$ and call any member of $\aleph(\mathfrak{A},\mathfrak{h})$ a <u>numerical</u> <u>assignment</u>. From now on the term <u>scale</u> will be used to denote the triple $(\mathfrak{A},\mathfrak{h},f)$ where $\mathfrak{A}$ is an empirical $\Sigma$-model, $\mathfrak{h}$ is a full numerical $\Sigma$-model and $f \in \aleph(\mathfrak{A},\mathfrak{h})$. Finally we will define for any $\Sigma$-model $\mathfrak{Y}$ with carrier Y the set of automorphisms $\mathfrak{A}(\mathfrak{Y}) = \{\phi : \phi$ is an isomorphism on Y onto Y$\}$.

If $\phi$ belongs to $\mathfrak{A}(\mathfrak{h})$ and f and g belong to $\mathfrak{H}(\mathfrak{G},\mathfrak{h})$ and $\phi \circ f = g$ then we will say that $\phi$ is a <u>scale transformation</u> of $(\mathfrak{G},\mathfrak{h},f)$ into $(\mathfrak{G},\mathfrak{h},g)$.

We wish to show that a group of scale transformations can be associated with any pair $(\mathfrak{G},\mathfrak{h})$.

<u>Lemma 3.3.1</u>. $\mathfrak{A}(\mathfrak{Y})$ is a group with respect to composition.

Proof: Since $\mathfrak{A}(\mathfrak{Y})$ is a subset of the group of all bijections of Y onto Y with respect to composition we need only verify that if $f, g \in \mathfrak{A}(\mathfrak{Y})$ then $f \circ g$ and $f^{-1}$ are also. Since the group operation is composition it is correct to use the same symbol $f^{-1}$ to denote the inverse of f as an element of $\mathfrak{A}(\mathfrak{Y})$ and to denote the inverse function to f. Since $f^{-1}$ is an isomorphism if and only if f is, it remains only to show that $f \circ g$ is an isomorphism. If $A \subseteq Y^r$ such that $f^r(A) \subseteq A$ and $g^r(A) \subseteq A$ then $(f \circ g)^r(A) = f^r(g^r(A)) \subseteq A$ so $f \circ g$ satisfies (H1) through (H4) of 3.2.5. If $\pi \subseteq \bar{S}_2 \times \bar{S}_1 \times \bar{S}_1$ such that $f^3(\pi) \subseteq \pi$ and $g^3(\pi) \subseteq \pi$ then $(f \circ g)^3(\pi) = f^3(g^3(\pi)) \subseteq f^3(\pi) \subseteq \pi$ and thus (H5) is satisfied. Now $f^{-1}$ and $g^{-1}$ are isomorphisms so by applying the preceding result $g^{-1} \circ f^{-1}$ is also a homomorphism, but $g^{-1} \circ f^{-1}$ is the inverse of $f \circ g$ so we have shown that $f \circ g$ is an isomorphism.

Let $\mathfrak{X} = (X; R_1, R_2; \psi_{11}, \psi_{12}, \ldots, \psi_{1h}, \psi_{21}, \psi_{22}, \ldots, \psi_{2i}; \sigma_{11}, \sigma_{12}, \ldots, \sigma_{1j}, \sigma_{21}, \sigma_{22}, \ldots, \sigma_{2k}; \tau)$ be an empirical $\Sigma$-model and

$\mathfrak{N} = (N; Q_1, Q_2; \omega_{11}, \omega_{12}, \ldots, \omega_{1h}, \omega_{21}, \omega_{22}, \ldots, \omega_{2i}; \rho_{11}, \rho_{12}, \ldots, \rho_{1j}, \rho_{21}, \rho_{22}, \ldots, \rho_{2k}; \pi)$ be a numerical $\Sigma$-model of the same type where $\bar{Q}_1$ denotes the real numbers and $\bar{Q}_2$ denotes the natural numbers.

Lemma 3.3.2. If $\phi \in \mathfrak{A}(\mathfrak{N})$ and $f \in \mathfrak{H}(\mathfrak{X}, \mathfrak{N})$ then $\phi \circ f \in \mathfrak{H}(\mathfrak{X}, \mathfrak{N})$.

Proof: By hypothesis f determines an isomorphism of $\mathfrak{X}$ onto $\mathfrak{N}'$, that sub-$\Sigma$-model of $\mathfrak{N}$ which is the image of $\mathfrak{X}$. Let $Y = f(X)$, $\bar{S}_1 = Y \cap \bar{Q}_1$, $\bar{S}_2 = \bar{Q}_2$, $\bar{\omega}_{nm} = \omega_{nm} | Y$, $\bar{\rho}_{nm} = \rho_{nm} | Y$, and $\bar{\pi} = \pi | Y$ then clearly $\mathfrak{N}' = (Y; S_1, S_2; \bar{\omega}_{11}, \bar{\omega}_{12}, \ldots, \bar{\omega}_{1h}, \bar{\omega}_{21}, \bar{\omega}_{22}, \ldots, \bar{\omega}_{2i}; \bar{\rho}_{11}, \bar{\rho}_{12}, \ldots, \bar{\rho}_{1j}, \bar{\rho}_{21}, \bar{\rho}_{22}, \ldots, \bar{\rho}_{2k}; \bar{\pi})$ and since f is an isomorphism we have

$$
\begin{aligned}
&(1)\quad f(\bar{R}_1) \subseteq \bar{S}_1, \\
&(2)\quad f(\bar{R}_2) \subseteq \bar{S}_2, \\
&(3)\quad f^{r+1}(\psi_{nm}) \subseteq \bar{\omega}_{nm}, \text{ where } r = a(\psi_{nm}), \\
&(4)\quad f^{r}(\sigma_{nm}) \subseteq \bar{\rho}_{nm}, \text{ where } r = a(\sigma_{nm}), \text{ and} \\
&(5)\quad f^{3}(\tau) \subseteq \bar{\pi}.
\end{aligned}
\tag{3.3.1}
$$

Let $Z = \phi(Y)$, $\bar{P}_1 = Z \cap \bar{Q}_1$, $\bar{P}_2 = \bar{Q}_2$, $\bar{\bar{\omega}}_{nm} = \omega_{nm} | Z$, $\bar{\bar{\rho}}_{nm} = \rho_{nm} | Z$, and $\bar{\bar{\pi}} = \pi | Z$ then $\mathfrak{N}'' = (Z; P_1, P_2; \bar{\bar{\omega}}_{11}, \bar{\bar{\omega}}_{12}, \ldots, \bar{\bar{\omega}}_{1h}, \bar{\bar{\omega}}_{21}, \bar{\bar{\omega}}_{22}, \ldots, \bar{\bar{\omega}}_{2i}; \bar{\bar{\rho}}_{11}, \bar{\bar{\rho}}_{12}, \ldots, \bar{\bar{\rho}}_{1j}, \bar{\bar{\rho}}_{21}, \bar{\bar{\rho}}_{22}, \ldots, \bar{\bar{\rho}}_{2k}; \bar{\bar{\pi}})$ and we must show that $\phi | Y$ determines an isomorphism of $\mathfrak{N}'$ onto $\mathfrak{N}''$ and that $\mathfrak{N}''$ is a sub-$\Sigma$-model of $\mathfrak{N}$. Now $\phi : N \to N$ so $Z \subseteq N$. Since $\phi$ is an isomorphism we have

(1) $\phi(\bar{Q}_1) = \bar{Q}_1$,

(2) $\phi(\bar{Q}_2) = \bar{Q}_2$,

(3) $\phi^{r+1}(\omega_{nm}) \subseteq \omega_{nm}$, where $r = a(\omega_{nm})$, (3.3.2)

(4) $\phi^{r}(\rho_{nm}) \subseteq \rho_{nm}$, where $r = a(\rho_{nm})$, and

(5) $\phi^{3}(\pi) \subseteq \pi$.

Since $\mathfrak{h}'$ is a sub-$\Sigma$-model by hypothesis Y admits $\omega_{nm}$ so $r = a(\omega_{nm})$, $\omega_{nm}(t, \vec{q})$ and $\vec{q} \in \bar{\bar{S}}_1^r$ imply $t \in \bar{\bar{S}}_1$ but then $\phi(t) \in \bar{\bar{P}}_1$ and $\phi^r(\vec{q}) \in \phi^r(\bar{\bar{S}}_1^r)$ and $\phi^r(\bar{\bar{S}}_1^r) = \bar{\bar{P}}_1^r$ since $\phi(\bar{\bar{S}}_1) = \bar{\bar{P}}_1$. Thus we have shown that if $\vec{q} \in \bar{\bar{S}}_1^r$ and $\omega_{nm}(t, \vec{q})$ then $\phi^{r+1}(t, \vec{q}) \in P_1^{r+1}$. It now follows that if $\vec{p} \in \bar{\bar{P}}_1^r$ and $\omega_{nm}(s, \vec{p})$ then $(s, \vec{p}) = (\phi(t), \phi^r(\vec{q}))$ for some $t \in \bar{\bar{S}}_1$ and $\vec{q} \in \bar{\bar{S}}_1^r$ and so $\omega_{nm}(s, \vec{p})$ if and only if $\omega_{nm}(\phi^{r+1}(t, \vec{q}))$. Moreover $(\phi^{-1})^{r+1}(\omega_{nm}) \subseteq \omega_{nm}$ since $\phi$ is an isomorphism and so $\omega_{nm}(\phi^{r+1}(t, \vec{q}))$ if and only if $\omega_{nm}(t, \vec{q})$ and then by the preceding argument $\phi^{r+1}(t, \vec{q}) \in \bar{\bar{P}}_1^{r+1}$ , . <u>i</u>.<u>e</u>., $s = \phi(t) \in \bar{\bar{P}}_1$ and Z admits $\omega_{nm}$. From the above argument, <u>mutatis mutandis</u>, we conclude that not only is

(1) $\phi^{r+1}(\bar{\omega}_{nm}) \subseteq \bar{\bar{\omega}}_{nm}$, but

(2) $\phi^{r}(\bar{\rho}_{nm}) \subseteq \bar{\bar{\rho}}_{nm}$, and (3.3.3)

(3) $\phi^{3}(\bar{\pi}) \subseteq \bar{\bar{\pi}}$.

