DYNAMICAL LOAD FACTOR OF IMPACT LOADED SHELL STRUCTURES

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SUMMARY

Dynamical loaded structures can be analysed by spectral representations, which usually lead to an enormous computational effort. If it is possible to find a fitting dynamical load factor, the dynamical problem can be reduced to a statical one. The computation of this statical problem is much more simpler. The disadvantage is that the dynamical load factor usually leads to a very rough approximation. In this paper it will be shown, that by combination of these two methods, the approximation of the dynamical load factor can be improved and the consumption of computation time can be enormously reduced.

If we consider a system with one degree of freedom the dynamical load factor is a function of the dynamical load and of the eigenfrequency of the system. If we have a continuum with many eigenfrequencies the dynamical load factor of the critical eigenfunction must be taken, which usually leads to very high dynamical load factors. For improvement a computation by spectral representation must be done: The solution of the dynamically loaded structure will be evaluated in terms of the eigenfunctions of vibrating structures. The terms of the series will be multiplied by the amplification factors of each eigenfunction and all terms are added. For practical solutions only a finite number of terms can be considered. If we take for granted the convergence of the series, we must additionally be sure that the number of terms is sufficient to get a good approximation of the solution. To prove this, we evaluate the statical problem in terms of the same eigenfunctions of the freely vibrating structure as the dynamical problem. The limit of the statical series is the exact statical solution, which easily can be computed. If we consider only N terms of the statical series, the quality of the solution can be seen as the difference to the exact statical solution. Now we consider a nonperiodical dynamical impact load. In this case the dynamical amplification factor is a number between 0 and 2, and so we can state that both series, the statical one and the dynamical one, behave in the same manner. This knowledge will help us to construct an improved approximation of the dynamical solution though only a few terms of the series are considered.

We consider N terms of the dynamical series and we get without additional computation the statical series with N terms, because the eigenfunctions and the evaluation constants of both series are the same. The remainder of the statical series—difference to the exact statical solution—will be multiplied by a dynamical load factor δ_R^* and will be added to the dynamical series with N terms. The value of δ_R^* ist the maximum of the frequencies ω_{N^*} with $N^* > N$. By this way δ_R^* decreases with increasing number N and runs to 1. As an example a clamped hemispherical shell with $h/a = 1/500$ is considered. The shell has an impact load at its dome. The time variation of the load is the well known function of aircraft impact. The critical frequency of this load function leads to a dynamical load factor $\delta_{max} = 1.83$. By the above mentioned method the dynamical load factor can be enormously reduced, though only a few terms of the dynamical series are considered. The same method (comparison of the dynamical series with the statical one) makes it possible to control the accuracy of the dynamical series with N terms.

1. Introduction The safeguard of modern buildings requires in many cases that dynamical loads, such as impact, explosion and earth quake, must be considered. When calculating the dynamical problem by spectral representation, the variables are evaluated in terms of the eigenmodes of the freely vibrating system. If the lowest eigenmodes of a structure are dominant (beams, plates), this method converges very rapidly. But considering shell problems higher eigenmodes are of greater influence; that is why in this case a very great number of eigenmodes must be considered to achieve a sufficient good approximation, and we have to expect high computational costs. Besides the computation of higher eigenmodes is not so accurate as that of the lower ones. If it is possible to find a fitting dynamical load factor, the cost wasting dynamical computation can be replaced by a relative simple statical one. Unfortunately the approximation by a dynamical load factor is a very rough one. In the example of the axisymmetrical loaded hemispherical shell we want to show, how the combination of both methods (spectral representation and dynamical load factor) leads to a very good approximation of the dynamical solution without too extensive calculations.

