

DETECTING TRUNCATION POINT IN STEADY-STATE SIMULATION USING CHAOS THEORY

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ABSTRACT

In the steady-state simulation, it is important to identify initialization bias for the correct estimates of the simulation model under study. In this paper, the methods from chaos theory are applied to the determination of truncation points in the simulation data for controlling the initial bias. Two methods are proposed and evaluated based on their effectiveness for estimating the average waiting time in $M/M/1(\infty)$ queueing model.

1 INTRODUCTION

A simulation method has been used broadly to evaluate the performance of the system under study, especially when the quantitative analysis using mathematical models is not applicable. The initial conditions of the stochastic simulated system may be different each time the simulation is run. And the estimation of true responses in the steady-state simulation is so complicated because of the possible presence of initial bias. So, in studies of the steady-state characteristics of a simulation model, it is important to identify initialization bias. Many researchers have proposed the methods for determining the truncation points (or warm up period) to control initial bias, but these methods seem not to work well as intended.

The survey by Gafarian et al. (1978) indicated that published procedures for identifying a truncation point appear not to exhibit very good behavior. Kelton and Law (1984, 1985) investigated the deletion effect of initialization bias for the three types of point estimator criteria using a particular stochastic model. Schruben et al. (1982, 1983) presented a family of tests for detecting initialization bias in the mean of a simulation output data using a hypothesis testing framework. Cash et al. (1992) evaluated the power of a family of tests for

initialization bias.

Variations of the system observations in the transient state are large and irregular due to initial bias as compared to those in the steady state. In this paper, we first propose how to measure the difference between the system variation in the transient state and the variation in the steady state and then develop two methods to determine truncation points that can be used in eliminating initial bias of the system.

We evaluate the performances of the two proposed methods based on their effectiveness in estimating the average waiting time using an $M/M/1(\infty)$ queueing model. These results are also compared with the method suggested in the literature.

This paper is organized as follows. Section 2 describes the state space reconstruction to estimate a dimension for a given time series. Section 3 explains the Lyapunov exponent which tests the stability of dynamic system and the sensitive dependence on initial conditions. In Section 4, we discuss the simulation output analysis. Section 5 presents the deletion strategies to decide how much data to be deleted in simulation output data. In Section 6 we summarize and discuss our experimental results. Finally in Section 7 we present our conclusions and recommendations for further work.

2 RECONSTRUCTION OF STATE SPACE

Until recently, the notions of determinism and randomness were seen as opposites, and were studied as separate subjects with little or no overlap. Complicated phenomena were assumed to result from complicated physics among the processes. Simple dynamic systems were assumed to produce simple phenomena, so only simple phenomena were modeled deterministically.

Chaos provides a link between deterministic systems and random processes. The chaotic dynamics in a deterministic system can amplify small differences. Also,

chaos implies that not all random-looking behavior is the product of complicated physics. An important consideration in chaos theory is the dimension of the dynamics, which is to estimate the fractal dimension of a hypothesized 'strange attractor', to define the asymptotic solution of a dynamic system, in a reconstructed state space. And a dimension of the dynamics counts the minimum number of degrees of freedom necessary to describe this motion. To estimate a dimension for a time series, we must first reconstruct a state space. The past behavior of a time series contains information about the present state $x(t)$. And so, if the delay time τ is assumed uniform, the state at time t $x(t)$ can be reconstructed as a delay vector of dimension m ,

$$X(t) = \{ x(t), x(t-1), \dots, x(t-(m-1)\cdot\tau) \} \quad (1)$$

where is called the embedding dimension.

If a time series is deterministic and of finite dimension, the estimated dimension of the reconstructed attractor should converge to the dimension of the strange attractor as the embedding dimension is increased. On the contrary, if a time series is random, the estimated dimension should be equal to the embedding dimension. The Dimension that is considered in this paper is expressed as follows[1] :

$$C(r) = \lim_{r \rightarrow 0} \frac{1}{N^2} \sum_{i=0}^N \sum_{j=1}^N H(r - |x_i - x_j|) \quad (2)$$

where

$$H(s) = \begin{cases} 1, & \text{if } s > 0 \\ 0, & \text{if } s < 0 \end{cases}$$

N : total number of points in the state space.

r : length of side for small cubes.

