THERMOELASTOPLASTIC SOLUTION OF A THICK-WALLED TUBE SUBJECTED TO TRANSIENT THERMAL AND PRESSURE LOADINGS

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SUMMARY

A literature survey indicates that the problem of elastoplastic thick-walled cylinders have been solved by almost all investigators based on cylinders subjected to either only on mechanical loading or only on thermal loading. Very little work has been done on the elastoplastic solutions for thick-walled cylinders subjected to both mechanical and transient thermal loadings. The equivalent stress in the thick-walled cylinder under pressure and axial loading has been proved to be a monotonic function with respect to the radius; hence, only one elastic-plastic boundary exists. However, for the problem of an elastoplastic thick-walled cylinder subjected to both mechanical and transient thermal loadings, the equivalent stress in the cylinder is not necessarily a monotonic function with respect to radius. Therefore, the number of elastic-plastic boundaries in the wall of the cylinder is not necessarily restricted to be one but may be two or more appearing at several places in the cylinder. The temperature distribution, temperature gradient, temperature and pressure loading history are very important factors in determining the locations of the elastic-plastic boundaries. Therefore, this problem is much more difficult than the problem of a thick-walled cylinder subjected only to mechanical loading.

On the basis of the Prandtl-Reuss stress-strain rule, the von Mises flow criterion, and temperature-dependent yield strength, the strain-hardening, and compressibility properties of a material, an efficient incremental solution technique proposed by Chu for solving elastoplastic problems of thick-walled tubes subjected to mechanical loading is here extended to include the effects of transient thermal loading. The finite-difference treatment of the elastoplastic problem of a thick-walled tube is normally based on the differential equations for the displacement vector; hence, determination of stresses and strains requires numerical differentiation. However, good results in differentiation are not provided by the computer unless a rather fine grid is used. With the alternative method, developed in this paper, incremental stresses and strains are directly used as variables; hence, numerical differentiation in the evaluation of stresses and strains is not required. Moreover, since no assumption is made on the displacement field, this method can provide better results than those of the finite-element methods. Since the consideration of stress, strain, loading and temperature history is involved in the analysis, the present theory is particularly suitable for predicting stress and strain distribution, and location of elastic-plastic boundaries of a thick-walled tube subjected to nonproportional mechanical and transient thermal loadings.

1. Introduction

A literature survey indicated that the problems of elasto-plastic thick-walled cylinders have been solved by almost all investigators on the basis of cylinders subjected to mechanical loadings [1-14]. Very little work has been done on the elastoplastic solutions for thick-walled cylinders subjected to transient thermal loadings which is very important for power plant pipe line design. The equivalent stress in a thick-walled tube under pressure and axial loading is a monotonic function with respect to the radius; hence, only one elastic-plastic boundary exists. However, for the problem of an elastoplastic thick-walled tube subjected to transient thermal loading, the equivalent stress in the cylinder is not necessarily a monotonic function with respect to the radius, but is strongly dependent upon the heat flow in the wall of the tube. Therefore, the number of elastic-plastic boundaries in the wall of the tube is not necessarily restricted to one, but may be two or more appearing at several places in the tube. Also, the temperature distribution, temperature gradient, and temperature history of the cylinder are very important factors in determining the locations of the elastic-plastic boundaries. Therefore, the problem of an elastoplastic thick-walled cylinder subjected to both mechanical and transient thermal loading, is much more difficult and more complicated than the problem of a thick-walled cylinder subjected only to mechanical loading. Moreover, no previous solution is available for this problem.