2. The dynamical load factor of a spring-mass-system

The method of the dynamical load factor is well known from systems of one degree of freedom. We consider a system of mass m and of spring constant c (figure 1), which is loaded by a force $F(t)$. The differential equation of the displacement $x(t)$ of the mass is:

$$m\ddot{x} + cx = F(t) \qquad \text{with} \quad (\;)^{\cdot} \mathrel{\hat{=}} \frac{d()}{dt} \tag{1}$$

If the mass m is in rest at the beginning ($x(t=0) = 0$, $\dot{x}(t=0) = 0$, the force $F(t)$ exists for $t > 0$) the solution of equation (1) is:

$$x(t) = \frac{\omega}{c}\int_0^t F(\tau)\sin\omega(t-\tau)\,d\tau. \tag{2}$$

ω is the eigenfrequency of the mass-spring-system. Now we compare the dynamical solution $x(t)$ from equation (2) with the statical displacement of the mass-spring-system under the maximum F_0 of the load $F(t)$.

$$X_{stat} = \frac{F_0}{c} \tag{3}$$

The ratio of the solutions (2) and (3) is the so called dynamical amplification $\delta(t)$.

$$\delta(t) = \frac{X(t)}{X_{stat}} = \frac{\omega}{F_0}\int_0^t F(\tau)\sin\omega(t-\tau)\,d\tau \tag{4}$$

The maximum of the dynamical amplification $\delta(t)$ is the dynamical load factor δ^*. If this dynamical load factor δ^* is known and if only the maximum of the dynamical solution is of interest, an approximation of the dynamical solution can be given by a statical calculation.

The dynamical load factor δ^* from eq. (4) is a function of the shape of F(t) and a function of the eigenfrequency ω of the considered mass-spring-system. If we consider a force F(t) which increases linearly to its maximum F_0 within the time t_1 and then remains constant (figure 2a), we get the following amplification:

$$\delta(t) = \frac{t}{t_1} - \frac{1}{\omega t_1} \sin \omega t \qquad \text{for} \quad t < t_1 , \tag{5a}$$

$$\delta(t) = 1 - \frac{1}{\omega t_1} \sin \omega t + \frac{1}{\omega t_1} \sin \omega (t - t_1) \qquad \text{for} \quad t > t_1 . \tag{5b}$$

The maximum amplification δ^* occurs for $t > t_1$ and is given over $\omega \cdot t_1$ in figure 2a .

$$\delta^* = 1 + \left| \frac{2}{\omega t_1} \sin \omega t_1/2 \right| \tag{6}$$

For the given shape of F(t) the factor δ^* is only a function of time t_1 and of the eigenfrequency ω . For low numbers of ωt_1 (low eigen-frequency ω or short time t_1) δ^* is nearly 2.0 and for great numbers of ωt_1 (high eigenfrequency ω or long time t_1) δ^* tends to 1 . If we consider another load function F(t), the dynamical load factor δ^* over $\omega \cdot t_1$ looks quite different.In figure 2b we have a sinusoidal load shape. The factor δ^* increases from 0 to its maximum of about 1.75 and then tends to 1 again for great numbers of ωt_1.

3. The dynamical loaded shell

For a continuum the calculation with help of the dynamical load factor is difficult, because the continuum has an infinite number of eigenfrequencies. A reliable approximation of the dynamical solution with the dynamical load factor requires the greatest factor (δ^*= 2.0 for F(t) of figure 2a, δ^*= 1.75 for F(t) of figure 2b). If we know the eigenfrequencies of the continuum, the eigenfrequency, which leads to the greatest dynamical load factor, must be regarded. Thereby the dynamical load factor can be reduced sometimes but in many cases the approximation is not very good. So mostly a dynamical calculation is necessary.

In the example of the dynamical loaded spherical shell we briefly want to explain the dynamical solution of a continuum by spectral representation. The governing equations of the dynamical loaded shell are:

$$L_1(u,v,w) = -\frac{1}{\omega_0^2}\ddot{u}$$
$$L_2(u,v,w) = -\frac{1}{\omega_0^2}\ddot{v}$$
$$L_3(u,v,w) = -\frac{1}{\omega_0^2}\ddot{w} - \frac{a^2}{D}q(\varphi,\vartheta,t) \qquad (7)$$

