ON SOME GENERAL RENEWAL THEOREMS FOR NONIDENTICALLY DISTRIBUTED VARIABLES

by

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1. Introduction

As a convenience, let us agree to call an infinite sequence $X_n \equiv X_1, X_2, X_3, \ldots,$ of independent random variables, a <u>renewal sequence</u>, and when all the random variables are identically distributed let us call $\{X_n\}$ a <u>renewal process</u>. If all the random variables are nonnegative let us say $\{X_n\}$ is a <u>positive</u> renewal sequence (process).

The renewal sequence (process) will be called <u>periodic</u> if there is a real $\omega > 0$ such that, with probability one, every random variable in the renewal sequence (process) is a multiple of ω. If the renewal sequence (process) is not periodic we shall call it <u>continuous</u>.

We shall write $S_n = X_1 + X_2 + \ldots = X_n$ with $n = 1,2,3, \ldots,$ for the partial sums of the renewal sequence, $F_n(x) = P\{S_n \leqq x\}$ for the distribution function of S_n, and $U(x) = P\{0 \leqq x\}$ for the so-called Heaviside unit function. We then define the random variable $N(x)$ as the number of partial sums S_n which satisfy the inequality $S_n \leqq x$:

$$(1.1) \qquad N(x) = \sum_{j=1}^{\infty} U(x - S_j) .$$

Thus, if $H(x) = E\{N(x)\}$, it follows from (1.1) that

$$(1.2) \qquad H(x) = \sum_{j=1}^{\infty} F_j(x).$$

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The function $H(x)$ is called the renewal function and is of prime interest in renewal theory; we refer to Smith [11] for an extensive account of it. A knowledge of its asymptotic behavior has proved very useful in establishing a variety of results about stochastic processes. However, with very few exceptions (one being the paper by Cox and Smith [5]) almost all the work done so far in this subject has referred to renewal processes (possibly with the trivial modification of allowing X_1 a different distribution from all the other X_n for $n > 1$) rather than the more general renewal sequences.

The crucial theorem for continuous renewal processes is due to Blackwell. It states that, if $0 < E\{X_n\} \leq \infty$, then for every fixed $\alpha > 0$

$$(1.3) \qquad H(x+\alpha) - H(x) \longrightarrow \frac{\alpha}{E\{X_n\}}, \quad \text{as } x \longrightarrow \infty.$$

This was proved first for positive renewal processes (Blackwell [1]) and later extended to the general case (Blackwell [2]). Note that in (1.3), and in similar contexts elsewhere, it is to be understood that if $EX_n = \infty$ then $1/(EX_n) = 0$.

There is, of course, a periodic analogue to (1.3); in particular, Erdös, Feller, and Pollard [6] gave a proof of this analogue for positive periodic renewal processes.

In addition to the papers mentioned so far, we should also draw attention to papers by Chung and Pollard [3], Chung and Wolfowitz [4], Karlin [8], Smith [10], and Kesten and Runnenburg [9]. These authors tackle various aspects of (1.3), some the continuous case, some the periodic case, some both such cases; some restrict themselves to positive processes, and so on. We must again refer to Smith [11] or, of course, to the original papers, for further details.

Let $\mathfrak{L}$ be the class of "kernel" functions $k(x)$ which vanish for $x < 0$ and which are nonnegative, nonincreasing, and integrable, over $(0, \infty)$. An alternative form of Blackwell's theorem (1.3) for positive continuous renewal processes, which was given by Smith [10], is

$$(1.4) \qquad \int_{-\infty}^{+\infty} k(x-z)\, dH(z) \longrightarrow \frac{1}{EX_n} \int_{-\infty}^{+\infty} k(z)\, dz, \quad \text{as } x \longrightarrow \infty,$$

for every $k(z) \in \mathfrak{L}$. This form is often more convenient for applications, although it is not hard to show that (1.3) and (1.4) are equivalent.

Our object in this paper is to establish general conditions under which a theorem like (1.4) will hold for renewal sequences (not necessarily positive ones) rather than for renewal processes. Before we discuss matters further, however, it will be as well if we introduce some more notation, and also the kernel class $\mathcal{K}$ which we shall use.

If k(x) is any absolutely integrable function then we write

$$\|k\| = \int_{-\infty}^{+\infty} k(x)\, dx ,$$

and we remark that $\|k\|$ may be negative.

If A(x) and B(x) are functions which possess a Stieltjes convolution then we write

$$A(x) * B(x) = \int_{-\infty}^{+\infty} A(x-z)\, dB(z) .$$

In dealing with these convolutions we shall occasionally assume, without comment, that A(x) * B(x) = B(x) * A(x), but only when it is easy to verify that this commutativity is valid. However, in this connection see our remarks at the end of section 6.

Definition 1. We write $k(x) \in \mathcal{K}$ if

($\mathcal{K}_1$) k(x) is Riemann-integrable in every finite interval;

$$(\mathcal{K}_2) \quad \sum_{n=-\infty}^{n=+\infty} \max_{n < x \leq n+1} |k(x)| < \infty .$$

We shall also write $\mathcal{K}_+$ for the subclass of functions k(x) such that $k(x) \geq 0$, all x. For future reference we note the following fact. Suppose $k(x) \in \mathcal{K}$, and define

$$k_1(x) = \max_{n < x \leq n+1} |k(x)| , \quad n < x \leq n+1,$$

for all n = . . ., -2, -1, 0, 1, 2, ... Then $k_1(x)$ and $k_1(x)+k(x)$ both belong to $\mathcal{K}_+$. Hence an arbitrary member of $\mathcal{K}$ can always be represented as the difference between two members of $\mathcal{K}_+$. Incidentally, note that $\mathcal{K}$ is a broader class of functions than $\mathfrak{L}$.

The main theorem of this paper is theorem 4, and all the later limit theorems in this paper are deduced from it. It states that certain very weak restrictions on the renewal sequence $\{X_n\}$ imply that $k(x) * H(x) \longrightarrow 0$ as $x \longrightarrow \infty$ for every $k(x) \in \mathcal{K}$ with $\|k\| = 0$. Thus, although theorem 4 does not state that $k(x) * H(x)$ necessarily tends to a limit for all $k(x) \in \mathcal{K}$, it does insist that if $k_1(x)$ and $k_2(x)$ both belong to $\mathcal{K}$, and if $\|k_1\| \neq 0$, $\|k_2\| \neq 0$, then as $x \longrightarrow \infty$

$$\frac{k_1(x) * H(x)}{\|k_1\|} - \frac{k_2(x) * H(x)}{\|k_2\|} \longrightarrow 0 \ .$$

The conditions involved in theorem 4 have been introduced to cover eventualities which just cannot arise when we restrict our attention to renewal processes. Thus, it is well known that for renewal processes, with $0 < E\{X_n\} \leqq \infty$, $H(x)$ is always finite and $H(x+1) - H(x)$ is uniformly bounded. These two properties of the renewal function do not appear to hold necessarily for quite general renewal sequences. In condition (a) of theorem 4 we simply suppose they do hold. However, we describe in section 6 of this paper a certain condition $\mathfrak{C}$ on $\{X_n\}$ which, if satisfied, automatically ensures the satisfaction of condition (a) of theorem 4. Condition $\mathfrak{C}$ relates to basic properties of the $\{X_n\}$ sequence and should not be difficult to verify in particular circumstances; for positive renewal sequences, corollary 3.1 shows that $\mathfrak{C}$ can be replaced by a much simpler condition.

In treating renewal <u>processes</u> the distinction between the periodic and the continuous cases is clear-cut. But when we turn our attention to renewal <u>sequences</u> a new and significant obstacle is found to bar the progress of our investigation. We have defined, simply enough, what we mean by a continuous renewal sequence, the trouble is that as we run through the series of random variables

$\{X_n\}$ sequentially they may "misbehave" and begin to look more and more like lattice variables. Thus, although we may "officially" be dealing with a "continuous" sequence, we may in fact be faced with a sequence which, in some vague sense which we will not bother to make precise, is "ultimately periodic." A major part of the present paper (sections 2, 3, 4, and 5) is devoted to a discussion of this matter, a matter which requires no discussion at all when dealing with renewal processes. Our primary object in this part of the paper is to determine weak restrictions on the sequence $\{X_n\}$ which will prevent its "misbehaving" too badly. We introduce the notions of <u>asymptotically lattice</u> sequences of random variables, and of <u>insistent meshes</u> of the renewal sequence $\{X_n\}$. These notions cannot be briefly described in this introduction; we shall content ourselves with the remark that it is the <u>insistent mesh</u> structure of $\{X_n\}$ which determines how well or badly it "behaves." Conditions (b), (c), and (d) of theorem 4 impose such restrictions on the insistent mesh structure of $\{X_n\}$ as were found necessary for our present methods of analysis to be successful. We do not believe these conditions to be necessary ones, but it certainly should be said that only extremely pathological renewal sequences fail to satisfy them. Roughly speaking, we regard $\{X_n\}$ as well behaved except when it contains arbitrarily long, uninterrupted runs of consecutive variables which are arbitrarily nearly like lattice-variables. Thus the <u>periodic</u> renewal sequence is ruled out from consideration for theorem 4; however, we explain in section 12 that there is a completely parallel theory for periodic renewal sequences.

With the exception of the papers by Chung and Pollard [3] and by Cox and Smith [5], it is probably true to say that all the published proofs of (1.3) and (1.4) are Tauberian in nature. That is, they utilize the knowledge that, for a special kernel function $k(x)$, the convolution $k(x) * H(x)$ tends to a certain limit, and succeed in deducing from this fact that similar limiting behavior operates when $k(x)$ is any member of such and such a class of functions. In fact, in the continuous case, the special function is always

$k(x) = U(x) - F_1(x)$; the well known integral equation of renewal theory shows that $\{U(x) - F_1(x)\} * H(x) = F_1(x)$, which tends to unity as x tends to infinity. Karlin [8] and Smith [10] actually go so far as to appeal to Wiener's general Tanberian theorem.

When we consider renewal sequences instead of processes, no convenient special kernel is available and the Tauberian kind of argument is no use. Thus we have had to develop a quite new attack. The methods of Cox and Smith [5] are unsuitable as they make heavy assumptions about $\{X_n\}$. In the sense that our chosen argument uses Fourier analysis it resembles that of Chung and Pollard [3], but the resemblance does not go very far. If our present argument is applied to renewal processes it will be found to be much simpler than the estimation methods employed by Chung and Pollard (who had to make an assumption about the characteristic function of X_1 which is now known to be unnecessary). The method we actually adopt utilizes some easily proved properties of the triangular probability density function and of its Fourier transform. We discuss these properties in section 7.

In section 9 we show that for continuous renewal processes, with $0 < E\{X_n\} \leq \infty$, it is an easy consequence of theorem 4 that

$$(1.5) \qquad k(x) * H(x) \longrightarrow \frac{\|k\|}{E\{X_n\}}, \quad \text{as } x \longrightarrow \infty,$$

for all $k(x) \in \mathcal{K}$. Section 10 takes up the question of the extent to which (1.5) will be true for renewal sequences. The notion of an ultimately stochastically stable sequence of random variables is introduced, and it is proved in theorem 6 that if the renewal sequence $\{X_n\}$ is such a sequence, and if its insistent mesh structure satisfies the conditions of theorem 4, then (1.5) will hold with $E\{X_n\}$ replaced by a certain constant of the sequence $\{X_n\}$. Thus theorem 6 provides the desired generalization of (1.4) from renewal processes to renewal sequences. If is useful, however, to have convenient necessary and sufficient conditions under which the renewal sequence will be ultimately stochastically stable. Such necessary and sufficient conditions are given by theorem 7 and are as follows.

There must be an integer N such that $E\{X_N\}$, $E\{X_{N+1}\}$, $E\{X_{N+2}\}$, . . ., ad infinitum, is a stable sequence (as defined by Cox and Smith [5]). There must also be two distribution functions $G_-(x)$, $G_+(x)$, each referring to a random variable of finite expectation, such that $G_-(x) \leq P\{X_r \leq x\}$, $r \geq 0$, $G_+(x) \geq P\{X_r \leq x\}$, $r \geq N$. Provided these two conditions are satisfied, and provided the asymptotic mesh structure of the renewal sequence is satisfactory, it follows therefore, from theorems 6 and 7, that the convolution $k(x) * H(x)$ convergees to a limiting value as $x \longrightarrow \infty$. Theorem 8 gives a version of theorem 6 for the situation when the $\{X_n\}$ predominantly have infinite positive expectations and $k(x) * H(x) \longrightarrow 0$ for all $k(x) \in \mathcal{K}$.

In section 11 we consider functions of the form $Q(x) = \Sigma_j a_j F_j(x)$, of which $H(x)$ is a special case. It is explained that, when the constants $\{a_n\}$ are bounded, theorem 4 holds for Q as well as for H. If the constants $\{a_n\}$ form a stable sequence with average a, then it is proved in theorem 9 that there is a suitable extension of theorem 6, that is, limits like (1.5) can be proved for Q instead of H:

$$k(x) * Q(x) \longrightarrow \frac{a\,\|k\|}{\mu}, \quad \text{as } x \longrightarrow \infty,$$

for all $k(x) \in \mathcal{K}$, where μ is a constant associated with the ultimately stochastically stable sequence $\{X_n\}$. Functions like $Q(x)$ have been considered previously by Cox and Smith [5].

Finally, in section 12, we discuss briefly two additional matters. First, we comment on the theory of periodic renewal sequences, which parallels the theory given in this paper, but which we do not develop in detail. Second, we discuss the case of dependent variables $\{X_n\}$ and introduce the idea of "structure - R." We show that a theorem like theorem 4 will hold for certain sequences of dependent variables if they have structure - R.

2. On insistent subsequences

In our study of the renewal sequence $\{X_n\}$ we will have to guard against its behaving too much like a sequence of lattice variables. To discuss this undesirable possibility we introduce the concepts of an *insistent subsequence* and of an *asymptotically lattice sequence* of random variables. This section is concerned with the first of these ideas.

Suppose we are given a certain subsequence $\{A_{n_\nu}\}$, $\nu=0,1,2,\ldots$, of some arbitrary infinite sequence of terms $\{A_n\}$, $n=0,1,2,\ldots$. For each value of the integer n define the integer ℓ_n as follows:

(a) If A_n does not appear in the subsequence $\{A_{n_\nu}\}$ then set $\ell_n = 1$.

(b) If A_n does appear in $\{A_{n_\nu}\}$ then let ℓ_n be the maximum integer such that $A_n, A_{n+1}, A_{n+2}, \ldots, A_{n+\ell_n-2}$ all appear in $\{A_{n_\nu}\}$.

We call ℓ_n the ℓ - sequence of $\{A_{n_\nu}\}$. If the sequence of integers $\{\ell_n\}$, $n=0,1,2,\ldots$, is unbounded then we shall sqy $\{A_{n_\nu}\}$ is an *insistent* subsequence of $\{A_n\}$. Thus, when we have an insistent subsequence of a given sequence, we can find arbitrarily long runs of successive terms in the given sequence, which terms all appear in the subsequence.

It is necessary to introduce various degrees of insistence of the subsequence $\{A_{n_\nu}\}$, If there is a $\delta > 0$ such that, although $\{A_{n_\nu}\}$ is insistent, $n^{\delta-1/3}\,\ell_n \longrightarrow 0$ as $n \longrightarrow \infty$, then we shall say $\{A_{n_\nu}\}$ is *weakly* insistent. If $\{A_{n_\nu}\}$ is insistent but not weakly insistent, and if $n^{-1}\,\ell_n \longrightarrow 0$ as $n \longrightarrow \infty$, then we shall say $\{A_{n_\nu}\}$ is *mildly* insistent. If $\{A_{n_\nu}\}$ is insistent, but neither weakly nor mildly so, then we shall say it is *strongly* insistent. It is to be emphasised that a subsequence $\{A_{n_\nu}\}$ can be "highly representative" of the given sequence $\{A_n\}$ without being in the least insistent (in the present sense). For instance, $\{A_{n_\nu}\}$ might consist of every term in $\{A_n\}$ with an even suffix; then $\ell_n = 1$ or 2 according as n is odd or even, and the subsequence is clearly not insistent. On the other hand, if $\{A_{n_\nu}\}$ is a strongly insistent subsequence of $\{A_n\}$ then there must be an $\epsilon > 0$ such that, for infinitely many values of n, all the terms A_r for $n \leqq r \leqq n(1+\epsilon)$ belong to the subsequence.

The various degrees of insistence are useful in connection with a certain method of summing series, which we shall use later, by aggregating successive terms of the series into blocks. We therefore describe the blocking procedure appropriate to the subsequence $\{A_{n_\nu}\}$. The first ℓ_0 terms of $\{A_n\}$ are assigned to the first black, B_1 say. Define $\lambda_1 = \ell_0$. Assign to B_2, the second block, the next remaining ℓ_{λ_1} terms of $\{A_n\}$. Thus B_2 starts with A_{ℓ_0} and runs to $A_{\ell_0+\lambda_1-1}$. Define $\lambda_2 = \ell_{\lambda_1}$ and assign to B_3 the next $\ell_{\lambda_1+\lambda_2}$ terms of $\{A_n\}$. Define $\lambda_3 = \ell_{\lambda_1+\lambda_2}$ and assign to B_4 the next $\ell_{\lambda_1+\lambda_2+\lambda_3}$ terms of $\{A_n\}$. It should be clear how this procedure is to be continued; its motivation is easy to grasp. Each block consists of a run of successive terms in the sequence $\{A_n\}$ of which all but the last belong to the subsequence $\{A_{n_\nu}\}$. The following two lemmas, needed later, refer to this "blocking procedure."

<u>Lemma 1. If $\{A_{n_\nu}\}$ is neither a mildly nor a strongly insistent subsequence of the infinite sequence $\{A_n\}$ then</u>

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n^{3/2}} < \infty .$$

<u>Proof.</u> If $\{A_{n_\nu}\}$ is not insistent then the ℓ-sequence $\{\ell_n\}$ and hence the λ-sequence $\{\lambda_n\}$ are bounded; the lemma is then trivial. Suppose therefore that $\{A_{n_\nu}\}$ is weakly insistent and that $\delta(>0)$ has been chosen so that $n^{\delta-1/3}\ell_n \longrightarrow 0$ as $n \longrightarrow \infty$. Thus, as $n \longrightarrow \infty$,

$$(\lambda_1 + \lambda_2 + \ldots + \lambda_{n-1})^{\delta-1/3}\lambda_n \longrightarrow 0 .$$

If we write $\Lambda_n = \lambda_1 + \lambda_2 + . . . + \lambda_n$, then the last limit can be rewritten

$$(\Lambda_{n-1})^{\delta-1/3}(\Lambda_n - \Lambda_{n-1}) \longrightarrow 0 .$$

But $\lambda_n \geqq 1$ for all n, so that $\Lambda_n > \Lambda_{n-1}$. Thus

$$\Lambda_{n-1}^{\delta-1/3}(\Lambda_n - \Lambda_{n-1}) > \Lambda_n^{\delta+2/3} - \Lambda_{n-1}^{\delta+2/3} > 0 ,$$

and we may conclude

$$\lim_{n\to\infty} (\Lambda_n^{\delta+2/3} - \Lambda_{n-1}^{\delta+2/3}) = 0 .$$

Define $\epsilon_1 = \Lambda_1^{\delta+2/3}$ and $\epsilon_n = \Lambda_n^{\delta+2/3} - \Lambda_{n-1}^{\delta+2/3}$, for $n > 1$.
Then $\epsilon_n \longrightarrow 0$ as $n \longrightarrow \infty$ and

$$\Lambda_n = (\epsilon_1 + \epsilon_2 + . . . + \epsilon_n)^{3/(2+3\delta)} .$$

If we further define $\bar{\epsilon}_n = n^{-1}(\epsilon_1 + \epsilon_2 + . . . + \epsilon_n)$ then, as a well-known consequence of $\epsilon_n \longrightarrow 0$, we have $\bar{\epsilon}_n \longrightarrow 0$ as $n \longrightarrow \infty$. If we now observe that

$$\frac{\Lambda_n}{n^{5/2}} = \frac{\bar{\epsilon}_n^{\,3/(2+3\delta)}}{n^{(4+15\delta)/(4+6\delta)}}$$

it follows immediately that

$$\sum_{n=1}^{\infty} \frac{\Lambda_n}{n^{5/2}} = \sum_{n=1}^{\infty} \frac{1}{n^{5/2}} \left(\sum_{m=1}^{n} \lambda_m\right)$$

is a convergent double series of positive terms. Rearrangement is legitimate and shows

$$\sum_{n=1}^{\infty} \lambda_n \left(\sum_{m=n}^{\infty} \frac{1}{m^{5/2}}\right)$$

to be convergent. Since

$$(2/3)\, n^{-3/2} \sim \sum_{m=n}^{\infty} m^{-5/2}, \text{ as } n \longrightarrow \infty,$$

the lemma is therefore proved.