3 LYAPUNOV EXPONENTS

Suppose one has the ability to measure a position with accuracy Δx and a velocity with accuracy Δv . Then in the position-velocity plane (known as the phase space) we can divide up the space into areas of size $\Delta x \cdot \Delta v$ as shown in Figure 1. If we are given initial conditions to the stated accuracy, we know the system is somewhere in the shaded box in the phase plane. But if the system is chaotic, this uncertainty grows in time to $N(t)$ boxes as shown in Figure 1.b.

The Size of uncertainty at time t , $N(t)$ can be expressed as follows :

$$N(t) \cong N(0) \cdot e^{ht} \quad (3)$$

where constant h is related to the concept of entropy in information theory and will also be related to another concept called the Lyapunov exponent. The test using Lyapunov exponent measures the sensitivity of the system to the change in the initial conditions.

Conceptually, one imagines a small ball of initial conditions in phase space and looks at its deformation

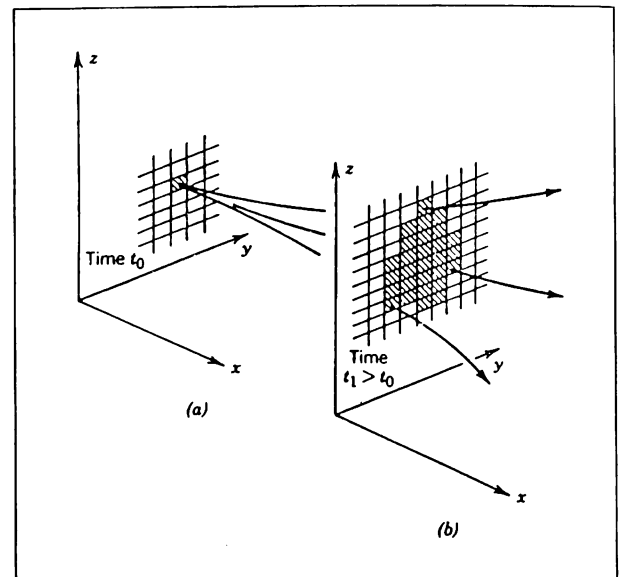


Figure 1 : The Growth of Uncertainty in a Dynamic System

into an ellipsoid under the dynamics of the system. If $d(t)$ is the maximum length of the ellipsoid at time t and $d(0)$ is the initial size of the initial condition sphere, the Lyapunov exponent λ is interpreted by the equation (4) in the reconstructed state space.

$$d(t) = d(0) \cdot 2^{\lambda t}$$

$$\lambda = \frac{1}{t} \log_2 \frac{d(t)}{d(0)} \quad (4)$$

There is a relationship between Lyapunov exponent which test the stability of chaotic system and eigenvalue which test the stability of dynamic system, and also a relationship between Lyapunov exponent which test the sensitive dependence on initial conditions and entropy which measure the growth of uncertainty.

The sign of Lyapunov exponent provides a qualitative picture of a system's dynamics such as

$$\begin{aligned} \lambda > 0 & : \text{chaotic motion} \\ \lambda \leq 0 & : \text{regular motion.} \end{aligned} \quad (5)$$

A chaotic system must have nonlinear elements or properties. A linear system cannot exhibit chaotic

vibrations. The notion of a Lyapunov exponent is a generalization of the idea of an eigenvalue as a measure of the stability of a fixed point. For a chaotic trajectory, it is not sensible to examine the instantaneous eigenvalue of a trajectory. The best quantity, therefore, is an eigenvalue averaged over the whole trajectory. The idea of measuring the average stability of a trajectory leads to the formal notion of a Lyapunov exponent.

Mathematically, Lyapunov exponent λ is defined by the equation (6).

$$\lambda = \frac{1}{t_M - t_0} \sum_{k=1}^M \log_2 \frac{L'(t_k)}{L(t_{k-1})} \quad (6)$$

where

- $L(t_{k-1})$: Euclidean distance between the initial point and the nearest neighbor.
- $L'(t_k)$: Euclidean distance between the initial point and the nearest neighbor at a later time.

A schematic representation of the evolution and replacement procedure is shown in Figure 2.