L_1, L_2, L_3 are differential operators; u, v, w are the displacements of the shell in the direction of the shell coordinates φ, ϑ, r ; $q(\varphi,\vartheta,t)$ is a normal load varying with time and space, $\omega_0 = \sqrt{E/\varrho a^2(1-\vartheta^2)}$ is a reference-frequency $D = Eh/(1-\nu^2)$ is the extensional rigidity; E is the modulus of elasticity, ϑ Poisson's ratio, ϱ the density of the shell material, a the radius and h the thickness of the spherical shell. The homogeneous solution - the freely vibrating shell - are the eigenfunctions u_n^*, v_n^*, w_n^* with the eigenfrequency ω_n and the orthogonality condition B_n.

$$\int_{\sigma}(u_n^* u_m^* + v_n^* v_m^* + w_n^* w_m^*)\,d\sigma \begin{cases} = 0 & \text{for } m \neq n \\ = B_n & \text{for } m = n \end{cases} \qquad (8)$$

Now we expand the variables u,v,w in terms of the eigenfunctions u_n^*, v_n^*, w_n^*.

$$u = \sum_n \phi_n(t)\, u_n^*(\varphi,\vartheta)$$
$$v = \sum_n \phi_n(t)\, v_n^*(\varphi,\vartheta)$$
$$w = \sum_n \phi_n(t)\, w_n^*(\varphi,\vartheta) \qquad (9)$$

$\phi_n(t)$ is the n-th generalized coordinate.
The functions u, v, w from (9) are put into the inhomogeneous diff. eq. system (7) and we get together with the orthogonality condition (8) a single diff. eq. for $\phi_n(t)$.

$$\ddot{\phi}_n + \omega_n^2\phi_n = -\frac{\int_{\sigma} q(\varphi,\vartheta,t)\, w_n^*\, d\sigma}{\varrho h B_n} \qquad (10)$$

If we assume that the local distribution of the load does not change with time

$$q(\varphi,\vartheta,t) = p(\varphi,\vartheta)\,F(t), \qquad (11)$$

we get:

$$\ddot{\phi}_n + \omega_n^2\phi_n = \omega_n^2 W_n F(t) \qquad (12)$$

with $$W_n = - \frac{\int_\sigma p(\varphi,\vartheta)\, w_n^* \, d\sigma}{\varrho\, h\, B_n\, \omega_n^2} \tag{13}$$

The solution of eq. (12) can easily be given:

$$\phi_n(t) = \phi_n(o) \cos\omega_n t + \frac{1}{\omega_n} \dot{\phi}_n(o) \sin\omega_n t + \omega_n W_n \int_0^t F(\tau) \sin\omega_n (t-\tau) d\tau \tag{14}$$

with

$$\phi_n(o) = \frac{1}{B_n} \int_\sigma (u(o)\, u_n^* + v(o)\, v_n^* + w(o)\, w_n^*)\, d\sigma \tag{15a}$$

$$\dot{\phi}_n(o) = \frac{1}{B_n} \int_\sigma (\dot{u}(o)\, u_n^* + \dot{v}(o)\, v_n^* + \dot{w}(o)\, w_n^*)\, d\sigma \tag{15b}$$

$u(o)$, $v(o)$, $w(o)$ are the displacements of the shell and $\dot{u}(o)$, $\dot{v}(o)$, $\dot{w}(o)$ the displacement "velocities" of the shell in the beginning at time $t = 0$.

4. Statical solution of the normally loaded spherical shell

If we omit the time dependent terms in the diff. equation system (7) we get the equations of the statically loaded shell.

$$\begin{aligned} L_1(u,v,w) &= 0 \\ L_2(u,v,w) &= 0 \\ L_3(u,v,w) &= -\frac{a^2}{D}\, p(\varphi,\vartheta) \end{aligned} \tag{16}$$

In the following we want to compare the dynamical solution with the statical one. That is why we expand the statical solution in terms of the eigenfunctions u_n^*, v_n^*, w_n^* of the freely vibrating shell, too.

$$\begin{aligned} u &= \sum_n \bar{W}_n\, u_n^* \\ v &= \sum_n \bar{W}_n\, v_n^* \\ w &= \sum_n \bar{W}_n\, w_n^* \end{aligned} \tag{17}$$

$\bar{W}_n$ is an unknown expansion coefficient.