Lemma 2. If $\{A_{n_\nu}\}$ is not a strongly insistent subsequence of of $\{A_n\}$ then

$$\sum_{n=1}^{\infty} \rho^n \lambda_n < \infty$$

for all ρ such that $0 < \rho < 1$.

Proof. The previous lemma proves the present one easily if $\{A_{n_\nu}\}$ is weakly insistent or not insistent at all.

If $\{A_{n_\nu}\}$ is mildly insistent then $\lambda_n/n \longrightarrow 0$ and so $\lambda_n/(\lambda_1 + \lambda_2 + \ldots + \lambda_{n-1}) \longrightarrow 0$ as $n \longrightarrow \infty$. Hence, given any $\epsilon > 0$, there is an integer $m(\epsilon)$ such that for all $n \geqq m(\epsilon)$,

$$0 < \lambda_n < \epsilon\, (\lambda_1 + \lambda_2 + \ldots + \lambda_{n-1}) .$$

Put $C = \lambda_1 + \lambda_2 + \ldots + \lambda_{m-1}$; then we can develop the following inequalities in a systematic fashion:

$$\begin{aligned} 0 < \lambda_m &< \epsilon\, C\ ; \\ 0 < \lambda_{m+1} &< \epsilon(C + \lambda_m) \\ &< \epsilon(1 + \epsilon)\, C\ ; \\ 0 < \lambda_{m+2} &< \epsilon(C + \lambda_m + \lambda_{m+1}) \\ &< \epsilon(1 + \epsilon)^2 C; \end{aligned}$$

and, generally, for all $r \geqq 0$, $0 < \lambda_{m+r} < \epsilon(1 + \epsilon)^r\, C$.

If ϵ is chosen small enough to make $\rho(1 + \epsilon) < 1$, which choice is always possible when $0 < \rho < 1$, the lemma follows directly from these last inequalities.

3. Asymptotically lattice sequences

Suppose $\{Y_n\}$ is a given sequence of random variables. Suppose it is possible to find two sequences of finite constants $\{a_n\}$, $\{h_n\}$, such that $h_n > 0$ for all n and (a) $h_n \longrightarrow h$ as $n \longrightarrow \infty$, where $0 < h \leqq \infty$; (b) if the random variable $\overline{Y}_n$ is defined by $-h_n/2 \leqq \overline{Y}_n < +h_n/2$, and $\overline{Y}_n \equiv Y_n - a_n (\text{mod } h_n)$ then $\overline{Y}_n \longrightarrow 0$ in probability as $n \longrightarrow \infty$. In this case we say Y_n is an <u>asymptotically lattice sequence</u>, and h is its <u>asymptotic mesh</u>.

Let us adopt the notation that if Y is any random variable then Y' and Y" are independent random variables distributed like Y. Write $Y^s = Y' - Y''$ for the symmetrization of Y. We then have

<u>Lemma 3. A necessary and sufficient condition for $\{Y_n\}$ to be asymptotically lattice is that $\{Y_n^s\}$ be so.</u>

<u>Proof.</u> The necessity part of the lemma is rather obvious; we prove (and need) only the sufficiency part. Suppose therefore that $\{Y_n^s\}$ is asymptotically lattice, with appropriate sequences $\{a_n^s\}$, $\{h_n^s\}$. Because the variables $\{Y_n^s\}$ are symmetrical we can clearly choose $a_n^s = 0$ for all n. Define $S_n(\epsilon)$ to be the set of all real numbers which differ from zero, or a positive or negative multiple of h_n^s, by less than ϵ. Then if, for some $\epsilon > 0$, we have

$$(3.1) \qquad P\{Y_n' - Y_n'' \in S_n(\epsilon)\} > 1 - \epsilon,$$

there must be a real constant a_n such that

$$P\{Y_n' - Y_n'' \in S_n(\epsilon) \mid Y_n'' = a_n\} > 1 - \epsilon.$$

But Y_n' and Y_n'' are independent, so the last inequality implies

$$(3.2) \qquad P\{Y_n' - a_n \in S_n(\epsilon)\} > 1 - \epsilon.$$

The fact that (3.1) implies (3.2) is enough to prove the lemma.

We remark that there is an alternative proof of the above lemma in terms of characteristic functions; this proof is, in some ways, more appealing; but it does not cover the interesting case $h = \infty$ which is embraced by the arguements we give. However, characteristic functions are useful when we restrict our attention to finite asymptotic meshes; there is then an alternative definition of asymptotically lattice sequences, which will be important in the sequel, and which is summarized in the following.

Theorem 1. *If $\mathcal{J}_n(\theta) = E\{e^{i\theta Y_n}\}$ for $n = 1, 2, 3, \ldots$, then a necessary and sufficient condition for $\{Y_n\}$ to be asymptotically lattice, with a finite asymptotic mesh, is that there should exist a sequence of angles $\{\theta_n\}$ and a $\hat{\theta}$, $0 < \hat{\theta} < \infty$, such that $\theta_n \longrightarrow \hat{\theta}$ and $|\mathcal{J}_n(\theta_n)| \longrightarrow 1$, as $n \longrightarrow \infty$.*

Proof. We prove the necessity part first. Suppose $\{Y_n\}$ is asymptotically lattice with the finite asymptotic mesh h. Let $\{a_n\}$, $\{h_n\}$, have the usual meanings. Then it is easy to show that

$$\lim_{n \to \infty} E\left\{\cos\left[\frac{2\pi(Y_n - a_n)}{h_n}\right]\right\} = 1 , \tag{3.3}$$

or

$$\lim_{n \to \infty} \mathcal{R}\, e^{-2\pi i a_n/h_n} \mathcal{J}_n(2\pi/h_n) = 1. \tag{3.4}$$

But $|\mathcal{J}_n(\theta)| \leqq 1$ for all θ, so we may deduce from (3.4) that

$$\lim_{n \to \infty} |\mathcal{J}_n(2\pi/h_n)| = 1 .$$

This proves the necessity of our condition; evidently we may take $\theta_n = 2\pi/h_n$ and $\hat{\theta} = 2\pi/h$.

To prove the sufficiency part of the condition, first put

$\vartheta_n(\theta_n) = \rho_n e^{-i\alpha_n}$, where ρ_n and α_n are real and chosen so that $0 \leq \rho_n \leq 1$, and $0 \leq \alpha_n < 2\pi$. Then $\rho_n \to 1$ as $n \to \infty$, or

$$(3.5) \qquad \lim_{n \to \infty} e^{-i\alpha_n} \vartheta_n(\theta_n) = 1.$$

On taking real parts of (3.5) we discover

$$(3.6) \qquad \lim_{n \to \infty} E\left\{\cos(\theta_n Y_n - \alpha_n)\right\} = 1.$$

Define $h_n = 2\pi / |\theta_n|$ and $a_n = \alpha_n / |\theta_n|$, and we see that (3.6) implies (3.3). The asymptotically lattice nature of $\{Y_n\}$ is an easy deduction from this last limit, (3.3), since $h_n \to 2\pi / |\hat{\theta}| = h$, say, and $0 < h < \infty$.

It is of interest to see why the present arguement fails for the case of infinite asymptotic mesh. The reason is that when $h_n \to \infty$ the fact that the limit (3.3) holds is not equivalent to an asymptotically lattice nature of the sequence $\{Y_n\}$.

4. Insistent meshes of the renewal sequence

We now apply the ideas of the preceding two sections to the renewal sequence $\{X_n\}$. To do this we shall need to employ the following convenient notation. If $\{A_n\}$ is any sequence, and if $\{A_{n_\nu}\}$ is some given subsequence of it, let $\{A_n / A_{n_\nu}\}$ denote the sequence obtained from $\{A_n\}$ by deleting those terms which are also in $\{A_{n_\nu}\}$. In other words $\{A_n / A_{n_\nu}\}$ is the subsequence which complements $\{A_{n_\nu}\}$ with respect to $\{A_n\}$.

If the renewal sequence $\{X_n\}$ possesses an insistent subsequence $\{X_{n_\nu}\}$ which is asymptotically lattice, with asymptotic mesh h, then we shall say that h is an insistent mesh (or, briefly, an I-mesh) of $\{X_n\}$. If h is an I-mesh of $\{X_n\}$ and if it is possible to find a subsequence $\{X_{n_\nu}\}$, say, such that h is not an I-mesh of the new sequence $\{X_n / X_{n_\nu}\}$, then we shall say $\{X_{n_\nu}\}$ annihilates h; alternatively, we may say $\{X_{n_\nu}\}$ is an annihilating subsequence. Notice that is is not necessary for the annihilating subsequence to be asymptotically lattice; it is obvious, however, that an annihilating subsequence must be an insistent subsequence. Another point to be noted is that the modified sequence $\{X_n / X_{n_\nu}\}$ may contain I-meshes which were not I-meshes of the original sequence $\{X_n\}$.

If h is an I-mesh which can be annihilated by a weakly insistent subsequence of $\{X_n\}$ then we shall say h is a weakly insistent mesh (briefly, an I_w-mesh). If h is an I-mesh, but not an I_w-mesh, and if h can be annihilated by a mildly insistent subsequence of $\{X_n\}$ then we shall say h is a mildly insistent mesh (an I_m-mesh). If h is an I-mesh which can only be annihilated by a strongly insistent subsequence of $\{X_n\}$ then we shall say h is a strongly insistent mesh (an I_s-mesh). In the latter case we may if necessary, regard the entire renewal sequence as an annihilating subsequence of itself, and we must adopt the view that finite (or empty) sequences of random variables possess no I-meshes. For example, if every random variable of the renewal sequence is, with probability one, an even integer, then 2 is a strongly insistent mesh; the I_s-mesh 2 can only be annihilated by subsequences $\{X_{n_\nu}\}$ such that $\{X_n / X_{n_\nu}\}$ contains but finitely many terms.

It is sometimes helpful to speak of degrees of insistence. Thus if h_1 is an I_m-mesh and h_2 is an I_w-mesh then we shall say h_1 is more highly insistent than h_2; if h_1 and h_2 are both I_m-meshes, say, then we shall call h_1 and h_2 similarly insistent; and so on. If h_1 and

and h_2 are distinct finite I-meshes such that h_1 / h_2 is a positive integer ($\geqq 2$), then it is not difficult to see that h_1 cannot be more highly insistent than h_2. Furthermore, if h is a finite I-mesh then h/2, h/3, h/4, . . ., are all I-meshes which by our previous remark, are at least as insistent as the I-mesh h. If we write $\mathcal{H}$ for the set of all I-meshes then it is obvious that, unless $\mathcal{H}$ is empty, it always has 0 as a limit point. Nevertheless, 0 never belongs to $\mathcal{H}$, since it can never be an asymptotic mesh.

<u>Theorem 2. If</u> $0 < h \leqq \infty$ <u>and if h is a limit point of $\mathcal{H}$, then $h \in \mathcal{H}$.</u>

<u>Proof</u>: Suppose $\{h_m\}$ is a sequence of I-meshes such that $h_m \to h$ as $m \to \infty$. For each I-mesh h_m we can find an insistent subsequence $\{X_{n_{m\nu}}\}$ of $\{X_n\}$, and real sequences $\{a_\nu^{(m)}\}$, $\{h_\nu^{(m)}\}$ for $\nu=1,2,3,\ldots$, such that, if $-h_\nu^{(m)}/2 \leqq \overline{X}_\nu^{(m)} < +h_\nu^{(m)}/2$ and $\overline{X}_\nu^{(m)} \equiv X_{n_{m\nu}} - a_\nu^{(m)} \pmod{h_\nu^{(m)}}$, then $\overline{X}_\nu^{(m)} \to 0$ in probability as $\nu \to \infty$.

Let $\{\epsilon_m\}$ be a decreasing sequence of real numbers such that $\epsilon_m \to 0$ as $m \to \infty$. Since $\{X_{n_{1\nu}}\}$ is insistent, we can find integers α_1, β_1, such that the sucessive terms of $\{X_{n_{1\nu}}\}$ from $\nu=\alpha_1$ to $\nu=\beta_1$ are also successive terms of $\{X_n\}$; such that $n_{1\beta_1} - n_{1\alpha_1} > \epsilon_1^{-1}$; and such that $|h^{(1)} - h_\nu^{(1)}| < \epsilon_1$ and $P\{|\overline{X}_\nu^{(1)}| > \epsilon_1\} < \epsilon_1$ for all ν such that $\alpha_1 \leqq \nu \leqq \beta_1$. Similarly we can α_2 and β_2 such that the successive terms of $\{X_{n_{2\nu}}\}$ from $\nu = \alpha_2$ to $\nu = \beta_2$ are also successive terms of $\{X_n\}$; such that $|h^{(2)} - h_\nu^{(2)}| < \epsilon_2$ and $P\{|\overline{X}_\nu^{(2)}| > \epsilon_2\} < \epsilon_2$ for all ν such that $\alpha_2 \leqq \nu \leqq \beta_2$. We can evidently arrange, moreover, to have $n_{1\beta_1} < n_{2\alpha_2}$, and it is clear how we can continue, on the lines we have described, to runs of consecutive terms from $\{X_{n_{3\nu}}\}$, $\{X_{n_{4\nu}}\}$, and so on.

Define a subsequence $\{X_{n_{0\nu}}\}$ of $\{X_n\}$ as follows. Let the first $n_{1\beta_1} - n_{1\alpha_1} + 1$ terms of $\{X_{n_{0\nu}}\}$ be the terms of $\{X_{n_{1\nu}}\}$ from $\nu = \alpha_1$ to $\nu = \beta_1$. Let the succeeding $n_{2\beta_2} - n_{2\alpha_2} + 1$ terms of $\{X_{n_{0\nu}}\}$ be the terms of $\{X_{n_{2\nu}}\}$ from $\nu = \alpha_2$ to $\nu = \beta_2$; and so on. Since ϵ_m decreases to 0 and $h_m \to h$ as $m \to \infty$ it is not difficult to see that $\{X_{n_{0\nu}}\}$ is an insistent subsequence of $\{X_n\}$ which is asymptotically lattice, with asymptotic mesh h. This proves the theorem.

An important consequence of theorem 2 is that if ∞ is not an I-mesh then $\mathcal{H}$ must be a bounded set. The following corollary shows further that if ∞ is not in $\mathcal{H}_m \cup \mathcal{H}_s$ then $\mathcal{H}_m \cup \mathcal{H}_s$ must be a bounded set. ($\mathcal{H}_w, \mathcal{H}_m, \mathcal{H}_s$, are the sets of I_w-, I_m-, I_s-, meshes).

Corollary 2.1. If $0 < h \leqq \infty$ and if h is a limit point of $\mathcal{H}_m \cup \mathcal{H}_s$, then $h \in \mathcal{H}_m \cup \mathcal{H}_s$.

Proof. The proof is similar to that for the main theorem. Suppose $\{h_m\}$ is a sequence of I-meshes in $\mathcal{H}_m \cup \mathcal{H}_s$ such that $h_m \to h$ as $m \to \infty$. For each I-mesh h_m we can find an asymptotically lattice, insistent subsequence $\{X_{n_{m\nu}}\}$, as before, but with the additional properly that if $\{\ell_n^{(m)}\}$ is the associated ℓ-sequence then, for every $\delta > 0$, $\{n^{\delta-1/3}\ell_n^{(m)}\}$ is an unbounded sequence. To see this last point, observe that there must be an $\epsilon > 0$ such that $n^{\delta/2 - 1/3}\ell_n^{(m)} > \epsilon$ for infinitely many values of n; thus $n^{\delta-1/3}\ell_n^{(m)} > \epsilon\, n^{\delta/2}$ for infinitely many values of n.

Let $\{\delta_\nu\}$ be a decreasing sequence of real numbers such that $\delta_\nu \to 0$ as $\nu \to \infty$. Then we can modifiy the construction of the main proof so that, in the notation of that proof

$$n_{\nu\beta_\nu} - n_{\nu\alpha_\nu} > \epsilon_\nu^{-1} \; n_{\nu\alpha_\nu}^{1/3-\delta_\nu},$$

for all ν. The rest of the argument holds with only trivial changes; in particular the fact that δ_ν decreases to zero ensures that $\{X_{n_{0\nu}}\}$ will not be weakly insistent.

We complete this section with two lemmas which show that the degree of insistence of an I-mesh h, say, cannot be lower than that of any asymptotically lattice, insistent subsequence with asymptotic mesh h.

<u>Lemma 4</u>. <u>If the renewal sequence $\{X_n\}$ contains an asymptotically lattice, insistent subsequence $\{X_{n_\nu}\}$ which is not weakly insistent and of which the asymptotic mesh is h, then h is not an I_w-mesh (although it is, of course, an I-mesh).</u>

<u>Proof</u>. We use the notation of section 2 and let $\{\ell_n\}$ be the ℓ-sequence associated with $\{X_{n_\nu}\}$. Then for every $\delta > 0$ we must have $n^{\delta-1/3}\,\ell_n \to 0$ as $n \to \infty$.

Suppose it is claimed that h is an I_w-mesh and that the weakly insistent subsequence $\{X_{m_\nu}\}$, say, is such that $\{X_n / X_{m_\nu}\}$ does not have h as an I-mesh. Let $\{\ell_n^*\}$ be the ℓ-sequence associated with $\{X_{m_\nu}\}$ and suppose $\delta^* > 0$ is such that $n^{\delta^* - 1/3}\ell_n^* \to 0$ as $n \to \infty$. Take $0 < \delta < \delta^*$. Then there must be an $\epsilon > 0$ such that $\ell_n > \epsilon n^{1/3-\delta}$ for infinitely many values of n; let n^* be such a value of n.

In the argument that follows regard the values of any functions of n^* or of n as being taken to the nearest integer. Then there is a run of $\epsilon n^{*1/3-\delta}$ successive terms in $\{X_{n_\nu}\}$; call this run R. We ask: how many terms in R also belong to $\{X_n / X_{m_\nu}\}$?