Then by the definition of $\bar{\bar{\omega}}_{nm}$ and $\bar{\bar{\rho}}_{nm}$ $\mathfrak{h}''$ is a sub-structure of $\mathfrak{h}'$ and therefore of $\mathfrak{h}$ and by assumption (see section 3.2 above) $\Sigma^*$ is hereditary so $\mathfrak{h}''$ is a sub-$\Sigma$-model of $\mathfrak{h}$.

Next we consider $\bar{\phi} = \phi | Y$ and $\phi' = \phi^{-1} | Z$. Clearly these are bijections and homomorphisms and $\phi' = (\bar{\phi})^{-1}$. We have finally to show that $\phi \circ f \in \mathfrak{H}(\mathfrak{X}, \mathfrak{n})$. Now $\phi \circ f = \bar{\phi} \circ f$ and $\bar{\phi} \circ f : X \to Z$ which is a composition of bijections and is therefore a bijection. Next we note that, since $\mathfrak{X}$ and $\mathfrak{n}''$ have the same type and $\bar{\phi}$ and f are homomorphisms, we have $(\bar{\phi} \circ f)^r(\psi_{nm})) = \bar{\phi}^r(f^r(\psi_{nm})) \subseteq \bar{\phi}^r(\bar{\omega}_{nm}) \subseteq \bar{\bar{\omega}}_{nm}$ for all operations $\psi_{nm}$ of $\mathfrak{X}$, where $r = a(\psi_{nm})$. Similarly $(f^{-1} \circ \phi')^r(\bar{\bar{\omega}}_{nm}) \subseteq \psi_{nm}$, $(\bar{\phi} \circ f)^e(\sigma_{nm}) \subseteq \bar{\bar{\rho}}_{nm}$ and $(f^{-1} \circ \phi')^e(\bar{\bar{\rho}}_{nm}) \subseteq \sigma_{nm}$, where $e = a(\sigma_{nm})$, and also $(\bar{\phi} \circ f)^3(\tau) \subseteq \bar{\bar{\pi}}$ and $(f^{-1} \circ \phi')^3(\bar{\bar{\pi}}) \subseteq \tau$. Thus the proof is complete.

Lemma 3.3.3. $\Phi = \{\phi \in \mathfrak{A}(\mathfrak{n}) : \exists f, g \in \mathfrak{H}(\mathfrak{X}, \mathfrak{n}) \text{ and } \phi \circ f = g\}$ is a group with respect to composition.

Proof: By definition $\Phi \subseteq \mathfrak{A}(\mathfrak{n})$ and if $\phi \in \Phi$ then $\exists f, g$ in $\mathfrak{H}(\mathfrak{X}, \mathfrak{n})$ such that $\phi \circ f = g$ and $\phi^{-1} \circ g = f$. If also $\psi \in \Phi$ then by lemma 3.3.2 $\phi \circ g = k$ belongs to $\mathfrak{H}(\mathfrak{X}, \mathfrak{n})$ so $(\psi \circ \phi) \circ f = \psi \circ g = k$ hence $(\psi \circ \phi) \in \Phi$.

Theorem 3.3.4. The set of all scale transformations forms a group with respect to composition and if $(\mathfrak{X}, \mathfrak{n}, f)$ and $(\mathfrak{X}, \mathfrak{n}, g)$ are any scales then the sub-$\Sigma$-model $\mathfrak{n}'$ with carrier $f(X)$ is isomorphic to the sub-$\Sigma$-model $\mathfrak{n}''$ with carrier $g(X)$.

Proof: The first statement follows from lemma 3.3.3. For all $y \in f(X)$ let us define $\mu$ by $\mu(y) = g(x)$ where $x \in X$ satisfies $y = f(x)$. This defines a function $\mu : f(X) \to g(X)$ as we shall show.

If $y = f(x_1) = f(x_2)$ and $g(x_1) \neq g(x_2)$ we have a contradiction since $f$ and $g$ are isomorphisms. Thus $\mu$ is well defined and $\mu \circ f = g$ and $\mu$ is a bijection. By a computation similar to that of lemma 3.3.2 $\mu$ is an isomorphism.

# 4. EXAMPLES

## 4.1. Extensive Quantities and Ratio Scales

Our objective is to describe an extensive quantity and formulate the fact that it contains an adequate set of standards in such a way that it is a $\Sigma$-model of a set $\Sigma$ of sentences of $\mathcal{P}(T, I)$ and to show later that it determines a scale. Since we have undertaken to formulate an elementary theory (Mostowski, 1955, pp. 5-7) in the first order predicate calculus we cannot naively assume that the system of natural numbers with all its usual properties will be available in our theory. Thus we must construct a formal analogue of the natural number system by means of elementary axioms and definitions which does not however need to be identical with the natural number system.

To this end we have adopted a technical device that amounts to assuming that the carrier of our T-structure is the union of the set $\bar{S}$ of degrees of the quantity and another set $\bar{N}$ which is the set of positive elements of an ordered integral domain. The latter has all the properties we need but we cannot show that it is order isomorphic to the natural number system without using notions of set theory which cannot be formulated in $\mathcal{P}(T, I)$. We need two unary relations denoted $N(x)$ and $S(x)$ which express the property that x belongs to $\bar{N}$ and to $\bar{S}$ respectively. One binary relation denoted $x < y$ expresses the order relation in $\bar{S}$. Also two operations, $+$ and $\cdot$, are primitive notions; the former is a binary operation in $\bar{S}$ and the latter is a binary operation expressing the action of elements of $\bar{N}$ on elements of $\bar{S}$. We interpret $n \cdot x = y$ as $y = x + x + \ldots + x$ where there are n terms on the right in the last equation. Our language is what is usually called the first order predicate calculus with equality so that we do not need axioms like $(x = z) \wedge (y = w) \wedge (u = v) \wedge (x + y = u) \Rightarrow (z + w = v)$ and will carry out substitutions without comment.

We introduce by definition, for elements y such that $N(y)$, the operations $\#$, $\times$, the binary relation $\ll$, the function $\sigma$ and the constant 1. Our definitions are motivated by the interpretation we wish to construct, that is we think of $n \in \bar{N}$ as a function on $\bar{S}$ to $\bar{S}$ defined by the equation $n(x) = n \cdot x = x + x + \ldots + x$ so that, for example, we define the "function" $n + m$ by the equation $\{n + m\}(x) = n(x) + m(x)$. By this means some desired properties of $\#$, $\times$, and $\ll$ become direct consequences of similar properties of $+$, $\cdot$ and $<$, and we need only two simple axioms concerning $\sigma$ and 1. These together with an induction axiom scheme suffice for us to show that $(N, \#, \times, \ll)$ has the properties we claimed.

Our claim that this is an elementary formulation is based on the following argument. Since we only wish to prove a finite list of theorems, the induction axiom scheme may be replaced by a finite list of axioms each one of which is obtained by substituting for the meta-linguistic variable P(x) in 4.1.4 E a specific formula which occurs in the proof of one of the theorems which use this axiom. Actually a distinct axiom is not needed for each theorem proved by induction. A similar argument for considering the axioms of identity as elementary axioms is presented by Hilbert and Ackermann (1950, pp. 107, 108).

One important benefit of this approach is that consistency and completeness of the set of axioms may be investigated by standard methods. Specifically we apply A. Robinson's test for model-completeness (Robinson, 1956, p. 16) and show further that any semi-group of the type M (to be defined later) has a prime (Robinson, 1956, p. 72) sub-semi-group and hence that its elementary theory is complete. We shall also show that such a semi-group determines a scale and that the group of all scale transformations associated with this scale is the group of homothetic transformations.

We shall follow the custom of writers on formal theories and give informal sketches of the proofs of the theorems below. However, it should be clearly understood that it is our claim that completely formal proofs could be constructed using these sketches as a guide. The statements of the axioms and theorems will likewise be formulated so as to be intuitively clear but could be easily translated into formulae of $P(T, I)$.

The first set of axioms expresses the properties of the unary relations N and S and their connections with the other relations and operations.

(A) The sets, $\bar{N}$ and $\bar{S}$, of elements which satisfy $N(x)$ and $S(x)$ respectively, are complementary sets.

(B) The relation $x < y$ holds only between elements of $\bar{S}$.

(C) The relation $x + y = z$ holds only between elements of $\bar{S}$.

(D) The relation $x \cdot y = z$ holds only if $x$ belongs to $\bar{N}$ and $y$ and $z$ belong to $\bar{S}$. (4.1.1)

(E) If $x$ and $y$ are any members of $\bar{S}$ then there is an element $z$ in $\bar{S}$ such that $x + y = z$.

(F) If $x$ is any member of $\bar{N}$ and $y$ is any member of $\bar{S}$ then there is an element $z$ in $\bar{S}$ such that $x \cdot y = z$.

(G) If $x$, $y$, $z$ and $w$ are any members of $\bar{S}$ such that $x + y = z$ and $x + y = w$ then $z = w$.

The second set of axioms asserts that $\bar{S}$ is fully ordered by $<$.

(A) If $x$, $y$ and $z$ are any elements of $\bar{S}$ such that $x < y$ and $y < z$ then $x < z$.

(4.1.2)

(B) If $x$ and $y$ are any elements of $\bar{S}$ such that $x = y$ then $\sim(x < y)$.

(C) If $x$ and $y$ are any elements of $\bar{S}$ such that $\sim(x = y)$ then $x < y$ or $y < x$.

The next set of axioms state that $(\bar{S}, +, <)$ is a naturally Archimedean ordered divisible abelian semi-group.

(A) If $x$, $y$, $z$, $u$, $v$ and $w$ are any elements of $\bar{S}$ such that $x + y = u$ and $u + z = v$ and $y + z = w$ then $x + w = v$.

(B) If $x$, $y$ and $z$ are any elements of $\bar{S}$ such that $x + y = z$ then $y + x = z$.

(C) If $x$ and $y$ are any members of $\bar{S}$ then $x < y$ if and only if there is $z$ in $\bar{S}$ such that $x + z = y$. (4.1.3)

(D) If $x$ and $y$ are any members of $\bar{S}$ then there is $z$ in $\bar{N}$ such that $y < z \cdot x$.

