ANALYSIS OF PIPING LOADS ON SHELLS OF REVOLUTION

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SUMMARY

The work of Bijlaard as presented by Wichman, *et al.*, Welding Research Council Bulletin 107 (1965), is widely used in industry to evaluate the effects of concentrated loads on cylinders and spheres. But many axially symmetric structures have configuration which preclude the use of this work. In such cases, the procedure given in this paper can greatly reduce the computational effort needed to obtain accurate stresses and deflections due to loads such as occur at piping connections.

Otherwise the analyst generally will have no recourse but to model a great deal of the structure and solve hundreds or thousands of equations at great expense. For example, if the load is expanded into a Fourier series, and the analyst wants accurate bending stresses, possibly a hundred harmonics or so might be needed before the remainder is negligible.

However, since for a given Fourier harmonic the load term and the structural response are predictable, one can obtain a high degree of accuracy by using a greatly abbreviated series plus a correction factor. This is so, assuming the analyst knows:

1. where to terminate the abbreviated series, and
2. what function will describe the structural response from the termination point onwards.

This function is assumed to be:

$$\Delta y = C \sum_{n=n_1}^{\infty} \frac{a_n}{n^b} \cos n\theta_0$$

where,

Δy	=	correction to the variable, displacement or stress, being evaluated at angle θ_0,
C	=	constant with respect to n, but a function of the variable being considered, structural stiffness, etc.,
a_n	=	term from Fourier decomposition of load,
n_1	=	cut off point of abbreviated Fourier series,
b	=	power to which n should be raised for given variable, structure, etc.,
$\cos n\theta_0$	=	term to evaluate variable at θ_0. Sin $n\theta_0$ is used if the variable is part of a sine series.

The two constants C and b can be evaluated from the last two terms in the abbreviated Fourier series of the variable y. If b is close to being an integer, the assumed function will be accurate from n_1 on. Also, it should be noted that in most cases Δy oscillates in sign as the summation progresses. Therefore, there are values of n at which the truncation error is zero.

Thus, the paper provides a method of calculating the truncation error due to terminating the Fourier expansion of a loading at a given point. Also for various common loading situations, it provides points where the truncation error is zero. To demonstrate the validity of this procedure, the "exact" maximum bending stress in a sphere under six equidistant, concentrated loads is compared with the result obtained from a series which stopped at one of these points of zero error. The error in fact is less than 1%.

1. Introduction

The work of Bijlaard as presented by Wichman, et.al., [1] , is widely used in industry to evaluate the effects of concentrated loads on cylinders and spheres. But many axially symmetric structures have configurations which preclude the use of this work, e.g. large R/t ratios, unusual geometries, boundary conditions, or reinforcements.

An analysis, if required, can be rather expensive since the effect of concentrated loads on shells of revolution may propagate over much of the structure. This situation requires either many finite elements, if a three dimensional model is used, or a long Fourier loading series, if a shell of revolution solution is used. Either way, normally, many equations must be solved in order to produce results of good accuracy.

However, if a shell of revolution is used, one can take advantage of the regularity of the Fourier series describing the loading and the applicable differential equations with respect to Fourier frequency increases, the terms involving the highest powers of the frequency will dominate so that a simple approximation is usually possible. This paper describes a procedure for evaluating solutions after the highest power dominates.

2. Error Equation

Because all axially symmetric structures become stiffer in varying degrees as the loading takes on more and more waves around the circumference (e.g. see section 4.1), the simplest approximation of the error resulting from truncating a Fourier series solution is taken to be:

$$\Delta y = C \sum_{n=N+1}^{\infty} \frac{a_n}{n^b} \cos n\theta_o \qquad (1)$$

where,

Δy = estimated error in variable

C = constant for particular variable, structure, etc.

a_n = amplitude of the nth term in Fourier loading series

N = last Fourier term used in full solution

b = power to which n should be raised

$\cos n\theta_o$ (or $\sin n\theta_o$) = Fourier function needed to evaluate at θ_o.

