A NUMERICAL METHOD FOR VIBRATION ANALYSIS OF COMPOSITE STRUCTURES WITH DIFFERENT DAMPING CAPACITIES

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SUMMARY

The most important computational problem for vibration analysis of structures is the solution of the eigenvalue problem:

$$K\delta + \lambda C\delta + \lambda^2 M\delta = 0 . \tag{1}$$

If the structure consists of materials with different damping capacities, it is not always possible to define a similarity transformation which simultaneously diagonalizes matrices K, C and M. In this case the solution of problem (1) is reduced to the solution of the problem:

$$Ay = \lambda y \tag{2}$$

where vector $y^T = [\delta^T, \lambda\delta^T]$, and A is a properly defined non-symmetric matrix.

Well-known numerical methods are available for the computation of the complete eigensystem of problem (2) which can be conveniently used for small-size problems.

For large-size problems, such as those arising from finite element applications, the number of eigenvalues and eigenvectors required is usually much smaller than the order of the matrices, so that the solution of problem (2) can be restricted to the evaluation of a few dominant modes.

This paper presents a numerical method for the computation of the first p eigenvalues of the smallest modulus of problem (1), and corresponding vectors.

This method is based on the simultaneous iteration concept, and has a convergence rate for the i-th eigenvalue depending on the ratio

$$\rho_i = \left| \frac{\lambda_i}{\lambda_{p+1}} \right|$$

where

$$|\lambda_1| \leqslant |\lambda_2| \leqslant \ldots \leqslant |\lambda_p| < |\lambda_{p+1}| \leqslant \ldots \leqslant |\lambda_n| .$$

The efficiency of this iterative procedure was improved using a complex initial guess, obtained by the solution of the undamped equation associated with problem (1).

A computer program based on this numerical model and using finite element method was coded to perform numerical experimentation.

In the cases considered the damping matrix C was obtained by assembling element matrices $C_e = \alpha_e k_e$ where k_e is the element stiffness matrix.

1. Introduction

This paper is concerned with the vibration analysis of structures composed of materials with different viscous damping capacities. This problem is amenable to the solution of the eigenvalue problem:

$$K\delta + \lambda C\delta + \lambda^2 M\delta = 0 \tag{1}$$

where K, C and M are symmetric definite positive matrices.

If matrix C satisfies the orthogonality condition:

$$X^T C X = D \tag{2}$$

where X is the matrix of the undamped modes, and D is a diagonal matrix, the problem (1) is ultimately reduced to the solution of undamped equation $KX=MX\Omega^2$ [1].

For structures composed of different materials, generally eq. (2) is not verified so that problem (1) is equivalent to a standard eigenvalue problem of the form:

$$AX = \lambda X \tag{3}$$

where A is a non-symmetric matrix. In this case, the eigenvectors δ are complex and each point of the structure may vibrate with a different phase angle.

Well-known numerical methods are available for computation of the complete eigensystem of problem (3), which can be conveniently used for small size problems.

For large size problems (matrix order $\geqslant$100), the number of eigenvalues and eigenvectors required is usually much smaller than the order of matrices, so that the solution of problem (3) can be restricted to the evaluation of a few modes of interest.

This poses the problem of specifying which eigenvalues and corresponding vectors are significant for practical purposes. For many structural problems the eigenvalues of minimum imaginary part are of interest but unfortunately the algorithms available for computation of selected modes can only determine the eigenvectors relevant to the eigenvalues having the smallest (or largest) modulus. Nevertheless, these numerical methods may be an useful tool in solving a large class of practical problems. In many cases indeed, defining the complex eigenvalues λ as follows:

$$\lambda = -\omega\nu \pm i\omega\sqrt{1-\nu^2} \qquad (|\lambda| = \omega)$$

the arrangement by increasing moduli turns out to be coincident with the arrangement by increasing frequencies, being the damping factor $\nu \ll 1$.

This paper presents a numerical method based on the simultaneous iteration concept for computation of the first p eigenvalues of the smallest modulus and corresponding vectors of problem(1)(*).

