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APPLICATION OF SHELL EQUATIONS TO UNSYMMETRICALLY LOADED COMPOUND SHELLS

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SUMMARY

A number of systems of equations are available in the literature for the stress analysis of shells of revolution under lateral loading. Although numerical integration of any of these systems is in principle straightforward, in practice, the application to shells, with negative curvature elements and complex boundary conditions, can give rise to difficulties and there are few cases documented of detailed results for unsymmetrically loaded shells.

Eight first order differential equations have been derived for the general axi-symmetric shell of revolution under lateral loading.

These have been integrated to find the stress distributions in a corrugated tube or bellows, under lateral loading. Details are given of the independent cases to be considered, the treatment of negative curvature elements and the boundary conditions assumed.

The results of loading tests on a model are compared with the theoretical curves of stress distributions for the complete shell. Although the results are complex in detail and not immediately obvious, the agreement of experiment and theory gives confidence in their accuracy. Particular care has been found to be necessary in order to get the correct number of independent cases for numerical integration. In addition, the signs of the variables at junctions between corrugations have required special consideration.

The formulation of the shell equations used is very much simpler than some which have been proposed.

Consideration is given to application to a sphere cylinder junction under lateral loading. For this the equations must be in terms of a co-ordinate of distance along the shell meridian. This enables the same equations to be used for cylinder and sphere.

It is taken that the boundary conditions at a suitable distance from the junction between the shells correspond to membrane forces but not membrane deflections. This enables the problem to be considered as a two point boundary problem in the same way as the corrugated tube. Lateral loading cases, consisting of either a moment or a shearing force applied to the cylinder, can be covered, or any combination of the two. Comparisons of stress distributions in both cylinder and sphere and the resulting Stress Concentration Factors are made with other published theoretical and experimental results.

1. Introduction

A stress analysis of the laterally loaded shell shown in Fig. 1 has been made using a Runge Kutta integration procedure.

Comparable experimental stress distributions have been obtained from loading tests on a specimen machined from a solid billet.

The first order differential equations used to analyse the shell shown in Fig. 1 were in terms of the independent variable ϕ, the angle between a perpendicular to the meridian and the axis.

By reformulating the equations in terms of S, a co-ordinate of distance along the shell meridian, a stress analysis of a sphere-cylinder intersection, with a lateral load applied to the cylinder, has been obtained.

2. Differential Equations

Equations 1-13 were derived from the equilibrium and displacement equations of Flügge [1] Chapter 6, in the simplified form for thin shells. They are for the case n = 1, where n is the number of the terms in the Fourier series, for the loads, displacements and stress resultants, for shells under general unsymmetrical loading.

For the case n = 1, all loads, displacements and stress resultants vary sinusoidally with θ, the co-ordinate of angular position in a plane perpendicular to the axis. The variables in equations 1-13 represent the maximum values of deflections and stress resultants, which occur at either $\theta = 0^{\circ}$ or $\theta = 90^{\circ}$.

$$\frac{dw}{d\phi} = d, \qquad \text{eq. (1)}$$

$$\frac{dd}{d\phi} = b^2\left[\frac{-M_\phi}{K} + \frac{\mu}{r}\left(\frac{w}{r} - \frac{d}{b}\cos\phi\right)\right] \qquad \text{eq. (2)}$$

$$\frac{dv}{d\phi} = b\left[\frac{-N_\phi}{D} - \frac{w}{b} - \frac{\mu}{r}(u + v\cos\phi + w\sin\phi)\right], \qquad \text{eq. (3)}$$

$$M_{\phi\theta} = \frac{K}{r}(1 - \mu)\left(\frac{w}{r}\cos\phi - \frac{d}{b}\right), \qquad \text{eq. (4)}$$

$$N_{\phi\theta} = N_1 + \frac{M_{\phi\theta}\sin\phi}{r}, \qquad \text{eq. (5)}$$