(E) If $x$ is any member of $\bar{S}$ and $y$ is any member of $\bar{N}$ then there is $z$ in $\bar{S}$ such that $x = y \cdot z$.

We turn next to a group of axioms that describe $\bar{N}$ and its action on $\bar{S}$. First two definitions will be stated which will clarify the intent of the axioms to follow.

(A) If $x$ is any member of $\bar{N}$ we will write $x = 1$ if and only if for each $y$ in $\bar{S}$ we have $x \cdot y = y$.

(4.1.4)

(B) If $x$ and $y$ are any members of $\bar{N}$ we will write $x = \sigma(y)$ if and only if for each $z$ in $\bar{S}$ we have $x \cdot z = y \cdot z + z$.

The axioms state, in terms of the concepts just introduced, properties closely related to the Peano axioms. The last axiom is a scheme in which $P(x)$ may be replaced by any formula in $\mathcal{P}(T, I)$ which contains just one free variable provided appropriate replacements are made for $P(\sigma(x))$ and $P(y)$ and it expresses an induction principle.

(C) There is an element $x$ in $\bar{N}$ such that for each $y$ in $\bar{S}$ we have $x \cdot y = y$.

(D) For each $y$ in $\bar{N}$ there is an $x$ in $\bar{N}$ such that $x \cdot z = y \cdot z + z$ holds for all $z \in \bar{S}$. (4.1.4) (cont.)

(E) If $P(1)$ and for each $x$ in $\bar{N}$, $P(x)$ implies $P(\sigma(x))$ then $P(y)$ holds for each $y$ in $\bar{N}$.

Operations and a relation can be defined in $\bar{N}$ so that it forms the set of positive elements of an ordered integral domain.

(A) If $a$, $b$ and $c$ are any members of $\bar{N}$ we will write $a \# b = c$ if and only if $a \cdot x + b \cdot x = c \cdot x$ holds for all $x$ in $\bar{S}$.

(B) If $a$, $b$ and $c$ are any members of $\bar{N}$ we will write $a \times b = c$ if and only if $a \cdot (b \cdot x) = c \cdot x$ for each $x$ in $\bar{S}$. (4.1.5)

(C) If $a$ and $b$ are any members of $\bar{N}$ we will write $a \ll b$ if and only if $a \cdot x < b \cdot x$ for each $x$ in $\bar{S}$.

The first sequence of theorems establishes the elementary properties of the order relation $\ll$ in $\bar{N}$.

<u>Theorem 4.1.1</u>. If $a$, $b$ and $c$ are any members of $\bar{N}$ such that $a \ll b$ and $b \ll c$ then $a \ll c$.

Proof: Applying definition 4.1.5 C to the hypotheses we obtain

$$(x)[a \cdot x < b \cdot x]$$

$$(x)[b \cdot x < c \cdot x]$$

which yield $(x)[a \cdot x < c \cdot x]$ but this is equivalent to the desired conclusion by 4.1.5 C.

<u>Theorem 4.1.2</u>. If a and b are any members of $\bar{N}$ such that $a = b$ then $\sim[a \ll b]$.

Proof: Using 4.1.2 B and the definition of the operation, $\cdot$, we obtain

$$a = b \Rightarrow \sim(x)[a \cdot x < b \cdot x].$$

From 4.1.5 C it follows that

$$(x)[a \cdot x < b \cdot x] \Leftrightarrow [a \ll b].$$

<u>Theorem 4.1.3</u>. If a, b, c, d, e and f are any members of $\bar{N}$ then

$$a \# b = c \Rightarrow b \# a = c,$$

and $[a \# b = d \;\&\; d \# c = e \;\&\; b \# c = f] \Rightarrow a \# f = e.$

Proof: Commutativity follows from 4.1.5 A and 4.1.3 B;

$$a \# b = c \Leftrightarrow (x)[a \cdot x + b \cdot x = c \cdot x],$$

$$b \# a = c \Leftrightarrow (x)[b \cdot x + a \cdot x = c \cdot x],$$

$$(x)[a \cdot x + b \cdot x = c \cdot x] \Leftrightarrow (x)[b \cdot x + a \cdot x = c \cdot x].$$

For the proof of associativity we apply 4.1.5 A and 4.1.3 A to obtain

$$a \# b = d \Rightarrow (x)[a \cdot x + b \cdot x = d \cdot x],$$

$$d \# c = e \Rightarrow (x)[d \cdot x + c \cdot x = e \cdot x],$$

$$b \# c = f \Rightarrow (x)[b \cdot x + c \cdot x = f \cdot x],$$

and

$$(x)[a \cdot x + b \cdot x = d \cdot x \;\&\; d \cdot x + c \cdot x = e \cdot x \;\&\; b \cdot x + c \cdot x = f \cdot x] \Rightarrow$$

$$(x)[a \cdot x + f \cdot x = e \cdot x]$$

but $$(x)[a \cdot x + f \cdot x = e \cdot x] \Rightarrow a \# f = e.$$

The next theorem states a pair of properties which could have been used to define the operation #:

$$\sigma(a) = a \# 1,$$

$$\sigma(a \# b) = \sigma(a) \# b.$$

Theorem 4.1.4. If $a$ and $b$ are any members of $\bar{N}$ then $b = \sigma(a)$ if and only if $b = a \# 1$. Furthermore if $a$, $b$, $c$, $d$ and $e$ are any members of $\bar{N}$ such that $a \# b = c$ and $\sigma(c) = d$ and $\sigma(a) = e$ then $d = e \# b$.

Proof: From 4.1.4 B we obtain

$$b = \sigma(a) \Leftrightarrow (x)[b \cdot x = a \cdot x + x],$$

and from 4.1.5 A

$$a \# 1 = b \Leftrightarrow (x)[a \cdot x + 1 \cdot x = b \cdot x],$$

but from 4.1.4 A and 4.1.4 C

$$(x)\, 1 \cdot x = x$$

and so the first result is proved.

From the hypothesis, $a \# b = c$, and theorem 4.1.3 we obtain $b \# a = c$. From the hypothesis, $\sigma(c) = d$, and the first part of this theorem we obtain $c \# 1 = d$. Similarly, $a \# 1 = e$. These three results satisfy the hypothesis of theorem 4.1.3 so we can conclude that $b \# e = d$ and then by another application of theorem 4.1.3 that $d = e \# b$, thus completing the proof of the second part.

<u>Theorem 4.1.5</u>. If a is any member of $\bar{N}$ such that $a \neq 1$ then there is a member b of $\bar{N}$ such that $a = \sigma(b)$.

Proof: Let $(\exists y)[\sim[x = 1] \Rightarrow x = \sigma(y)]$ be denoted $P(x)$. Then $P(1)$ is true since $1 \neq 1$ is false. Now assume $P(k)$. We show that each of $k \neq 1$ and $k = 1$ imply $P(\sigma(k))$. If $k \neq 1$ then by the induction hypothesis $k = \sigma(y)$ and $\sigma(k) = \sigma(\sigma(y))$. On the other hand if $k = 1$ then $\sigma(k) = \sigma(1)$. Both of these arguments depend on the fact that $\sigma$ is a function. In either case $P(\sigma(k))$ follows so by 4.1.4 E the theorem is proved.

<u>Theorem 4.1.6</u>. If a, b and c are any members of $\bar{N}$ such that $a \# c = b$ then $a \ll b$.

Proof: If $a \# c = b$, by 4.1.5 A we have $(x)[a \cdot x + c \cdot x = b \cdot x]$. Applying 4.1.3 C to this we have $(x)[a \cdot x < b \cdot x]$ but by 4.1.5 C this implies $a \ll b$.

<u>Theorem 4.1.7</u>. If a and b are any members of $\bar{N}$ such that $a \neq b$ then there is c in $\bar{N}$ such that either $a \# c = b$ or $a = c \# b$.

Proof: If $a \neq 1$, by theorem 4.1.5 there is a c in $\bar{N}$ such that $a = \sigma(c)$ and, by theorem 4.1.4, $a = c \# 1$. Thus if $P(b)$ denotes the formula $(a)(c)[a \neq b \Rightarrow [a \# c = b \text{ or } a = c \# b]]$ then we have shown that $P(1)$ is true. Now suppose that $P(k)$ is true and that $x \neq \sigma(k)$ then either $x = k$ or $x \neq k$. If $x \neq k$ then by the induction hypothesis we have $x \# c = k$ or $x = c \# k$ for some c in $\bar{N}$.

From $x \# c = k$ we can conclude $(x \# c) \# 1 = k \# 1$; that is, $x \# \sigma(c) = \sigma(k)$. On the other hand if $x = c \# k$ we consider two cases: $c = 1$ and $c \neq 1$. In case $c = 1$ then $x = \sigma(k)$ which contradicts our hypothesis above. In case $c \neq 1$ then there is $d$ in $\bar{N}$ such that $c = \sigma(d) = d \# 1$ so $x = c \# k$ leads to $x = (d \# 1) \# k$ or $x = d \# \sigma(k)$. This completes the argument for the case $x \neq k$. Now if $x = k$ then, since $\sigma(k) = k \# 1$ we have $\sigma(k) = x \# 1$ and so in each case $P(\sigma(k))$ follows so by 4.1.4E the theorem is proved.

<u>Theorem 4.1.8</u>. If $a \neq b$ then $a \ll b$ or $b \ll a$.

Proof: This result is an immediate corollary of theorems 4.1.6 and 4.1.7.

We turn next to a sequence of four theorems which are the first four Peano Postulates.

<u>Theorem 4.1.9</u>. There is a member 1 in $\bar{N}$. For each a in $\bar{N}$ there is a b in $\bar{N}$ such that $b = \sigma(a)$.

Proof: This is an immediate consequence of 4.1.4A, B, C and D.

<u>Theorem 4.1.10</u>. For each a in $\bar{N}$, $\sigma(a) \neq 1$. For each a in $\bar{N}$, $\sigma(a) \gg a$.

Proof: Suppose $b = \sigma(a)$ then by theorem 4.1.4 $b = a \# 1$ and thus by theorem 4.1.6 $b \gg 1$. Taking the contrapositive of theorem 4.1.2 $b \gg 1 \Rightarrow b \neq 1$ hence combining these results $\sigma(a) \neq 1$.