The variables u, v, w of (17) are put into the equation system (16) and we get after some manipulations under consideration of the orthogonality condition (8) the constant $\bar{W}_n$.

$$\bar{W}_n = - \frac{\int_\sigma p\, w_n^* \, d\sigma}{\varrho\, h\, B_n\, \omega_n^2} \tag{18}$$

Looking at (13) we can see, that the coefficient $\overline{W}_n$ coinsides with W_n, the coefficient of the dynamical solution. The expansion (17) is the complete solution of the statical problem, because the chosen functions u_n^*, v_n^*, w_n^* satisfy all boundary conditions.

5. The dynamic load factor of shell structures

We consider the n-th term of the dynamical displacement:

$$w_n^D = \phi(t)\, w_n^* \tag{19}$$

and the n-th term of the statical displacement:

$$w_n^{St} = W_n\, w_n^* \; . \tag{20}$$

The ratio of both displacements is the dynamical amplification of the n-th term.

$$\delta_n(t) = \frac{w_n^D}{w_n^{St}} = \frac{\phi_n}{W_n} = \omega_n \int_0^t F(\tau) \sin \omega_n (t-\tau)\, d\tau \tag{21}$$

The maximum of the amplification $\delta_n(t)$ is the dynamical load factor δ_n^* of the n-th mode. For a given load $F(t)$ the factor δ_n^* is a function of the eigenfrequency ω_n .

6. Approximation of the dynamical solution

For practical calculations each series must be ended after a finite number of terms. If we assume that the dynamical series is convergent, we also must be sure, that the considered terms are sufficient to describe the final solution. To proof this we first calculate the exact statical solution. The statical series from chapter 4 must tend to this solution. If we compare the exact statical solution with our statical series with N terms we are informed about the quality of this series with its N terms. The difference between both solutions is the rest of the series.

If we have a nonperiodical impact load, the dynamical amplification must be between 0 and 2. For this case the dynamical series must converge, too, and must behave in the same manner as the statical solution. If we end the dynamical series after N terms $w_{1\div N}^D$, we get without additional computation the statical series with N terms $w_{1\div N}^{St}$, because the evaluation coefficients W_n of the statical and the dynamical series are the same. By a comparison of the statical series $w_{1\div N}^{St}$ with the exact statical solution w_{St} we get the rest of the statical series, which will be multiplied with a dynamical load factor δ_R^* and will be added to the dynamical series $w_{1\div N}^D$.

$$W^{D} = W^{D}_{1 \div N} + \delta^{*}_{R} \left(W_{st} - W^{st}_{1 \div N} \right) \tag{22}$$

Thereby δ^{*}_{R} is the greatest dynamical load factor of frequencies ω_n with $n > N$.