$$\frac{du}{d\phi} = b\left(\frac{2N_{\phi\theta}}{D(1-\mu)} + \frac{v + u\cos\phi}{r}\right), \qquad \text{eq. (6)}$$

$$M_\theta = K\left[\frac{1}{r}\left(\frac{d}{b}\cos\phi - \frac{w}{r}\right) + \frac{\mu}{b^2}\frac{dd}{d\phi}\right], \qquad \text{eq. (7)}$$

$$N_\theta = D\left[\frac{u + v\cos\phi + w\sin\phi}{r} + \frac{\mu}{b}\left(\frac{dv}{d\phi} + w\right)\right] \qquad \text{eq. (8)}$$

$$\frac{dM_\phi}{d\phi} = \frac{b\cos\phi}{r}(M_\theta - M_\phi) + b\left(Q_1 - \frac{2M_{\phi\theta}}{r}\right), \qquad \text{eq. (9)}$$

$$\frac{dN_1}{d\phi} = \frac{b}{r}\left(N_\theta - 2N_1\cos\phi - \frac{M_{\phi\theta}\cos\phi}{r}\frac{a}{b} - \frac{M_\theta\sin\phi}{r}\right), \qquad \text{eq. (10)}$$

$$Q_\phi = Q_1 - \frac{M_{\phi\theta}}{r}, \qquad \text{eq. (11)}$$

$$\frac{dN_\phi}{d\phi} = \frac{b}{r}\left[(N_\theta - N_\phi)\cos\phi - N_{\phi\theta}\right] + Q_\phi, \qquad \text{eq. (12)}$$

$$\frac{dQ_1}{d\phi} = -\frac{b}{r}(Q_1\cos\phi + N_\theta\sin\phi) - N_\phi + \frac{b}{r^2}(M_\theta - 2\cos\phi M_{\phi\theta}), \qquad \text{eq. (13)}$$

The displacements w, v and u, and the stress resultants N_ϕ, N_θ, M_ϕ, M_θ, Q_ϕ and $M_{\phi\theta}$ are as defined by Flügge [1].

b is the radius of curvature of the meridian, which for this case is constant and always +ve.

r is the radius in the θ plane at right angles to the axis.

$D = \frac{Eh}{1-\mu^2}$, $K = \frac{Eh^3}{12(1-\mu^2)}$, E being Young's Modulus, μ Poisson's Ratio and h the shell thickness.

N_1 and Q_1 are the resultant wall shearing forces in the directions of $N_{\phi\theta}$ and Q_ϕ respectively and are defined as

$$N_1 = N_{\phi\theta} - \frac{M_{\phi\theta}\sin\phi}{r} \qquad \text{eq. (14)}$$

$$\text{and } Q_1 = Q_\phi + \frac{M_{\phi\theta}}{r} \qquad \text{eq. (15)}$$

Thus the dependent variables in equations 1-13 are w, d, v, u, M_ϕ, N_1, N_ϕ and Q_1, which correspond directly to the terms defining boundary conditions.

It may be noted that Q_θ has been eliminated from these equations and that $M_{\theta\phi}$ is taken to be equal to $M_{\phi\theta}$, and that d is defined by equation (1).

The resultant solution, for the given shell, is given by the superposition of a suitable number of independent cases. These are integrated separately, starting with arbitrary values at one end of a shell segment.

3. Independent Cases Integrated

4 segments were taken and integration was performed over 90° of φ.

The starting and finishing angles were:

segment 1 : 0° - 90°
segment 2 : 180° - 270°
segment 3 : 270° - 360°
segment 4 : 90° - 180°

For each segment, 6 independent cases were integrated. For each case, the initial values of each variable were all set at zero except one, which was set at the arbitrary, but convenient, value of 1.

Non-Zero Starting Values

Case 1 $d = \frac{dw}{d\phi} = 1$: this corresponds to a rotation of a tangent to the meridian.