This algorithm differs from Bauer's bi-iteration in that a simplified rule is used for the computation of the left-hand iterates and bi-orthogonalization of the trial vectors is performed by using the eigensystem of a p x p matrix. In this way convergence can be reached

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even if some of the first p eigenvalues have the same modulus. To explore the influence of the initial guess on the convergence rate different strategies were used for the computation of the starting trial vectors and their efficiency was compared. Simple numerical tests were carried out with the aim of evaluating the numerical method rather than solving a significant problem.

2. Numerical Method

The computation of the first p eigenvalues of smallest modulus and corresponding vectors of problem (1) by the simultaneous iteration method, is based on the reduction of eq.(1) to the standard form

$$A X_i = \mu_i X_i \qquad (i= 1,2 n) \tag{4}$$

where:

$$A = \begin{bmatrix} 0 & I \\ -R^{-T}MR^{-1} & -R^{-T}CR^{-1} \end{bmatrix}; \quad X_i = \begin{bmatrix} R\,\delta_i \\ \mu_i\, R\delta_i \end{bmatrix}; \quad \mu_i = \frac{1}{\lambda_i} \tag{5}$$

R being the triangular factor of the Cholesky decomposition of matrix K of order n x n.

In this way the problem is reduced to the computation of the first p dominant eigenvalues μ_i of problem (4) and corresponding vectors, assuming:

$$|\mu_1| \geq |\mu_2| \geq \cdots \geq |\mu_p| > |\mu_{p+1}| \geq \cdots \geq |\mu_{2n}| \tag{6}$$

The simultaneous iteration algorithms are extensively used to compute the dominant modes of symmetric matrices [2, 3, 4]. These methods involve iterations with p orthogonal trial vectors which are premultiplied by the matrix, and ultimately converge to the first dominant p eigenvectors. For non-symmetric problems the iterations are generally performed with two sets of bi-orthogonal trial vectors (right-hand iterates $X^{(j)}$ and left-hand iterates $Y^{(j)}$) which are premultiplied by matrix A and by its traspose A^T and converge respectively to the first p right and left-hand dominant vectors of matrix A [5, 6]. If only the right-hand vectors are of interest, a simplified iteration rule can be used avoiding the premultiplication of the left-hand interates $Y^{(j)}$ by the matrix A^T.

Indeed it can be proved [7] that the right-hand iterates $X_i^{(j)}$ (i=1,p) converge to the dominant vectors X_i if and only if the iterates $Y_i^{(j)}$ converge to a set of p vectors $\tilde{Y}_i$ such that:

$$\tilde{Y}_k^T X_i = \delta_{ik} \qquad (i,\ k = 1,p) \tag{7}$$

Eq. (7) does not obviuosly imply that $\tilde{Y}_i$ are the first p dominant left-hand vectors of matrix A.

Such a simplified iteration can be convenient for large-size eigenvalue problems since the most expensive contribution to the total computing time is the matrix-vector multiplication step.

The main steps of the numerical scheme proposed are the following:

a) Start with trial matrices $Y^{(1)}$ and $X^{(1)}$ of order 2n x p such that:

$$Y^{(1)T} X^{(1)} = I$$

Start with j=1

b) Performe the "power step":

$$Z^{(j)} = A\, X^{(j)}$$

c) Compute the matrix of order p x p

$$B^{(j)} = Y^{(j)T}\, Z^{(j)}$$

d) Solve the eigenvalue problem (Q R method)

$$B^{(j)}\, Q^{(j)} = Q^{(j)}\, D^{(j)}$$

where $D^{(j)} = \mathrm{diag}\,(d_{ii}^{(j)})$ and $\left|d_{11}^{(j)}\right| \geq \dots \geq \left|d_{pp}^{(j)}\right|$

e) Compute the iterates:

$$X^{(j+1)} = Z^{(j)}\, Q^{(j)}\, D^{(j)^{-1}}$$

f) Compute the iterates:

$$Y^{(j+1)} = Y^{(j)}\, Q^{(j)-T}$$

g) Test for termination and go to b) if required.

It can be proven that

$$\lim_{j\to\infty} d_{ii}^{(j)} = \mu_i \quad ; \qquad \lim_{j\to\infty} X_i^{(j)} = X_i \qquad\qquad (i=1,p)$$

where $X_i^{(j)}$ and X_i are the i-th columns of matrices $X^{(j)}$ and X.