It may be supposed that n^* is so large that $\ell_n^* < \epsilon n^{1/3-\delta^*}$

for all $n \geqq n^*$. Thus, if R contains a subrun of consecutive terms in $\{X_n\}$ which also belong to $\{X_{m_\nu}\}$, then this subrun can contain no more than $\epsilon(n+\epsilon n^{1/3-\delta})^{1/3-\delta*}$ terms. Each such subrun of R must be followed by a term of $\{X_n / X_{m_\nu}\}$. Thus R must contain at least

$$\frac{\epsilon n^{*1/3-\delta}}{\epsilon(n^* + \epsilon n^{*1/3-\delta})^{1/3-\delta^*}} = \rho_{n^*},$$

say, terms of $\{X_n / X_{m_\nu}\}$. Plainly $\rho_{n^*} \sim n^{*\delta^*-\delta}$ which $\rightarrow \infty$ as $n^* \rightarrow \infty$. Thus $\{X_n / X_{m_\nu}\}$ contains arbitrarily long runs of consecutive terms which belong also to the asymptotically lattice subsequence $\{X_{n_\nu}\}$. Thus h is an I-mesh of $\{X_n / X_{m_\nu}\}$, and this contradiction of our hypothesis proves the lemma.

We bring this section to a close with the following lemma which, while barely used in the sequel, is of value in analyzing particular renewal sequences.

<u>Lemma 5</u>. <u>If</u> $\{X_n\}$ <u>has an asymptotically lattice, strongly insistent subsequence</u> $\{X_{n_\nu}\}$ <u>with asymptotic mesh</u> h, <u>then h is an</u> I_s<u>-mesh.</u>

<u>Proof</u>. The proof can be constructed on lines similar to the ones adopted for the proof of lemma 4. Lemma 4 shows that h cannot be an I_w-mesh. Suppose it is claimed that h is an I_m-mesh and that the mildly insistent subsequence $\{X_{m_\nu}\}$ annihilates h. Define the ℓ-sequences as in the last proof and put $\epsilon_n = \ell_n/n$ and $\epsilon_n^* = \ell_n^*/n$. Then there is an $\epsilon > 0$ such that $\epsilon_n > \epsilon^{1/2}$ for infinitely many values of n, while $\epsilon_n^* < \epsilon$ for all sufficiently large values of n. For arbitrarily large values of n we can find sums of more than $n\,\epsilon^{1/2}$ consecutive terms in $\{X_{n_\nu}\}$ which are also consecutive terms in $\{X_n\}$. In such a run, R, say, the subruns which are consecutive terms in

$\{X_{m_\nu}\}$ cannot contain more than $\epsilon(n+n\,\epsilon^{1/2})$ terms. Thus R contains at least $1/[\epsilon^{1/2}(1+\epsilon^{1/2})]$ terms of $\{X_n / X_{m_\nu}\}$. Since ϵ can be arbitrarily small, we have proved that $\{X_n / X_{m_\nu}\}$ contains arbitrarily long runs of consecutive terms which also fall in the asymptotically lattice subsequence $\{X_{n_\nu}\}$. Thus h is an I-mesh of $\{X_n / X_{m_\nu}\}$ and the lemma is proved by this contradiction of our hypothesis.

5. On sums of characteristic functions

Let $\{X_n\}$ be the renewal sequence of independent random variables and let $\phi_n(\theta) = E\{e^{i\theta x_n}\}$ be the characteristic function of X_n. Write $\psi_n(\theta) = \phi_1(\theta)\,\phi_2(\theta)\,\ldots\,\phi_n(\theta)$ for the characteristic function of the partial sum S_n. Recall that $\mathcal{H}_w, \mathcal{H}_m, \mathcal{H}_s$, are the sets of all weakly, mildly, strongly insistent meshes, respectively.

If $\theta = \pm\, 2\pi/h$, where $h \in \mathcal{H}$, then we shall call θ an <u>insistent angle</u> (I-angle). If $\theta = \pm\, 2\pi/h$, where $h \in \mathcal{H}_w$, we shall call θ a <u>weakly insistent angle</u> (I_w-angle); similarly for mildly, and strongly, insistent angles (I_m- and I_s-angles). Note that 0 can be an insistent angle if ∞ is an insistent mesh.

<u>Lemma</u> 6. <u>If J is a bounded closed interval which</u>
<u>(a) does not include 0;</u>
<u>(b) contains no strongly insistent angles;</u>
<u>(c) contains no more than a finite number of mildly insistent angles;</u> <u>then</u> $\Sigma_1^\infty |\psi_j(\theta)|$ <u>is boundedly convergent in J.</u>

<u>Proof</u>. Suppose first that J contains exactly one I_m-angle, $\hat\theta$, say. Let $\{X_{n_\nu}\}$ be a mildly insistent subsequence of $\{X_n\}$ which annihilates the I_m-mesh $2\pi/|\hat\theta|$, and let $\{\ell_n\}$ be the corresponding ℓ-sequence. Then $\ell_n/n \to 0$ as $n \to \infty$. Let $\{\epsilon_n\}$ be a decreasing

sequence such that $\epsilon_n > \ell_n/n$ for all n, and $\epsilon_n \to 0$ as $n \to \infty$.

Since $\{X_{n_\nu}\}$ is mildly insistent, for every $\delta > 0$ there is some $\epsilon > 0$ such that $\ell_n > \epsilon n^{1/3-\delta}$ for infinitely many values of n. Hence $\epsilon_n > \epsilon / n^{2/3+\delta}$ for infinitely many values of n. Write

$$L = \lim_{k=\infty} \sup_{\theta \in J} \sup_{n \geq k} \inf_{n \leq j \leq n(1+\epsilon_n^{1/2})} |\phi_j(\theta)| .$$

It is obvious that $0 \leq L \leq 1$, but our wish is to prove that $L < 1$. To this end, suppose $L = 1$. This implies the existence of an unbounded increasing sequence of integers $\{n_\nu\}$ and of a sequence of angles in J, $\{\theta_\nu\}$ such that if we write

$$L_\nu = \inf_{n_\nu \leq j \leq n_\nu(1 + \epsilon_{n_\nu}^{1/2})} |\phi_j(\theta_\nu)|$$

then $L_\nu \to 1$ as $\nu \to \infty$. Because J is a bounded closed set we can (by selecting a suitable subsequence if necessary) arrange for $\{\theta_\nu\}$ to be a convergent sequence with a limit point $\theta^* \neq 0$.

Define a subsequence S, say , of $\{X_n\}$ by assigning X_j to S if $n_\nu \leq j \leq n_\nu (1 + \epsilon_{n_\nu}^{1/2})$ for some ν. Since $L_\nu \to 1$ as $\nu \to \infty$, while $\theta_\nu \to \theta^* \neq 0$, it is an immediate consequence of theorem 1 that S is asymptotically lattice, with asymptotic mesh $2\pi/|\theta^*|$. Furthermore, if $\{\ell_n^*\}$ is the ℓ-sequence corresponding to S then the definition of S shows that $\ell_n^* / n \geq \epsilon_n^{1/2}$ for infinitely many values of n. By our choice of $\{\epsilon_n\}$ it follows that, for every $\delta > 0$, there is an $\epsilon > 0$ such that $\ell_n^*/ n^{1/3-\delta} \geq n^{1/3+\delta/2} \epsilon^{1/2}$, for infinitely many values of n. Thus S is not weakly insistent, and an appeal to lemma 4 establishes that $2\pi/|\theta^*|$ is either an I_m-mesh or an I_s-mesh. But our hypothesis is that J contains no I_s-angles and exactly one mildly insistent angle, namely

$\hat{\theta}$. Thus $\theta^* = \hat{\theta}$.

Finally, we make use of the annihilating subsequence $\{X_{n_\nu}\}$. The modified sequence $\{X_n / X_{n_\nu}\}$ does not have $2\pi/|\hat{\theta}|$ as an I-mesh. However, if we employ the kind of argument used in the proofs of lemmas 4 and 5, we can show that, for infinitely many values of n, $\{X_n / X_{n_\nu}\}$ must contain runs of more than

$$\frac{n\,\epsilon_n^{1/2}}{(n + n\,\epsilon_n^{1/2})\,\epsilon_{n(1+\epsilon^{1/2})}} = \rho_n,$$

say, successive terms which also belong to S. But $\{\epsilon_n\}$ is a decreasing sequence, so that

$$\rho_n \geqq \frac{1}{(1 + \epsilon_n^{1/2})\,\epsilon_n^{1/2}} .$$

This last inequality shows, since $\epsilon_n \longrightarrow 0$, that $\{X_n / X_{n_\nu}\}$ contains arbitrarily long runs of consecutive terms which also belong to the symptotically lattice sequence S, whose asymptotic mesh is $2\pi/|\hat{\theta}|$. Thus $2\pi/|\hat{\theta}|$ is an I-mesh of the modified sequence $\{X_n / X_{n_\nu}\}$. But $\{X_{n_\nu}\}$ is supposed to have annihilated this particular I-mesh. Thus we have a contradiction and must conclude that $L < 1$.

Since $L < 1$, there must be an integer k_0, and a number ρ with $0 < \rho < 1$, such that

$$(5.1) \qquad \inf_{n \leqq j \leqq n(1 + \epsilon_n^{1/2})} |\phi_j(\theta)| < \rho$$

for all $n \leqq k_0$ and all $\theta \in J$.

With no loss of generality we can assume $k \geqq 2$. We then construct a subsequence Φ, say , of $\{\phi_n(\theta)\}$ as follows:

(a) for $j=1,2,\ldots,k_0-1$, $\phi_j(\Theta)$ does <u>not</u> belong to Φ;

(b) for $k_0 \leqq j \leqq k_0(1 + \epsilon_{k_0}^{1/2}) - 1$, $\phi_j(\Theta)$ belongs to Φ;

(c) if k_1 is the least integer exceeding $k_0(1 + \epsilon_0^{1/2}) - 1$, then $\phi_{k_1-1}(\Theta)$ does <u>not</u> belong to Φ;

(d) for $k_1 \leqq j \leqq k_1(1 + \epsilon_{k_1}^{1/2}) - 1$, $\phi_j(\Theta)$ belongs to Φ;

(e) if k_2 is the least integer exceeding $k_1(1 + \epsilon_{k_1}^{1/2}) - 1$, then $\phi_{k_2-1}(\Theta)$ does not belong to Φ; and so on ad infinitum.

It is apparent that Φ is not a strongly insistent subsequence of $\{\phi_n(\Theta)\}$ because $\epsilon_n \to 0$ as $n \to \infty$. Let us assign the terms of $\{\phi_n(\Theta)\}$ to blocks in accordance with the blocking procedure described in section 2. The first (k_0-1) blocks will each contain just one member. Block B_{k_0}, however, contains all $\phi_j(\Theta)$ for which $k_0 \leqq j \leqq k_0(1+\epsilon_{k_0}^{1/2})$. Therefore, if we employ the λ - notation of section 2 for the number of terms in the blocks, the number of terms in B_{k_0} is $\lambda_{k_0} \leqq k_0\epsilon_{k_0}^{1/2}$. More generally, B_{r+k_0} contains all $\phi_j(\Theta)$ for which $k_r \leqq j \leqq k_r(1+\epsilon_{k_r}^{1/2})$, and so contains $\lambda_{r+k_0} \leqq k_r\,\epsilon_{k_r}^{1/2}$ terms.

In view of (5.1), and the fact that the modulus of a characteristic function never exceeds unity, we find that

$$\left| \prod_{\phi_j \in B_{r+k_0}} \phi_j(\Theta) \right| < \rho$$

for all $\Theta \in J$ and $r = 0,1,2,\ldots$. Therefore, if $n \geqq k_m$,

$$|\psi_n(\Theta)| \leqq \prod_{r=0}^{m-1} \left| \prod_{\phi_j \in B_{r+k_0}} \phi_j(\Theta) \right| < \rho^m ,$$

and so

$$\sum_{k_m \leq n < k_{m+1}} \left| \psi_n(\theta) \right| < \lambda_{m+k_0} \rho^m .$$

The present case of the lemma is now seen to be a consequence of lemma 2.

To deal with the case when J contains several I_m-angles (but only a finite number, of course) is now easy. We merely represent J as the union of a finite number of bounded closed subintervals each of which contains exactly one I_m-angle. The previous argument can be applied to each subinterval.

To complete the proof of the lemma we must discuss the case when J contains no I_m-angles. The argument needed for this case is similar to but somewhat simpler than the one we have given for the case when J contains exactly one I_m-angle. We define L as before, but with $\epsilon_n = n^{-2/3}$, say. We can then deduce from the hypothesis L = 1 the following facts: (a) $\{X_n\}$ contains an asymptotically lattice subsequence $\{X_{n_\nu}\}$ with asymptotic mesh $2\pi/|\hat{\theta}|$, say, where $\hat{\theta} \in J$; (b) if $\{\ell_n\}$ is the ℓ-sequence associated with $\{X_{n_\nu}\}$, then $\ell_n \geq n^{2/3}$ for infinitely many values of n. From (b) it is clear that $\{X_{n_\nu}\}$ is insistent, but not weakly insistent. An appeal to lemma 4 shows that $2\pi /|\hat{\theta}|$ is either an I_m-mesh or an I_s-mesh; since $\hat{\theta} \in J$ we have, in either event, a contradiction because J is supposed to contain neither I_m-angles nor I_s-angles. Thus L < 1 and the remainder of the proof proceeds exactly as before.

The lemma we have just proved gives us vital information about the behavior of the series $\Sigma\, \psi_j(\theta)$ in closed intervals which do not contain 0. It is also necessary, however, to examine the behavior of this series in a neighborhood of 0. The next two lemmas consider this problem.

Lemma 7. If X is a symmetric random variable with characteristic function $\phi(\theta)$; if, for some fixed small $\theta \neq 0$, $\epsilon^2 \geqq [1-\phi(\theta)]/\theta^2$ where ϵ is small; and if $A(\theta, \epsilon)$ is the set of real numbers which lie within $(3\epsilon)^{1/2}$ of an integral multiple of $2\pi/\theta$ (positive, negative, and zero multiples allowed); then

$$P\{X \in A(\theta, \epsilon)\} > 1 - \epsilon .$$

Proof. Denote by B the set of x-values for which $1-\cos\theta x \leqq \epsilon\,\theta^2$, and let B^c be the set of x-values complementary to B. Since X is symmetric,

$$\epsilon^2 \geqq \frac{1-\phi(\theta)}{\theta^2}$$

$$= E\left\{\frac{1-\cos\theta X}{\theta^2}\right\}$$

$$> \epsilon\, P\{X \in B^c\} .$$

Thus $P\{X \in B\} > 1-\epsilon$, and the lemma will be proved if we can show $B \subset A(\theta, \epsilon)$.

The set B consists of an infinite sequence of congruent intervals, of width 2κ, say, centered on the integral multiples of $2\pi/\theta$. Since ϵ is supposed small, κ is also small; we find an upper bound for κ by observing that, for all small x, we have $(1-\cos\theta x)/\theta^2 > x^2/3$, so that $\kappa^2/3 < \epsilon$, that is, $\kappa < (3\epsilon)^{1/2}$. Hence $B \subset A(\theta, \epsilon)$, and the lemma is proved.

Lemma 8. If ω is neither a strongly nor a mildly insistent mesh of the renewal sequence $\{X_n\}$, then for all sufficiently small $\eta > 0$

$$\int_0^\eta \theta^2 \sum_1^\infty |\psi_j(\theta)|\, d\theta < \infty .$$

Proof. By corollary 2.1 the sets $\mathcal{H}_m$ and $\mathcal{H}_s$ must be bounded. Thus we can choose $\eta > 0$ so that the interval $[0, \eta]$ contains neither I_m nor I_s-angles. By lemma 6, incidentally, $\Sigma|\psi_j(\theta)|$ will converge for all $0 < \theta \leqq \eta$.

Write J for the open interval $(0,\pi)$, $\bar{J}$ for its closure, and, for $\delta > 0$,

$$L_\delta = \lim_{k\to\infty} \inf_{\theta\in J} \inf_{n \geq k} \sup_{n \leq j \leq n + n^{1/3-\delta}} \left(\frac{1 - |\phi_j(\theta)|^2}{\theta^2}\right) .$$

We shall prove that $L_\delta > 0$ for some sufficiently small $\delta > 0$. To establish this let us make the hypothesis that $L_\delta = 0$ for all $\delta > 0$. Then there exist three sequences: (a) an unbounded increasing sequence of integers n_ν ; (b) a sequence of angles $\{\theta_\nu\}$ in J which may, with no loss of generality, be assumed convergent to a limit θ^*, say, in $\bar{J}$; (c) a decreasing sequence of real numbers $\{\delta_\nu\}$ which converges to 0. These three sequence, moreover, will have the property that, if we write

$$(5.2) \qquad \epsilon_\nu^2 = \sup_{n_\nu \leq j \leq n_\nu + n_\nu^{1/3-\delta_\nu}} \left(\frac{1 - |\phi_j(\theta_\nu)|^2}{\theta_\nu^2}\right),$$

then $\epsilon_\nu \to 0$ as $\nu \to \infty$.

Employ the notation of lemma 7 and write A_ν for the set $A(\theta_\nu, \epsilon_\nu)$. Observe, furthermore, that $|\phi_j(\theta)|^2$ is the characteriestic function of the symmetrized variable X_j^s. Then (5.2) and lemma 7 combine to show that

$$(5.3) \qquad 1 - \epsilon_\nu \leq \inf_{n_\nu \leq j \leq n_\nu + n_\nu^{1/3-\delta_\nu}} P\{X_j^s \in A_\nu\} .$$

Since $\epsilon_\nu \to 0$, (5.3) proves that $\{X_j^s\}$ contains an asymptotically lattice subsequence with asymptotic mesh $2\pi/|\theta^*|$ (which may be ∞ if $\theta^* = 0$). Lemma 3 then shows that the corresponding subsequence of $\{X_n\}$ has similar asymptotically lattice properties. But, by (5.3), this subsequence is insistent, and since $\delta_\nu \to 0$ it is not weakly insistent. Thus lemma 4 proves θ^* to be either an I_m-angle or an or an I_s-angle. This contradicts the hypothesis that $\bar{J}$ contains no such angles. Thus we must conclude that $L_\delta > 0$ for some $\delta > 0$.

It may be pointed out that if it could be supposed that $\theta^* \neq 0$ then the argument from (5.2) onward could be greatly simplified, and lemma 7 dispensed with. However, $\theta^* = 0$ is a possible value for an insistent angle and it is this possibility which makes the present argument necessary.

Having proved that $L_\delta > 0$ for some $\delta > 0$, we can infer the existence of an integer k_o and a real number $\rho > 0$ such that

$$\sup_{n \leqq j \leqq n + n^{1/3-\delta}} \frac{1 - |\phi_j(\theta)|^2}{\theta^2} > \rho$$

for all $\theta \in J$ and all $n \geqq k_o$. Thus, for each value of $\theta \in J$ there is a j such that $n \leqq j \leqq n + n^{1/3-\delta}$ and

$$1 - |\phi_j(\theta)|^2 > \rho\, \theta^2 ,$$

or

$$|\phi_j(\theta)| < 1 - \rho\, \theta^2 .$$

It appears, therefore, that for all $\theta \in J$ and all $n \geqq k_o$,

$$\left| \prod_{n \leqq j \leqq n + n^{1/3-\delta}} \phi_j(\theta) \right| < 1 - \rho\theta^2 . \tag{5.4}$$

Construct a subsequence Φ of $\{\phi_n(\theta)\}$ exactly as was done in the proof of lemma 6, but with the term $n(1 + \epsilon_n^{1/2})$, which is used in the earlier construction, replaced by the term $n + n^{1/3-\delta}$. The subsequence Φ so constructed is weakly insistent.