The two constants C and b can be determined from the last two terms of the full solution. Since

$$y_N = C \frac{a_N}{N^b} \quad \text{and} \quad y_{N-1} = C \frac{a_{N-1}}{(N-1)^b} \qquad (2)$$

then,

$$b = \frac{\ln (a_N \cdot y_{N-1}/y_N \cdot a_{N-1})}{\ln(N/N-1)} \qquad (3)$$

In general, if b is to be constant or nearly so, it must be an integer or nearly so and must be stable, i.e. it should not make much difference which two terms near the end of the full solution that one uses in eq.(3). Under these conditions eq.(1) yields a very accurate estimate of the error. This error of course can be added to the truncated value to closer approach the true answer.

3. Evaluation of Error

3.1 Idealized Concentrated Force

Since the Fourier series for a concentrated load has constant values of a_n after the n=0 term, the terms in the error equation do not change sign if the load is applied at θ=0 and the variable is to be evaluated at $\theta_o=0$. Therefore a straight forward summation is somewhat tedious. However, if the series converges, i.e. b>1, the summation can be done through integration.

$$\Delta y = \int_N^\infty \frac{C}{n^b}\, dn = \left[\frac{Cn^{1-b}}{1-b}\right]_N^\infty = \frac{C}{N^b}\;\frac{N}{b-1} = y_N\;\frac{N}{b-1} \qquad (4)$$

3.2 Force Distributed over Finite Region

If the load is applied as a constant force over an arc length from θ= −α to +α, the general Fourier amplitude will be $a_n=\overline{P}\sin n\alpha/n\alpha$, where, $\overline{P}$ is a constant scale factor. Thus, even if the error is to be evaluated at $\theta_o=0$, the signs of the terms in the error equation are going to change back and forth as n increases. This makes the integration more difficult; but since the value of the error changes sign as the summation progresses, there obviously are values of n for which the error is zero. For example

$$\int_0^\infty \frac{\sin x}{x}\, dx = \int_0^{1.927} \frac{\sin x}{x}\, dx = \frac{\pi}{2}$$

Thus, if b=0 for some variable being evaluated at $\theta_o=0$ and the solution is complete up to N=1.927/α, the value of the variable is correct, i.e. Δy=0, assuming that b remains constant from N=1.927/α to infinity. If b is still a function of n at N=1.927/α, then the solution can be completed up to a point where b is relatively constant and eq.(3) summed up to N=4.899/α where the remainder of the summation is again zero. Thus one never has to sum more than the terms between these points of zero error.

However, since the zero points generally must be evaluated by trial and error using a table of definite integrals from x to infinity, some common cases have been included in table I. These series can be used to approximate

a. radial load which is constant from θ= −α to +α, with evaluation of the error at θ=0 and ±α. (Case 1 and 2).

b. radial load as above but which linearly builds up to the maximum over an arc of α on each side, with evaluation of error at θ=0, ±α, and ±2α. (Case 6, 7, and 8).

c. circumferential moment loading which varies linearly from $\theta = -\alpha$ to $+\alpha$, with evaluation at $\theta = \pm\alpha$. (Case 3).

d. circumferential moment loading as above but which linearly builds up to the maximum over an arc of α on each side, with evaluation of the error at $\theta = \pm\alpha$ and $\pm 2\alpha$. (Case 4 and 5).

Thus, if one can estimate about how many terms will be required for b to approach a constant integer and what this integer will be for the variable being considered, then the maximum Fourier harmonic used in the solution can be evaluated from the next zero point as $N = x_i/\alpha$.

4. Applications

4.1 Concentrated Radial Load on a Ring

To illustrate the above procedure and how it works, consider a ring loaded with a concentrated radial load of P at $\theta=0$ resisted by a shear force $S = P \sin\theta/\pi r$ applied at a reference radius on the ring. If all variables are decomposed into Fourier series, the equilibrium equations become