For the convergence rate one has

$$\left|d_{ii}^{(j)} - \mu_i\right| = O\left(\left|\frac{\mu_{p+1}}{\mu_i}\right|^{j}\right) ; \qquad \left\| AX_i^{(j)} - X_i^{(j)}\mu_i \right\|_2 = O\left(\left|\frac{\mu_{p+1}}{\mu_i}\right|^{j/2}\right) \quad (i=1,p)$$

3. Computational aspects of the iterative scheme for free damped vibration problems

If matrix A is defined as in (5), the main steps of the algorithm described in sect. 2 assume the following form.

Perform the Cholesky decomposition of matrix K:

$$K = R^T R$$

a) Start with trial matrices $u^{(1)}, v^{(1)}, \varphi^{(1)}, \psi^{(1)}$ of order n x p such that:

$$\varphi^{(1)T}\, u^{(1)} + \psi^{(1)T}\, v^{(1)} = I$$

Start with j = 1

b) Solve the following linear systems:

$$R\, s^{(j)} = u^{(j)}$$

$$R\, t^{(j)} = v^{(j)}$$

$$R^T\, w^{(j)} = -\, Ms^{(j)} - Ct^{(j)}$$

c) Compute the matrix of order p x p:

$$B^{(j)} = \varphi^{(j)T}\, v^{(j)} + \psi^{(j)T}\, w^{(j)}$$

d) Solve the eigenvalue problem (QR method)

$$B^{(j)}\, Q^{(j)} = Q^{(j)}\, D^{(j)}$$

where $D^{(j)} = \mathrm{diag}\,(d_{ii}^{(j)})$ and $\left|d_{11}^{(j)}\right| \geq \dots \geq \left|d_{pp}^{(j)}\right|$

e) Compute the right-hand iterates:

$$u^{(j+1)} = v^{(j)}Q^{(j)}D^{(j)-1}$$

$$v^{(j+1)} = w^{(j)}Q^{(j)}D^{(j)-1}$$

f) Compute the left-hand iterates:

$$\varphi^{(j+1)} = \varphi^{(j)}Q^{(j)-T}$$

$$\psi^{(j+1)} = \psi^{(j)}Q^{(j)-T}$$

g) Test for termination and go to b) if required.

When convergence is reached, the eigenvectors δ_i of problem (1) relevant to the eigenvalues $\lambda_i (i=1,p)$ can be obtained by solving the linear systems:

$$R\,\delta_i = u_i \qquad (u_i \text{ is the i-th column of matrix } u^{(j)})$$

Even if the numerical scheme described does not provide the first p left-hand vectors Y_i of matrix A, they can be computed since the following relation holds:

$$Y = GX$$

where:

$$G = \begin{bmatrix} -R^{-T}MR^{-1} & 0 \\ 0 & I \end{bmatrix} ; \quad X = \begin{vmatrix} u \\ v \end{vmatrix}$$

the following remarks are worth making:

- In accordance with the converge properties, higher efficiency can be obtained if iterations are performed with p vectors in order to compute $r < p$ eigenvectors. Obviously p must not be much larger than r to have acceptable running times.
- The algorithm does not converge if $|\lambda_{p+1}| = |\lambda_p|$. For this reason p must be an even number in order to reach convergence in the case of complex conjugate eingenvalues.
- Advantages can be taken from the sparse structure of matrices K, C and M. Indeed if matrix K is large and sparse, computer storage and arithmetic can be saved if using some sparsity-oriented Gaussian elimination technique for the solution of the linear systems in step b). In particular the "frontal technique" can be conveniently used for finite element applications. The sparse structure of matrices M and C can be fully retained because they are only used to perform products with some vectors.
- The choice of a convenient guess to start the iterative process may reduce the number of iterations, and consequently the computing time. This imply that some information must be available on the problem considered. If an approximation $\tilde{C}$ to the damping matrix C is available such that $\tilde{C} = \alpha K + \beta M$, the starting guess may be obtained by computing the first p/2 modes of the undamped problem

$$KX_i = \omega_i^2 MX_i \quad (X_k^T KX_i = \delta_{ki} \;;\; k,i = 1,p/2) \qquad (8)$$

and assuming:

$$u_{2i-1}^{(1)} = u_{2i}^{(1)} = RX_i$$

$$v_{2i-1}^{(1)} = \frac{1}{\lambda_i} RX_i \; ; \quad v_{2i}^{(1)} = \frac{1}{\bar{\lambda}_i} RX_i$$

$$(i = 1, p/2) \qquad (9)$$

$$\varphi_{2i-1}^{(1)} = - u_{2i-1}^{(1)} \frac{\lambda_i^2}{\omega_i^2 - \lambda_i^2} \quad ; \quad \varphi_{2i}^{(1)} = \bar{\varphi}_{2i-1}^{(1)}$$

$$\psi_{2i-1}^{(1)} = v_{2i-1}^{(1)} \frac{\lambda_i^2 \cdot \omega_i^2}{\omega_i^2 - \lambda_i^2} \quad ; \quad \psi_{2i}^{(1)} = \bar{\psi}_{2i-1}^{(1)}$$

where λ_i, $\bar{\lambda}_i$ are the complex conjugate solutions of the equation

$$\lambda_i^2 + \lambda_i \, (\alpha\omega_i^2 + \beta) + \omega_i^2 = 0$$

and $\bar{\varphi}$, $\bar{\psi}$ are the complex conjugates of φ and ψ.

4. Numerical results

This section reports some results obtained with a finite element programme set up to perform vibration analysis of three-dimensional networks.

Since our aim is to prove the efficiency of the numerical scheme described rather than to solve a significant physical problem, we restrict ourselves to considering a very simple structure. The structure examined is a clamped beam composed of two materials having different damping capacities.

The damping matrix C for the entire structure was obtained as direct sum of element damping matrices $C_e = \alpha_e k_e$ when k_e is the element stiffness matrix and α_e is a parameter whose value is material dependent. The geometry and material properties of the beam considered are shown in fig. 1.

To evaluate the efficiency of the numerical procedure, two finite element idealizations were considered respectively with 100 and 200 degrees of freedom and the first 4 eigenvalues and eigenvectors and their complex conjugates were computed.

In order to show the influence on the computing time of the initial guess constructing procedure, for each finite element idealization, the following computations were performed:

a) the initial matrices $u^{(1)}$, $v^{(1)}$, $\varphi^{(1)}$, $\psi^{(1)}$ are formed by unit matrix sections multiplied by proper coefficients to guarantee the bi-orthogonality of the right and left-hand iterates.

b) the initial guess is computed according to the strategy described in sect. 3 with $\alpha = \beta = 0$. The solution of the undamped problem (8) is performed with the Rutishauser simultaneous iteration method [3,4] .

c) the initial guess is computed as in case b) assuming $\beta = 0$ and $\alpha = (\alpha_1 + \alpha_2)/2$ where α_1 and α_2 are the damping coefficients of the two materials which form the structure.

The estimate of the computational efficiency of the different strategies used to start the iterative process is based on the number of operations required in each case, according to the convention to count only multiplications and divisions.

The number of operations is of the form:

$$N_{SI} = N_G + N_1 + \sum_{j=1}^{L} N_3^{(j)} \tag{10}$$

where:

$$N_G = \sum_{j=1}^{L_G} N_2^{(j)} \quad \text{(number of operations for the computation of the initial guess)}$$

$$N_1 = \frac{b}{6}(3\,n - 2\,b) + b\,(2\,n - b)\,r$$

$$N_2^{(j)} = (p_G - t_j)\left[b\,(2\,n-b) + \sum_{i=1}^{E} g_i^2\right] + 2\,n\,(p_G - t_j)^2$$

$$N_3^{(j)} = (p - t_j)\left[3\,b(2\,n-b) + 4\sum_{i=1}^{E} g_i^2\right] + 8\,n\,p^2 + 16\,n\,(p - t_j)^2$$

and:

n = number of degrees of freedom

$2b-1$ = total band-width of matrix K

p_G = number of trial vectors used in the computation of the starting guess

p = number of trial vectors used in the iterative process.

r = number of eigenvectors required ($r \leq p$)

E = number of elements in the structure

g_i = number of degrees of freedom in the i^{th} element

L_G = number of iterations performed for the solution of the undamped problem to compute the starting guess

L = number of iterations performed for the computation of the free damped frequencies and modes

t_j = number of eigenvectors which have fulfilled the acceptance test up to iteration j.