Bland [17] analyzed the problem of a thick-walled tube subjected to pressure and a steady-state temperature gradient. On the basis of the Tresca yield criterion, the incompressibility of a material, and the plane-strain assumption, Weiner and Huddleston [18] proposed a thermal stress analysis of heat-treated cylinders. By use of von Mises' yield criterion, Prandtl-Reuss' flow rule of plasticity, isotropic-isothermal hardening rule, and compressibility of a material, Chu [16] proposed a numerical thermoelastoplastic solution of a thick-walled tube. In all of the analysis above, the thermal stresses were induced by thermal expansion. However, the temperature effect on the yield strength of a material was not considered. Hence, these solutions can be applied to only the case in which the temperature in the material is lower than certain values. The yield strength of a material decreases with temperature increases, and it decreases dramatically if the temperature of the material reaches a certain high level. At the present time, no solution is available for the elasto-plastic problems of thick-walled tubes subjected to both mechanical and transient thermal loading with the consideration of temperature yield strength of a material and nonisothermal flow rules of plasticity. The purpose of this investigation was to solve this complicated problem.

2. Development of an Incremental Solution

In the development of an incremental inelastic theory for a thick-walled tube, a cylindrical coordinate system (r, θ, z) is used with the z axis coincident with the axis of the tube. The cross section of a thick-walled tube is divided into n rings by radii $r_1 = a$, $r_2 = ,\ldots, r_k = c,\ldots, r_{n+1} = b$, where $r_k = c$ is an elastic-plastic boundary.

For any point in the inelastic region, the governing equations for stresses, increments of stresses, and strains can be derived as follows:

Since Prandtl-Reuss' flow rule is assumed to be valid, two independent incremental stress-strain equations are derived as

$$-\frac{1}{E}[(2\sigma_z-\sigma_r-\sigma_\theta) + \nu(2\sigma_r-\sigma_\theta-\sigma_z)]d\sigma_r + \frac{\nu}{E}[(2\sigma_z-\sigma_r-\sigma_\theta) - (2\sigma_r-\sigma_\theta-\sigma_z)]d\sigma_\theta$$

$$+\frac{1}{E}[(2\sigma_r-\sigma_\theta-\sigma_z) + \nu(2\sigma_z-\sigma_r-\sigma_\theta)]d\sigma_z + (2\sigma_z-\sigma_r-\sigma_\theta)d\varepsilon_r$$

$$= (2\sigma_r-\sigma_\theta-\sigma_z)d\varepsilon_z + \beta[(2\sigma_z-\sigma_r-\sigma_\theta) - (2\sigma_r-\sigma_\theta-\sigma_z)]dT \quad (1)$$

where β is the thermal expansion coefficient. All notations in the paper are the same as in Ref. (16), except those defined in the paper.

$$\frac{\nu}{E}[(2\sigma_z-\sigma_r-\sigma_\theta) - (2\sigma_\theta-\sigma_z-\sigma_r)]d\sigma_r - \frac{1}{E}[(2\sigma_z-\sigma_r-\sigma_\theta) + \nu(2\sigma_\theta-\sigma_z-\sigma_r)]d\sigma_\theta$$

$$+\frac{1}{E}[\nu(2\sigma_z-\sigma_r\sigma_\theta) + (2\sigma_\theta-\sigma_z-\sigma_r)]d\sigma_z + (2\sigma_z-\sigma_r-\sigma_\theta)d\varepsilon_\theta$$

$$= (2\sigma_\theta-\sigma_z-\sigma_r)d\varepsilon_z + \beta[(2\sigma_z-\sigma_r-\sigma_\theta) - (2\sigma_\theta-\sigma_z-\sigma_r)]dT \quad (2)$$

The surface used to define the elastic limit is referred to as the yield surface. For strain-hardening nonisothermal material, the subsequent yield surface or loading function can be represented as

$$\{\frac{1}{2\bar{\sigma}}(2\sigma_r-\sigma_\theta-\sigma_z) + \frac{(1+\nu)}{E}\phi\,[(2\varepsilon_r-\varepsilon_\theta-\varepsilon_z) - \frac{(1+\nu)}{E}(2\sigma_r-\sigma_\theta-\sigma_z)]\}d\sigma_r$$

$$+\{\frac{1}{2\bar{\sigma}}(2\sigma_\theta-\sigma_z-\sigma_r) + \frac{(1+\nu)}{E}\phi\,[(2\varepsilon_\theta-\varepsilon_z-\varepsilon_r) - \frac{(1+\nu)}{E}(2\sigma_\theta-\sigma_z-\sigma_r)]\}d\sigma_\theta$$