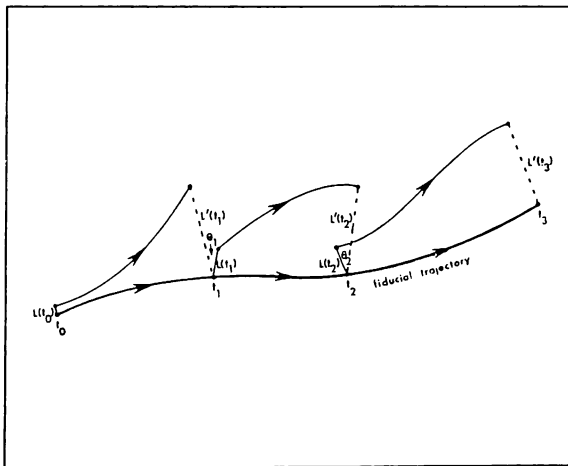


Figure 2 : A Schematic Representation of the Evolution and Replacement Procedure

The magnitudes of the Lyapunov exponent quantify an attractor's dynamics in information theoretic terms. The exponent λ measures the rate at which system processes create or destroy information; thus the exponents are expressed in bits of information per unit time or bits per orbit for a continuous system and bits per iteration for a discrete system.

4 SIMULATION OUTPUT ANALYSIS

When the goal of a simulation experiment is to estimate the value of steady state parameters, the initial conditions of the simulation usually bias the estimators. This problem is particularly troublesome when several independently seeded runs of the simulation are made and the results are used to construct confidence intervals. The frequency with which confidence intervals based on biased outputs include the true performance value generally decreases as more runs are made. This is caused by the intervals shrinking about an inaccurate point estimator. The literature on simulation methodology contains some techniques for controlling initialization bias as follow. These techniques are often too elaborate and offer no assurance that initialization bias will be effectively controlled.

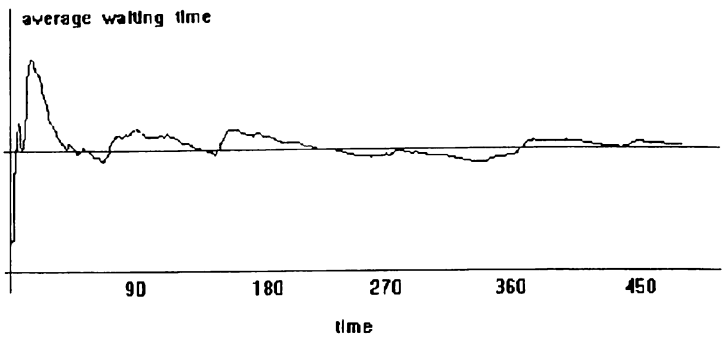
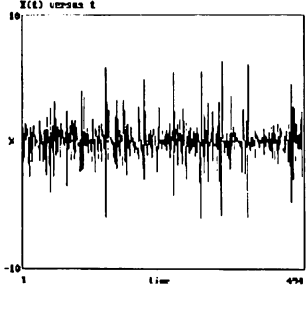
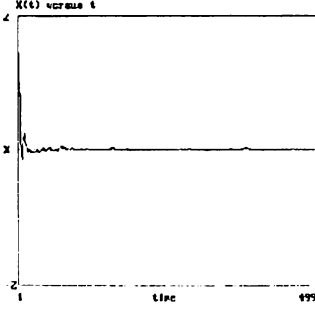
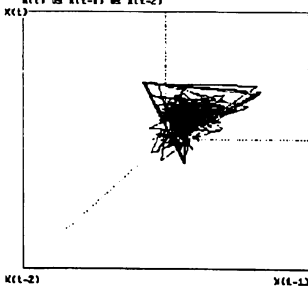
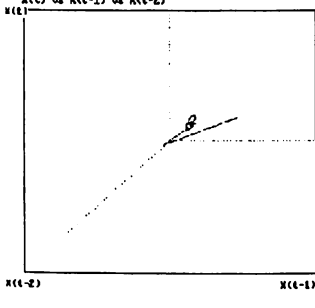
Variations of the system observations in the transient state are large and irregular due to initial bias as compared to those in the steady state. In this research, we first propose how to measure the difference between the system variation in the transient state and the variation in the steady state and then develop two methods to determine truncation points that can be used in eliminatinnng initial bias of the system.

The equations like (7) or (8) in Table 1 measures the level of variations of the system observations over time. The level of varitions is then analyzed using the chaos theory to ascertain whether or not the difference in the varitions is significant. The chaos theory used for this purpose has some good features. One good feature is that it can classify dynamic characteristics of the system into the regular motion, the chaotic motion, and the random motion without knowing any information about the input parameters. This classification process is done according to reconstruction state space, dimension, and Lyapunov exponent that are based on the time series of the system of under study.

To test the effectiveness of equation (7) and (8), the equation are applied to given simulation output data in Table 1. The obtained time series data, reconstructed state space, dimension using equation (2), and Lyapunov exponent using equation (6) for two equations are also shown in the Table 1.