If we form blocks of terms in the same way as before and employ the same notation, then we find from (5.4) that

$$\sum_{k_m \leqq n < k_{m+1}} |\psi_n(\theta)| < \lambda_{m+k_o} (1 - \rho\, \theta^2)^m, \text{ for all } \theta \in J. \tag{5.5}$$

However, if we suppose η so small that $\rho\,\eta^2 < 1$, we have

$$\int_0^{\eta} \theta^2(1 - \rho\,\theta^2)^m\,d\,\theta = \frac{1}{2\rho^{3/2}}\int_0^{\rho\eta^2} u^{1/2}(1 - u)^m du$$

$$\leq \frac{1}{2\rho^{3/2}}\int_0^1 u^{1/2}(1 - u)^m\,du$$

$$= \frac{1}{2\rho^{3/2}}\quad\frac{\Gamma(\frac{3}{2})\;\Gamma(m + 1)}{\Gamma(m + \frac{5}{2})}$$

An appeal to Stirling's theorem then shows that

$$(5.6)\qquad \int_0^{\eta} \theta^2(1 - \rho\,\theta^2)^m\,d\theta = 0\;(m^{-3/2})\;.$$

The integral of an infinite sum of positive terms equals the sum of the integrals of the individual terms. The λ-sequence in the present case is derived from a weakly insistent subsequence. The lemma is therefore proved by (5.5), (5.6), and an appeal to lemma 1.

To close this section we introduce one further definition. Suppose $\hat{\theta}$ is a strongly insistent angle; and suppose further that, for all θ in some sufficiently small neighborhood of $\hat{\theta}$, the partial sums $\Psi_N(\theta) = \Sigma_{j=1}^{N}\,\psi_j(\theta)$ oscillate boundedly as $N \longrightarrow \infty$; in other words, suppose there are finite numbers η and B such that $|\Psi_N(\theta)| < B$ for all N and all θ in the interval $(\hat{\theta} - \eta,\ \hat{\theta} + \eta)$. Then, under these conditions, we shall say that $\hat{\theta}$ is removable; the reason for this terminology will appear later.

A strongly insistent mesh h is called removable if the I_s-angle $2\pi/h$ is removable.

It is to be noted that a removable I_s-angle can arise in the theory of continuous renewal processes. For $\hat{\theta} \neq 0$ to be such an angle, since $\phi_n(\theta)$ is independent of n, we must have $|\phi_n(\hat{\theta})| = 1$. Suppose therefore that for some real a we have $\phi_n(\hat{\theta}) = e^{i\hat{\theta}a}$; then it follows (Kolmogorov and Gnedenko [7] p. 59) that the X_n are, with probability one, restricted to values on the lattice $a + 2k\pi/\hat{\theta}$, where k takes integer values. Set $h = 2\pi/|\hat{\theta}|$. Then a and h must be incommensurable or the renewal process would be periodic; therefore a $\hat{\theta}$ cannot be a multiple of 2π and so $\phi_n(\hat{\theta}) \neq 1$. However, it is easy to show that $|\Psi_N(\theta)| < 2/|1-\phi(\theta)|$, for all N, so that (since characteristic function are continuous) $|\Psi_N(\theta)|$ is bounded uniformly in N in some sufficiently small neighborhood of $\hat{\theta}$. We have therefore proved

<u>Lemma 9.</u> <u>If $\{X_n\}$ is a continuous renewal process with $0 < E\{X_n\} \leqq \infty$, then the strongly-insistent meshes, if any, must form a sequence of the form h, h/2, h/3, ..., and they are all removable.</u>

The restriction on $E\{X_n\}$ rules out the silly possibility $P\{X_n = 0\} = 1$, which would make 0 a strongly insistent angle (which would not be removable).

Notice that weakly and mildly insistent angles cannot arise in connection with a renewal process.

It would be desirable to obtain some general structural properties of the renewal sequence $\{X_n\}$ which would ensure that a strongly insistent angle be removable. Such a problem seems very difficult indeed; it is not unrelated to problems of estimation of trigonometric sums, such as have been considered by Vinogradov [13]. We have been unable to obtain any worthwhile results here.

6. The uniformly bounded variation property of the renewal function

When $\{X_n\}$ is a renewal process it is well known that the associated renewal function has the property that for every $\epsilon > 0$ there is a finite $\delta(\epsilon)$ such that

$$H(x+\epsilon) - H(x) < \delta(\epsilon) \tag{6.1}$$

for all x. The proof of (6.1) is quite simple; for the case of a positive renewal process there is a proof in the paper of Blackwell [1], and the "positive" restriction is easy to remove from his proof.

We shall call (6.1) the uniformly bounded variation (U.B.V.) property of H(x); it will play a crucial role in our proof of the main theorem (theorem 4). The object of this section is to discuss conditions which will ensure that quite general renewal functions have the U.B.V. property.

$$(6.2) \qquad \Lambda_{pq}(x) = \sup_{n \geqq q} P\left\{X_{n+1} + X_{n+2} + \dots + X_{n+p} \leqq x\right\}.$$

Definition 2. If, for some values of p and q, there is a distribution function $K(x) \geqq \Lambda_{pq}(x)$ for all x, which is the distribution function of a random variable with a strictly positive (but possibly infinite) expectation, and if $E\{|\min(0,X_n)|\} < \infty$, for $n = 1,2,\dots, q-1$, then we shall say $\{X_n\}$ satisfies the condition $\mathfrak{A}$.

Theorem 3. If the renewal sequence $\{X_n\}$ satisfies condition $\mathfrak{A}$ then the associated renewal function has the U.B.V. property and is finite for all x.

Proof: Suppose first that we can put q = 0 and find p so that the function $\Lambda_{po}(x)$ of (6.2) is dominated by K(x), the distribution function of a variable Z, say, where $0 < E\{Z\} \leqq \infty$.

Fix a semiclosed interval $J \equiv (a,b]$, and let N be the number of partial sums S_n which fall in J. For each integer p satisfying

$0 \leqq p \leqq (k-1)$, let N_p be the number of partial sums of the kind S_{p+jk} $(j=0,1,2,\ldots)$ which fall in J. Then $N=N_0+N_1+\ldots+N_{k-1}$ and, since the expectation of a finite sum equals the sum of the expectations $E\{N\} = E\{N_0\} + E\{N_1\} + \ldots + E\{N_{k-1}\}$.

Let us fix p, $0 \leqq p \leqq (k-1)$, and write $K_{pn}(x) = P\{S_{p+nk} - S_{p+(n-1)k} \leqq x\}$. Then, by (6.2), $K_{pn}(x) \leqq K(x)$ for all x.

Suppose $\{Z_n\}$ is a renewal process with associated distribution function K(x). Define the random variable Y_{pn} as the greatest lower bound of numbers y such that $K_{pn}(y) \geqq K(Z_n)$. We shall assume, temporarily, that K(x) is a continuous function; in view of this assumed continuity, and of the monotone character of distribution functions, the inverse function $K^{-1}(x)$ is uniquely defined on $0 \leqq x \leqq 1$. It can therefore be seen that $Y_{pn} \leqq x$ if and only if $Z_n \leqq K^{-1}[K_{pn}(x)]$, so that $P\{Y_{pn} \leqq x\} = K\{K^{-1}[K_{pn}(x)]\}$. Thus the sequence $\{Y_{pn}\}$, for $n=1,2,3,\ldots$ represents a realization of the sequence $\{S_{p+nk} - S_{p+(n-1)k}\}$, for $n=1,2,3,\ldots$. Moreover, by the construction we have adopted, $Z_n \leqq Y_{pn}$ for all n.

Let $A_{pr}(J)$ be the event that $r(< \infty)$ is the least value of j for which $S_{p+jk} \in J$; let $A_{p\infty}(J)$ be the event that no $A_{pr}(J)$ happens; and let $\sim A_{p\infty}(J)$ be the event complementary to $A_{p\infty}(J)$. Then, trivially, $E\{N_p \mid A_{p\infty}(J)\} = 0$.

Suppose $A_{pr}(J)$ occurs; then N_p cannot exceed unity plus the total number of partial sums $Y_{r+1}, Y_{r+1}+Y_{r+2}, Y_{r+1}+Y_{r+2}+Y_{r+3}, \ldots$, and so on, which are $\leqq b$. As an example, if $Y_{p,r+1}+Y_{p,r+2} > b$, then necessarily $S_{p+(r+2)k} > a + b$.

Thus, since $Z_n < Y_{pr}$ for all n, N_p cannot exceed unity plus the total number of partial sums $Z_{r+1}, Z_{r+1} + Z_{r+2}, Z_{r+1} + Z_{r+2} + Z_{r+3}, \ldots$, and so on, which are $\leqq b$.

If L(x) is the renewal function of the renewal process $\{Z_n\}$ then it appears that

$$E\{N_p \mid A_{pr}(J)\} \leqq 1 + L(b),$$

for all r. Therefore, since $E\{N_p \mid A_{p\infty}(J)\} = 0$, we find

$$E\{N_p\} \leqq P\{\sim A_{p\infty}(J)\}\left[1 + L(b)\right],$$

and so

$$(6.3) \qquad E\{N\} \leqq \left[1 + L(b)\right] \sum_{p=0}^{k-1} P\{\sim A_{p\infty}(J)\}.$$

If H(x) is the renewal function associated with $\{X_n\}$ then (6.3) implies

$$H(a+b) - H(a) \leq k\left[1 + L(b)\right].$$

Recalling that $E\{Z\} > 0$ we remark that the "conventional" renewal function L(b) is necessarily finite, and the U.B.V. property of H(x) is then proved. But we assumed K(x) to be continuous and we must now show that this assumption does not matter.

Suppose therefore that K(x), the distribution function of Z, is discontinuous. Choose a small $\eta > 0$ and let the random variable Z_o be independent of Z and have a rectangular distribution over $(0, \eta)$. Let $\overline{K}(x)$ be the distribution function of $Z - Z_o$. Then $\overline{K}(x)$ is continuous; $\overline{K}(x) \geqq K_{pn}(x)$ for all p, n, x; and, if η is sufficiently small, $0 < E\{Z-Z_0\} < \infty$. We can thus use $\bar{K}(x)$ instead of $K(x)$ in the preceding argument. Furthermore, we remark that all of this argument will hold good if we take J to be the semi-infinite interval $(-\infty, x)$; thus we have proved incidentally that $H(x)$ is finite for all x. Combining the fact that $H(o)$ is finite and that $H(x)$ has the U.B.V. property it also appears that $H(x) = O(x)$ as $x \to \infty$.

To conclude the proof we consider the case when it is necessary to take $q > 0$ before a suitable value of p can be found. Let $H_q(x)$ be the renewal function associated with the sequence $X_q, X_{q+1}, X_{q+2}, \ldots$, ad infinitum. The foregoing discussion applies to $H_q(x)$; and we shall see later in this section that the condition $E\{|\min(0, X_n)|\} < \infty$, for $n=1,2,\ldots,q-1$, will ensure the existence of $H_q(x) * F_{q-1}(x)$. But

(6.4) $$H(x) = \sum_{j=1}^{q-1} F_j(x) + H_q(x) * F_{q-1}(x),$$

and from this equation it is easy to deduce that $H(x)$ has the requisite properties.

The case of the <u>positive</u> renewal sequence deserves special attention. Let us say that a sequence of random variables is <u>null</u> if it converges to zero in probability; then we have

<u>Corollary 3.1</u> <u>If $\{X_n\}$ is a positive renewal sequence which contains no insistent null subsequence then the associated renewal function has the U.B.V. property, and is finite for all x.</u>

<u>Proof</u>. Write

$$Q_\epsilon = \sup_k \inf_n P\{X_n + X_{n+1} + \ldots + X_{n+k} > \epsilon\}$$

and suppose $Q_\epsilon = 0$ for all $\epsilon > 0$. Then there must be a sequence $\{\epsilon_\nu\}$ where $\epsilon_\nu \to 0$ as $\nu \to \infty$, and two unbounded increasing sequence of of integers $\{n_\nu\}$, $\{k_\nu\}$, such that

$$(6.5) \qquad \lim_{\nu=\infty} P\{X_{n_\nu} + X_{n_\nu+1} + \ldots + X_{n_\nu+k_\nu} > \epsilon_\nu\} = 0.$$

Since the $\{X_n\}$ are nonnegative, (6.5) implies the presence of an insistent null sequence. Thus we are foreed to conclude that, for some $\epsilon>0$, $Q_\epsilon > 0$. We can therefore find an integer k, and a real number $\rho > 0$, such that

$$(6.6) \qquad P\{X_n + X_{n+1} + \ldots + X_{n+k}\} > \rho$$

for all n.

Define $K(x) = 0$, for $x < 0$

$= 1 - \rho/2$, for $0 \leqq x < \epsilon$,

$= 1$, for $\epsilon \leqq x$.

Then, if $\Lambda_{pq}(x)$ is the function defined in (6.2), we have $K(x) \geqq \Lambda_{kQ}(x)$ for all x. But K(x) is clearly the distribution function of a random variable with mean $\rho\epsilon/2 > 0$. Thus $\{X_n\}$ satisfies condition $\mathfrak{T}$ and the corollary is proved.

To bring the discussion of the U.B.V. property to a close, let us draw attention to the following simple consequences of assuming that H(x) is finite for all x and that it has the U.B.V. property.

(a) Since $\Sigma F_N(0)$ is convergent and $F_n(x) \leqq F_n(0)$ for all $x \leqq 0$, it follows that $\Sigma F_n(x)$ is uniformly convergent for $x \leqq 0$. But $F_n(x) \to 0$ as $x \to -\infty$, and so, since $H(x) = \Sigma F_n(x)$, it follows that $H(x) \to 0$ as $x \to -\infty$.

(b) We have seen that $H(x) = O(x)$, as $x \to \infty$, so there is a constant C such that $H(x) \leqq C\,x$ for all $x \geqq 1$. Thus, if $A(x)$ is any distribution function it is easy to see that

$$H(x) * A(x) \leq H(1) \int_{x-1}^{\infty} dA(z) + C\,|x|\,A(x-1) + C \int_{-\infty}^{x-1} |z|\, dA(z), \text{ and so}$$

$H(x) * A(x)$ will be necessarily finite if

$$\int_{-\infty}^{0} |z|\, dA(z) < \infty \ . \tag{6.7}$$

On the other hand, if we assume that $H(x) \geqq \epsilon x$ for some fixed $\epsilon > 0$ and for all large positive x, then it is equally easily shown that (6.7) must hold if $H(x) * A(x)$ is finite. Since $\liminf_{x = \infty} H(x)/x$ is usually positive it will be be appreciated why, to ensure that all the convolutions we encounter will be finite, we occasionally impose conditions like (6.7) on distribution functions (or equivalent conditions on the corresponding random variables). Moreover, if we are granted that $H(x) * A(x)$ is finite then it is an immediate consequence of Fubini's theorem that $H(x) * A(x) = A(x) * H(x)$, since both convolutions equal

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} U(x - z - y)\, dH(y)\, dA(z).$$

Thus not only does (6.7) ensure the finiteness of $H(x)*A(x)$ but it also allows us to write $H(x) * A(x) = A(x) * H(x)$.

7. Some special functions

This section is devoted to the discussion of certain special functions, closely related to the triangular distribution, which will be used in our proof of the main theorem.

Define the function $\Delta(x)$ by

$$\Delta(x) = 0 \quad \text{for } |x| > 1,$$
$$= 1 - x \text{ for } |x| \leqq 1.$$

For any $a > 0$ write $\Delta_a(x)$ for the function $a^{-1}\Delta(x\,a^{-1})$.

We shall denote the Fourier transform of an L_1-function $g(x)$, say, thus:

$$g^{\dagger}(\theta) = \int_{-\infty}^{+\infty} e^{i\theta x}\, g(x)\, dx.$$

Lemma 10.

$$\Delta_a^{\dagger}(\theta) = \frac{\sin^2(a\theta/2)}{(a\theta/2)^2} .$$

Proof: This result is well known and can be obtained by direct computation. Notice that $\Delta_a^{\dagger}(\theta) \geqq 0$ for all θ.

Let us next define the function

$$\delta_{a0}^{\dagger}(\theta) = \Delta_a^{\dagger}(\theta) \text{ for } |\theta| \leqq \frac{2\pi}{a} ,$$

$$= 0 \quad \text{otherwise;}$$

and for $n = 1,2,3,\ldots$, the functions

$$\delta_{an}^{\dagger}(\theta) = \Delta_a^{\dagger}(\theta) \text{ for } \frac{2n\pi}{a} \leqq |\theta| \leqq \frac{2(n+1)\pi}{a}$$

$$= 0 \quad \text{otherwise .}$$

Then, trivially,

$$(7.1) \qquad \Delta_a^{\dagger}(\theta) = \sum_{n=0}^{\infty} \delta_{an}^{\dagger}(\theta).$$

Less trivially, we have

Lemma 11. The functions $\{\delta_{an}^{\dagger}(\theta)\}$, for $n=0,1,2,\ldots$, are Fourier transforms of certain absolutely integrable functions $\{\delta_{an}(x)\}$; furthermore there is a finite constant C, which depends on a, such that

$$\left|\delta_{ao}(x)\right| \leqq \frac{C}{1+x^2},$$

for all x while for $n \geqq 1$,

$$\left|\delta_{an}(x)\right| \leq \frac{C}{n^2(1+x^2)},$$

for all x.

Proof: If we write $\delta_{an}(x)$ for the inverse Fourier transform of $\delta^{\dagger}_{an}(\theta)$ then, for $n \geqq 1$,

$$(7.2) \qquad \delta_{an}(x) = \frac{1}{\pi} \int_{\frac{2n\pi}{a}}^{\frac{2(n+1)\pi}{a}} (\cos \theta x) \frac{\sin^2 (a\theta/2)}{(a\theta/2)^2} d\theta,$$

$$= \frac{2}{a\pi} \int_{n\pi}^{(n+1)\pi} \left(\cos \frac{2\theta x}{a}\right) \frac{\sin^2\theta}{\theta^2} d\theta .$$

Thus, from (7.2),

$$(7.3) \qquad \left|\delta_{an}(x)\right| \leqq \frac{2}{a\pi} \int_{n\pi}^{(n+1)\pi} \frac{d\theta}{\theta^2} \leqq \frac{2}{a\pi^2 n^2} .$$

If we return to (7.2) and twice integrate by parts, we find

$$(7.4) \; \delta_{an}(x) = -\frac{a}{\pi x^2} \int_{n\pi}^{(n+1)\pi} \left(\cos \frac{2\theta x}{a}\right) \left\{ \frac{\cos^2\theta}{\theta^2} - \frac{\sin^2\theta}{\theta^2} - \frac{4\sin\theta\cos\theta}{\theta^3} + \frac{3\sin^2\theta}{\theta^4} \right\} d\theta.$$

For all $|\theta| \geqq \pi$, the integrand of (7.4) is dominated by a function A/θ^2, where A is some absolute constant. Thus we can infer from (7.4) that

$$(7.5) \quad |\delta_{an}(x)| \leqq \frac{a}{\pi x^2} \int_{n\pi}^{(n+1)\pi} \frac{A}{\theta^2}\, d\theta \leqq \frac{Aa}{\pi^2 n^2 x^2} .$$

Inequalities (7.4) and (7.5) combine to show the existence of a finite constant C, depending on a, such that

$$(7.6) \qquad |\delta_{an}(x)| < \frac{C}{n^2(1+x^2)} .$$

Incidentally, (7.6) proves that $\delta_{an}(x)$, like $\delta^{\dagger}_{an}(\theta)$, is an absolutely integrable function. Thus we can conclude that $\delta^{\dagger}_{an}(\theta)$ is the Fourier transform of $\delta_{an}(x)$, and the lemma is proved for the cases when $n \geqq 1$.