If $b = \sigma(a)$ then by theorem 4.1.4 $b = a \# 1$ and thus by theorem 4.1.6 $b \gg a$ or $\sigma(a) \gg a$.

<u>Theorem 4.1.11</u>. If a and b are any members of $\bar{N}$ and x is any member of $\bar{S}$ then $a \cdot x = b \cdot x$ implies $a = b$.

Proof: If $b \neq 1$ then by theorem 4.1.5 there is c in $\bar{N}$ such that $b = \sigma(c)$. Now $\sigma(c) \cdot x = c \cdot x + x$ by 4.1.4 B and thus $b \cdot x = c \cdot x + x$. Thus if $a \cdot x = b \cdot x$ it follows that $a \cdot x = c \cdot x + x$ and thence by 4.1.4 B $a = \sigma(c) = b$. It only remains to be shown that the conclusion follows also if $b = 1$. In this case $b \cdot x = x$ by 4.1.4 A and if $a \cdot x = b \cdot x$ then $a \cdot x = x$ so by 4.1.4 A we conclude $a = 1 = b$.

<u>Theorem 4.1.12</u>. If a and b are any members of $\bar{N}$ then $a = b$ if and only if $\sigma(a) = \sigma(b)$.

Proof: If $a = b$ then $a \cdot z + z = b \cdot z + z$ and by 4.1.4 D there is c in $\bar{N}$ such that $c \cdot z = a \cdot z + z$ and d in $\bar{N}$ such that $d \cdot z = b \cdot z + z$. Combining these we have $c \cdot z = d \cdot z$ for all z in $\bar{S}$. Applying theorem 4.1.11 yields $c = d$ but $c \cdot z = a \cdot z + z$ implies $c = \sigma(a)$ and $d \cdot z = b \cdot z + z$ implies $d = \sigma(b)$ by 4.1.4 B so finally $\sigma(a) = \sigma(b)$. For the converse suppose $\sigma(a) = c$, $\sigma(b) = d$ and $c = d$. From $\sigma(a) = c$ we infer $a \cdot z = c \cdot z + z$ and from $\sigma(b) = d$ we infer $b \cdot z = d \cdot z + z$ by 4.1.4 B. From $c = d$ we infer $c \cdot z + z = d \cdot z + z$ and it follows by transitivity that then

$a \cdot z = b \cdot z$. Applying theorem 4.1.11 we obtain $a = b$.

The proof of the first four Peano Postulates is now complete. Our axiom 4.1.4 E is an induction postulate which appears to be weaker (J. Robinson, 1949) than the usual set theoretic formulation.

The elementary properties of the operation, $\times$, in $\bar{N}$ must be established next to justify the claim that $(\bar{N}, \#, \times)$ is a reasonable facsimile of the natural number system.

Theorem 4.1.13. If $a$ is any member of $\bar{N}$ and $x$ and $y$ are any members of $\bar{S}$ then $a \cdot x + a \cdot y = a \cdot (x + y)$.

Proof: Let $a \cdot (x + y) = a \cdot x + a \cdot y$ be denoted by $P(a)$. Clearly $1 \cdot (x + y) = x + y = 1 \cdot x + 1 \cdot y$ so $P(1)$ is true. Suppose that $k \cdot (x + y) = k \cdot x + k \cdot y$ then $\sigma(k) \cdot (x + y) = k \cdot (x + y) + (x + y) = (k \cdot x + x) + (k \cdot y + y) = \sigma(k) \cdot x + \sigma(k) \cdot y$. Thus we have shown that $P(k)$ implies $P(\sigma(k))$ and by 4.1.4 E the proof is complete.

The next theorem states a pair of properties which could have been used to define the operation $\times$.

Theorem 4.1.14. If $a$ and $b$ are any members of $\bar{N}$ then $a \times 1 = a = 1 \times a$ and $a \times \sigma(b) = (a \times b) \# a$.

Proof: By 4.1.4 A and 4.1.4 C $1 \cdot x = x$ for any $x$ in $\bar{S}$ and then $a \cdot (1 \cdot x) = a \cdot x$. By 4.1.5 B this implies $a \times 1 = a$. By 4.1.4 A and C $1 \cdot (a \cdot x) = a \cdot x$ and by 4.1.5 B this implies $1 \times a = a$.

By 4.1.5 B $a \times b = d$ if and only if $a \cdot (b \cdot x) = d \cdot x$ for all $x$ in $\bar{S}$ and by 4.1.5 A $d \# a = c$ if and only if $d \cdot y + a \cdot y = c \cdot y$ for all $y$ in $\bar{S}$. By 4.1.4B $\sigma(b) = e$ if and only if $b \cdot z + z = e \cdot z$ for all $z$ in $\bar{S}$ and by 4.1.5B $a \times e = c$ if and only if $a \cdot (e \cdot w) = c \cdot w$ for all $w$ in $\bar{S}$. Let us suppose that $\sigma(b) = e$, $a \times e = c$ and $a \times b = d$, then $a \cdot (b \cdot w + w) = c \cdot w$ for all $w$ in $\bar{S}$. By theorem 4.1.13 $a \cdot (b \cdot w + w) = a \cdot (b \cdot w) + a \cdot w$ then by combining $a \cdot (b \cdot w) = d \cdot w$ with the two preceding equations we have $d \cdot w + a \cdot w = c \cdot w$ and thus $d \# a = c$ or $a \times e = d \# a$ or finally $a \times \sigma(b) = (a \times b) \# a$.

Theorem 4.1.15. If $a$ and $b$ are any members of $\bar{N}$ then $a \times b = b \times a$.

Proof: If we let $P(b)$ denote $a \times b = b \times a$ then $P(1)$ follows from theorem 4.1.14 so let us suppose that $P(k)$ is true. Also let $\sigma(k) = c$, $c \times a = h$ and $k \times a = g$. Now by 4.1.4B $\sigma(k) = c$ if and only if $c \cdot y = k \cdot y + y$ for all $y$ in $\bar{S}$ and by 4.1.5B $c \times a = h$ if and only if $c \cdot (a \cdot x) = h \cdot x$ for all $x$ in $\bar{S}$ and also $k \times a = g$ if and only if $k \cdot (a \cdot z) = g \cdot z$ for all $z$ in $\bar{S}$. Finally by 4.1.5A $g \# a = h$ if and only if $g \cdot w + a \cdot w = h \cdot w$ for all $w$ in $\bar{S}$. We derive by substitution:

$$c \cdot y = k \cdot y + y$$

$$c \cdot (a \cdot x) = k \cdot (a \cdot x) + a \cdot x$$

$$h \cdot x = g \cdot x + a \cdot x$$

$$g \# a = h$$

$$(k \times a) \# a = c \times a$$

By induction hypothesis $k \times a = a \times k$ so we obtain

$$(a \times k) \# a = \sigma(k) \times a$$

but by theorem 4.1.14

$$a \times \sigma(k) = (a \times k) \# a$$

hence $$a \times \sigma(k) = \sigma(k) \times a.$$

Thus we have shown that $P(k) \Rightarrow P(\sigma(k))$ and by 4.1.4E the proof is complete.

<u>Theorem 4.1.16</u>. If a, b and c are any members of $\bar{N}$ then $a \times (b \times c) = (a \times b) \times c$.

Proof: Let us define d, e, f and g as follows and apply 4.1.5B to each of them.

$$b \times c = d \Leftrightarrow (x)[b \cdot (c \cdot x) = d \cdot x]$$

$$a \times d = e \Leftrightarrow (x)[a \cdot (d \cdot x) = e \cdot x]$$

$$a \times (b \times c) = e \Leftrightarrow (x)[a \cdot (b \cdot (c \cdot x)) = e \cdot x]$$

$$a \times b = f \Leftrightarrow (x)[a \cdot (b \cdot x) = f \cdot x]$$

$$f \times c = g \Leftrightarrow (x)[f \cdot (c \cdot x) = g \cdot x]$$

$$(a \times b) \times c = g \Leftrightarrow (x)[a \cdot (b \cdot (c \cdot x)) = g \cdot x]$$

From these definitions we easily infer that $(x)[e \cdot x = g \cdot x]$ and by theorem 4.1.11 that $e = g$ or $a \times (b \times c) = (a \times b) \times c$.

<u>Theorem 4.1.17</u>. If a, b and c are any members of $\bar{N}$ then $a \times (b \# c) = (a \times b) \# (a \times c)$.

Proof: Let us define u, v, w, y and z as follows and apply 4.1.5 A or 4.1.5 B to each of them.

$$b \# c = u \Leftrightarrow (x)[b \cdot x + c \cdot x = u \cdot x]$$

$$a \times u = z \Leftrightarrow (x)[a \cdot (u \cdot x) = z \cdot x]$$

$$a \times b = v \Leftrightarrow (x)[a \cdot (b \cdot x) = v \cdot x]$$

$$a \times c = w \Leftrightarrow (x)[a \cdot (c \cdot x) = w \cdot x]$$

$$v \# w = y \Leftrightarrow (x)[v \cdot x + w \cdot x = y \cdot x]$$

Then $z \cdot x = a \cdot (u \cdot x) = a \cdot (b \cdot x + c \cdot x)$ by substitution and applying theorem 4.1.13 and further substitutions we obtain

$$a \cdot (b \cdot x + c \cdot x) = a \cdot (b \cdot x) + a \cdot (c \cdot x) = v \cdot x + w \cdot x = y \cdot x.$$

Applying transitivity and theorem 4.1.11 we have $(x)[z \cdot x = y \cdot x] \Rightarrow z = y$. Thus $a \times (b \# c) = a \times u = z = y = v \# w = (a \times b) \# (a \times c)$.

<u>Theorem 4.1.18</u>. If a, b and c are any members of $\bar{N}$ then $[a \times c = b \times c] \Rightarrow a = b$.