As seen in the time series of the equation in Table 1, equation (7) does not illustrate the system varition sufficiently. Also, since the reconstruction attractor has an infinite dimension and Lyapunov exponent results in a positive value, the variation rate evaluated using equation (7) is under the random motion. This implies that equation (7) fails to distinguish the transient state from the steady state.

Table 1 : An Example of a Dynamic Characteristics for given Simulation Output Data

<p>Simulation Output Data</p>		
<p>Equations</p>	$\log_2 \frac{ x_{i+1} - x_i }{ x_i - x_{i-1} } \quad (7)$	$\log_2 \frac{x_{i+1}}{x_i} \quad (8)$
<p>Modified Time Series by Equations</p>		
<p>Reconstructed State Space</p>		
<p>Dimension</p>	<p style="text-align: center;">∞</p>	
<p>Lyapunov Exponent</p>	<p style="text-align: center;">0.587</p>	<p style="text-align: center;">-0.002</p>

On the contrary, when equation (8) is used, the time series of the equation indicates the system variation correctly. Since the reconstruction attractor has a finite dimension and Lyapunov exponent results in a negative value, the variation rate seems under the regular motion. This also states that the difference of variation rates between the transient state and the steady state is

significant. Based on these results, we propose two methods M1 and M2 using equation (8) for the determination of truncation points. The underlying concept of the two methods is that they determine truncation points based on the system variations in order to control initial bias. This is quite different from the concept of the existing

methods in which truncation points are decided using the mean and the variance of the output data.

5 DELETION STRATEGIES

Initialization bias can be a major source of error in estimating the steady state value of a simulated system performance measure. Lyapunov exponent λ , which decides whether convergence or divergence of a trajectory is applied to deciding how much data to be deleted which is called truncation points in simulation output data. The methods to be used in the experiments are as follow.

1) Method 1 (M1)

The first method uses λ_i for deciding truncation point represented by equation (9).

$$\lambda_i = \frac{1}{k} \sum_{i=1}^k \log_2 \frac{x_{i+1}}{x_i} \quad , \quad k=1,2,\dots,n-1 \quad (9)$$

where

- x_i : average waiting time in queue for up to the i -th customer from the first customer in M/M/1(∞) with arrival rate λ , service rate μ .
- n : run length (total number of observations in simulation output data).

The criteria for λ_i to decide the truncation points are as follow.

- criterion 1 (C1) : $|\lambda_i|$ must be less than the specified value.

The specified values for variation rate 5% is obtained to be 0.036 as follows,

$$\log_2 \frac{[a(1 \pm 0.025)]}{a} = 0.036$$

- criterion 2 (C2) : λ_i must not have the positive values(or negative values) more than 30 times continuously.

A variation rate is distributed normally if system's behavior comes to be the steady state.

2) Method 2 (M2)

Partition n simulation data x_1, x_2, \dots, x_n into b nonoverlapping batches in which each batch has m

observations such that $n=b \cdot m$, and define the following functions of the original data for $i=1,2,\dots,b$.

$$\lambda_i = \frac{1}{m} \sum_{j=1}^m \log_2 \frac{x_{(i-1)m+j}}{x_{(i-1)m+1}} \quad (10)$$

where

$x_{(i-1)m+j}$: average waiting time in queue for up to the j -th customer in i -th batch.

- criterion 3 (C3) : $|\lambda_i|$ must be less than the specified value.

This criterion C3 is equal to the criterion C1.

3) Method 3 (M3)

The deletion strategy suggested by Cash et al.(1992) is as follow. First, compute the below values for $i=1,2,\dots,b$ and $j=1,2,\dots,m$. And perform the F test with $f=0.25$, the fraction of the batches used to compute the variance estimator ; if the null hypothesis of no initial condition bias is rejected, delete the first 25% of the data and apply the test again to the remaining data. If the null hypothesis is accepted, retest at $f=0.5$ and next at $f=0.75$.