We can treat $\delta_{ao}(x)$ in a similar way; define it as the inverse Fourier transform of $\delta^{\dagger}_{ao}(\theta)$:

$$\delta_{ao}(x) = \frac{1}{2\pi} \int_{\frac{-2\pi}{a}}^{\frac{+2\pi}{a}} (\cos\theta x) \frac{\sin^2(a\theta/2)}{(a\theta/2)^2}\, d\theta$$

$$= \frac{1}{a\pi} \int_{-\pi}^{+\pi} (\cos \frac{2\theta x}{a}) \frac{\sin^2\theta}{\theta^2}\, d\theta .$$

From (7.7) we have, since $\sin^2\theta \leqq \theta^2$, that

$$(7.8) \qquad |\delta_{ao}(x)| \leqq \frac{1}{a\pi} \int_{-\pi}^{+\pi} d\theta = \frac{2}{a} \; .$$

Moreover, if we twice integrate (7.7) by parts we find

$$(7.9)\, \delta_{ao}(x) = -\frac{a}{2\pi x^2} \int_{-\pi}^{+\pi} (\cos\frac{2x\theta}{a}) \left\{ \frac{\cos^2\theta}{\theta^2} - \frac{\sin^2\theta}{\theta^2} - \frac{4\sin\theta\cos\theta}{\theta^3} + \frac{3\sin^2\theta}{\theta^4} \right\} d\theta \; .$$

The expression in the braces under the integral sign in (7.9) is approximately $-1/3$ for small θ and so this expression is a bounded function of θ. Thus we can deduce from (7.9) that, for some absolute constant A,

$$(7.10) \qquad |\delta_{ao}(x)| \leqq \frac{a}{2\pi x^2} \int_{-\pi}^{+\pi} A \; d\theta = \frac{Aa}{x^2} \; .$$

The remainder of the proof uses (7.8) and (7.10) and proceeds as before.

Lemma 12.

$$\Delta_a(x) = \sum_{n=o}^{\infty} \delta_{an}(x) \; .$$

Proof. From (7.1) we have, formally,

$$\Delta_a(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\theta x} \Delta_a^{\dagger}(\theta) \, d\theta$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\theta x} \left\{ \sum_{n=0}^{\infty} \delta_{an}^{\dagger}(\theta) \right\} d\theta$$

$$= \sum_{n=0}^{\infty} \delta_{an}(x).$$

The inversion of summation and integration is justified by bounded convergence, since

$$\int_{-\infty}^{+\infty} \delta_{an}^{\dagger}(\theta)\, d\theta = O\left(\frac{1}{n^2}\right) .$$

Lemma 13. For every $n \geqq 0$, the function $\delta_{an}(x)$ can be differentiated any number of times, and the derivatives so obtained are bounded functions of x.

Proof: Equations (7.2) and (7.7) show that $\delta_{an}(x)$ can always be represented as a trigonometric integral over a finite range; consequently differentiation under the integral sign is easily justified. For $n \geqq 1$ one obtains, for example, that

$$\delta'_{an}(x) = -\frac{4}{a^2\pi} \int_{n\pi}^{(n+1)\pi} \left(\sin \frac{2\theta x}{a}\right) \frac{\sin^2\theta}{\theta}\, d\theta .$$

Thus

$$|\delta'_{an}(\theta)| \leqq \frac{4}{n\, a^2 \pi} ,$$

for all x. A similar argument will apply to higher derivatives.

8. A general renewal theorem

We are now in a position to prove our main theorem.

Theorem 4. Let $\{X_n\}$ be a renewal function. Suppose that

(a) H(x) is always finite, and has the U.B.V. property (which will be the case if $\{X_n\}$ satisfies condition $\mathcal{J}$ of section 6);

(b) ∞ is neither a mildly nor a strongly insistent mesh;

(c) there is no limit point of $\mathcal{H}_m \cup \mathcal{H}_s$, except, possibly, 0;

(d) Any meshes in $\mathcal{H}_s$ are removable; then, if $k_1(x)$ and $k_2(x)$ are arbitrary members of $\mathcal{K}$,

$$\|k_2\|\, k_1(x) * H(x) - \|k_1\|\, k_2(x) * H(x) \longrightarrow 0,$$

as $x \to \infty$.

Proof: By lemma 11, for $n \geqq 1$,

$$\left| \delta_{an}(x) * H(x) \right| \leqq \frac{6}{n^2} \int_{-\infty}^{+\infty} \frac{dH(z)}{1 + (x-z)^2},$$

and so, because $H(x)$ has the U.B.V. property, there must be a finite constant C_1, say, such that

(8.1) $$\left|\delta_{an}(x) * H(x)\right| \leqq \frac{C_1}{n^2},$$

for all $n \geqq 1$, all x.

An appeal to lemma 12, the inequalities (8.1), and the theorem on bounded convergence, shows that

(8.2) $$\left|\Delta_a(x) * H(x)\right| = \sum_{n=0}^{\infty} \delta_{an}(x) * H(x).$$

But (8.1) shows, in addition, that the series on the right hand side of (8.2) is uniformly convergent. Thus, if the limits exist,

(8.3) $$\lim_{x=\infty} \left\{ \Delta_a(x) - \delta_{ao}(x) \right\} * H(x) = \sum_{n=1}^{\infty} \lim_{x\to\infty} \delta_{an}(x) * H(x),$$

That these limits do indeed exist is a consequence of

Lemma 14. Under the conditions of theorem 4, for all $n \geqq 1$ $\delta_{an}(x) * H(x) \longrightarrow 0$, as $x \longrightarrow \infty$.

Proof: If we define, for large N,

$$H_N(x) = \sum_{j=N+1}^{\infty} F_j(x),$$

then the variation of $H_N(x)$ over an interval is clearly not greater than the corresponding variation of $H(x)$. Thus, for $n \geqq 1$ and some large number M, by lemma 11,

$$(8.4) \quad \left| \int_{|x-z|>M} \delta_{an}(x-z)\, d\, H_N(z) \right| < \frac{C}{n^2} \int_{|x-z|\geq M} \frac{d\, H(z)}{1+(x-z)^2}$$

$$< \epsilon$$

if we choose M, independently of N, sufficiently large and appeal to the U.B.V. property of $H(x)$.

However, since $\Sigma_1^{\infty} F_j(x)$ converges for all x we can find N, depending upon M and x, so that

$$\sum_{j=N+1}^{\infty} F_j(x+M) < \frac{\epsilon}{M} .$$

Thus, again appealing to lemma 11,

$$(8.5) \quad \left| \int_{|x-z|\leqq M} \delta_{an}(x-z)\, d\, H_N(z) \right| < \frac{C}{2n^2} \cdot 2M \cdot \frac{\epsilon}{M} = \frac{C\epsilon}{n^2} .$$

From (8.4) and (8.5) it follows that

$$\int_{-\infty}^{+\infty} \delta_{an}(x-z)\, d\, H_N(z) \longrightarrow 0,$$

as $N \to \infty$, and so, since

$$H(x) = \sum_{j=1}^{N} F_j(x) + H_N(x) ,$$

we have that

$$(8.6) \qquad \delta_{an}(x) * H(x) = \lim_{N=\infty} \sum_{j=1}^{N} \delta_{an}(x) * F_j(x).$$

In view of lemma 13 it is easy to show that

$$(8.7) \qquad \sum_{j=1}^{N} \delta_{an}(x) * F_j(x)$$

has a bounded derivative everywhere; thus, in particular, in any small interval (8.7) is continuous and of bounded variation. The Fourier transform of (8.7) is evidently $\delta^{\dagger}_{an}(\theta) \sum_{j=1}^{N} \psi_j(\theta)$, which, since it is bounded and vanishes outside a finite interval, is absolutely integrable. Thus we can apply a familiar Fourier inversion theorem (for example, Titchmarsh [11], p. 42) to deduce that, for all x,

$$(8.8) \quad \sum_{j=1}^{N} \delta_{an}(x) * F_j(x) = \frac{1}{2\pi} \int_J e^{-i\theta x} \delta^{\dagger}_{an}(\theta) \left\{ \sum_{j=1}^{N} \psi_j(\theta) \right\} d\theta ,$$

where J is the bounded set $2n\pi/a \leqq |\theta| \leqq 2(n+1)\pi/a$.

Suppose J contains ν strongly insistent angles. By condition (c) of theorem 4, the number ν must be finite, for otherwise $\mathcal{H}_m \cup \mathcal{H}_s$ would contain a nonzero limit point. Let us enclose each of the ν strongly insistent angles in open intervals $J_1, J_2, \ldots, J_\nu$, each of width η, say. Since the strongly insistent angles are removable, we can make η small enough for there to be a finite constant C_2 such that $\left|\sum_{j=1}^{N} \psi_j(\theta)\right| < C_2$ for all N and all $\theta \in J_1 \cup J_2 \cup \ldots \cup J_\nu$. Now enclose the ν I_s-angles in open intervals $J_1', J_2', \ldots, J_\nu'$ within the intervals $J_1, J_2, \ldots, J_\nu$, respectively, and having width $\epsilon/(\nu C_2)$. Let $J' = J_1' \cup J_2' \cup \ldots \cup J_\nu'$. Then, using lemma 11, we find that

$$\left|\frac{1}{2\pi}\int_{J'} e^{-i\theta x}\delta_{an}^{\dagger}(\theta)\left\{\sum_{j=1}^{N}\psi_j(\theta)\right\}d\theta\right| < \frac{1}{2\pi}\cdot\frac{C}{2n^2}\cdot C_2\cdot\frac{\epsilon}{C_2} = \frac{C\epsilon}{4n^2\pi}.$$

By taking ϵ sufficiently small, therefore, we can make

$$\frac{1}{2\pi}\int_{J'} e^{-i\theta x}\delta_{an}^{\dagger}(\theta)\left\{\sum_{j=1}^{N}\psi_j(\theta)\right\}d\theta \tag{8.9}$$

as small as we please, uniformly with respect to both x and N.

The set J-J' consists of a finite number of closed intervals. Let J" be a typical such interval. Then J" contains no strongly insistent angles, and at most finitely many mildly insistent angles (otherwise $\mathcal{H}_m \cup \mathcal{H}_s$ would have a nonzero limit point again); furthermore J" does not contain 0. Thus we can appeal to lemma 6 to infer that $\Sigma_1^\infty|\psi_j(\theta)|$ is boundedly convergent in J". By the theorem on bounded convergence it follows that

$$\lim_{N\to\infty}\frac{1}{2\pi}\int_{J''} e^{-i\theta x}\delta_{an}^{\dagger}(\theta)\left\{\sum_{j=1}^{N}\psi_j(\theta)\right\}d\theta = \frac{1}{2\pi}\int_{J''} e^{-i\theta x}\delta_{an}^{\dagger}(\theta)\left\{\sum_{j=1}^{\infty}\psi_j(\theta)\right\}d\theta.$$

But $\Sigma_1^\infty\psi_j(\theta)$ and $\delta_{an}^{\dagger}(\theta)$ are bounded functions of θ in the interval J", so we can appeal to the Riemann-Lebesgue lemma to deduce that

$$\lim_{x\to\infty}\lim_{n\to\infty}\frac{1}{2\pi}\int_{J''} e^{-i\theta x}\delta^{+}_{an}(\theta)\left\{\sum_{j=1}^{N}\psi_j(\theta)\right\}d\theta = 0 .$$

Since J-J' is the union of finitely many intervals like J" it follows that

$$(8.10)\quad \lim_{x\to\infty}\lim_{N\to\infty}\frac{1}{2\pi}\int_{J-J'} e^{-i\theta x}\delta^{+}_{an}(\theta)\left\{\sum_{j=1}^{N}\psi_j(\theta)\right\}d\,\theta = 0.$$

From (8.8), (8.10), and the fact that (8.9) can be made arbitrarily small, uniformly with respect to x and N, we can deduce

$$\lim_{x\to\infty}\lim_{N\to\infty}\sum_{j=1}^{N}\delta_{an}(x) * H(x) = 0 .$$

The lemma can now be proved by an appeal to (8.6).

Lemma 14 and (8.3) provide us with the important result that, as $x \to \infty$,

$$(8.11)\quad \left\{\Delta_a(x) - \delta_{ao}(x)\right\} * H(x) \to 0.$$

A consequence of (8.11), trivially easy to deduce, is that if c is any real constant then, as $x \to \infty$,

$$(8.12)\quad \left\{\Delta_a(x-c) - \delta_{ao}(x-c)\right\} * H(x) \to 0.$$

Let $\mathcal{G}$ be the class of nonnegative functions which can be represented as finite sums of the form

$$(8.13)\qquad g(x) = \sum_{j=1}^{\gamma} w_j\,\Delta_{a_j}(x - c_j) ,$$

where γ is an integer, $w_1, w_2, \ldots, w_\gamma$ are positive weights, and $a_1, a_2, \ldots, a_\gamma, c_1, c_2, \ldots, c_\gamma$ are finite real numbers (the a_j being positive).

Let $\mathcal{G}_0$ be the subclass of functions g(x) which belong to $\mathcal{G}$ and have the additional property that

$$(8.14) \qquad \int_{-\infty}^{+\infty} x\, g(x)\, dx = 0; \int_{-\infty}^{+\infty} g(x)\, dx = 1.$$

<u>Lemma 15</u>. <u>If</u> $g(x) \in \mathcal{G}_0$ <u>and has the representation</u> (8.13) <u>then, as</u> $x \to \infty$,

$$\left\{ \delta_{ao}(x) - \sum_{j=1}^{\gamma} w_j\, \delta_{a_j o}(x-c_j) \right\} * H(x) \to 0.$$

<u>Proof</u>. Write

$$(8.15) \qquad \Omega(x) = \delta_{ao}(x) - \sum_{j=1}^{\gamma} w_j\, \delta_{a_j o}(x-c_j).$$

Then, on taking Fourier transforms,

$$(8.16) \qquad \Omega^\dagger(\theta) = \delta^\dagger_{ao}(\theta) - \sum_{j=1}^{\gamma} w_j\, \delta^\dagger_{a_j o}(\theta)\, e^{i\theta c_j}.$$

Plainly, $\Omega^\dagger(\theta)$ is a bounded function of θ which vanishes outside some bounded interval containing 0. Moreover, for θ sufficiently small we can write

$$(8.17) \qquad \Omega^\dagger(\theta) = \Delta^\dagger_a(\theta) - g^\dagger(\theta)$$

because, for each value of a, $\delta^\dagger_a(\theta) = \Delta^\dagger_a(\theta)$ for all sufficiently small θ. Thus, near 0, $\Omega^\dagger(\theta)$ equals the difference between two characteristic functions of distributions with zero means and finite variances. We may therefore infer that, for some finite constant C_3, say, (8.18) $$\left|\Omega^{+}(\theta)\right| < c_3\,\theta^2$$

for all sufficiently small θ.

The results (8.6) and (8.8) were actually proved only for $n \geqq 1$, but slight modifications of their proofs (in respect of certain bounds) will extend them to the case $n = 0$. It then follows that, since $\Omega(x)$ is only a finite weighted sum of the functions $\delta_{ao}(x)$, $\delta_{a_1o}(x)$, . . ., and so on,

(8.19) $$\Omega(x) * H(x) = \lim_{N\to\infty} \frac{1}{2\pi} \int_{J_o} e^{-i\theta x}\,\Omega^{+}(\theta)\left\{\sum_{j=1}^{N} \psi_j(\theta)\right\} d\theta,$$

where J_o is some bounded closed interval containing 0.

The set $\mathcal{H}_m \cup \mathcal{H}_s$ must be bounded, for otherwise ∞ would be a limit point of $\mathcal{H}_m \cup \mathcal{H}_s$ and so, by corollary 2.1, would itself belong to $\mathcal{H}_m \cup \mathcal{H}_s$ in contradiction to condition (b) of theorem 4. Thus by lemma 8 we can find a small $\eta > 0$ such that

$$\int_0^{\eta} \theta^2 \left\{\sum_{1}^{\infty} \left|\psi_j(\theta)\right|\right\} d\theta < \infty .$$

In view of this result, (8.18), and the fact that $|\psi_j(-\theta)| = |\psi_j(\theta)|$ for all real θ, we can appeal to the theorem on dominated convergence to infer that

(8.20) $$\lim_{N\to\infty} \frac{1}{2\pi}\int_{-\eta}^{+\eta} e^{-i\theta x}\,\Omega^{+}(\theta)\left\{\sum_{j=1}^{N} \psi_j(\theta)\right\}d\theta = \frac{1}{2\pi}\int_{-\eta}^{+\eta} e^{-i\theta x}\,\Omega^{+}(\theta)\left\{\sum_{j=1}^{\infty} \psi_j(\theta)\right\}d\theta .$$

We can further state that $\Omega^{+}(\theta)\left\{\sum_{j=1}^{\infty} \psi_j(\theta)\right\}$ is absolutely integrable over the interval $(-\eta, +\eta)$, and then deduce from (8.20), by another appeal to the Riemann-Lebesgue lemma, that

$$(8.21) \quad \lim_{x\to\infty} \lim_{N\to\infty} \frac{1}{2\pi} \int_{-\eta}^{+\eta} e^{-i\theta x} \Omega^{\dagger}(\theta) \Big\{ \sum_{j=1}^{N} \psi_j(\theta) \Big\} d\theta = 0.$$

Let J_o' be the set of points in J_o but not in $(-\eta, +\eta)$. Clearly $\Omega^{\dagger}(\theta)$ is bounded on J_o', and so we can treat each of the two intervals comprising J_o' much as we treated the interval J earlier in this section, and deduce

$$(8.22) \quad \lim_{x\to\infty} \lim_{N\to\infty} \frac{1}{2\pi} \int_{J_o'} e^{-i\theta x} \Omega^{\dagger}(\theta) \Big\{ \sum_{j=1}^{N} \psi_j(\theta) \Big\} d\theta = 0 .$$

The lemma is proved by (8.19), (8.21), and (8.22).

Let us now return to the proof of theorem 4. If $g(x) \in \mathcal{G}_o$ and has representation (8.13) then it follows from (8.12) that, as $x \to \infty$,

$$(8.23) \quad \Big\{ g(x) - \sum_{j=1}^{\gamma} w_j \, \delta_{a_j o}(x - c_j) \Big\} * H(x) \to 0.$$

Hence, from (8.11), (8.23), and lemma 15 we find that, as $x \to \infty$,

$$(8.24) \quad \Big\{ \Delta_a(x) - \frac{g(x)}{\|g\|} \Big\} * H(x) \to 0,$$

for every $g(x) \in \mathcal{G}_o$.

Next suppose $g(x)$ belongs to $\mathcal{G}$, but not necessarily to $\mathcal{G}_o$. Write

$$\alpha = \int_{-\infty}^{+\infty} g(x)\, dx \; ; \quad \beta = \int_{-\infty}^{+\infty} x\, g(x)\, dx \quad .$$

Since functions in $\mathcal{G}$ are nonnegative we may assume $\alpha > 0$. Define, for some small $\epsilon > 0$,

$$\bar{g}(x) = \frac{1}{(\alpha+\epsilon)}\left\{g(x) + \epsilon\,\Delta_1\left(x + \frac{\beta}{\epsilon}\right)\right\} .$$

It may be verified that $\bar{g}(x) \in \mathcal{C}_{\gamma_0}$, and so

$$(8.25) \qquad \left|\left\{\Delta_a(x) - \bar{g}(x)\right\} * H(x)\right| < \epsilon$$

for all sufficiently large x. But, because H(x) has the U.B.V. property, there must be a finite constant C_4 such that, for all x,

$$\left|\Delta_1\left(x + \frac{\beta}{\epsilon}\right) * H(x)\right| < C_4$$

and

$$\left| g(x) * H(x)\right| < C_4 .$$

(recall that g(x) is a finite linear combination of "triangle" functions.)