Case 2 $u = 1$: this corresponds to change in diameter in a plane perpendicular to the axis.

Cases 3, 4, 5 and 6 : initial values of 1 given to M_ϕ, N_1, N_ϕ and Q_1, respectively.

Thus 24 integrations of equations 1-13 were made and as each integration proceeded the values of N_ϕ, N_θ, M_ϕ and M_θ were stored at 10° intervals of φ.

A 1° interval of integration was used throughout, and from each set of starting values, a standard Runge Kutta procedure evaluated, by finding a weighed average slope, the values of all variables, step by step, over 90 steps.

The final values at the end of the integration paths were used to find the constants, which determined the proportions of each of the six independent cases, which were required to satisfy the boundary conditions of each segment.

4. Boundary Conditions

There are two deflection conditions causing stress:

(a) Radial displacement $\delta = v\cos\phi + w\sin\phi + u$ eq. (16)

(b) Rotation of the tangent

$$\beta = \frac{d}{b} - \frac{w\cos\phi}{r} + v\left(\frac{\sin\phi}{r} - \frac{1}{b}\right) \qquad \text{eq. (17)}$$

These expressions for δ and β have eliminated rigid body movements, corresponding to sideways movement of the shell perpendicular to the axis and rotation of the axis. Hence for all joints between segments, displacement boundary conditions require that δ and β, for adjacent segments, must be the same. Also, the four stress resultants must be equated at the joints.

At the point of application of the load it was assumed that a thin circular plate was present such that:

(a) It was rigid in its plane $\therefore\ \delta = 0$

(b) It was perfectly flexible under bending $\therefore M_\phi = 0$

and $N_\phi = 0$

(c) The total load of 1000 lb. was reacted by an unknown proportion of N_1 and Q_1, distributed according to the n = 1 assumption, so that: $\pi r(N_1 + Q_1) = -1000$ lb. eq. (18)
where r is the local radius = 6 in.

At the base of the shell it was assumed that the support was completely rigid $\therefore\ \delta = \beta = 0$.

Hence Joints 2, 3 and 4 give rise to 6 equations each, Joint 5 to 4 equations and Joint 1 to 2 equations.

Thus the boundary conditions give rise to 24 equations, which correspond to the 24 constants for the 6 independent cases integrated for the 4 segments.

5. Signs of the Variables at the Joints

In equating δ and β it is important to note that the signs of w, v, M_ϕ and N_1 may be different on either side of a joint.

6. Calculation of Resultant Stress Distributions

Suppose the constants for the six integrations for a segment are $K_1 - K_6$. Then if the values of M_ϕ, say, at 40° from the start of integration are $M_{\phi 40}$ as stored for each case then:

Resultant M_ϕ at $40^\circ = K_1 \times M_{\phi 40\ \text{case 1}} + K_2 \times M_{\phi 40\ \text{case 2}} + \ldots K_6\ M_{\phi 40\ \text{case 6}}$ eq. (19)

Similarly the resultant M_θ, N_ϕ and N_θ values can be found for all the angles (10° intervals), for which values from the integration cases have been stored.

Then by dividing the force resultants by h, the shell thickness and the moments by $\frac{h^2}{6}$, the direct and bending stress at all angles can be found.

7. Experimental Tests

A specimen was machined from a blank of Alcan GB-26SM, Aluminium Alloy, for which

Young's Modulus $E = 10.1 \times 10^6$ lb/sq.in. and Poisson's Ratio $\mu = 0.33$. Actual $h = 0.095$ in. 88 strain gauges were used to measure stresses on the inside and outside surfaces of the shell.

8. Results

Examples of the comparison between theoretical and experimental results are shown in Fig. 2 and 3.

Table 1 shows a comparison of theoretical and experimental values of N_1. The confirmation of the reverseal of N_1 at Joint 2 may be noted.