$$\bar{x}_{ij} = \frac{1}{j} \sum_{t=1}^j x_{(i-1)m+t}$$

$$S_{ij} = \bar{x}_{im} - \bar{x}_{ij}$$

$$\hat{K}_i = \max \{ j \cdot S_{ij} \} \quad , \quad \text{for } 1 \leq j \leq m$$

$$\hat{S}_i = \hat{K}_i \cdot \hat{S}_{i\hat{K}_i}$$

$$Q_{\max} = \sum_{i=1}^b \frac{m \cdot \hat{S}_i^2}{\hat{K}_i \cdot (m - \hat{K}_i)}$$

$$V_{\max} = \frac{Q_{\max}}{3b}$$

6 EXPERIMENTAL RESULTS

M/M/1(∞) queuing model was used to evaluate the performance of methods. In order to gather the data, we executed the simulation using the following conditions. Simulations are conducted with the utilities $\rho=0.5(1/\lambda=20.00, 1/\mu=10.00)$, $\rho=0.7(1/\lambda=14.29, 1/\mu=10.00)$ and $\rho=0.9(1/\lambda=11.11, 1/\mu=10.00)$. Each experiment involved 10 independently seeded replications in which each has 500 observations of customer waiting times in the queue. Each of the observations is gathered every 30 customers who have been served. To evaluate the effectiveness of the methods, the criterion in equation (11) was used.

$$\frac{|W_q^* - W_q|}{W_q^*} \tag{11}$$

where

$$W_q^* = \frac{1}{k} \sum_{j=1}^k \bar{x}_{.j}$$

$$\bar{x}_{.j} = \frac{1}{n-l} \sum_{i=1}^n x_{ij} \quad , \quad j = 1, 2, \dots, k.$$

- W_q^* : theory value of average waiting time.
- x_{ij} : average waiting time of i-th customer in j-th replication.
- n : run length of each run.
- l : truncation point in each run.
- k : number of replications.

The cases of experimentals for the deletion strategies are shown in Table 2.

Experimental results can be summarized as follow.

Table 2 : Cases of Experiments for Each Deletion Strategy

Deletion Strategy	Cases	
Method 1	A1	$\rho=0.5$
	A2	$\rho=0.7$
	A3	$\rho=0.9$
Method 2	B1	$\rho=0.5$ batch size=30
	B2	$\rho=0.5$ batch size=30
	B3	$\rho=0.5$ batch size=30
	B4	$\rho=0.7$
	B5	$\rho=0.7$
	B6	$\rho=0.7$
	B7	$\rho=0.9$
	B8	$\rho=0.9$
	B9	$\rho=0.9$
Method 3	D1	$\rho=0.5$
	D2	$\rho=0.7$
	D3	$\rho=0.9$

1) In Case of $\rho=0.5$

With respect to point estimators and equation (11), B3 produced good results among the cases of the method M2 but the difference in the cases of M2 is not significant. And the difference among the results of three methods is not also significant. Comparisons of A1(M1), B3(M2), D1(M3) for each run are shown in Figure 3.

2) In Case of $\rho=0.7$

In this case, B6 produced good results among the cases of the method M2 but the difference in the cases of M2 is

not significant. And the difference among the results of three methods is not also significant. Comparisons of A2(M1), B6(M2), D2(M3) for each run are shown in Figure 4.

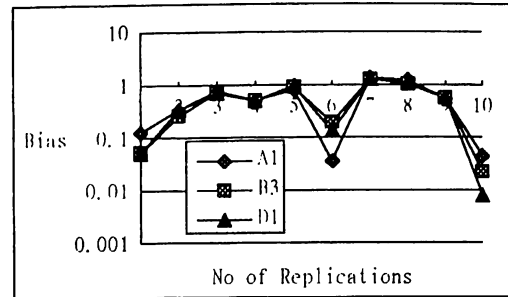


Figure 3 : Comparisons of Deletion Strategies for $\rho=0.5$

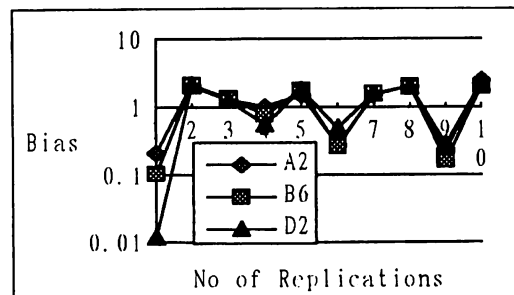


Figure 4 : Comparisons of Deletion Strategies for $\rho=0.7$

3) In Case of $\rho=0.9$

In this case, system behavior fails to reach steady state in some cases, with the given present run length. The method M1 and M3 detect a truncation point regardless of the system state. But the method M2 can determine correctly the transient state and B8 yields the best outcome regarding to point estimator and equation (11). Throughout the evaluation of three methods, we observed that the method M2 showed the best results. Comparison of A3(M1), B8(M2), D3(M3) for each run are shown in Figure 5.