Further,

$$\bar{g}(x) - \frac{1}{(\alpha+\epsilon)}\,g(x) = \frac{\epsilon}{(\alpha+\epsilon)}\,\Delta_1\left(x + \frac{\beta}{\epsilon}\right) ,$$

so that, for all x,

$$(8.26) \quad \left|\bar{g}(x) * H(x) - \frac{1}{(\alpha+\epsilon)}\,g(x) * H(x)\right| < \frac{C_4\epsilon}{(\alpha+\epsilon)} .$$

If we now appeal to (8.25) and (8.26) we discover that for all sufficiently large x,

$$\left|\Delta_a(x) * H(x) - \frac{1}{\alpha}\,g(x) * H(x)\right|$$

$$< \epsilon + \frac{C_4\epsilon}{(\alpha+\epsilon)} + \frac{\epsilon}{\alpha(\alpha+\epsilon)}\left|g(x) * H(x)\right|$$

$$< \epsilon + \frac{C_4\epsilon}{(\alpha+\epsilon)} + \frac{C_4\epsilon}{\alpha(\alpha+\epsilon)} .$$

Since ϵ can be chosen arbitrarily small we conclude that (8.24) holds for all $g \in \mathcal{G}$.

Next suppose $g(x)$ is a step function; in particular, suppose there are finite constants a, b, such that

$$g(x) = 1 \quad \text{if } a < x \leqq b,$$
$$= 0 \quad \text{otherwise.}$$

Choose a large integer N and put $\eta = (b-a)/2N$. Define

$$g_+(x) = \eta \sum_{r=0}^{2N} \Delta_\eta (x - a - r\eta),$$

$$g_-(x) = \eta \sum_{r=1}^{2N-1} \Delta_\eta (x - a - r\eta) .$$

Then

$$g_+(x) \geqq g(x) \geqq g_-(x) \geqq 0,$$

for all x, so that

$$g_+(x) * H(x) \geqq g(x) * H(x) \geqq g_-(x) * H(x), \tag{8.27}$$

for all x.

The functions $g_+(x)$ and $g_-(x)$ are both in $\mathcal{G}$ so we have, for all sufficiently large x,

$$\begin{aligned} g_+(x) * H(x) &< \|g_+\| \Delta_a(x) * H(x) + \eta, \\ g_-(x) * H(x) &> \|g_-\| \Delta_a(x) * H(x) - \eta . \end{aligned} \tag{8.28}$$

However, by the U.B.V. property of $H(x)$, there must be a finite constant C_5 such that $\Delta_a(x) * H(x) < C_5$ for all x. Also,

$\|g_+\| = \|g\| + \eta$ and $\|g_-\| = \|g\| - \eta$. Thus, for all sufficiently large x, by (8.27) and (8.28),

$$\|g\| \Delta_a(x) * H(x) - (C_5+1)\eta < g(x) * H(x) < \|g\| \Delta_a(x) * H(x) + (C_5+1)\eta \ .$$

We may take N very large and thereby make η arbitrarily small. Thus (8.24) must hold when g(x) is a step function. It is a straightforward matter to extend this result and show that (8.24) holds when g(x) is a simple function, that is, when g(x) is a finite linear combination of step functions.

Next suppose k(x) vanishes for all $|x|$ greater than some large N, and is bounded, nonnegative, and Riemann-integrable in $[-N, +N]$. Then, for any prescribed $\epsilon>0$, we can determine simple functions $g_+(x)$, $g_-(x)$, such that $g_-(x) \leqq k(x) \leqq g_+(x)$, for all x, and $\|g_+ - g_-\| < \epsilon$. An argument similar to the one which we have just used to show that (8.24) holds for step functions will now prove that (8.24) also holds for functions like k(x) in the place of g(x).

Finally, suppose k(x) is an arbitrary member of $\mathcal{K}_+$. Let us write

$$k_N(x) = k(x) \qquad \text{for } |x| \leqq N \ ,$$
$$= 0 \qquad \text{otherwise;}$$

and let us put $k^N(x) = k(x) - k_N(x)$. By the property $\mathcal{K}_2$ of the class $\mathcal{K}$, given any $\epsilon > 0$ we can choose N so large that

$$\sum_{n=-\infty}^{+\infty} \max_{n< x \leqq (n+1)} k^N(x) < \epsilon \ .$$

By the U.B.V. property of H(x) we can find a finite C_6 such that

$H(x+1) - H(x) < C_6$ for all x. Thus

$$(8.29) \quad k^N(x) * H(x) = \sum_{n=-\infty}^{+\infty} \int_{x-n-1}^{x-n} k^N(x-z)\, d\, H(z)$$

$$< C_6\, \epsilon .$$

We have already proved that (8.24) holds for functions like $k_N(x)$ in the place of $g(x)$. Thus, for all sufficiently large x,

$$\|k_N\|\, \Delta_a(x) * H(x) - \epsilon < k_N(x) * H(x) < \|k_N\|\, \Delta_a(x) * H(x) + \epsilon .$$

Hence, using (8.29) and the fact that $k(x) = k_N(x) + k^N(x)$, we have, for all sufficiently large x,

$$(8.30) \quad \|k_N\|\, \Delta_a(x) * H(x) - \epsilon < k(x) * H(x) < \|k_N\| \Delta_a(x) * H(x) + (1+C_6)\epsilon .$$

But we must have $\|k^N\| < \epsilon$ because of the way we chose N and so $\|k_N\| < \|k\| < \|k_N\| + \epsilon$. Therefore we can deduce from (8.30) that, for all sufficiently large x,

$$(8.31) \quad \|k\|\, \Delta_a(x) * H(x) - (1+C_7)\epsilon < k(x) * H(x) < \|k\|\, \Delta_a(x) * H(x) + (1+C_6)\epsilon ,$$

where C_7 is a finite constant such that $\Delta_a(x) * H(x) < C_7$ for all x. The theorem is proved by (8.31) for all $k(x) \in \mathcal{K}_+$ and, since an arbitrary member of $\mathcal{K}$ can always be represented as the difference between two functions in $\mathcal{K}_+$, its validity for all $k(x) \in \mathcal{K}$ is immediately deducible.

9. Application of the general theorem to renewal processes

As an example of the application of theorem 4 let us consider the case of a continuous renewal process $\{X_n\}$ with $0 < E\{X_n\} \leqq \infty$. We know that the associated renewal function $H(x)$ satisfies condition (a) of theorem 4, and lemma 9 shows that the remaining conditions of theorem 4 are also satisfied. Thus the conclusion of theorem 4 applies to the present renewal function.

Suppose $F(x)$ is the distribution function of the $\{X_n\}$. If $0 < E\{X_n\} < \infty$, and if we write $k_1(x) = U(x) - F(x)$, then it is easy to show that $\|k_1\| = E\{X_n\}$ and that $k_1(x) \in \mathcal{K}$. But the integral equation of renewal theory (see for example, Smith [10]) states that $k_1(x) * H(x) = F(x)$, which implies that $k_1(x) * H(x) \to 1$ as $x \to \infty$. Thus, by theorem 4, for any $k(x) \in \mathcal{K}$ we have

$$k(x) \quad H(x) \to \frac{\|k\|}{E\{X_n\}}, \quad \text{as} \quad x \to \infty.$$

If $E\{X_n\} = \infty$ we modify our argument slightly. Choose a large positive N, and put

$$\begin{aligned} k_{1N}(x) &= k_1(x) \qquad \text{for } x \leqq N, \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

The integral equation of renewal theory still applies and, since $k_{1N}(x) * H(x) \leqq k_1(x) * H(x)$ for all x, it enables us to infer that

$$\limsup_{x \to \infty} \quad k_{1N}(x) * H(x) \leqq 1 \quad . \tag{9.1}$$

But, as is easy to see, $k_{1N}(x) \in \mathcal{K}$ and so, from theorem 4 and (9.1), we have that for any $k(x) \in \mathcal{K}_+$

$$\lim_{x \to \infty} \sup \quad k(x) * H(x) \leq \frac{\|k\|}{\|k_{1N}\|}. \tag{9.2}$$

However, since $E\{X_n\} = \infty$, we can make $\|k_{1N}\|$ arbitrarily large by choice of N. Since $k(x) * H(x) \geqq 0$ we can thus infer from (9.2) that $k(x) * H(x) \to 0$ as $x \to \infty$. We have therefore proved

Theorem 5. If $H(x)$ is the renewal function of a continuous renewal process $\{X_n\}$ with $0 < E\{X_n\} \leqq \infty$, then $k(x) * H(x) \to k / E\{X_n\}$, as $x \to \infty$, for every $k(x) \in \mathcal{K}$.

Theorem 5 generalizes the renewal theorem 1 of Smith [9] by broadening the kernel class and removing the restriction to positive renewal processes. There is nothing surprising in either of these improvements. However, in another paper of Smith [10] an attempt was made to show that theorem 5 would be false if $\mathcal{K}$ were replaced by $L_1(-\infty, +\infty)$. Unfortunately, the counterexample was incorrectly described in [10] (the kernel as described there was a member of $\mathcal{K}$). Since it is of some interest to see that the kernel class $\mathcal{K}$ of theorem 5 cannot be much widened, we give here a correct counter-example. Suppose, as in [10], that the $\{X_n\}$ take the values 1 and $\sqrt{2}$ with equal probabilities of one-half. Then $\{X_n\}$ is a continuous renewal process such that $H(x)$ can only increase at x-values of the form $r + s\sqrt{2}$, where r and s are positive integers. Write A for the set of all numbers in the interval (0,1) of the form $r + s\sqrt{2}$, where here r and s are positive or negative integers or zero. The set A is countable and so has measure zero. Define $k(x)$ to be zero on A and on the complement of (0, 1); define $k(x)$ to be unity where it is not zero. Then $k(x)$ is in $L_1(-\infty, +\infty)$ and

$$\liminf_{x \to \infty} k(x) * H(x) = 0.$$

Thus theorem 5 does not hold. It is possible to show that the points of A are everywhere dense in (0, 1) so that k(x) is discontinuous at every point of (0, 1). A necessary and sufficient condition for a bounded function to be Riemann-integrable in an interval is that its points of discontinuity in that interval shall form a set of measure zero. Thus the k(x) we have just defined is not Riemann-integrable.

10. A renewal theorem for stochastically stable sequences

The general renewal theorem, theorem 4, covers a very wide class of renewal sequences. In this section we shall demonstrate that by restricting this class somewhat we can draw a firmer conclusion than theorem 4 allows. More precisely, we shall establish conditions under which, if $k(x) \in \mathcal{K}$ and H(x) is the renewal function, then

$$\lim_{x \to \infty} k(x) * H(x) \tag{10.1}$$

exists.

Cox and Smith [5] introduced the notion of a <u>stable sequence</u> of constants. If $\{A_n\}$ is a sequence of real numbers such that

$$\frac{1}{p}(A_{n+1} + A_{n+2} + \cdots + A_{n+p})$$

tends to some finite limit A, uniformly in n, as $p \to \infty$, then $\{A_n\}$ is <u>stable with average A</u>. It is proved in [5] that a stable sequence is necessarily bounded. We wish to extend the notion of a stable sequence to cover sequences of random variables.

<u>Definition 3. Suppose $\{X_n\}$ is a renewal sequence, and</u> μ <u>some finite constant; suppose we write, for</u> $\xi \geq 0$,

$$\Pi_p(\xi) = \sup_{n \geq 0} P\left\{ \left| \frac{X_{n+1} + X_{n+2} + \cdots + X_{n+p}}{p} - \mu \right| \geq \xi \right\}. \tag{10.2}$$

<u>If μ can be chosen so that</u>

$$(10.3) \qquad \int_0^\infty \Pi_p(\xi)\, d\xi \longrightarrow 0$$

<u>as</u> p $\longrightarrow \infty$, <u>then</u> $\{X_n\}$ <u>is stochastically stable (S.S.) with average μ.</u>

Clearly $\Pi_p(\xi)$ is a nonincreasing function so that, for any fixed $\epsilon > 0$, $\int_0^\infty {}_p(\xi)\, d\xi > \epsilon \Pi_p(\epsilon)$. Hence, if $\{X_n\}$ is S. S. with average μ, for every fixed $\epsilon > 0$ we have

$$P\left\{\left|\frac{X_{n+1} + X_{n+2} + \cdots + X_{n+p}}{p} - \mu\right| \geq \epsilon\right\} \longrightarrow 0,$$

as p $\longrightarrow \infty$, uniformly with respect to n.

It is to be remarked that if $\{X_n\}$ is S.S. then $\int_0^\infty \Pi_p(\xi)\, d\,\xi)$ is necessarily finite for all p. This may be seen as follows. There must be a q such that $\int_0^\infty \Pi_p(\xi)\, d\xi\}$ is finite for all $p \geq q$. Suppose that $m < q$. Choose any fixed ν such that $\Pi_q(\nu) < 1/2$. Then, by the independence of the $\{X_n\}$,

$$2P\left\{X_{n+1} + \cdots + X_{n+m+q} - (m+q)\mu \geq (m+q)\xi\right\}$$

$$> 2P\left\{X_{n+1} + \cdots + X_{n+m} - m\mu \geq (m+q)\xi + q\nu\right\}$$

$$\times P\left\{X_{n+m+1} + \cdots + X_{n+m+q} - q\mu \geq -q\nu\right\}$$

$$> P\left\{X_{n+1} + \cdots + X_{n+m} - m\mu \geq (m+q)\xi + q\nu\right\}.$$

Similarly,

$$P\left\{X_{n+1} + \cdots + X_{r+m} - m\mu \leq -(m+q)\xi - q\nu\right\}$$

$$< 2P\left\{X_{n+1} + \cdots + X_{n+m+q} - (m+q)\mu \leq -(m+q)\xi\right\}.$$

Thus, for all n,

$$P\left\{\left|\frac{X_{n+1}+\cdots+X_{n+m}}{m} - \mu\right| \geq \left(1+\frac{q}{m}\right)\xi + \frac{q}{m}\nu\right\} \leq 2\Pi_{m+q}(\xi),$$

and so

$$\Pi_m\left(\left\{1+\frac{q}{m}\right\}\xi + \frac{q}{m}\right) \leq 2\Pi_{m+q}(\xi).$$

This proves that $\int_0^{\infty}\Pi_m(\xi)\,d\xi$ must be finite.

Definition 4. The renewal sequence $X_1, X_2, X_3, \ldots,$ ad infinitum, is said to be ultimately stochastically stable (U.S.S.) with average μ if the following two conditions hold

(U_1) for some integer N the renewal sequence $X_N, X_{N+1}, X_{N+2}, \ldots,$ ad infinitum, is stochastically stable with average μ,

(U_2) $E\{|\min(0, X_i)|\} < \infty$ for $i = 1,2,\ldots, N-1$.

Condition (U_2) in this definition is to ensure that all the convolutions we encounter shall be finite, a point discussed briefly at the end of section 6. Notice that there is no point in defining ultimate stability for sequences of constants; if such a sequence were ultimately stable then it could be proved to be stable. With sequences of random variables however, some early members may have infinite expectations; this would spoil stability, but not ultimate stability.

Theorem 6. Let $\{X_n\}$ be a renewal sequence whose I-mesh structure satisfies conditions (b), (c), and (d) of theorem 4, and whose renewal function is H(x). Then, if $\{X_n\}$ is U.S.S. with a finite average $\mu > 0$,

$$k(x) * H(x) \to \frac{\|k\|}{\mu}, \text{ as } x \to \infty,$$

for every $k(x) \in \mathcal{K}$.

<u>Proof</u>: Let us suppose, to begin with, that $\{X_n\}$ is S.S. (as opposed to being U.S.S.).

For $\xi \geqq 0$ let us define

$$\Pi_p^{(+)}(\xi) = \sup_{n \geqq 0} P\left\{\frac{X_{n+1} + X_{n+2} + \cdots + X_{n+p}}{p} \leqq \mu - \xi\right\},$$

$$\Pi_p^{(-)}(\xi) = \sup_{n \geqq 0} P\left\{\frac{X_{n+1} + X_{n+2} + \cdots + X_{n+p}}{p} \geqq \mu + \xi\right\}.$$

Clearly $\Pi_p^{(+)}(\xi) \leqq \Pi_p(\xi)$, $\Pi_p^{(-)}(\xi) \leqq \Pi_p(\xi)$, so that if we define

$$\rho_p^{(+)} = \int_0^\infty \Pi_p^{(+)}(\xi)\, d\xi,$$

$$\rho_p^{(-)} = \int_0^\infty \Pi_p^{(-)}(\xi)\, d\xi,$$

then we have $\rho_p^{(+)} \to 0$ and $\rho_p^{(-)} \to 0$, as $p \to \infty$.

Next define distribution functions $K_p^{(+)}(x)$, $K_p^{(-)}(x)$ as follows:

$$K_p^{(+)}(x) = \Pi_p^{(+)}\left(\mu - \frac{x}{p}\right) \quad \text{for } x < p\mu,$$

$$= 1 \quad \text{for } x \geqq p\mu;$$

$$K_p^{(-)}(x) = 0 \quad \text{for } x < p\mu,$$

$$= 1 - \Pi_p^{(-)}\left(\frac{x}{p} - \mu\right) \quad \text{for } x \geqq p\mu.$$

Then for all n, all x,

$$(10.4) \quad K_p^{(+)}(x) \geqq P\{X_{n+1} + X_{n+2} + \cdots + X_{n+p} \leqq x\} \geqq K_p^{(-)}(x).$$

Computation also establishes that the means of the distribution functions $K_p^{(+)}(x)$, $K_p^{(-)}(x)$ are $p(\mu - \rho_p^{(+)})$ and $p(\mu + \rho_p^{(-)})$ respectively.

Since $\rho_p^{(+)}$ can be made arbitrarily small, $p(\mu - \rho_p^{(+)})$ can be made strictly positive, by choice of a suitable p. Thus the sequence $\{X_n\}$ satisfies condition $\mathfrak{T}$ of section 6. (The distribution function $K_p^{(+)}(x)$ plays the role of the K(x) in definition 2.) The sequence $\{X_n\}$ therefore satisfies all the conditions of theorem 4. Thus in the special case when $\|k\| = 0$ the present theorem is proved.

Write $G_n(x)$ for the distribution function of X_n. Then (10.4) is equivalent to

$$(10.5) \quad K_p^{(+)}(x) \geqq G_{n+1}(x) * G_{n+2}(x) * \ldots * G_{n+p}(x) \geqq K_p^{(-)}(x),$$

which holds, therefore, for all n and all x.