Proof: Let us define u and v and apply 4.1.5 B as follows:

$$a \times c = u \Leftrightarrow (x)[a \cdot (c \cdot x) = u \cdot x],$$

$$b \times c = v \Leftrightarrow (x)[b \cdot (c \cdot x) = v \cdot x].$$

Then $u = v$ implies $u \cdot x = v \cdot x$ for all x and thus $(x)[a \cdot (c \cdot x) = b \cdot (c \cdot x)]$. But for each x, $a \cdot (c \cdot x) = b \cdot (c \cdot x)$ implies $a = b$ by theorem 4.1.11.

Stoll (1963, p. 136) asserts that the fourteen properties he lists characterize the integers to within an order-isomorphism.

At this point we have verified all of those properties which apply to the positive integers, except for well-ordering. This last property cannot be stated in full generality without using set theoretic concepts which cannot be formulated in the first order predicate calculus. However the proof of the next theorem uses a similar technique.

Theorem 4.1.19. For each $u$ and $v$ in $\bar{S}$ there is $k$ in $\bar{N}$ such that $v < k \cdot u$ and $[v < m \cdot u \Rightarrow k \ll m \vee k = m]$.

Proof: Let $P(n)$ denote $[v < n \cdot u]$ and $Q(n)$ denote $[v < m \cdot u \Rightarrow n \ll m \vee n = m]$.

(1). $Q(1)$ is true. This follows from the fact that $(j)[1 = j$ or $1 \ll j]$. We prove this as follows. If $j \neq 1$ then by theorem 4.1.5 there is $i$ in $\bar{N}$ such that $\sigma(i) = j$. Then by 4.1.4 B $\sigma(i) \cdot x = j \cdot x + x$, by 4.1.3 C $j \cdot x + x > x$ and by 4.1.4 A $x = 1 \cdot x$ hence $\sigma(i) \cdot x > 1 \cdot x$. Then using 4.1.5 C we deduce $1 \ll \sigma(i)$ but $j = \sigma(i)$. Thus $1 \neq j \Rightarrow 1 \ll j$.

(2). $(\exists k) Q(k) \wedge \sim Q(\sigma(k))$ is true, for if not then $(k) Q(k) \Rightarrow Q(\sigma(k))$ but this and the preceding result would yield $(k) Q(k)$. Now by 4.1.3 D $(\exists j)[v < j \cdot u]$ and $Q(\sigma(j))$ *i.e.* $[v < j \cdot u] \Rightarrow [\sigma(j) \ll j \vee \sigma(j) = j]$, could be combined to yield $[\sigma(j) \ll j \vee \sigma(j) = j]$, a contradiction of theorem 4.1.10 since $\bar{N}$ is fully ordered.

(3). $P(k)$ is true, where $k$ is the element whose existence was shown in (2), for if not then $\sim P(k) \Rightarrow (j)[P(j) \Rightarrow k \ll j]$ and thus $(j)[P(j) \Rightarrow \sigma(k) = j \vee \sigma(k) \ll j]$ since $k \ll j \Rightarrow k \# p = j$ and $1 \ll p$ or $1 = p$; so if $p = 1$, $\sigma(k) = j$, but if $1 \ll p$ then $p = \sigma(r) = r \# 1$; so $k \# p = (k \# r) \# 1 = (k \# 1) \# r$ and $\sigma(k) \ll j$. Now $[P(j) \Rightarrow \sigma(k) = j \vee \sigma(k) \ll j]$ is just $Q(\sigma(k))$ which contradicts (2).

Corollary 4.1.20. $(m)(\exists n) u < m \cdot v \Rightarrow n \cdot u \leq m \cdot v < \sigma(n) \cdot u$.

This corollary which is an easy consequence of the preceding theorem and the fact that $\bar{S}$ is fully ordered makes possible, as we shall see, the assertion that any degree of the quantity can be approximated arbitrarily closely by a rational multiple of a degree $u$ chosen as a unit.

The following theorems lay a foundation for the construction of a set, $Q^+$, of formal quotients of members of $\bar{N}$.

Theorem 4.1.21. If $x$, $y$ and $z$ are any members of $\bar{S}$ then $x < y \wedge y = z \Rightarrow x < z$.

Proof: A simple indirect argument establishes this result.

Theorem 4.1.22. If $x$, $y$ and $z$ are any members of $\bar{S}$ then $x < y \Rightarrow x + z < y + z$.

Proof: If $x < y$ then $x + w = y$ for some $w$ in $\bar{S}$ by 4.1.3C. From the same postulate it follows that $x + z < (x + z) + w$. Using 4.1.3A and B we can show that $(x + z) + w = w + (x + z) = (w + x) + z = (x + w) + z = y + z$. Hence if $x < y$, $x + z < y + z$.

<u>Theorem 4.1.23</u>. If $x$, $y$ and $z$ are any members of $\bar{S}$ then

$x + z = y + z \Rightarrow x = y$.

Proof: If $x \neq y$ then $x < y$ or $y < x$ but by theorem 4.1.22 this would imply either $x + z < y + z$ or $y + z < x + z$.

<u>Theorem 4.1.24</u>. $\bar{S}$ is uniquely divisible.

Proof: If $n \cdot x = n \cdot y$ and $x \neq y$ then $x < y$ or $y < x$. Hence the theorem will follow if we can show $x < y$ implies $n \cdot x < n \cdot y$. So we let $P(n)$ denote $x < y \Rightarrow n \cdot x < n \cdot y$. Clearly $P(1)$ is true, so let us suppose that $P(k)$ is true. Now by the induction hypothesis $k \cdot x < k \cdot y$ and by theorem 4.1.22 $k \cdot x + x < k \cdot x + y$ and $k \cdot x + y < k \cdot y + y$ so by 4.1.2 A we have $k \cdot x + x < k \cdot y + y$ or $\sigma(k) \cdot x < \sigma(k) \cdot y$, by 4.1.4 B. Thus $P(k) \Rightarrow P(\sigma(k))$ and by 4.1.4 E the proof is complete.

We have shown that $\bar{S}$ is a fully ordered and uniquely divisible commutative cancellative semi-group and it is an immediate consequence of 4.1.3 C and 4.1.2 B that none of its elements is idempotent. Such a semi-group will be called a semi-group of <u>type M</u>. Now we can apply the result of Whitney (1968, pp. 122-124) and obtain $Q^+$ by a similar construction. Of course $Q^+$ consists of ordered pairs of members of $\bar{N}$ (denoted $m/n$ where $m \in \bar{N}$ and $n \in \bar{N}$).

It will be convenient to have a statement of the circumstances under which subtraction is possible in $\bar{S}$ and in $Q^+$.

Definition 4.1.25. If $x$, $y$ and $u$ are any members of $\bar{S}$ then,

$$y - x = u \text{ if and only if } x + u = y.$$

Corollary 4.1.26. If $x$ and $y$ are any members of $\bar{S}$ such that $x < y$ then,
$$(y - x) + x = y.$$

Theorem 4.1.27. If $x$, $y$ and $z$ are any members of $\bar{S}$ such that $z < y$ then,
$$(x + y) - z = x + (y - z).$$

Proof: Using the associative law and the corollary above we see that

$$(x + (y - z)) + z = x + ((y - z) + z) = x + y = ((x + y) - z) + z.$$

Applying the cancellation law we obtain the desired result.

Since $Q^+$ is easily seen to be, with respect to addition, a semi-group of type M, a similar definition and theorem hold in $Q^+$ also.

Theorem 4.1.28. If $p$ and $q$ are any members of $Q^+$ and $\alpha$ is any member of $\bar{S}$ such that $p > q$ then $p \cdot \alpha - q \cdot \alpha = (p - q) \cdot \alpha$.

Proof: First note that $p \cdot \alpha = ((p - q) + q) \cdot \alpha$ and $((p - q) + q) \cdot \alpha = (p - q) \cdot \alpha + q \cdot \alpha$ and thus applying the definition of subtraction we have the desired result.

As is usually the case for a theory which has no finite models we must be satisfied with a relative consistency theorem (Stoll, 1963, p. 237). The fundamental theorem of Holder (1901, p. 39) states that every Archimedean, naturally fully ordered cancellative semi-group is order isomorphic to a sub-semi-group of the additive semi-group of all non-negative real numbers. This representation theorem thus establishes the existence of models for the elementary theory of extensive quantities. Since the consistency of the real number system is still an open question we can only assert that if this question is answered in the affirmative then our theory also will be consistent. This hypothesis will underlie the rest of our investigation of the axioms for extensive quantities.

In order to establish the completeness of the elementary theory of extensive quantities we shall apply some theorems of Abraham Robinson which we will state here for the convenience of the reader. If S is a sub-semi-group of S' then S' will be called an _extension_ of S. The set $\Sigma$ referred to in the theorems below is a subset of $\wp(T, I)$.

_Theorem 4.1.29_. In order that a non-empty consistent set of statements $\Sigma$ be model-complete, it is necessary and sufficient that for every pair of models of $\Sigma$, S and S' such that S' is an extension of S, any primitive statement Y which is defined in S can hold in S' only if it holds in S.

A structure $S_0$ is said to be a _prime_ model of $\Sigma$ if $S_0$ is a $\Sigma$-model and if every model S' of $\Sigma$ contains a sub-$\Sigma$-model S such that S is isomorphic to $S_0$.

Theorem 4.1.30. Let $\Sigma$ be a model-complete set of statements which possesses a prime model $S_o$ then $\Sigma$ is complete.

A completely divisible ordered abelian group will be called a group of type DO.

Theorem 4.1.31. The elementary theory of a group of type DO which contains at least two different elements is model-complete.

Theorem 4.1.32. The elementary theory of a group of type DO which contains at least two different elements is complete.

The additive group of rational numbers provides a prime model and theorem 4.1.32 follows from theorems 4.1.30 and 4.1.31. The formulation of the elementary theory of groups of type DO is relevant to this discussion since it will be shown that any semi-group of type M can be embedded in a group of type DO so that every sentence in the theory of type M semi-groups is a sentence in the theory of type DO groups. Hence if the theory of semi-groups of type M failed to be negation complete, the same would be true for groups of type DO, contradicting Robinson's theorem, 4.1.32.

The following definitions will simplify the comparison between the theory of type M semi-groups developed here and the theory of type DO groups presented in Robinson (1956, p. 36):

$$\text{(A)}\quad E(x,y) \Leftrightarrow x = y \qquad (4.1.6)$$

(B) $S(x, y, z) \Leftrightarrow x + y = z$

(C) $Q(x, y) \Leftrightarrow (x < y) \vee (x = y)$ (4.1.6) (cont.)