$$+\{\frac{1}{2\bar{\sigma}}(2\sigma_z-\sigma_r-\sigma_\theta) + \frac{(1+\nu)}{E}\phi\,[(2\varepsilon_z-\varepsilon_r-\varepsilon_\theta) - \frac{(1+\nu)}{E}(2\sigma_z-\sigma_r-\sigma_\theta)]\}d\sigma_z$$

$$-\phi[(2\varepsilon_r-\varepsilon_\theta-\varepsilon_z) - \frac{(1+\nu)}{E}(2\sigma_r-\sigma_\theta-\sigma_z)]d\varepsilon_r$$

$$-\phi[(2\varepsilon_\theta-\varepsilon_z-\varepsilon_r) - \frac{(1+\nu)}{E}(2\sigma_\theta-\sigma_z-\sigma_r)]d\varepsilon_\theta$$

$$= \phi[(2\varepsilon_z-\varepsilon_r-\varepsilon_\theta) - \frac{(1+\nu)}{E}(2\sigma_z-\sigma_r-\sigma_\theta)]d\varepsilon_z + \phi\beta dT + \frac{d\sigma(T)}{dT}dT \quad (3)$$

in which

$$\bar{\sigma} = \frac{1}{\sqrt{2}}[(\sigma_r-\sigma_\theta)^2 + (\sigma_\theta-\sigma_z)^2 + (\sigma_z-\sigma_r)^2]^{1/2} \quad (4)$$

$$\bar{\varepsilon}^p = \frac{\sqrt{2}}{3}[(\varepsilon_r^p-\varepsilon_\theta^p)^2 + (\varepsilon_\theta^p-\varepsilon_z^p)^2 + (\varepsilon_z^p-\varepsilon_1^p)^2]^{1/2} \quad (5)$$

and

$$\phi = \frac{2}{9}\frac{\eta E}{1-\eta}\frac{1}{\bar{\varepsilon}^p} \quad (6)$$

where η is the strain-hardening factor for the material, $\sigma(T)$ is the temperature-dependent yield strength of the material, and ε_r^p is the plastic strain component in the r direction.

For any point in the elastic region, the Duhamel-Neumann stress-strain-temperature relations are assumed to be satisfied:

$$d\varepsilon_r = \frac{1}{E}\, d\sigma_r - \frac{\nu}{E} d\sigma_\theta - \frac{\nu}{E} d\sigma_z + \beta dT \tag{7}$$

$$d\varepsilon_\theta = - \frac{\nu}{E} d\sigma_r + \frac{1}{E} d\sigma_\theta - \frac{\nu}{E} d\sigma_z + \beta dT \tag{8}$$

$$d\varepsilon_z = - \frac{\nu}{E} d\sigma_r - \frac{\nu}{E} d\sigma_\theta + \frac{1}{E} d\sigma_z + \beta dT \tag{9}$$

The equation of compatibility and the equation of equilibrium are valid for both the elastic and the inelastic regions of a thick-walled tube. The finite-difference forms of these two equations are given by [12]:

$$- (r_{i+1}-r_i)(d\varepsilon_r)_i + (r_{i+1}-r_i)(d\varepsilon_\theta)_i + r_i(d\varepsilon_\theta)_{i+1}$$

$$= (r_{i+1}-r_i)(\varepsilon_r-\varepsilon_\theta)_i - r_i[(\varepsilon_\theta)_{i+1} - (\varepsilon_\theta)_i] \tag{10}$$

for the equation of compatibility, and

$$(r_{i+1}-2r_i)(d\sigma_r)_i - (r_{i+1}-r_i)(d\sigma_\theta)_i + r_i(d\sigma_r)_{i+1}$$

$$= (r_{i+1}-r_i)(\sigma_\theta-\sigma_r)_i - r_i[(\sigma_r)_{i+1} - (\sigma_r)_i] \tag{11}$$

for the equation of equilibrium.