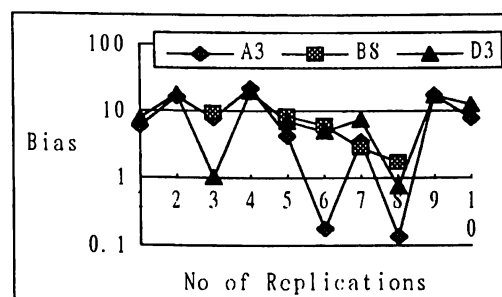


Figure 5 : Comparisons of Deletion Strategies for $\rho=0.9$

The experimental results of the deletion strategies for $\rho=0.5, \rho=0.7$ and $\rho=0.9$ are summarized in Table 3. When method M2 is applied in case of $\rho=0.9$, if the system's behavior shows a transient state, the run length which must be added to reach steady state is calculated as follows.

$$n = \left(\frac{Z_{\alpha/2} \times \sigma}{e} \right)^2 \quad (12)$$

where

- $Z_{\alpha/2}$: the value of the standard normal distribution leaving an area of $\alpha/2$ to the right
- σ : standard deviation
- e : a specified amount for absolute deviation between \hat{W}'_q and W'_q

Table 3 : Performance of Deletion Strategy

Cases	Deletion Strategy	\hat{W}'_q	$ \hat{W}'_q - W'_q /W'_q$	W'_q	
$\rho=0.5$	A1	M1	10.398	0.040	10.00
	B1	M2	10.357	0.036	
	B2		10.328	0.033	
	B3		10.317	0.032	
	D1		M3	10.338	
$\rho=0.7$	A2	M1	23.483	0.007	23.310
	B4	M2	23.537	0.010	
	B5		23.474	0.007	
	B6		23.423	0.005	
	D2	M3	23.412	0.004	
$\rho=0.9$	A3	M1	97.134	0.078	90.090
	B7	M2	96.661	0.073	
	B8		88.851	0.014	
	B9		91.618	0.017	
	D3	M3	95.467	0.060	

7 CONCLUSIONS

Initialization bias is the most troublesome problem in evaluating system's responses correctly using simulation output data. Two methods have been proposed to handle this problem using chaos theory.

The suggested methods have been compared with the method by Cash et al.(1992) using M/M/1(∞). We could find that the method M2 of which batch size is 40 among the three methods produced the best results with respect to $\rho=0.5, \rho=0.7$ and $\rho=0.9$ especially. Truncation point cannot be found in some cases. This means that system behavior fails to reach steady state, with the given the present run length. For these cases, we could naturally think that the run length should be increased to get more data. Some statistical work is required to determine the most appropriate criteria to be used in the proposed methods.

REFERENCES

- Cash, C.R. et al. 1992. Evaluation of Tests for Initialization Condition Bias. *Proceedings of the 1992 Winter Simulation Conference* : 577-585.
- Farmer, J.D. 1982. Chaotic Attractors of an Infinite-Dimensional Dynamical System. *Physica 4D* : 366-393.
- Gafarian, A.V. et al. 1978. Evaluation of Commonly Used Rules for Detecting Steady - State in Computer Simulation. *Naval Research Logistics Quarterly* 25 : 511-529.
- Kelton, W.D. and A.M. Law. 1984. An Analytical Evaluation of Alternative Strategies in Steady-State Simulation. *O.R.* 32 : 169-184.
- Kelton, W.D. and A.M. Law. 1985. The Transient Behavior of The M/M/s Queue, with Implications for Steady-State Simulation. *O.R.* 33 : 378-396.
- Moon, F.C. 1987. *Chaotic Vibrations*. John Wiley & Sons.
- Morrison, F. 1991. *The Art of Modeling Dynamic Systems*. John Wiley & Sons.
- Schruben, L. 1982. Detecting Initialization Bias in Simulation Output. *O.R.* 30 : 569-590.
- Schruben, L. et al. 1983. Optimal Tests for Initialization Bias in Simulation Output. *O.R.* 31 : 1167-1178.
- Shaw, R. 1981. Strange Attractors, Chaotic Behavior and Information Flow. *Z. Naturforsch* 36A : 80.
- Tufillaro, N.B. et al. 1992. *Nonlinear Dynamics and Chaos*. Addison Wesley.
- Wolf, A. et al. 1985. Determining Lyapunov Exponent from a Time Series. *Physica 16D* : 285-317.
- Wright, J. 1984. Method for Calculating a Lyapunov Exponent. *Physical Review A* 29 : 2924-2927.

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