(D) $S_n(x, y) \Leftrightarrow n \cdot x = y$

Using this notation the theory of type DO groups has been formulated in the first order predicate calculus by Robinson as follows. The first set of postulates, which was a part of the language in the formulation of type M semi-group theory, deals with equality and substitution.

(A) $(x) E(x, x)$

(B) $(x)(y)[E(x, y) \Rightarrow E(y, x)]$

(C) $(x)(y)(z)[E(x, y) \wedge E(y, z) \Rightarrow E(x, z)]$ (4.1.7)

(D) $(u)(v)(w)(x)(y)(z)[S(u, v, w) \wedge E(u, x) \wedge E(v, y) \wedge E(w, z) \Rightarrow S(x, y, z)]$

(E) $(x)(y)(z)(w)[Q(x, y) \wedge E(x, z) \wedge E(y, w) \Rightarrow Q(z, w)]$

The second set of postulates deals with the order relation.

(A) $(x)(y)(z)[Q(x, y) \wedge Q(y, z) \Rightarrow Q(x, z)]$

(B) $(x)(y)[Q(x, y) \vee Q(y, x)]$ (4.1.8)

(C) $(x)(y)[Q(x, y) \wedge Q(y, x) \Leftrightarrow E(x, y)]$

The third set of postulates describe an abelian group.

(A) $(x)(y)(\exists z)S(x, y, z)$

(B) $(x)(y)(z)(w)[S(x, y, z) \wedge S(x, y, w) \Rightarrow E(z, w)]$ (4.1.9)

(C) $(u)(v)(w)(x)(y)(z)[S(u, v, w) \wedge S(w, x, y) \wedge S(v, x, z) \Rightarrow S(u, z, y)]$

(D) $(x)(y)(z)[S(x, y, z) \Rightarrow S(y, x, z)]$ (4.1.9) (cont.)

(E) $(x)(y)(\exists z) S(x, z, y)$

The last set is a miscellaneous collection of postulates.

(A) $(\exists x)(\exists y)[\sim E(x, y)]$

(B) $(x)(\exists y) S_n(y, x)$ (4.1.10)

(C) $(x)(y)(z)(v)(w)[S(x, y, z) \wedge S(x, v, w) \wedge Q(y, v) \Rightarrow Q(z, w)]$

The major difference between the theory described in 4.1.1 to 4.1.5 and that described in 4.1.7 to 4.1.10 is that in the former a set, $\bar{N}$, is contained in the carrier and thus only one operation, $\cdot$, is needed whereas in the latter a countably infinite set of relations, $S_n$, is used. The former approach was necessary so that the Archimedean property, 4.1.3D, could be stated. In the latter formulation it would be $(x)(y)(w)(\exists S_n) S_n(x, w) \wedge Q(y, w) \wedge \sim E(y, w)$ which is not a sentence of the first order predicate calculus since a relation symbol is quantified. By means of 4.1.6 either theory could be developed in terms of either set of predicates leaving out those developments which depend on the Archimedean property.

The order isomorphism, whose existence is asserted by Holder's theorem, will be denoted by f and for any semigroup of type M its image, $f(\bar{S})$, in R together with O and the set of negatives, $-f(\bar{S})$, constitute, as we shall see, a group, G, of type DO. One can then easily construct another group, order isomorphic to G, which actually contains $\bar{S}$ as a sub-semi-group, if desired. The logical postulates for equality were included in the first order predicate calculus with equality so equations 4.1.7 all hold for G. Using 4.1.2A and theorem 4.1.21 we can easily prove 4.1.8A. From 4.1.2B and C and 4.1.6C we easily obtain 4.1.8B and C. From the fact that G is a subset of the real number system it readily follows that equations 4.1.9 are all satisfied. Postulate 4.1.10C is an immediate consequence of theorem 4.1.22. It follows from 4.1.1A, 4.1.2B and 4.1.3C that $\bar{S}$ and its image, $f(\bar{S})$, are countably infinite and thus contain at least two distinct members. The operation can be extended by the definition

$$n\cdot(-x) = -(n\cdot x) \tag{4.1.11}$$

so that G is divisible. Thus we have shown that a semi-group of type M can be embedded in a group of type DO.

We have already seen in theorem 3.3.4 that the set of all scale transformations forms a group under a very general definition of scale and scale transformation. In the special case of a scale for the measurement of extensive quantities we can show that this group is the group of similarity transformations. (For the definition of similarity transformation see Suppes and Zinnes, 1963, p. 11.)

The fundamental representation theorem of Holder quoted above can be proved by constructing a function $g$ as follows. Let $u$ be an arbitrary element of A, the carrier of an empirical $\Sigma$-model, where $\Sigma$ is the set of axioms, presented in equations 4.1.1 through 4.1.5, for an extensive quantity. For any other element $x$ of A we define the set $B(x) = \{m/n : n \cdot x \leq m \cdot u \wedge n \in \bar{N} \wedge m \in \bar{N}\}$ and $g(x) =$ g.l.b. $(B(x))$. We think of $\bar{N}$ and $Q^+$ as being embedded in the reals and so $g(x)$ is a real valued function. Holder's theorem is then proved by showing that $g$ is an (order) isomorphism. If $h$ is any other (order) isomorphism then we claim that $h(x) = h(u)g(x)$. Suppose, by way of contradiction, that $h(x) < h(u)g(x)$ then there is a rational $p/q$ such that

$$h(x)/h(u) < p/q < g(x)$$

but from the definition of $g$ it follows then that $p \cdot u \leq q \cdot x$. Now since $h$ is an (order) isomorphism we can infer

$$p \cdot h(u) \leq q \cdot h(x)$$

or
$$p/q \leq h(x)/h(u).$$

Similarly by assuming $h(x) > h(u)g(x)$ we obtain a contradiction. This result is sometimes called a <u>uniqueness theorem</u> (Suppes and Zinnes, 1963, p. 43) and shows that any two scales for an extensive quantity differ by a multiplicative factor. If two objects, $x$ and $y$, are measured on two scales we have,

$$\frac{h(x)}{h(y)} = \frac{h(u)g(x)}{h(u)g(y)} = \frac{g(x)}{g(y)}$$

and thus ratios are preserved under a change of scale. Thus it has been shown that extensive quantities determine ratio scales.

## 4.2. Alternative Interpretations of the Theory

It was shown in section 4.1 that a theory could be formulated in the first order predicate calculus, and interpreted in the set of physical objects so that a description of the process of measuring extensive quantities was the result. The notion of interpretation was used informally without discussion. At this point it will be profitable to use this notion explicitly. Any T-structure, $\mathfrak{a}$, provides the basis for a correspondence called an interpretation between the symbols used in $\mathcal{P}(T, I)$ and the subsets, relations and operations of $\mathfrak{a}$. The same symbol has been used to denote itself in formulae of $\mathcal{P}(T, I)$ and to denote the corresponding set, relation or operation in $\mathfrak{a}$, and this practice will be continued. A more detailed discussion of interpretation may be found in Stoll (1963, pp. 399-401).

In discussing the theory of measurement two models have been used: an empirical $\Sigma$-model and a full numerical $\Sigma$-model for the theory $\Sigma$ of a quantity. The interpretation may be changed for either model producing changes in the group of scale transformations. Both cases will be illustrated.

For the sake of concreteness let us consider a method which could be used to measure temperature. A pair of Seebeck circuits (Weld, 1948, p. 323) which could be connected to a galvanometer as illustrated in Figure 2 could be used to determine two binary relations in the set, $O \times O$, of ordered pairs of objects. If the galvanometer shows no deflection then $(a,b)E(c,d)$ would denote the fact that the temperature difference between $a$ and $b$ was equivalent to that between $c$ and $d$. If the galvanometer showed a deflection in one direction then $(a,b)L(c,d)$ would denote the fact that the temperature difference between $a$ and $b$ was less than that between $c$ and $d$. A pair of Seebeck circuits which could be connected to a galvanometer as illustrated in Figure 3 could be used to verify empirically that $((a,b)P(b,c))E(a,c)$ thus defining a binary operation in $O \times O$, denoted by the symbol $P$. Of course the physical apparatus suggested would be usable for only a limited range of temperature. The mathematical properties which we would like to preserve in an ideal instrument will be abstracted as we describe the empirical structure and we shall see that 4.2.1, 4.2.2 and 4.2.4 together with a few theorems derivable from them suffice to show that this abstraction is a $\Sigma$-model for the theory of section 4.1.