At each point $r = r_i$, six incremental quantities $d\sigma_r$, $d\sigma_\theta$, $d\sigma_z$, $d\varepsilon_\theta$, and $d\varepsilon_z$ are present that have to be determined for each step of the variation of loading (this includes the variation of temperature distribution). Since the axial strain ε_z is independent of r, if the incremental of axial strain, $d\varepsilon_z$, is specified in each increment of loads, then only five incremental unknowns are present at each point. Hence, a total of 5(n+1) unknowns exist that must be determined for each increment of loading. Five equations listed above can be formulated at each point (except r = b), either in the elastic region or in the plastic region. At the outer surface, r = b, of a thick-walled tube, some information concerning quantities in equations (10) and (11) is unavailable. Hence, the total number of equations is 5(n+1)-2. To solve 5(n+1) unknowns, two additional equations resulting from the boundary conditions are:

$$(d\sigma_r)_{r=a} = -\Delta P_i \quad \text{and} \quad (d\sigma_r)_{r=b} = -\Delta P_o \tag{12}$$

3. Solution Procedure

To obtain a numerical elastoplastic solution of a thick-walled tube subjected to both mechanical and transient thermal loading, one must first determine the relationship of yield strength of the material vs. the temperature. Usually, the yield strength of a material decreases dramatically if the temperature reaches a certain high level. Test data of yield strength vs. temperature for chromium-molybdenum-vanadium (Cr-Mo-V) steel are shown in Figure 1 by open circles. A fourth-degree polynomial approximation for test data, as shown in Figure 1 by solid curve, is

$$\sigma(T) = 131{,}000 - 10.89047T + 0.06483536T^2 - 0.0001666979T^3 + 0.00000005984565T^4 \qquad (13)$$

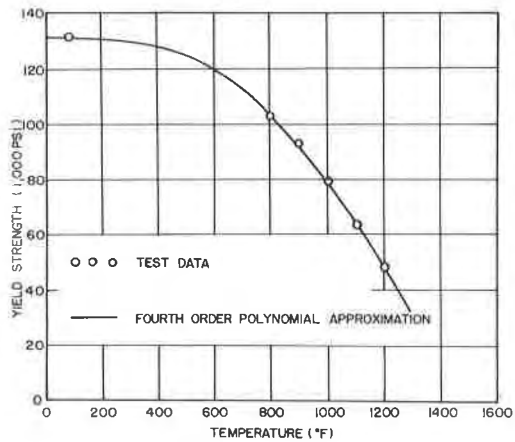

Fig. 1 Yield strength at elevated temperature for chromium molybdenum vanadium steel

This relation exhibits a very good approximation of test data.

To determine the transient thermal loading, one must first solve the transient temperature distribution. If the thermal properties of a material are assumed to be independent of the temperature, the heat flow in a thick-walled tube is governed by the well-known Fourier heat-conduction equation [19]

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad a<r<b,\ t>o \qquad (14)$$

where $\alpha>o$ denotes thermal diffusivity. In addition, the following initial and radiation-convection boundary conditions are specified:

$$T(r,o) = g(r) \qquad a<r<b,\ t=o \qquad (15)$$

$$K\frac{\partial T}{\partial r} = -h_1(t)[T_g(t)-T_w] - \gamma F[T_g^4(t)-T_w^4] \qquad r=a \qquad (16)$$

$$K\frac{\partial T}{\partial r} = -h_2(t)[T_w-T_a(t)] - \gamma F[T_w^4-T_a(t)] \qquad r=b \qquad (17)$$

Where $h_1(t)$ and $h_2(t)$ are the convection coefficients at the bore and outer surfaces, respectively, γ is the Stefan-Boltzman constant, K is thermal conductibility, and F is the interchange factor.