$$-nT_n + Q_n = -rq_n$$

$$-T_n + nQ_n = -rp_n$$

and $$rQ + nM_n = 0$$

The strain equations become

$$e_n = (nV_n + W_n)/r$$

and $$k_n = n(V_n + nW_n)/r^2 \qquad (6)$$

The force-strain relationships become

$$M_n = EIk_n + EA\bar{z}e_n$$

and $$T_n = EAe_n + EA\bar{z}k_n \qquad (7)$$

so that

$$W_n = (a_2 q_n - a_3 p_n) r^2/EA\,(a_2^2 - a_1 a_3) \qquad (8)$$

$$V_n = (a_2 p_n - a_1 q_n) r^2/EA\,(a_2^2 - a_1 a_3)$$

and thus the forces can be determined by substituting eq.(8) into eq.(6) and eq.(7). The variables in these equations are:

n = Fourier frequency

T_n, Q_n = thrust and radial shearing force

M_n = moment (about reference radius)

p_n, q_n = radial and shearing forces/unit length (applied at reference radius).

r = reference radius

e_n = strain (at reference radius)

k_n = curvature change

W_n, V_n = radial and tengential displacements

E = Young's modulus

A = area

I = moment of inertia about reference radius

$\bar{z}$ = distance from reference radius to section centroid

$a_1 = 1 + 2n^2\bar{z}/r + n^4 I/Ar^2$

$a_2 = \left[1 + (1+n^2)\bar{z}/r + n^2 I/Ar^2 \right] n$

$a_3 = \left[1 + 2\bar{z}/r + I/Ar^2 \right] n^2$

As can be readily seen, these equations involve a variety of powers of n, and it would appear that the dominate power of n will depend on the relative magnitudes of A, $A\bar{z}/r$, and I/r^2. For example if $\bar{z}=0$, and in addition $q_n=0$, then the above equations reduce to

$$W_n = \left(\frac{r^2}{I} + \frac{1}{A}\right) \frac{p_n r^2}{E(n^2-1)^2} \qquad V_n = -\left(\frac{r^2}{I} + \frac{n^2}{A}\right) \frac{p_n r^2}{En(n^2-1)^2}$$

$$T_n = -p_n r/(n^2-1) \qquad M_n = p_n r^2/(n^2-1)$$

Now it is easy to see that as soon as $n^2 >> 1$ that b=4 when solving for W and b=2 when solving for T and M. But in the case of V the power will depend on the properties of the ring. When $r^2/I > n^2 A$ the power of n is 5, but ultimately b must be 3. This change is even more pronounced when $\bar{z} \neq 0$. For example, let A=3, I=36, $\bar{z}=3$, r=100., and E=30,000,000., then early in the series

$$b = \frac{\ln(V_3/V_4)}{\ln(4/3)} = 4.83$$

However at n=10 things are different, as can be seen below.

$y(\theta)$	M(0)	T(0)	$W(0)\times10^3$	$V(\pi/4)\times10^3$
$\Sigma n=0$ to 10	6546.	84.52	53.05	-25.27
b	2.02	2.02	4.04	3.58
estimated error	1010.	-10.10	0.13	.0
true error	954.	-9.52	0.12	.0

The power factor b was calculated by comparing the values for n=9 and 10. The error for M, T, and W was calculated using eq.(4), and the error for V was calculated using the third zero point from table I for the function $\sin x/x^3$.

4.2 Two Opposing Radial Loads on a Cylindrical Shell

In order to evaluate the effects of concentrated loads on unreinforced shells, consider a simply supported cylinder pinched by two opposing loads at the center of its length.

radius	=	10.	Young's modulus	=	10^7
thickness	=	0.1	Poisson's ratio	=	0.28
length	=	31.416	load (each)	=	31.416

A full solution for the first 10 even harmonics yielded a radial deflection under the loads

$$W = \sum_{n=0}^{14} W_n + W_{16} + W_{18} = (6.111 + .12918 + .09185) x\ 10^{-3} = 6.332 \times 10^{-3}$$

The estimated truncation error, ΔW, was calculated from eq.(4) as

$$\Delta W = \frac{1}{2} \int_{18}^{\infty} \frac{C}{n^3}\, dn = .09185 \left(\frac{1}{2}\right)\left(\frac{18}{2}\right) = .413$$

since, from eq.(3),

$$b = \frac{\ln(.12918/.09185)}{\ln(18/16)} = 2.90$$

The extra $\frac{1}{2}$ accounts for the summation of every other term. Thus the radial deflection for a full series can be estimated as 6.745×10^{-3} which differs by only 0.4% from Vlasov's [2] solution of 6.718×10^{-3}.