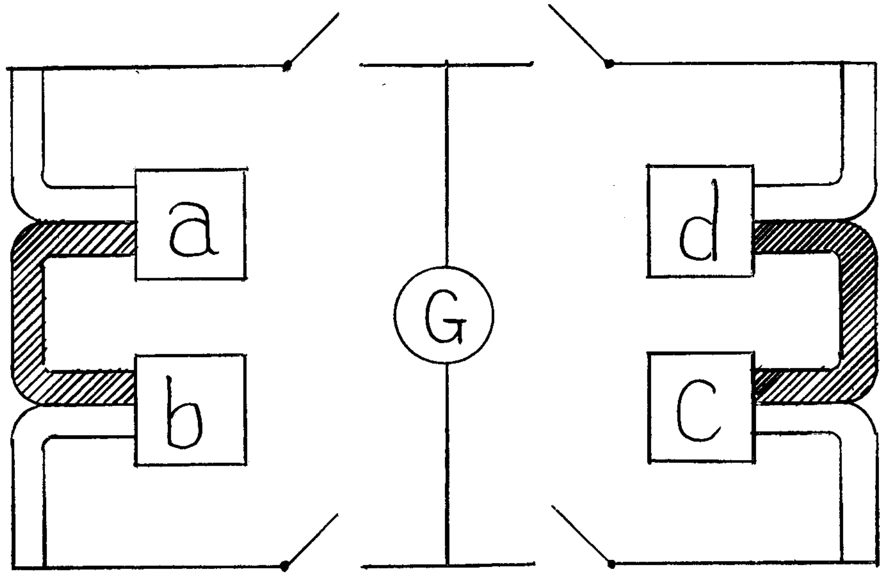


Figure 2. Definition of L and E for temperature

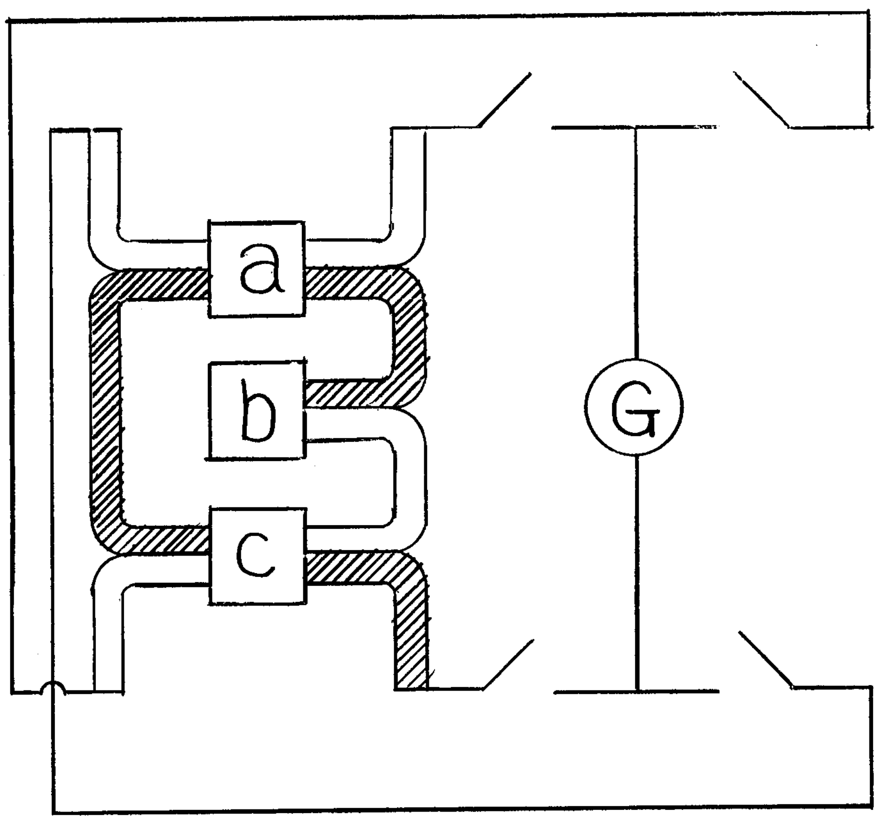


Figure 3. Definition of P for temperature

(A) $(a)(b)(c)(\exists d) \quad (a, b)E(c, d)$

(B) $(a)(b)(c)(d) \quad (a, b)E(c, d) \wedge (a=c) \Rightarrow (b=d)$ (4.2.1)

(C) $(a)(b)(c)(d) \quad (a, b)E(c, d) \Rightarrow (a, c)E(b, d)$

<u>Theorem 4.2.1</u>. $(a)(b) \quad (a, a)E(b, b)$

Proof: From 4.2.1A we have $(\exists c)\ (a, a)E(b, c)$ and from 4.2.1C $(a, a)E(b, c) \Rightarrow (a, b)E(a, c)$ but from 4.2.1B $(a, b)E(a, c) \Rightarrow (b=c)$, hence substituting in the first relation completes the proof.

<u>Theorem 4.2.2</u>. $(a)(b) \quad (a, b)E(a, b)$

Proof: Applying 4.2.1C to the result of the preceding theorem we have $(a, a)E(b, b) \Rightarrow (a, b)E(a, b)$.

(A) $(a)(b)(c)(d) \quad (a, b)E(c, d) \Rightarrow (c, d)E(a, b)$

(B) $(a)(b)(c)(d)(e)(f) \quad ((a, b)E(c, d)) \wedge$
$((c, d)E(e, f)) \Rightarrow (a, b)E(e, f)$

(C) $(a)(b)(c) \quad ((a, b)P(b, c))E(a, c)$ (4.2.2)

(D) $(a)(b)(c)(d)(e)(f) \quad ((a, b)E(c, d)) \Rightarrow$
$(((a, b)P(e, f))E((c, d)P(e, f)))$

(E) $(a)(b)(c)(d) \quad ((a, b)P(c, d))E((c, d)P(a, b))$

<u>Theorem 4.2.3</u>. $(a)(b)(c)(d)(\exists e) \quad ((a, b)P(c, d))E(a, e)$

Proof: By 4.2.1A $(\exists e)\ (c, d)E(b, e)$ and by 4.2.2D $((c, d)P(a, b))E((b,e)P(a, b))$. Applying 4.2.2A and B to this result we obtain $((a, b)P(c, d))E((a, b)P(b, e))$. Now by 4.2.2C $((a, b)P(b, e))E(a, e)$ and combining these by 4.2.2B we have the desired result.

Theorem 4.2.4. $(a)(b)(c)(d)(e)(f)(g)(h)\ ((a, b)\, E(c, d)) \wedge ((e, f)\, E(g, h)) \Rightarrow (((a, b)\, P(e, f))\, E((c, d)\, P(g, h)))$.

Proof: From 4.2.2 D we have $((a, b)\, P(e, f))\, E((c, d)\, P(e, f))$ and from 4.2.2 D and E, $((c, d)\, P(e, f))\, E((c, d)\, P(g, h))$. Combining these by 4.2.2 B we get $((a, b)\, P(e, f))\, E((c, d)\, P(g, h))$.

Theorem 4.2.5. $(a)(b)(c)(d)(e)(f)$

$((a, b)\, P((c, d)\, P(e, f)))\, E(((a, b)\, P(c, d))\, P(e, f))$.

Proof: Let $x$ and $y$ be chosen, by 4.2.1 A, so that $(c, d)\, E(b, x)$ and $(e, f)\, E(x, y)$. Then by theorem 4.2.4 we have

$$((c, d)\, P(e, f))\, E((b, x)\, P(x, y)),$$

and by 4.2.2 C, $\quad (b, x)\, P(x, y)\, E(b, y)$

so by 4.2.2 B we obtain

$$((c, d)\, P(e, f))\, E(b, y).$$

Then $\quad ((a, b)\, P((c, d)\, P(e, f)))\, E((a, b)\, P(b, y))$

and $\quad ((a, b)\, P(b, y))\, E(a, y)$

so we have reduced the left hand side to $(a, y)$. Similarly we obtain $\quad ((a, b)\, P(c, d))\, E((a, b)\, P(b, x))$

$$((a, b)\, P(b, x))\, E(a, x)$$

$$((a, x)\, P(e, f))\, E((a, x)\, P(x, y))$$

$$((a, x)\, P(x, y))\, E(a, y)$$

and the right hand side has also been reduced to $(a, y)$.

The properties 4.2.4 E and F make use of an operation, $\cdot$, defined by

$$1\cdot(a,b)E(a,b)$$
$$(n+1)\cdot(a,b)E(n\cdot(a,b)P(a,b)) \qquad (4.2.3)$$

for every $n \in \bar{N}$, a set which satisfies 4.1.1 to 4.1.5 and for any members, a and b, of O.

(A) $(a)(b)(c)(d)(e)(f)\ ((a,b)L(c,d)) \wedge ((c,d)L(e,f)) \Rightarrow ((a,b)L(e,f))$

(B) $(a)(b)(c)(d)\ ((a,b)E(c,d)) \Rightarrow \sim((a,b)L(c,d))$

(C) $(a)(b)(c)(d) \sim((a,b)E(c,d)) \Rightarrow ((a,b)L(c,d) \vee (c,d)L(a,b))$ (4.2.4)

(D) $(a)(b)(c)(d)(\exists e)\ ((a,b)L(c,d)) \Leftrightarrow ((a,b)P(b,e))E(c,d) \wedge ((a,a)L(b,e))$

(E) $(a)(b)(c)(d)(\exists n)\ ((a,a)L(c,d)) \Rightarrow (a,b)Ln\cdot(c,d)$

(F) $(a)(b)(c)(d)(\exists n) \sim(c=d) \Rightarrow (a,b)E(n\cdot(c,d))$

Theorem 4.2.6. $(a)(b)(c)(d)(e)(f)\ (a,b)E(c,d) \wedge (c,d)L(e,f) \Rightarrow (a,b)L(e,f)$

Proof: A simple indirect argument using the contrapositive of 4.2.4 B establishes this result.

The operation P and relations E and L defined in $O \times O$ induce an operation, +, and order relation, <, in a set, $\bar{S}$, of equivalence classes of the relation E as follows.

(A) $(\overline{a,b}) = \{(c,d):(a,b)E(c,d)\}$

(B) $\bar{S} = \{(\overline{a,b}): a \in O \wedge b \in O \wedge (a,a)L(a,b)\}$ (4.2.5)

(C) $(\overline{a, b}) + (\overline{c, d}) = (\overline{e, f}) \Leftrightarrow ((a, b)P(c, d))E(e, f)$

(D) $(\overline{a, b}) < (\overline{c, d}) \Leftrightarrow (a, b)L(c, d)$ (4.2.5) (cont.)

It is easy to see that $\bar{S}$, + and < with $\bar{N}$ satisfy axioms 4.1.1 to 4.1.5 and thus by the results of section 4.1 we have an order isomorphism g mapping $\bar{S}$ into a semi-group of type M embedded in the positive reals. From this we will construct a mapping from O into the reals which is an interval scale for temperature.