In previous formulae the number of operations involved in the computation of the eigensystem of matrix $B^{(j)}$ (see sect.2) was not taken into account since the computational cost of this step is negligible with respect to the total cost of one iteration if $n \gg p$.

The numerical experimentation was performed by using $p=12$ and $p_G=8$.

The natural frequencies and damping factors of the first four modes are shown in table I; the four complex modes are illustrated with amplitudes and phase-angles in fig.2.

Table II quotes the number of operations computed according to eq. (10).

The results quoted in Table II indicate that the use of the undamped modes for the computation of the initial guess have proved very efficient.

The most significant gain (57% over case a)) is obviously achieved in case c) since this guess provides a more accurate approximation of the complex modes than the one obtained in case b). However, if the undamped modes are assumed as initial trial vectors, a time savings of 33% over case a) is achieved.

Even if at this stage the numerical experimentation available is too limited to draw general conclusions, yet the results obtained suggest that the procedure used to construct the initial guess can be successfully applied to enhance the convergence rate.

5. Conclusions

This paper has presented a numerical method for the computation of the free damped frequencies and complex modes of structures.

This method, based on the simultaneous iteration concept, seems to be efficient for large-size problems especially if proper strategies are applied for the choice of the initial guess. Further investigations are necessary to improve the algorithm especially on account of its application to eigenvalue problems with real matrices. Indeed the need to perform the iterations with complex vectors to compute complex conjugate eigenvectors produces a computing time waste which may possibly be avoided.

References

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TABLE I

NATURAL FREQUENCIES AND DAMPING FACTORS

	Frequencies $rad \cdot s^{-1}$	Damping factor (%)
	$f = \omega\sqrt{1 - \nu^2}$	ν
1	0.38	0.03
2	2.38	1.59
3	6.68	5.12
4	13.26	9.74

TABLE II

NUMBER OF OPERATIONS - SPECIFIC NUMERICAL VALUES

Type of guess	n	b	N_G	N_{SI}
a)	100	4	0	$61.26\ 10^5$
b)	100	4	$1.62\ 10^5$	$40.59\ 10^5$
c)	100	4	$1.62\ 10^5$	$26.17\ 10^5$
a)	200	4	0	$121.54\ 10^5$
b)	200	4	$3.25\ 10^5$	$78.52\ 10^5$
c)	200	4	$3.25\ 10^5$	$50.65\ 10^5$

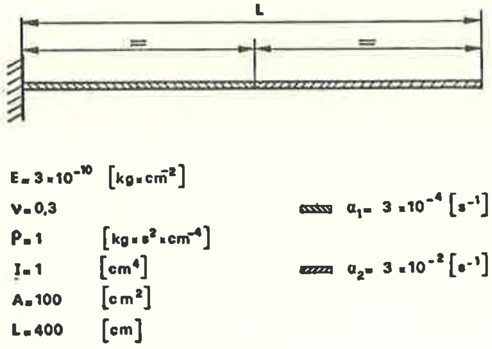

Fig. 1 Clamped beam

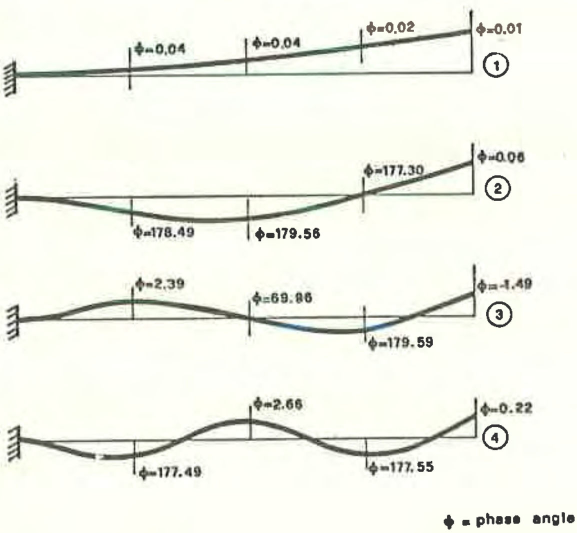

Fig. 2 Damped natural modes