An analysis of the non-zero values at the center line indicates that the power hierarchy for this type of problem is

b = 0	Q = radial shear
1	M_x, M_θ = moments
2	β_θ = circumferential slope change
3	N_x, N_θ = forces
	W = radial deflection
4	V = tangential deflection

Thus the series for moments does not converge to a finite value as is the case when flat plates are loaded with concentrated loads.

4.3 Sphere Under Six Equidistance Loads over Finite Regions

In order to verify the procedure using loads over finite regions, six loads, each 90° from its nearest neighbor, were applied to a sphere. Four loads were applied at the equator and these were expanded into a Fourier series. These loads were constant over a polygon formed by 16 equal sides with a distance across points of 15° arc, The two loads at the poles were also constant and were applied over a circle with an opening angle of 7.476°, so that the area of the polygon and circle would be equal. Because of the planes of symmetry around the equator, the Fourier frequencies n=1, 2, and 3 are not involved in the solution. Therefore, at the pole, all variables in all harmonics are identically zero except for the n=0 harmonic. Thus the results at the center of the loads on the equator can be compared with an axially symmetric solution at the pole which is essentially exact. The parameters of the problem are

radius	=	100	Young's modulus	=	30×10^6
thickness	=	2	Poisson's ratio	=	.3
load intensity	=	100			

The results for longitudinal moment, longitudinal force, and radial deflection are:

	M_ϕ	N_ϕ	W
$\Sigma n=0,4,\ldots,N_1$	1777.7	3159.6	.01060
$\Sigma n=0,4,\ldots,N_2$	1784.7	3159.6	.01059
pole	1781.8	3221.1	.01067

In this problem the calculation of b is clouded somewhat by having used three different Fourier series to describe the changing width of the loaded area along the meridan. But since this problem involves a pressure of finite magnitude instead of a concentrated shearing force, the power of n should be two higher than in section 4.2. Thus the zeros of the function $\sin x/x^3$ were used for the moment (N_1 = 2.322/.1309 = 17.74 and N_2 = 5.177/.1309=39.55) and the zeros of $\sin x/x^5$ were used for the force and radial displacement (N_1=19.46 and N_2=41.12, from table I, case 1, b=4). The series were stopped at the nearest n included in the solution, i.e., the nearest multiple of 4.

4.4 <u>Concentrated Load on Cylinder: Experimental Results</u>

In order to confirm that this type of analysis using thin shell theory is capable of predicting the stiffness of shell structures subjected to piping loads, Chicago Bridge and Iron Co. has conducted some tests on a simply supported cylindrical vessel with the following properties

radius	=	155.4"	Young's modulus	=	29,000,000 psi
thickness	=	0.296"	Poisson's ratio	=	.3
length	=	134.4"			

In one test, a short piece of 8"ϕ schedule 40 pipe was fitted through the cylindrical wall 54" from the base and welded. A radial load, longitudinal moment, and a circumferential moment were applied to the pipe. The experimental results compare quite well with the values calculated using linear thin shell of revolution theory, as can be seen below.

	Circumferential Moment	Longitudinal Moment	Radial Load	Radial Load
Load Magnitude	12000 in#	14400 in#	-2000#	-400#
Exp.Defl.,Side 1	.048"	.037"		
Exp.Defl.,Side 2	-.069"	-.046"		
Exp.Ave.Defl.	±.058"	±.041"	-.134"	-.024"
Linear Theory	±.056"	±.041"	-.126"	-.025"

Almost all of the difference between the theoretical and experimental deflections is due to large deflection effects. This can be seen by comparing the 400# and 2000# radial load cases and also by comparing the inward and outward deflections due to moments. The loaded region here is so small that the power factors start out as though it were an idealized concentrated load as in section 4.2 and very gradually change to the situation in section 4.3. In the theoretical solutions a zero close to N=100 was chosen because it took almost that long for b to settle down to a constant. The points of zero error were determined using table I with b=4 as in section 4.3. For the circumferential moment, from case 3, N_1 = 2.790/.02775 = 100. For the longitudinal moment, from case 1, N_1 = 2.548/.02775 = 92. For the radial load the deflection at the top of the pipe would be from case 1. To evaluate the deflection at the side, from case 2, N_2 = 2.692/.02775 = 97. So N=94 was used, and the values at the top and side were averaged.