To make our argument clear we shall briefly consider the case $p = 2$. We have

$$H(x) = G_1(x) + G_1(x) * G_2(x) + G_1(x) * G_2(x) * G_3(x) + G_1(x) * G_2(x) * G_3(x) * G_4(x) + \ldots$$

and, in view of (10.5), it appears therefore that

$$H(x) \leqq G_1(x) + K_2^{(+)}(x) + G_1(x) * K_2^{(+)}(x) + G_1(x) * G_2(x) * K_2^{(+)}(x) + \ldots,$$

that is,

$$H(x) \leqq G_1(x) + K_2^{(+)}(x) + K_2^{(+)}(x) * H(x) .$$

It should not be difficult to see that, for a general value of p, we have

$$(10.6) \quad H(x) \leqq \sum_{j=1}^{p-1} G_1(x) * G_2(x) * \ldots * G_j(x) + K_p^{(+)}(x) + K_p^{(+)}(x) * H(x).$$

Similarly, we can show that

(10.7) $$H(x) \geq \sum_{j=1}^{p-1} G_1(x) * G_2(x) * \cdots * G_j(x) + K_p^{(-)}(x) + K_p^{(-)}(x) * H(x) \;.$$

If we write $k_1(x) = U(x) - K_p^{(+)}(x)$ then (10.6) shows that

(10.8) $$k_1(x) * H(x) \leq K_p^{(+)}(x) + \sum_{j=1}^{p-1} G_1(x) * \cdots * G_j(x),$$

and hence that

(10.9) $$\limsup_{x \to \infty} k_1(x) * H(x) \leq p \;.$$

But $k_1(x) \in \mathcal{K}$ and $\|k_1\| = p(\mu - \rho_p^{(+)})$. Thus we can infer from (10.9) and theorem 4 that if $k(x)$ is an arbitrary member of $\mathcal{K}_+$,

(10.10) $$\limsup_{x \to \infty} k(x) * H(x) \leq \frac{\|k\|}{\mu - \rho_p^{(+)}} \;.$$

Similarly we can show, starting from (10.7), that

(10.11) $$\liminf_{x \to \infty} k(x) * H(x) \geq \frac{\|k\|}{\mu + \rho_p^{(-)}} \;.$$

If we let $p \to \infty$ in (10.10) and (10.11) and use the facts that $\rho_p^{(+)} \to 0$ and $\rho_p^{(-)} \to 0$, then the theorem is proved for the case $k(x) \in \mathcal{K}_+$. The proof of the theorem for the case when $k(x)$ is an arbitrary member of $\mathcal{K}$ is now a simple matter.

However, we have assumed in the preceding argument that $\{X_n\}$ is S.S. and we must now show how to deal with the case when $\{X_n\}$ is only U.S.S.

If $\{X_n\}$ is U.S.S. there will be a finite N such that

$X_N, X_{N+1}, X_{N+2}, \ldots$, ad infinitum, is S.S. Thus, if we write

$$H_N(x) = G_N(x) + G_N(x) * G_{N+1}(x) + G_N(x) * G_{N+1}(x) * G_{N+2}(x) + \ldots,$$

the preceding argument will apply to $H_N(x)$, and will show that

$$(10.12) \qquad k(x) * H_N(x) \longrightarrow \frac{\|k\|}{\mu}, \quad \text{as } x \longrightarrow \infty,$$

for all $k(x) \in \mathcal{K}$. But

$$H(x) = \sum_{j=1}^{N-1} F_j(x) + H_N(x) * F_{N-1}(x),$$

the convolution necessarily being finite, because of condition (U_2), Thus

$$(10.13) \quad k(x) * H(x) = \sum_{j=1}^{N-1} k(x) * F_j(x) + k(x) * H_N(x) * F_{N-1}(x).$$

Since $k(x)$ is a bounded function which tends to zero as $x \longrightarrow \infty$, then $k(x) * F_j(x) \longrightarrow 0$ as $x \longrightarrow \infty$ (lemma 1 of Smith [9]). It is possible to show, by appealing to condition $\mathcal{K}_2$ of the class $\mathcal{K}$ and to the U.B.V. property of $H_N(x)$, that $k(x) * H_N(x)$ is also a bounded function. Thus, in view of (10.12), we can also deduce (by another appeal to lemma 1 of Smith [9]) that

$$k(x) * H_N(x) * F_{N-1}(x) \longrightarrow \frac{\|k\|}{\mu}, \text{ as } x \longrightarrow \infty.$$

On referring to (10.13) we see that this last observation proves the theorem.

Theorem 6 shows that the convolution $k(x) * H(x)$ will converge to a limiting value, as $x \longrightarrow \infty$, if the renewal sequence $\{X_n\}$

satisfies two distinct conditions. The first of these, relating to the lattice structure of the renewal sequence, is non-probabilistic, and is concerned solely with arithmatical properties of the set of values taken by the random variables. This aspect of renewal sequences has been given much discussion in earlier sections. The second condition of theorem 6 is probabilistic and is concerned with the frequency with which the random variables assume very large values. It is desirable to relate stochastic stability to characteristics of the renewal sequence which are more immediately verifiable than those introduced in definition 3. The next theorem accomplishes this desired simplification.

Theorem 7. Let $\{X_n\}$ be a renewal sequence and let $\{G_n(x)\}$ be the corresponding sequence of distribution functions. A necessary and sufficient condition for $\{X_n\}$ to be stochastically stable is that there exist two distribution functions $G_+(x)$, $G_-(x)$, such that

(a) $G_+(x)$ and $G_-(x)$ are both distribution functions of random variables with finite mean values, so that

$$(10.14) \qquad \int_{-\infty}^{+\infty} \left| U(x) - G_{\pm}(x) \right| dx < \infty;$$

(b) for all x and all r,

$$(10.15) \qquad G_-(x) \leq G_r(x) \leq G_+(x);$$

(c) $E\{X_1\}, E\{X_2\}, E\{X_3\}, \ldots$, ad infinitum, is a stable sequence with average μ.

Proof: We prove the necessity part first. Thus we suppose $\{X_n\}$ to be stochastically stable and it then follows from an earlier discussion that $\int_0^\infty \Pi_1(\xi)\, d\xi$ is finite.

Let us define

$$G_+(x) = \Pi_1(\mu - x) \qquad \text{for } x < \mu,$$

$$= 1 \qquad \text{for } x \geq \mu;$$

and

$$G_-(x) = 0 \qquad \text{for } x < \mu$$

$$= 1 - \Pi_1(x-\mu) \qquad \text{for } x \geq \mu.$$

The distribution functions $G_{\pm}(x)$ thus defined have all the requisite properties. For example, if $x \geq \mu$,

$$\begin{aligned} G_n(x) &= P\left\{X_n \leq x\right\} \\ &\geq 1 - P\left\{|X_n - \mu| > x - \mu\right\} \\ &\geq 1 - \Pi_1(x-\mu) \\ &\geq G_-(x). \end{aligned}$$

Furthermore it is evident that the finiteness of $\int_0^\infty \Pi_1(\xi)\, d\xi$ ensures the satisfaction of (10.14).

Since (10.14) and (10.15) have been proved, it is clear that

$$\int_{-\infty}^{+\infty} \left| U(x) - G_n(x) \right| dx < \infty$$

for all n. Thus $\mu_n = EX_n$ is finite for all n.

Now, because $\{X_n\}$ is S.S., for any $\epsilon > 0$ we can find a $p_0(\epsilon)$ such that $\epsilon > \int_0^\infty \Pi_p(\xi)\, d\xi$ for all $p > p_0(\epsilon)$. Hence for all n and all $p > p_o(\epsilon)$,

$$\epsilon > \int_0^\infty P\left\{\left|\frac{X_{n+1}+\cdots+X_{n+p}}{p} - \mu\right| \geqq \xi\right\} d\xi$$

$$= E\left\{\left|\frac{X_{n+1}+\cdots+X_{n+p}}{p} - \mu\right|\right\}.$$

But $|a - b| \geqq a - b$, always, so we can infer that

$$\epsilon > \frac{\mu_{n+1}+\cdots+\mu_{n+p}}{p} - \mu .$$

Similarly

$$\epsilon > \mu - \frac{\mu_{n+1}+\cdots+\mu_{n+p}}{p} .$$

Thus, for all n and all $p > p_o(\epsilon)$, we have shown

$$\left|\frac{\mu_{n+1}+\cdots+\mu_{n+p}}{p} - \mu\right| < \epsilon.$$

This proves $\{\mu_n\}$ to be a stable sequence, and completes the necessity part of the theorem.

We begin the sufficiency part of the proof with some computations concerning the function

$$\gamma(x) = 1 - G_-(x) + G_+(x),$$

and our first observation is that an easy and well-known consequence of (10.14) is that

(10.16) $$x\,\gamma(x) \longrightarrow 0 \text{ and } \int_x^\infty \gamma(y)\,dy \longrightarrow 0, \text{ as } x \longrightarrow \infty .$$

Thus, if we select some arbitrarily small $\epsilon > 0$, then we can find an $x_o(\epsilon)$ such that

(10.17) $$x\gamma(x) < \epsilon, \int_x^\infty \gamma(y)\,dy < \epsilon, \text{ for all } x > x_o(\epsilon).$$

Our second observation is that, for any fixed $p > 0$, one can prove by elementary computation that

$$(10.18) \quad \int_{\epsilon}^{\infty} \frac{1}{\xi^2} \left\{ \int_0^{p\xi} x\, \gamma(x)\, dx \right\} d\xi = \frac{1}{\epsilon} \int_0^{p\xi} x\, \gamma(x)\, dx + \int_{p\epsilon}^{\infty} \gamma(x)\, dx .$$

Moreever, since strict convergence implies Cesàro convergence it follows from (10.16) that

$$\frac{1}{x} \int_0^x y\, \gamma(y)\, dy \longrightarrow 0 \text{ , as } x \longrightarrow \infty .$$

Thus we can deduce from (10.18) that

$$(10.19) \qquad \lim_{p \to \infty} \frac{1}{p} \int_{\epsilon}^{\infty} \frac{1}{\xi^2} \left\{ \int_0^{p\xi} x\, \gamma(x)\, dx \right\} d\xi = 0 .$$

With these preparatory computations accomplished, we can turn to the main part of the sufficiency proof. In our argument we shall employ a well-known truncation device of Kolmogorov and an inequality used by him in dealing with the weak law of large numbers (see, for example, Gnedenko and Kolmogorov [7] p. 106). Define $\mu_n = EX_n$ and $Z_n = X_n - \mu_n$, choose $\xi \geq \epsilon$ and $p > \epsilon^{-1} x_o(\epsilon)$, and then define

$$\begin{aligned} Z_n' &= Z_n , \quad \text{if } -p\xi < Z_n \leq p\xi , \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Write $S_{np} = Z_{n+1} + Z_{n+2} + \ldots + Z_{n+p}$ and $S_{np}' = Z_{n+1}' + Z_{n+2}' + \ldots + Z_{n+p}'$.

The distribution function of Z_r is $G_r(x + \mu_r)$. Since $\{\mu_n\}$ is a stable sequence it is also a bounded sequence. Thus, by resorting to some finite translations of the distribution functions $G_{\pm}(x)$ if necessary, we may suppose that (10.15) holds even when $G_r(x)$ is replaced by $G_r(x+\mu_r)$.

Plainly, $EZ_r = 0$, so that

$$EZ_r' = -\int_{p\xi}^{\infty} x \, d\, G_r(x+\mu_r) - \int_{-\infty}^{-p\xi} x \, d\, G_r(x+\mu_r) \, .$$

Some integrations by parts will then prove that

$$EZ_r' = -p\xi \left\{ 1 - G_r(p\xi + \mu_r) \right\} - \int_{p\xi}^{\infty} \left\{ 1 - G_r(x+\mu_r) \right\} dx$$

$$-p\xi \quad G_r(-p\xi + \mu_r) \qquad + \int_{-\infty}^{-p\xi} G_r(x+\mu_r) \, dx.$$

An appeal to (10.15) then shows that

$$|EZ_r'| < p\xi \left\{ 1 - G_-(p\xi) \right\} + \int_{p\xi}^{\infty} \left\{ 1 - G_-(x) \right\} dx$$

$$+ \quad p\xi \; G_+(-p\xi) \qquad + \int_{-\infty}^{-p\xi} G_+(x) \, dx,$$

so that

$$|EZ_r'| < p\xi \; \gamma \, (p\xi) + \int_{p\xi}^{\infty} \gamma \, (x) \, dx \, .$$

But $p\xi > x_o(\epsilon)$, so we may deduce from (10.17) that

(10.20) $\qquad |EZ_r'| < 2\epsilon \quad , \quad$ for all r.

The second moment of Z_r' must also be studied. Some more integration by parts will prove that

$$E\left\{ (Z_r')^2 \right\} = -p^2\xi^2 \left\{ 1 - G_r(p\xi + \mu_r) + G_r(-p\xi + \mu_r) \right\}$$

$$+ 2\int_0^{p\xi} x \left\{ 1 - G_r(x + \mu_r) + G_r(-x + \mu_r) \right\} dx,$$

and from this equation we can deduce the inequality

$$(10.21) \qquad E\left\{ (Z_r')^2 \right\} < 2\int_0^{p\xi} x\,\gamma(x)\,dx, \quad \text{for all } r.$$

The familiar inequality of Chebysher shows that

$P\left\{|S'_{np}| > p\xi\right\} < E\left\{(Z'_{n+1} + \ldots + Z'_{n+p})^2\right\}/p^2\xi^2$ and hence, if we use the independence of the $\{Z_n'\}$, we can easily prove that

$$P\left\{|S'_{np}| > p\xi\right\} < \frac{\sum_{r=1}^{p} E\left\{(Z'_{n+r})^2\right\} + \left(\sum_{r=1}^{p} E\{Z'_{n+r}\}\right)^2}{p^2\xi^2} .$$

On appealing to (10.20) and (10.21), it thus transpires that

$$(10.22) \qquad P\left\{|S'_{np}| > p\xi\right\} < \frac{4\epsilon^2}{\xi^2} + \frac{2}{p\xi^2}\int_0^{p\xi} x\gamma(x)\,dx.$$

Let us now define B_{np} as the event $\left\{Z'_{n+r} = Z_{n+r} \text{ for all } r = 1,2,\ldots,p\right\}$, and let B^c_{np} be the event complementary to B_{np}. Then

$$P\left\{B^c_{np}\right\} \leq \sum_{r=1}^{p} P\left\{Z'_{n+r} \neq Z_{n+r}\right\}$$

$$\leq \sum_{r=1}^{p} \left\{1 - G_{n+r}(p\xi + \mu_{n+r}) + G_{n+r}(-p\xi + \mu_{n+r}) \right. ,$$

and so, by (10.15),

$$(10.23) \qquad P\left\{B^c_{np}\right\} \leq p\,\gamma(p\xi).$$

Now

$$P\left\{|S_{np}| > p\xi\right\} = P\left\{B_{np} \& |S'_{np}| > p\xi\right\} + P\left\{B^c_{np} \& |S_{np}| > p\xi\right\}$$

$$\leq P\left\{|S'_{np}| > p\xi\right\} + P\left\{B^c_{np}\right\},$$

and therefore, by (10.22) and (10.23),

(10.24) $$P\left\{\left|\frac{X_{n+1}+\cdot\cdot\cdot+X_{n+p}}{p} - \frac{\mu_{n+1}+\cdot\cdot\cdot+\mu_{n+p}}{p}\right| > \xi\right\} \leq \gamma_p(\xi),$$

uniformly in n, where

(10.25) $$\gamma_p(\xi) = p\gamma(p\xi) + \frac{4\epsilon^2}{\xi^2} + \frac{2}{p\xi^2}\int_0^{p\xi} x\gamma(x)\,dx .$$

At this point it is to be noticed that, by (10.25),

$$\int_\epsilon^\infty \gamma_p(\xi)\,d\xi = \int_{p\epsilon}^\infty \gamma(u)\,du + 4\epsilon + \frac{2}{p}\int_\epsilon^\infty \frac{1}{\xi^2}\left\{\int_0^{p\xi} x\gamma(x)\,dx\right\} d\xi,$$

and so, on appealing to (10.16) and (10.19), it follows that

(10.26) $$\limsup_{p\to\infty} \int_\epsilon^\infty \gamma_p(\xi)\,d\xi \leq 4\epsilon .$$

Since $\{\mu_n\}$ is stable with average μ, there must be a $p_o(\epsilon)$ such that, for all $p > p_o(\epsilon)$, $|\mu - (\mu_{n+1}+\cdot\cdot\cdot+\mu_{n+p})/p| < \epsilon$ uniformly in n. Thus, from (10.24), for all $\xi \geq \epsilon$ and all $p > \max(p_o, x_o/\epsilon)$, $P\{|\mu-(X_{n+1}+\cdot\cdot\cdot+X_{n+p})/p| > \epsilon+\xi\} \leq \gamma_p(\xi)$ uniformly in n. Therefore, if we introduce the function $\Pi_p(\xi)$ of (10.2), it appears that $\Pi_p(\xi+\epsilon) \leq \gamma_p(\xi)$, and so, from (10.26),

$$\limsup_{p \to \infty} \int_{2\epsilon}^{\infty} \pi_p(\xi)\, d\xi \leq 4\epsilon .$$

However, $0 \leq \pi_p(\xi) \leq 1$ for all ξ; and ϵ can be chosen arbitrarily small. Thus it plainly follows that $\int_0^{\infty} \pi_p(\xi)\, d\xi \to 0$ as $p \to \infty$, and the theorem is proved.

It is to be remarked that if the $\{X_n\}$ are known to be bounded below by some finite constant c, say, then one can take $G_+(x) = U(x-c)$. Thus, for instance, for the case of a positive renewal sequence there is no difficulty in determining a suitable $G_+(x)$. Similar remarks apply if the $\{X_n\}$ are known to be bounded above; $G_-(x)$ is then easily determined.

Corollary 7.1 Let $\{X_n\}$ be a renewal sequence and let $\sigma(x)$ be a non-decreasing, non-negative function of x such that, for some sufficiently large Δ,

$$\int_{\Delta}^{\infty} \frac{dx}{\sigma(x)} < \infty ,$$

If $E\{X_1\}$, $E\{X_2\}$, $E\{X_3\}$, . . ., ad infinitum, is a stable sequence with average μ, and if, for some finite constant c, $E\{\sigma(|X_n|)\} < c$ for all n, then $\{X_n\}$ is stochastically stable with average μ.

Proof. Choose Δ appropriately and then define

$$\begin{aligned} G_+(x) &= c/\sigma(-x) && \text{for } x < -\Delta , \\ &= 1 && \text{for } x \geq -\Delta ; \\ G_-(x) &= 0 && \text{for } x < \Delta , \\ &= 1 - c/\sigma(x) && \text{for } x \geq \Delta . \end{aligned}$$

Then the functions $G_+(x)$ and $G_-(x)$ are both distribution functions of random variables with finite mean values. Moreover, from the monotone character of $\sigma(x)$ and the fact that $E\{\sigma(|X_n|)\} < c$ for all n, it follows that for all $x \geqq \Delta$

$$P\{|X_n| \geqq x\} < c/\sigma(x) . \tag{10.27}$$

It is an easy matter to verify, using (10.27), that the inequalities (10.15) hold for $G_+(x)$ and $G_-(x)$ as presently defined. The corollary then follows from theorem 7.

As a simple example of a situation to which corollary 7.1 conveniently applies, we mention the case when the variances of the $\{X_n\}$ are bounded.

Theorems 6 and 7 discuss the situation when most of the random variables $\{X_n\}$ have finite mean values and $k(x) * H(x)$ converges to a nonzero limit. To round off this section, let us prove the following theorem, which covers the case when many of the random variables have finite means, and $k(x) * H(x)$ converges to zero.