To provide all the calculations of the temperature distribution and the temperature history for this given problem, a finite-element digital computer code was developed [20]. For example, a three-layer compound tube with insulating material in the middle layer under cyclic heating condition is considered. The material properties of the tube are given by:

	$\rho(lb/ft^3)$	C(Btu/lb-°F)	K(Btu/ft-hr-°F)
$.0495' \leq r < 0.0510'$	490	0.10	26.0
$0.0570' \leq r \leq 0.0597'$	36	0.25	0.1
$0.0597' < r \leq 0.0677'$	490	0.10	26.0

The duration of each heating cycle is considered to be 100 milliseconds. The variations of the heating gas temperature, $T_g(t)$, and gas convection coefficient $h_1(t)$ are shown in Figure 2. $T_a = 100°F$, $h_2 = 5$ Btu/hr ft^2-°F, and $T(r,o) = 100°F$.

The bore temperature for the first five-heating cycles is shown in Figure 3. The maximum bore temperature is indicated at approximately 1.2 milliseconds after each heating cycle started.

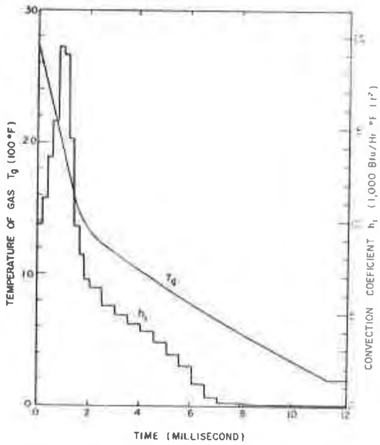

Fig. 2 Variations of heating gas temperature and gas convection coefficient

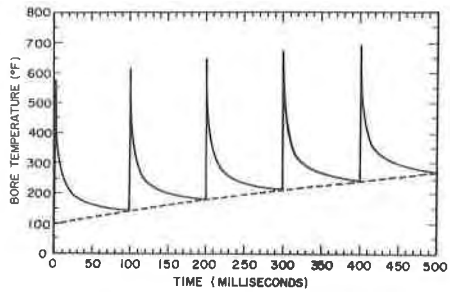

Fig. 3 Bore temperature history of a tube subjected to time dependent convection and coefficient and radiation and time dependent heating gas temperature

Since the stability and nonoscillation characteristics of the finite-element transient temperature solution were reported by Yalamanchili and Chu [21] in great detail, the discussion of instability and oscillation for the transient temperature numerical solution is omitted here.

The numerical computation procedure for obtaining thermo-elastoplastic solutions for thick-walled tubes can now be stated as follows: The computation starts with given loads (including thermal loading). The loading path is divided into a number of increments. At the beginning of each increment of loading, the distribution of stresses and strains is assumed to be known.

Step 1. Specify the values of $(d\sigma r)_{r=a} = -\Delta P_i$, $(d\sigma_r)_{r=b} = -\Delta P_o$, and temperature variation $dT_i = \Delta T_i$ at each nodal point i.

Step 2. Assume a value for $d\varepsilon_z$ (independent of r).

Step 3. Calculate $(d\sigma_r)_i$, $(d\sigma_\theta)_i$, $(d\sigma_z)_i$, $(d\varepsilon_r)_i$, and $(d\varepsilon_\theta)_i$ (for i=1, 2, ..., n+1) from 5(n+1) x 5(n+1) matrix equation formed by use of equations (1), (2), (3), (7), (8), (9), (10), (11), and (12).

Step 4. Compute $(\sigma_r)_i = (\sigma_r)_i|_{\text{before increase in loads}} + (d\sigma_r)_i$; $(\sigma_\theta)_i$, $(\sigma_z)_i$, $(\varepsilon_r)_i$, $(\varepsilon_\theta)_i$, and $(\varepsilon_z)_i$ are computed in the same way.

Step 5. Compute the axial load, P_c, by using Simpson's rule and compare it with the actual applied axial load P_a. If the difference between P_c and P_a falls within certain allowable limits, the computed stresses and strains in Step 4 are considered to be acceptable. Then, refer back to Step 1. Another set of new increments of mechanical load, variation of temperature distribution in the tube, and axial strain will be assigned to compute a new stress-and-strain distribution.

Step 6. If the difference between P_c and P_a is greater than some allowable limit, a new $d\varepsilon_z$ has to be assumed. Then, go back to Step 2.