7. Numerical example

We consider a clamped hemisphere with $h/a = 1/500$, which is partially loaded at its apex (figure 3a). The statical solution (bending theory of thin shells) is given over the coordinate φ in figure 3b. For an approximate dynamical solution the statical values have to be multiplied by a fitting dynamical load factor, which depends on the time variety of the load. The time dependence of the load is given in figure 4, which leads to dynamical load factors δ^{*} as given over the frequency ω in figure 5. If we want to deduce the maximum value of the dynamical solution from the statical calculation, we have to multiply the statical values from figure 3b with the greatest dynamical load factor of figure 5, $\delta^{*}_{max} = 1.83$. But the value $\delta^{*}_{max} = 1.83$ must only by taken, if the shell has frequencies, which are lying in the region of δ^{*}_{max} of figure 5. That is why sometimes a more accurate dynamical load factor can be found if the eigenfrequencies of the shell are known. In this case only the greatest dynamical load factor of the shell frequencies is of interest, in our example $\delta^{*}_{1} = 1.42$ (dynamical load factor of the first eigenfrequency). For further improvement the above mentioned method will be applied. If we calculate the radial displacement at shell apex, e.g., the statical solution yields the value $\frac{w\,D}{p a^2} = 121.48$. This exact statical solution will be compared with the statical series given in the 3rd column of table I , in which the first column shows how many terms N are considered. In the second column we have the related eigenfrequency ω_N/ω_0 of the N-th mode. For example, if we consider two terms, the eigenmodes of the first frequency 0.705 and of the second frequency 0.877 are considered, we get a statical series solution of 7.12, which will be compared with the exact statical value 121.48. The difference of both values will be multiplied by the greatest dynamical load factor, which occurs for frequencies $\omega > \omega_N$ (here greater than 0.877). The result will be added to the maximum of the dynamical series solution with two modes (4th column). So we achieve the approximate dynamical solution 149.96 (5th column). The last column shows the dynamical load factor of the approximate dynamical solution. The results of table I are graphically shown in figure 6 over N, the number of the considered terms. We can see, that by this method a good approximate dynamical solution can be achieved with a relative small number of eigenmodes - , e.g., the series with 10 terms only represents 70% of the final solution, but the approximate dynamical solution is satisfying with its 6% over the final value.

For the membrane forces at apex the same calculation is done and shown over numer N in figure 7 . We get similar results.

8. Conclusion

The dynamical series solution of an impact loaded spherical shell shows that in the case of shells many terms of the series must be considered in order to get sufficient practical solutions.
If we expand the statical solution in terms of the eigenmodes, too, and compare the statical series with the exact statical value, we are informed about the convergence of the statical series solution. If we are only interested in the maximum value of the dynamical solution, we can achieve a very good dynamical load factor by the above mentioned method, whereby only a few terms of eigenmodes must be considered.

References:

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McGraw-Hill Book Comp., Inc., New York/Toronto/London

[2] Hammel, J.: "Stoß auf eine Kugelschale"
Dissertation, Darmstadt 1974

[3] Hammel, J.: "Eine Abschätzung für die Beanspruchung dynamisch belasteter Kugelschalen"
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[4] Kraus, H.: "Thin Elastic Shells"
John Wiley & Sons, Inc., 1967, New York/London/Sidney

Table I: series solutions of radial displacement w at apex under consideration of N terms (ω_o = 537.1 $^1/s$; $^h/a$ = $^1/500$)
w_{St} = 121.48

N	$^{\omega_N}/\omega_o$	stat. sol. (N terms) w^{st}_{4+N}	dyn. sol. (N terms) w^{D}_{4+N}	approximate dyn. sol. w^{D}	DLF
1	0.705	2.81	3.96	174.47	1.436
2	0.877	7.12	8.19	146.96	1.210
4	0.928	20.98	22.85	144.23	1.187
6	0.942	40.52	43.39	141.13	1.162
8	0.957	62.22	65.68	137.27	1.130
10	0.980	83.63	87.14	132.83	1.093
12	1.017	102.74	106.06	128.77	1.060
14	1.072	117.83	121.53	125.96	1.037
16	1.150	128.17	132.19	125.49	1.033
18	1.251	133.94	137.77	125.31	1.032
20	1.378	135.97	139.80	125.31	1.032
22	1.627	134.96	138.70	125.26	1.031

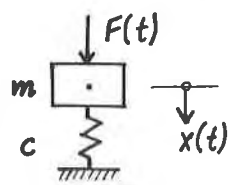

figure 1: mass-spring-system

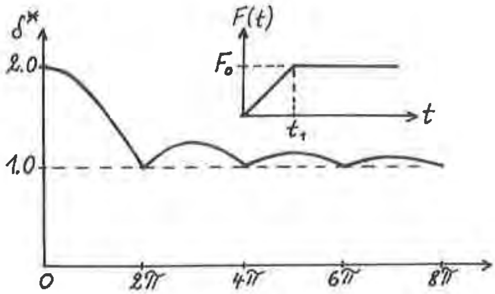

figure 2a: δ^* dynamical load factor (DLF) over ωt_1 for a linearly increasing load function F(t)