9. Sphere-Cylinder Intersection

The following geometry was considered:

Sphere radius = 5in, thickness = 0.1 in:

Cylinder radius = 1.5 in, thickness = 0.06 in:

a lateral load of 1000 lb was considered to be applied to the cylinder, 2.5 in from the intersection.

It was assumed that membrane forces, but not membrane deflections, existed at the point of application of the load and at an angle of $\phi = 30^\circ$ from the junction. Integration for both shells was inwards towards the junction. Three integration cases were then required for both cylinder and sphere corresponding to non-zero starting values of d, u and known membrane values of N_ϕ and N_1, respectively. At the junction, equating of δ, β, M_ϕ and N_1 enabled 4 equations to be obtained for the 4 constants corresponding to the 2 deflection cases each, for the cylinder and the sphere.

Hence superposition, of the stresses due to the three cases, enables resultant distributions to be found.

10. Conclusions

By eliminating rigid body movements, considering ϕ as varying over 360°, r_1 as always +ve, with appropriate signs for all variables, unsymmetrically loaded compound shells have been analysed. Boundary conditions have been given in terms of 'Kirchoff' shearing forces N_1 and Q_1 and relatively simple equations, with the assumption $M_{\phi\theta} = M_{\theta\phi}$, have given results, which have been well confirmed by experiment.

References

[1] FLÜGGE, W., "Stresses in Shells," Springer-Verlag, (1967).

[2] SUTCLIFFE, W. J., "Application of shell equations to an unsymmetrically loaded corrugated shell of revolution," Int. J. Mech. Sci., Vol. 14, (1972).

Table I

(+ve N_1 corresponds to N_1 reacting the load of 1000 lb)

Joint	Total force due to N_1 (lb)	
	Theory	Experiment
1	-	+ 330
2	- 1440	- 1390
3	+ 1007	+ 900
4	+ 1846	+ 1730
5	+ 1000	+ 1110

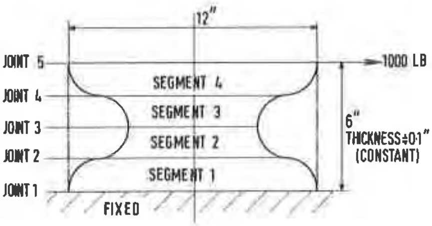

Fig. 1 Outline elevation of shell considered

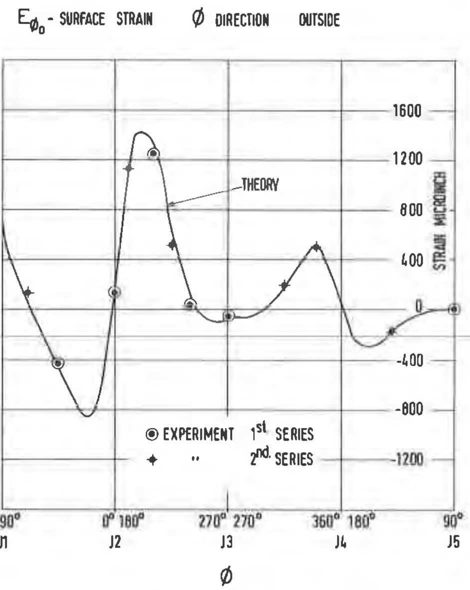

Fig. 2 Outside surface strains - ϕ direction ($\theta = 0^{\circ}$)

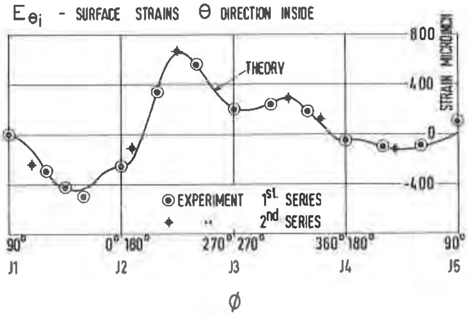

Fig. 3 Inside surface strains - θdirection ($\theta = 0^{\circ}$)