Step 7. The elastic-plastic boundary is determined by the condition that the effective stress $\bar{\sigma}$ is equal to the temperature-dependent yield strength, i.e.,

$$\bar{\sigma} = \frac{1}{\sqrt{2}} [(\sigma_r-\sigma_\theta)^2 + (\sigma_\theta-\sigma_z)^2 + (\sigma_z-\sigma_r)^2]^{1/2}\Big|_T = \sigma(T) \tag{15}$$

In the elastic-plastic problem, the elastic-plastic boundary was allowed to be moved by an increment, at most equal to the thickness of a volume element.

For each step of the changing of mechanical loading or temperature distribution, a system of 5(n+1) linear algebraic equations, with nonzero terms clustered about the main diagonal, will be obtained. This type of matrix is known as a band matrix and can be solved quite rapidly on a digital computer. In the computer code that has been developed, the Gaussian elimination method is used to solve these equations.

4. Application

To illustrate the solution technique on the elastoplastic problem of a thick-walled tube subjected to both mechanical and thermal loading, a thick-walled tube is considered that has been subjected to internal pressure and heat input at bore surface with radiating and convecting boundary conditions on inner and outer surfaces. Without loss of generality, the initial temperature of the tube is assumed to be uniform with a value of zero. The radiation-convection boundary conditions were given in equations (18) and (19).

To obtain a numerical solution, the following values were assigned:

α = 4.3530 (ft^2/hr), γ = 0.1714x10^{-8}(Btu/hr-ft-R), K = 219 (Btu/ft-hr-F), C_p specific heat = 0.0915 (Btu/16-F), F = 0.2, T_g = 3000°F, T_a = 0°F, h_1 = 100 (Btu/ft^2-hr-F), h_2 = 100 (Btu/ft^2-hr-F), b/a = 2.0, ν = 0.3, E = 30x10^6 psi, and P_o = 0, and η = 0.05.

The cross section of the tube was divided into twenty volume-elements by radii ρ_1 = 1.00, ρ_2 = 1.05,... ρ_{21} = 2.0. For each time increment Δt, the variation of temperature distribution for that particular time increment is computed, first by use of a finite-difference computer code; then this temperature variation and a increment of internal pressure ΔP_i are used as load inputs to compute all corresponding incremental stresses and strains by the method proposed in the previous section. For this particular example, plane strain assumption is considered to be valid. The distribution of stresses and strains, temperatures, and temperature-dependent yield strength at several times are given in Figures 4 - 7.

The elastic-plastic boundary in the tube is located by use of equation (15).

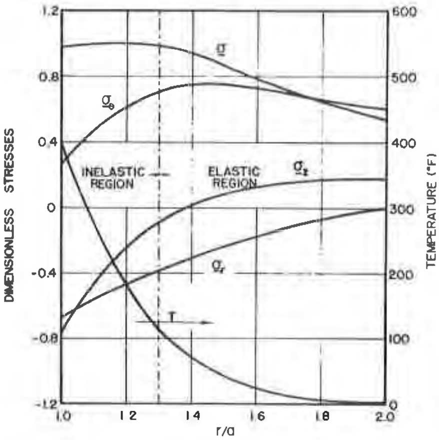

Fig. 4 Distribution of axial, radial, and circumferential stresses and temperature. (t=61.30 sec.)

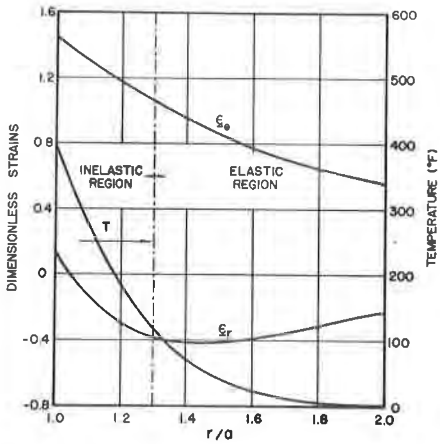

Fig. 5 Distribution of radial and circumferential strains and temperature. (t=61.30 sec.)