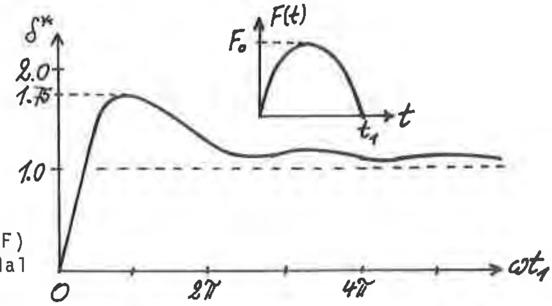

figure 2b: δ^* dynamical load factor (DLF) over ωt_1 for a sinusoidal load function

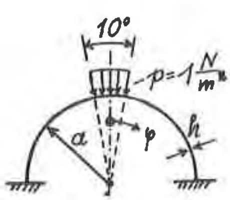

figure 3a: clamped hemisphere ($^h/a$ = $^1/500$) loaded at its apex

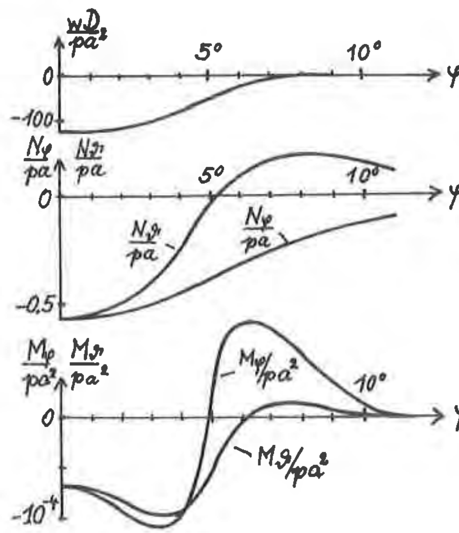

figure 3b: statical results (radial displacement w , membrane forces N_φ , N_ϑ , couple forces M_φ , M_ϑ) over meridian φ

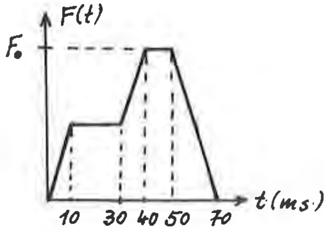

figure 4:
time variety of the load F(t)

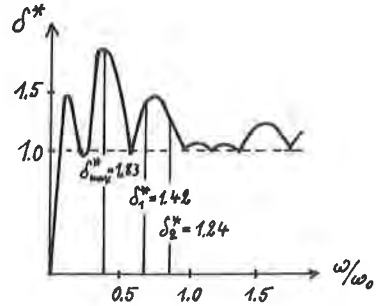

figure 5: DLF δ^* over ω/ω_0
for F(t) of figure 4

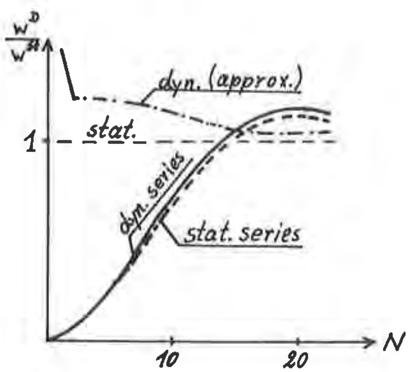

figure 6: approximate dynamical solution of the radial displacement at apex over number N

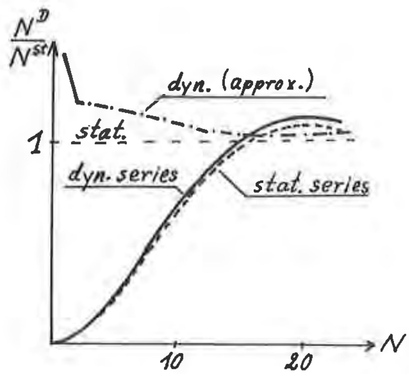

figure 7: approximate dynamical solution of the membrane force at apex over N