<u>Theorem 8. Let $\{X_n\}$ be a renewal sequence whose I-mesh structure satisfies conditions (b), (c), and (d) of theorem 4. Suppose that, for any arbitrarily large number M, we can find integers p and q such that the function $\Lambda_{pq}(x)$ of (6.2) is dominated by the distribution function $K(x)$ of a random variable whose expectation exceeds M_p, and suppose further that $E\{|\min(0, X_i)|\} < \infty$ for $i=1,2,\ldots,q-1$. If $H(x)$ is the renewal function of $\{X_n\}$, and $k(x) \in \mathcal{K}$, then $k(x) * H(x) \to 0$, as $x \to \infty$.</u>

<u>Proof</u>. If we give M any strictly positive value, then it is obvious that $\{X_n\}$ satisfies condition $\mathfrak{T}$ and hence all the conditions of theorem 4. The proof now closely parallels part of the proof of theorem 6 and we

shall merely give a sketch.

Suppose first that we can take $q = 0$ and find a value of p and a distribution function $K(x)$ with the properties described in the enunciation. Then

$$K(x) \geqq G_{n+1}(x) * G_{n+2}(x) * \ . \ . \ . * G_{n+p}(x)$$

for all n and x. On proceeding as before we find that, if $k_1(x)=U(x)-K(x)$, then $\| k_1 \| > M_p$ and $k_1(x) \in \mathcal{K}$ and

(10.29) $$\limsup_{x \to \infty} \quad k_1(x) * H(x) \leqq p \ .$$

Thus, if $k(x) \in \mathcal{K}_+$ we have, trivially,

(10.30) $$\liminf_{x \to \infty} \quad k(x) * H(x) \geqq 0 \ ,$$

and from (10.29) and theorem 4,

(10.31) $$\limsup_{x \to \infty} \quad k(x) * H(x) \leqq \frac{\|k\|}{M} \ .$$

But M can be chosen arbitrarily large and so, from (10.30) and (10.31), the theorem is proved when $k(x) \in \mathcal{K}_+$. The extension to $k(x)$ is then trivial.

If it is necessary to have $q > 0$, before a suitable value of p can be found, then we can adopt the same argument as was used at the end of the proof of theorem 6, namely, the argument which extended a result proved for a S.S. sequence to the case of an U.S.S. sequence.

11. On certain generalizations of renewal theory

Cox and Smith [5] discussed, in addition to the conventional renewal function, certain more general functions of the form

(11.1) $$Q(x) = \sum_{j=1}^{\infty} a_j F_j(x),$$

where $\{a_n\}$ is a sequence of real constants. The function Q(x) has a simple interpretation. Suppose that, for every n, we are to receive a "prize" of a_n units of currency if $S_n \leq x$ (if a_n is negative the "prize" becomes a "loss"). Our total prize is then

$$\sum_{j=1}^{\infty} a_j U(x - S_j)$$

and it is easily seen that Q(x) is the expected value of this total prize. Similarly, if we suppose that, for every n, we are to receive a prize of a_n units if $x_1 < S_n \leq x_2$, then our expected total prize is $Q(x_2)-Q(x_1)$. The renewal function H(x) refers to the special case when all the individual prizes equal one unit. Thus, if the $\{a_n\}$ are bounded, $|a_n| \leqq A$, say, for all n, it is trivial that (a) $|Q(x)| \leq A\, H(x)$ for all x; (b) for every x_1, x_2, $x_1 < x_2$, $|Q(x_2) - Q(x_1)| < A\{H(x_2) - H(x_1)\}$. The following lemma is now obvious.

Lemma 16: If (a) $\{a_n\}$ is a bounded sequence; (b) H(x) is finite for all x; (c) H(x) has the U.B.V. property; then Q(x) is finite for all x, and Q(x) has the U.B.V. property, in the sense that for every $\epsilon > 0$ we can find a finite $\delta(\epsilon)$ such that $\int_{\alpha}^{\alpha+\epsilon} |dQ(x)| < \delta(\epsilon)$, for all α.

If we suppose $\{a_n\}$ to be bounded, the work of section 5 on the behavior of $\Sigma|\psi_n(\theta)|$ applies equally well, after some obvious and easy changes, to the series $\Sigma|a_n\psi_n(\theta)|$. The only point needing serious attention is that we must either redefine what we mean by a removable I_s-angle or simply suppose that I_s-angles do not arise. We shall take here the latter and simpler course of action. With the proviso, therefore that $\mathcal{H}_s$ is empty, and in view of lemma 16 above, the proof of theorem 4 applies with slight modifications to the function Q(x) instead of H(x). In one or two places, however, the proof of theorem 4 depends upon H(x) being nondecreasing. We can arrange Q(x) to be nondecreasing by supposing $a_n \geqq 0$ for all n. Once we have proved theorem 4 for such a monotone Q(x) the extension to the general is trivial. Thus we have shown that if $\{a_n\}$ is a bounded sequence, and if the conditions of theorem 4 are satisfied, and if $\mathcal{H}_s$ is empty, then $k(x) * Q(x) \to 0$, as $x \to \infty$, for every $k(x) \in \mathcal{H}$ such that $\|k\| = 0$.

Of particular interest is the following extension of theorem 6 (compare with theorems 3 and 4 of Cox and Smith [5]).

<u>Theorem</u> 9. <u>Suppose that</u>

(a) $\{X_n\}$ <u>is a renewal sequence which is ultimately stochastically stable with a finite, strictly positive, average</u> μ;

(b) <u>the I-mesh structure of</u> $\{X_n\}$ <u>satisfies the conditions (b), (c), and (d) of theorem 4, together with the additional condition that</u> $\mathcal{H}_s$ <u>be empty</u>;

(c) $\{a_n\}$ <u>is a stable sequence of constants with average</u> a; <u>then if</u> Q(x) <u>is defined as in (11.1), and if</u> $k(x) \in \mathcal{H}$,

$$k(x) * Q(x) \to \frac{a\|k\|}{\mu}, \text{ as } x \to \infty .$$

<u>Proof</u>: We shall suppose $\{X_n\}$ is S.S., once the theorem has been proved for this case it can be extended to cover the case when $\{X_n\}$ is U.S.S. by reasoning similar to that employed in the proof of theorem 6.

Since $\{a_n\}$ is stable, it is also bounded, that is $|a_n| < A$ for all n. Let us put $a_n^* = A + a_n$, $Q^*(x) = \Sigma\, a_n^* F_n(x)$. Then $\{a_n^*\}$ is a positive stable sequence, and $Q^*(x) = AH(x) + Q(x)$. But theorem 9 is known to be true for $H(x)$ in the place of $Q(x)$, and therefore it will be true for $Q(x)$ if we can prove if for $Q^*(x)$. In other words there will be no loss of generality if in proving the theorem for $Q(x)$, we assume $a_n > 0$ for all n (and $Q(x)$ nondecreasing).

In the proof that follows we shall several times use the fact that if $A_1(x) \geqq A_2(x) \geqq 0$ for all x, and if $B(x)$ is a distribution function, then $A_1(x) * B(x) \geqq A_2(x) * B(x)$ for all x. Note also that if $B_1(x)$ and $B_2(x)$ are distribution functions of random variables Z_1, Z_2, say, then $\|B_1 - B_2\| = E\{Z_2\} - E\{Z_1\}$.

Let $\epsilon > 0$ be chosen arbitrarily small. Since $\{a_n\}$ is stable we can choose p so that

(11.2) $$\left| \frac{a_n + a_{n+1} + \cdots + a_{n+p}}{(p+1)} - a \right| < \epsilon$$

for all n.

We shall use the functions $K_p^{(+)}(x)$, $K_p^{(-)}(x)$ of section 10, and we note for reference that, by (10.5),

(11.3) $$K_r^{(-)}(x) * F_s(x) \leqq F_{r+s}(x) \leqq K_r^{(+)}(x) * F_s(x),$$

for all values of the integers r and s. Thus

(11.4) $$K_r^{(-)}(x) * Q(x) \leqq \sum_{n=1}^{\infty} a_n F_{n+r}(x).$$

If we write

(11.5) $$L_p^{(-)}(x) = \frac{U(x) + K_1^{(-)}(x) + \cdots + K_p^{(-)}(x)}{(p+1)},$$

then computation based on (11.4) shows that

$$(11.6) \qquad L_p^{(-)}(x) * Q(x)$$

$$\leqq \frac{a_1}{p+1} F_1(x) + \frac{a_1+a_2}{p+1} F_2(x) + \ldots + \frac{a_1+a_2+\ldots+a_p}{p+1} F_p(x)$$

$$+ \sum_{n=1}^{\infty} \frac{a_n+a_{n+1}+\ldots+a_{n+p}}{(p+1)} F_{n+p}(x) \quad .$$

Let $A(x)$ be the distribution function of a random variable with a large negative mean $-\Delta$, chosen so that $A(x) \geqq K_p^{(-)}(x)$ for all x. From (11.6) it follows that

$$(11.7) \quad A(x) * L_p^{(-)}(x) * Q(x)$$

$$\leqq \frac{a_1}{p+1} F_1(x) * A(x) + \ldots + \frac{a_1+a_2+\ldots+a_p}{p+1} F_p(x) * A(x)$$

$$+ \sum_{n=1}^{\infty} \frac{a_n+a_{n+1}+\ldots a_{n+p}}{p+1} F_{n+p}(x) * A(x).$$

If we define

$$(11.8) \qquad L_p^{(+)}(x) = \frac{K_p^{(+)}(x) + K_{p+1}^{(+)}(x) + \ldots + K_{2p}^{(+)}(x)}{(\ p+1\ )} \quad ,$$

then it may similarly be shown, using (11.5), that

$$(11.9) \quad L_p^{(+)}(x) * Q(x) \geqq \frac{a_1}{p+1} F_{p+1}(x) + \frac{a_1+a_2}{p+1} F_{p+2}(x) + \ldots$$

$$+ \frac{a_1+a_2+\ldots+a_p}{p+1} F_{2p}(x) + \sum_{n=1}^{\infty} \frac{a_n+a_{n+1}+\ldots+a_{n+p}}{(p+1)} F_{n+2p}(x).$$

On subtracting (11.9) from (11.7) we find that

$$(11.10)\quad \left\{ A(x) * L_p^{(-)}(x) - L_p^{(+)}(x) \right\} * Q(x)$$

$$\leqq \frac{a_1}{(p+1)} \left\{ F_1(x) * A(x) - F_{p+1}(x) \right\} + \ldots$$

$$+ \frac{a_1+a_2+\ldots+a_p}{(p+1)} \left\{ F_p(x) * A(x) - F_{2p}(x) \right\}$$

$$+ \sum_{n=1}^{\infty} \frac{a_n+a_{n+1}+\ldots+a_{n+p}}{(p+1)} \left\{ F_{n+p}(x) * A(x) - F_{n+2p}(x) \right\}.$$

However, if we refer to (11.3) again, we see that

$$(11.11)\quad F_{n+p}(x) * A(x) - F_{n+2p}(x) \leqq F_{n+p}(x) * \left\{ A(x) - K_p^{(-)}(x) \right\} .$$

Since, by choice of A(x), the right side of (11.11) is nonnegative, we can use (11.11) in the summation of (11.10) and at the same time appeal to (11.2) to deduce that

$$(11.12)\quad \left\{ A(x) * L_p^{(-)}(x) - L_p^{(+)}(x) \right\} * Q(x)$$

$$\leqq \frac{a_1}{(p+1)} \left\{ F_1(x) * A(x) - F_{p+1}(x) \right\} + \ldots$$

$$+ \frac{a_1+a_2+\ldots+a_p}{(p+1)} \left\{ F_p(x) * A(x) - F_{2p}(x) \right\}$$

$$+ (a+\epsilon) \left\{ A(x) - K_p^{(-)}(x) \right\} * \left\{ \sum_{n=1}^{\infty} F_{n+p}(x) \right\} .$$

At this stage let us note that:

(a) $\Sigma_1^\infty F_{n+p}(x)$ is a renewal function whose behavior is covered by theorem 6 (the absence of the first p terms which are usually present makes no difference);

(b) $\{A(x) - K_p^{(-)}(x)\} \in \mathcal{K}$ and $\| A(x) - K_p^{(-)}(x) \| = \Delta - p(\mu + \rho_p^{(-)})$.

Thus we can infer from (11.12) that

$$\limsup_{x \to \infty} \left\{ A(x) * L_p^{(-)}(x) - L_p^{(+)}(x) \right\} * Q(x) \leqq \frac{(a + \epsilon)(\Delta - p\mu - p\rho_p^{(-)})}{\mu} .$$

However, $A(x) * L_p^{(-)}(x) - L_p^{(+)}(x) \in \mathcal{K}$ also, and computation shows that

$$\| A(x) * L_p^{(-)}(x) - L_p^{(+)}(x) \| = \Delta + \frac{1}{p}\mu - \frac{1}{(p+1)} \sum_{j=1}^{p} \rho_j^{(-)} - \frac{1}{(p+1)} \sum_{j=p}^{2p} \rho_j^{(+)}$$

$$= \Delta + \mu_p ,$$

say.

We have already explained earlier in this section that under the conditions of the present theorem, the conclusion of theorem 4 can be applied to Q(x) as well as to H(x). Thus, for every $k(x) \in \mathcal{K}$, we must have

$$(11.13) \quad \limsup_{x \to \infty} k(x) * Q(x) \leqq \frac{(a+\epsilon)(\Delta - p\mu - p\rho_p^{(-)}) \| k \|}{(\Delta + \mu_p)\mu} .$$

If, on the right side of (11.13) we let $\Delta \to \infty$, keeping p fixed, and then let $\epsilon \to 0$, we finally achieve

$$(11.14) \qquad \limsup_{x \to \infty} k(x) * Q(x) \leqq \frac{a \| k \|}{\mu} .$$

A similar argument, which we spare the reader, will also show that

$$(11.15) \qquad \liminf_{x \to \infty} k(x) * Q(x) \geq \frac{a \|k\|}{\mu} \quad .$$

The theorem follows from (11.14) and (11.15).

There is, of course, a similar generalization of theorem 8 to cover the case when $k(x) * Q(x) \rightarrow 0$. It is very easy to prove, but we omit details.

12. Some concluding observations

The whole theory in this paper is concerned with what we have called continuous renewal sequences. Nevertheless it is to be expected that a parallel theory will exist for periodic renewal sequences. This is, in fact, so; the theory for the periodic case is, indeed, simpler in some respects. With no loss of generality we may suppose the period concerned to be unity. Thus the random variables X_n are all positive or negative integers, or zero. We define I-meshes, I-angles, and so on, much as before; however, we need only bother now with I-meshes which exceed unity and with I-angles in the interval $[-\pi, +\pi]$. The theory of sections 2 to 6 then goes much as for the continuous case. In sections 7 and 8 one must work with Fourier series in place of Fourier integrals. Instead of discussing $k(x) * H(x)$ one considers averages like $\sum k_{x-y} u_y$, where u_x is the expected number of partial sums S_n which equal the integer x. These averages can be represented as trigonometrical integrals over $[-\pi, +\pi]$. However, for the present we shall say no more on this topic, except to state that the sort of theorems one expects to find true in the periodic case-theorems, that is, which are analogous to the ones we have proved in this paper for the continuous case-are indeed true.

Lastly there is the question of to what extent the results of this paper will be valid if the variables X_n are dependent in some way. This is a considerable and a difficult question; its answer obviously hinges to a great degree on the nature of the dependence. For the main theorem, theorem 4, our proof basically depends on : (a) $H(x)$ having the U.B.V. property; (b) $\Sigma_1^\infty \psi_n(\theta)$ behaving itself. It should not be difficult to verify (a) in any special circumstances, but a discussion of (b) could become quite tricky;

Definition 5. We say the sequence of random variables $\{X_n\}$ has structure R if we can write $X_n = Y_n + Z_n$ for all n, where

(a) $\{Y_n\}$ is a sequence of (possibly) dependent random variables;

(b) $\{Z_n\}$ is a renewal sequence which is independent of the variables $\{Y_n\}$, that is the Z_n are mutually independent and also independent of the Y_n;

(c) The I-mesh structure of $\{Z_n\}$ satisfies the conditions (b), (c), and (d) of theorem 4 together with the additional condition that it have no I_s-mesh.

It is possible to show that $\Sigma_1^\infty \psi_n(\theta)$ has suitable good behavior if $\{X_n\}$ has structure R. For let $\omega_n(\theta)$ be the characteristic function of $Z_1 + Z_2 + \ldots + Z_n$. Then it is clear that, if $\{X_n\}$ has structure R, $|\psi_n(\theta)| \leq |\omega_n(\theta)|$ for all θ. But, except in dealing with removable I_s-meshes, the results in section 5 all involved absolute convergence. Thus, in the present circumstances, $\Sigma|\omega_n(\theta)|$ is well-behaved and, therefore, so is $\Sigma |\psi_n(\theta)|$.

When $\{X_n\}$ has structure R and the variables $\{Y_n\}$ and $\{Z_n\}$ are all nonnegative, it is not too difficult to discover whether H(x) has the U.B.V. property. Without these nonnegativity assumptions, however, there seems to be no convenient general prescription which will ensure U.B.V.

Notice that structure R is not quite such a restriction as it may, at first sight, appear. Nearly all the $\{Z_n\}$ may be zero. All that is needed is for a nonzero, nonlattice-like, Z_n to appear every 10^{10} terms, or so.

When we consider the limit theorems which follow theorem 4, however, our methods seem quite unsuitable for dealing with dependent variables. All our proofs have involved the construction of kernels k(x) for which the limiting behavior of $k(x) * H(x)$ could be estimated. These constructions all break down once the independence assumption is surrendered.

REFERENCES

[1] D. BLACKWELL, "A renewal theorem," Duke Math.J., Vol. 15(1948), pp. 145-150.

[2] _______, "Extension of a renewal theorem," Pacific J. Math., Vol. 3, (1953), pp. 315-320.

[3] K. L. CHUNG and H. POLLARD, "An extension of renewal theory," Proc. Amer. Math. Soc., Vol. 3(1952), pp. 303-309.

[4] K. L. CHUNG and J. WOLFOWITZ, "On a limit theorem in renewal theory," Ann. Math., Vol. 55(1952), pp. 1-6.

[5] D. R. COX and W. L. SMITH, "A direct proof of a fundamental theorem of renewal theory," Skand.Akt., Vol. 36(1953), pp. 139-150.

[6] P. ERDÖS, W. FELLER, and H. POLLARD, "A property of power series with positive coeffieients," Bull. Amer. Math. Soc., Vol. 55(1949), pp. 201-204.

[7] B. V. GNEDENKO and A. N. KOLMOGOROV, Limit Distributions for Sums of Independent Random Variables, Translated from the Russian by K. L. Chung, Cambridge, Mass, Addison-Wesley, 1954.

[8] S. KARLIN, "On the renewal equation," Pacific J. Math., Vol. 5(1955), pp. 229-257.

[9] H. KESTEN and J. TH. RUNNENBURG, "Some elementary proofs in renewal theory with applications to waiting times," Math. Centrum Amsterdam Statist. Afdeling Rep. S203(1956).

[10] W. L. SMITH, "Asymptotic renewal theorems," Proc.Roy. Soc. Edinburgh A, Vol. 64 (1954), pp. 9-48.

[11] _________, "Renewal theory and its ramifications," J. R. Statist. Soc.B, Vol. 20(1958), pp. 243-302.

[12] E. C. TITCHMARSH, Introduction to the Theory of Fourier Integrals, 2d ed., Oxford, Oxford Univ. Press, 1950.

[13] I. M. VINOGRADOV, The Method of Trigonometrical Sums in the Theory of Numbers.Translated from the Russian, revised and annotated by K. F. Roth and Anne Davenport, London, New York,Interscience Publishers, 1954.