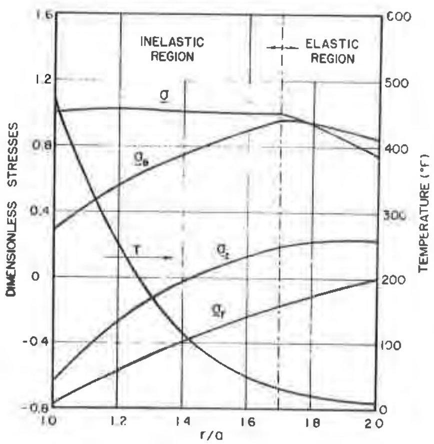

Fig. 6 Distribution of axial, radial, and circumferential stresses and temperature. (t=90.80 sec.)

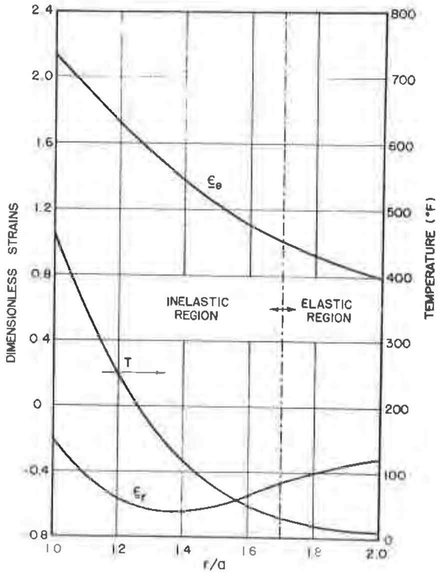

Fig. 7 Distribution of radial and circumferential strains and temperature. (t=90.80 sec.)

5. Conclusions

On the basis of the Prandtl-Reuss stress-strain rule, the von Mises flow criterion, and temperature-dependent yield strength, the strain-hardening, and compressibility properties of a material, an incremental theory is proposed in this paper for solving the problem of elastoplastic thick-walled tubes subjected to both mechanical and transient thermal loadings. Since the consideration of stress, strain, loading, and temperature history is involved in the analysis, the present approach is particularly suitable for predicting stress and strain distribution, and location of elastic-plastic boundaries of a thick-walled tube subjected to nonproportionate mechanical and transient thermal loadings.

A literature survey indicated that the closed-form elasto-plastic solutions of thick-walled tubes [1,2,7,8] were obtained by use of deformation theory of plasticity. On the basis of the deformation theory, the strains at any state depend on the instantaneous state of stress and strain, and not on how that stress and strain system is reached. Taylor pointed out the inaccuracy of deformation theory, particularly for a member subjected to nonproportionate loading. More rigorous elasto-plastic solutions can be obtained only on the basis of the flow theory of plasticity. And, in such situations, one has to rely on numerical methods. Both the finite-element method and the finite-difference method are commonly used for solving elastoplastic problems.

To use the finite-element method [10, 11] for solving elastoplastic problems, a functional has to be formulated on the basis of approximation expressions of the displacement field and the energy approach. Since the displacement field cannot be accurately determined at the beginning, the governing equations derived from the approximated functional do not exactly prescribe the inelastic behavior of a structural component. Hence, if a finite-element method must be used, some inherent errors cannot be avoided. The finite-difference treatment [3,5,6] of the elastoplastic problem of a thick-walled tube is usually based on the differential equations for the displacement vector and, hence, determination of stresses and strains requires numerical differentiation. However, the computer usually does not provide good results in differentiation unless a rather fine grid is used. With an alternative method developed in this investigation, incremental stresses and strains are directly used as variables and, hence, numerical differentiation in the evaluation of stresses and strains is not required. Moreover, since no assumption is made on the displacement field, this method can provide better results than those of the finite-element method.

References

[1] Cook, G., "The Stresses in Thick-Walled Cylinders of Mild Steel Overstrained by Internal Pressure", Proc. Inst. Mech. Engrs., Vol. 126, 1934.