<u>Theorem 4.2.7</u>. $(a)(b) \sim (a=b) \Rightarrow$

$(((a, a)L(a, b)) \vee ((a, a)L(b, a)))$

Proof: By 4.2.1B $(a, a)E(a, b) \Rightarrow (a=b)$

contradicting the hypothesis so by 4.2.4C

$$((a, a)L(a, b)) \vee ((a, a)L(b, a)).$$

By theorem 4.2.7 we can define $h(a, b) = g(\overline{a, b})$ if $(a, a)L(a, b)$ and $h(a, b) = -g(\overline{a, b})$ if $(a, b)L(a, a)$ and $h(a, a) = 0$, thus defining a function on $O \times O$ into the real number system. Let x be an arbitrary member of O and define $f(a) = h(a, x)$. Clearly there is a natural induced order in O, and any other scale for temperature, say f', will be a monotone function with respect to this order and the usual order in the reals. Moreover $h' : O \times O \to R$, defined by $h'(a, b) = f'(a) - f'(b)$, is an order isomorphism and hence by the uniqueness theorem for ratio scales of section 4.1 there is an ordered pair (u, v) such that

$$h'(a, b) = h'(u, v)h(a, b)$$

for all a,b in O and thus

$$f'(a) - f'(x) = h'(u, v)h(a, x)$$

or $$f'(a) = h'(u, v)f(a) + d'(x), \qquad (4.2.6)$$

and so we have shown that any temperature scale may be transformed into any other by an affine transformation (called a linear transformation by some authors). It follows readily from 4.2.6 that

$$\frac{f'(a) - f'(b)}{f'(c) - f'(d)} = \frac{f(a) - f(b)}{f(c) - f(d)}$$

and thus we have shown that ratios of intervals in this development are preserved under scale transformation and this is the characteristic property of interval scales (Suppes and Zinnes, 1963, p. 9).

It should be clear that other quantities may be measured by a similar procedure. Familiar examples are date, position and utility. The prerequisite is a procedure for determining equality or inequality of ordered pairs of objects and an operation combining pairs of pairs. By interpreting members of $\bar{S}$ as equivalence classes of ordered pairs of objects, and the order relation, <, and operation, +, as illustrated above we can see that another empirical model of the theory of section 4.1 has been described. Moreover changing the empirical model in this way has made it possible to define a useful scale which however possesses a different transformation group.

The second type of reinterpretation, *i.e.*, using a different numerical $\Sigma$-model, is quite simple to illustrate. As Suppes and Zinnes point out (1963, p. 44), one way to do this is to define $R^* = \{y : y = \exp(x) \wedge x \in R\}$ and use $R^* \cup Z$ as the carrier and multiplication as the operation for $\mathfrak{n}^*$, the full numerical $\Sigma$-model. When we used $\mathfrak{n}$ the scale transformation group was that of the similarity transformations but when $y = \exp(x)$ if $x$ is transformed to $kx$ then $y$ will transform to $y^k$ so the group consists in this case of the power transformations.

### 4.3. Multiple Interpretations

In this section we call attention to a theory which includes that of extensive measurement as a special case and illustrates the use of several models each of which determines a scale. Our use of the set O of objects was quite restricted and might have been extended in two directions. First, some properties of an object or system change with time under various external influences and a more sophisticated abstraction would be the ensemble of states of systems. Second, an object or system has more than one property so that in the theory at least we could conceive of several scales, determined simultaneously. These might be mappings into different numerical $\Sigma$-models of the same full numerical $\Sigma$-model. The primitive concepts of the theory of thermodynamics proposed by

Giles (1964, p. 25) are state, addition of states, and natural process with initial and final states. Thus the structure under consideration consists of a set $\bar{S}$ of states, a binary operation + and a binary relation $\rightarrow$.

Leaving out those axioms that deal with $\bar{N}$ we may state a set of axioms, which are an improvement on those of Giles, given by Roberts, F.S. and Luce, R.D. (1968, pp. 315-316), as an indication of the nature of this theory. For convenience we define $a \sim b \Leftrightarrow (a \rightarrow b \wedge b \rightarrow a)$.

$$\begin{aligned}&\text{(A)}\quad (a)(b)\quad a+b \sim b+a\\ &\text{(B)}\quad (a)(b)(c)\quad a+(b+c) \sim (a+b)+c\end{aligned} \tag{4.3.1}$$

Thus $\bar{S}$ is an abelian semi-group and the next axioms concern $\rightarrow$ which Roberts and Luce call a conditionally connected order.

$$\begin{aligned}&\text{(A)}\quad (a)\quad a \rightarrow a\\ &\text{(B)}\quad (a)(b)(c)\quad (a \rightarrow b \wedge b \rightarrow c) \Rightarrow (a \rightarrow c)\\ &\text{(C)}\quad (a)(b)(c)\quad (a \rightarrow b \wedge a \rightarrow c) \Rightarrow (b \rightarrow c \vee c \rightarrow b)\\ &\text{(D)}\quad (a)(b)(c)\quad (a \rightarrow b) \Leftrightarrow (a+c \rightarrow b+c)\end{aligned} \tag{4.3.2}$$

In 4.3.1 and 4.3.2 it has been assumed that the elements a, b and c belonged to $\bar{S}$ but in the remaining axioms we must be more explicit.

$$\begin{aligned}&\text{(A)}\quad (a)(b)(c)\quad S(a) \wedge S(b) \wedge N(c)\\ &\qquad \wedge\, c \cdot a \rightarrow c \cdot b \Rightarrow a \rightarrow b\end{aligned} \tag{4.3.3}$$

(B) $(a)(b)(c)(d)(\Sigma e)\ S(a) \wedge S(b) \wedge S(c) \wedge S(d) \wedge N(e) \wedge (a \rightarrow b) \wedge \sim(b \rightarrow a) \wedge ((c \rightarrow d) \vee (d \rightarrow c)) \Rightarrow e \cdot a + c \rightarrow e \cdot b + d$ (4.3.3) (cont.)

Now 4.3.3 B is a generalization of the Archimedean property (Roberts and Luce, 1968, p. 316). Some motivation for this theory will be provided by the following quotation from Roberts and Luce (1968, p. 312, 313).

> In classical thermodynamic theory, there are assigned to each state of a system certain parameters such as volume, internal energy, and the number of molecules of each of several chemically pure components; the values of these parameters are preserved under state transition. Giles calls these parameters components of content. The motivation is, in a given thermodynamic situation, to find sufficiently many of these components of content so that state transition between states a and b [$a \rightarrow b$] can occur if and only if a and b have the same value on all components of content. Then, one more parameter is needed to describe whether the transition a to b or the transition b to a is the naturally occurring one. This is the entropy [$\Phi$], which by convention never decreases. Thus, the idea is that a system can pass from one state to another if and only if the components of content are all conserved and the entropy does not decrease....The main feature of the conditional connectedness property is clear...it partitions [$\bar{S}$] into subclasses...of states having identical values on all components of content, and each subclass is weakly ordered by entropy.

The main result of Roberts and Luce states that if $(\bar{S}, +, \rightarrow)$ satisfies the axioms stated above then there exists a real valued function $\Phi$ on $\bar{S}$

(A) $(a)(b)\ S(a) \wedge S(b) \Rightarrow \Phi(a+b) = \Phi(a) + \Phi(b)$ (4.3.4)

(B) $(a)(b)\quad S(a) \wedge S(b) \Rightarrow ((a \rightarrow b) \Leftrightarrow \Phi(a) \leq \Phi(b))$
and for every component of content Q, (4.3.4)
$Q(a) = Q(b)$ (cont.)

Conversely if there is a $\Phi$ satisfying 4.3.4 then $(\bar{S}, +, \rightarrow)$ satisfies 4.3.1 to 4.3.3. This and an earlier theorem establishing the existence of components of content are what we have called representation theorems, and thus it is easy to see that this theory could be formalized in the first order predicate calculus with elementary axioms by adjoining our axioms stated in section 4.1 for $\bar{N}$. The major difference between this generalization and the earlier theory is that the semi-group is no longer fully-ordered and this was crucial for the results used to establish completeness.

## 5. SUMMARY AND CONCLUSIONS

We propose the following criterion: a quantity has been defined if and only if all of the following have been specified,

(i) a set $\Sigma$ of axioms for the theory of measurement of this quantity,

(ii) an empirical $\Sigma$-model and

(iii) a numerical $\Sigma$-model.

The definition may still be inconvenient, irrelevant to what scientists do, or suffer other defects but an alleged definition which fails to specify clearly any of (i), (ii) or (iii) is not worthy of critical analysis. It may be difficult or even impossible to fully satisfy this criterion for quantities which have not long been in use. Historically a theory has usually developed informally before any axiom set has been proposed for it. Until a theory has been formalized it is difficult to describe models of it. Nevertheless, at least as an ideal toward the achievement of which our efforts may be directed, this criterion should be useful.

The usefulness of this criterion should be tested by applying it to various definitions. It seems reasonable to hope that attempts to satisfy it will lead to reduction of confusion and disputation among those using emerging concepts in such disciplines as psychology, ecology, and numerical taxonomy and will provide a firm foundation for the development of general system theory.

In this paper the concept of quantity and scale have been defined quite generally so that in theorem 3.3.4, for the first time, the existence of a scale transformation group has been demonstrated. In the formulation of this theorem no restriction was placed on the empirical $\Sigma$-model or the numerical $\Sigma$-model and in the examples this freedom of choice has been emphasized by showing how different choices lead to different scale transformation groups.

The best developed theory, because it describes the quantities used for many years by physicists and chemists, is that of extensive quantities. In our exposition, unlike that of most authors, we have given equal prominence to quantities and to scales and stressed the importance of basing theoretical development on physically meaningful properties. However some of the fundamental assumptions involve the extended operations obtained by iterating the basic operation of conjunction or concatenation. Thus we have contributed a theoretical formulation which explicitly states the interconnection between measurement and the counting numbers. We have shown how this could be done in a completely formalized but elementary framework in which the Archimedean and divisibility properties could be stated.

The example discussed in section 4.2 was devised to show how a direct or fundamental measurement procedure could lead to the construction of an interval scale and thus demonstrates that there is not an equivalence between fundamental measurement procedures and ratio scales as Campbell seems to have believed. This special result suggests the need to examine the category of non-extensive quantities, such as temperature, and either to develop a theory of non-extensive quantities or, more likely, to show that this category should be subdivided.

Finally we have shown that the elementary theory of extensive quantities is complete and thus any property which can be stated in the symbolism of the theory and demonstrated to hold in any model must therefore hold in every model. It should be clear that the set of axioms developed here is not independent. While an independent set of axioms is possible it is hoped that those we have presented will better lend themselves to verification by the working scientist.

Perhaps the most interesting unsolved problem discovered during the course of this study is that of the completeness of the theory of thermodynamics discussed in section 4.3.

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