5. Conclusion

The procedure described herein for evaluating errors due to truncating Fourier solutions depends only on the fact that all axially symmetric structures become increasingly stiff as the Fourier frequency increases. Therefore it is applicable to any axially symmetric structure and any loading. The only restriction is that the power factor b which is a function of the type of loading, structural configuration, and variable being considered must be relatively constant so the eq.(1) is applicable.

REFERENCES

[1] WICHMAN, K.R., HOPPER, A.G., MERSHON, J.L., "Local Stresses in Spherical and Cylindrical Shells due to External Loadings", Welding Research Council Bulletin 107, August, 1965.

[2] VLASOV, V.Z., "General Theory of Shells and its Applications in Engineering", NASA Tech. Translation TTF-99,p.394, April, 1964.

Table I: Zero Integrals for Common Error Functions

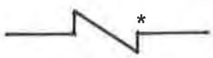
Case	Function and Evaluation Point*	b	X_1	X_2	X_3
1	 $\int_{X_i}^{\infty} \frac{\sin x}{x^{b+1}}\, dx = 0$	1	2.155	5.045	8.080
		2	2.322	5.177	8.181
		3	2.447	5.287	8.727
		4	2.548	5.382	8.353
		5	2.623	5.466	8.426
2	 $\int_{X_i}^{\infty} \frac{\sin x \cos x}{x^{b+1}}\, dx = 0$	1	1.078	2.520	4.040
		2	1.162	2.589	4.090
		3	1.225	2.645	4.134
		4	1.272	2.692	4.175
		5	1.313	2.733	4.213
3	 $\int_{X_i}^{\infty} \frac{\sin x}{x^{b}} \left(\frac{\sin x}{x^2} - \frac{\cos x}{x}\right) dx = 0$	1	3.135	3.148	6.271
		2	2.818	3.594	5.909
		3	2.786	3.738	5.850
		4	2.790	3.833	5.831
		5	2.805	3.902	5.831
4	 $\int_{X_i}^{\infty} \sin x \left(\frac{2\sin x - \sin 2x}{x^{b+2}}\right) dx = 0$	1	1.423	2.661	3.773
		2	1.492	2.692	3.817
		3	1.546	2.724	3.861
		4	1.590	2.755	3.899
		5	1.624	2.783	3.933
5	 $\int_{X_i}^{\infty} \sin 2x \left(\frac{2\sin x - \sin 2x}{x^{b+2}}\right) dx = 0$	1	.952	1.869	2.852
		2	1.112	1.916	2.843
		3	1.203	1.954	2.843
		4	1.263	1.982	2.849
		5	1.307	2.004	2.859
6	 $\int_{X_i}^{\infty} \cos 2x \left(\frac{\cos x - \cos 2x}{x^{b+2}}\right) dx = 0$	1	1.480	3.515	7.210
		2	1.583	3.594	7.314
		3	1.663	3.567	7.398
		4	1.725	3.710	7.471
		5	1.772	3.751	7.537

Table I: Continued

Case	Function and Evaluation Point*	b	X_1	X_2	X_3
7	 $\int_{X_i}^{\infty} \cos x \left(\frac{\cos x - \cos 2x}{x^{b+2}}\right) dx = 0$	1	1.266	2.554	4.040
		2	1.360	2.623	4.065
		3	1.429	2.680	4.090
		4	1.489	2.724	4.109
		5	1.533	2.765	4.128
8	 $\int_{X_i}^{\infty} \cos 2x \left(\frac{\cos x - \cos 2x}{x^{b+2}}\right) dx = 0$	1	1.175	1.832	2.683
		2	1.200	1.857	2.730
		3	1.219	1.885	2.765
		4	1.241	1.913	2.796
		5	1.257	1.942	2.824