[2] Sokolovsky, W. W., Theory of Plasticity, (in Russian and German), Chap. III, Gosudarstvennoe Izdatelstvo Tekhniko Teorefihicheskoi, Moscow, U.S.S.R., 1946.

[3] Hill, R., Lee, E. H., and Tupper, S. J., "The Theory of Combined Plastic and Elastic Deformation With Particular Reference To A Thick Tube Under Internal Pressure," Proc. Roy. Soc. Lond., Ser. A., Vol. 1971, 1947.

[4] MacGregor, C. V., Coffin, Jr., L. F., and Fisher, J. C., "Partially Plastic Thick-Walled Tubes," J. Franklin Inst. , Vol. 245, 1948.

[5] Hodge, P. C., and White, G. N., "A Quantitative Comparison of Flow and Deformation Theories of Plasticity", J. Appl. Mech., Vol. 72, 1950.

[6] Hill, R., Lee, E. H., and Tupper, S. J., "Plastic Flow in a Closed-End Tube With Internal Pressure", Proc. First U. S. Nat. Cong. Appl. Mech., Chicago, IL, 1951.

[7] Allen, D. N., and Sopwith, D. G., "The Stresses and Strains in a Partly-Plastic Thick Tube Under Internal Pressure and End Load," Proc. Roy. Soc. Lond., Ser. A., Vol. 205 1951.

[8] Steele, M. C., "Partially Plastic Thick-Walled Cylinder Theory", J. Appl. Mech., Vol. 74, 1952.

[9] Marcel, P. V., "A Note on the Elastic-Plastic Thick Cylinder With Internal Pressure in the Open and Closed-End Conditions," Int. J. Mech. Sci., Vol. 7, 1965.

[10] Meijers, P., "Elastic-Plastic Deformation of Thick-Walled Cylinders", First Int. Conf. on Pressure Vessel Technology, Part I, Design and Analysis, ASME, 1969.

[11] Chen, P. C. T., "The Finite Element Analysis of Elastic-Plastic Thick-Walled Tubes," Army Symposium on Solid Mechanics, Ocean City, Maryland, 1972.

[12] Chu, S. C., "A More Rational Approach to the Problem of Elasto-plastic Thick-Walled Cylinders", J. of Franklin Inst., Vol. 294, 1972.

[13] Chu, S. C., and Chang, T. Y., "Inelastic Behavior of Thick-Walled Cylinders Made of Strain-Hardening Materials Subjected to Repeated Loading", Army Symposium on Solid Mechanics, Ocean City, Maryland, 1972.

[14] Chu, S. C., and Vasilakis, J. O., "Inelastic Behavior of Thick-Walled Cylinders Subjected to Nonproportionate Loading", Experimental Mechanics, March, 1973.

[15] Chu, S. C., "A Numerical Solution of an Elastoplastic Thick-Walled Tube Subjected to Thermal Loading", AIAA Paper, No. 73-256, Jan. 1973.

[16] Chu, S. C., "A Numerical Thermoelastoplastic Solution of a Thick-Walled Tube", to be published in the AIAA Journal.

[17] Bland, D. R., "Elastoplastic Thick-Walled Tubes of Work-Hardening Material Subject to Internal and External Pressures and to Temperature Gradient", J. of Mech. and Phys. of Solids, Vol. 4, 1956.

[18] Weiner, J. H., and Huddleston, J. V., "Transient and Residual Stresses in Heat-Treated Cylinders", J. of Appl. Mech., Vol. 26.

[19] Carslaw, H. S., and Jaeger, J. C., Conduction of Heat in Solids, 2nd Edition, 1959, Oxford.

[20] Beckett, R. E., and Chu, S. C., "Finite-Element Method Applied to Heat Conduction in Solids With Nonlinear Boundary Conditions", Journal of Heat Transfer, Vol. 95, No. 1, Feb. 1973.

[21] Yalamanchili, R. V. S., and Chu, S. C., "Stability and Oscillation Characteristic of Finite-Element, Finite-Difference, and Method of Weight Residuals for Transient Two-Dimensional Heat Conduction in Structures", Journal of Heat Transfer, May 1973, pp. 235-239.