SOLUTION OF PLANE STRESS AND PLATE BENDING PROBLEMS BY BOUNDARY INTEGRAL EQUATIONS

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Summary

A method is developed for the direct solution of the problems of plane stress and plate bending in an infinite sheet with a curvilinear hole. The solution of the biharmonic equation is expressed as a boundary integral in terms of the stress function or the transverse deflection. The method differs from those found in the literature in that the Green's function satisfies the boundary conditions at the hole so that the unknown boundary values of the stress function or transverse deflection are eliminated in the integral equation. The stress function or transverse deflection is then found by integration of the boundary integral.

As examples, for an infinite plate with a circular hole, the plane stress problem of uniform tension and uniform heat flow, and the bending problem of uniform twisting moment are solved exactly by analytic integration. A general computer program for the solution of any plane stress or bending problem using numerical integration is discussed. The numerical and analytical results are compared.

1. Introduction

Integral equation formulation of the boundary value problems in the theory of elasticity is well known, see for example, the classical paper by Rizzo [1]. The application of the integral equation method to the stress concentration problems has attracted wide attention in the past few years, and has been discussed by many authors in the literature, for example, Christiansen and Hansen [2], and Hansen [3], to cite only these two references relevant to the present paper. In reference [2], the free space Green's function for the biharmonic operator was used to derive an integral expression for the Airy stress function. By applying the boundary conditions at the unloaded hole, they obtained two integral equations which are then solved numerically.

In this paper, a method of direct determination of the Airy stress function for the plane stress problems, or the transverse deflection for the plate bending problem is discussed. The method differs from those in the literature in that Green's function satisfies the boundary conditions at the hole so that unknown boundary values of the stress function or the transverse deflection are eliminated in the equation. The solution is then reduced to relatively simple exact analytical or numerical integration.

As examples, the plane stress problems of uniform tension and uniform heat flow in an infinite plate with a circular hole, and the plate bending problem of uniform twisting moment are solved exactly by analytical integration. A computer program for the solution of the problem using numerical integration is also discussed. The numerical and analytical results are compared.

2. Basic Equations

The solution of the biharmonic equation

$$\Delta\Delta\phi = 0 \qquad (1)$$

in a region R can be written in an integral form by multiplying eq. (1) by an appropriate Green's function G and repeated application of Green's identity,

$$\int_R [V\Delta U - U\Delta V]da = \int_C [UV,_n - VU,_n]ds \qquad (2)$$

where C denotes the boundary of R and a comma with n denotes the normal derivative on C. It can be shown that

$$\phi(P) = \int_C [\Delta G,_n(P,Q)\phi(Q) - \Delta G(P,Q)\phi,_n(Q) - G(P,Q)\Delta\phi,_n(Q) + G,_n(P,Q)\Delta\phi(Q)]ds \qquad (3)$$

where P is a point in R and Q is a point on the boundary C. The Green's function satisfies

$$\Delta\Delta G = \delta(P-Q) \qquad (4)$$

with appropriate boundary conditions at the hole depending on the problem under consideration. Here δ denotes the Dirac delta function.

Since the stresses are determined from second derivatives of ϕ, some arbitrariness in the choice of ϕ and its derivatives is to be expected. However, without loss of generality, the boundary condition at the hole contour C can be written as

$$\phi = 0 \qquad (5)$$

$$\phi,_n = 0 \qquad (6)$$

In addition to these, for an infinite sheet with a hole, one requires

$$\sigma_{ij} \to \sigma_{ij}{}^{p} \text{ at infinity} \qquad (7)$$

where σ_{ij} is the stress tensor in the sheet with a hole and $\sigma_{ij}{}^p$ is the stress tensor in the sheet without a hole under the same loading.

Let

$$\phi = \phi_h + \phi_p \tag{8}$$

where ϕ_p is for the solution of an infinite sheet without a hole, and ϕ_h denotes the perturbed solutions due to the presence of a hole. The boundary conditions for ϕ_h at the hole contour are then

$$\phi_h = - \phi_p \tag{9}$$

$$\phi_{h,n} = - \phi_{p,n} \tag{10}$$

3. Plane Stress Problems

For an infinite plane with a circular hole, the Green's function G satisfies eq. (4) and boundary conditions

$$G = G,_n = 0 \quad \text{on } C \tag{11}$$

Using the well known method of Muskhelishivili [4], the solution of eq. (4) with boundary conditions (11) yields

$$G(z,z_o) = G_e(z,z_o) + G_o(z,z_o) - G_i(z,z_o) + G_R(z,z_o)$$

$$- [(2+\ell n z_o\bar{z}_o)/4 - (1/4)(z_o\frac{\partial}{\partial z} + \bar{z}_o\frac{\partial}{\partial \bar{z}})]G_h(z,z_o) \tag{12}$$

where $z = x+iy$, $z_o = x_o+iy_o$ are complex variables representing points in R and

$$G_e(z,z_o) = (1/16\pi)(z-z_o)(\bar{z}-\bar{z}_o)\ell n(z-z_o)(\bar{z}-\bar{z}_o)$$

$$G_o(z,z_o) = (1/16\pi)(z-z_o)(\bar{z}-\bar{z}_o)\ell n z\bar{z}$$

$$G_i(z,z_o) = (1/16\pi)(z-z_o)(\bar{z}-\bar{z}_o)\ell n(z\bar{z}_o-1)(\bar{z}z_o-1)$$

$$G_h(z,z_o) = (1/4\pi)\ell n z\bar{z}$$

$$G_R(z,z_o) = (1/16\pi)\{(z-z_o)(\bar{z}-\bar{z}_o)\ell n z\bar{z}_o-(z\bar{z}_o-1)(\bar{z}z_o-1)\ell n z_o\bar{z}_o$$

$$-z_o\bar{z}_o(1-z\bar{z})\ell n z\bar{z}_o + z\bar{z}_o + \bar{z}z_o\} \tag{13}$$

For the problem of an infinite plane with a hole, since ϕ_h satisfies the biharmonic equation, by using eq. (3) and boundary conditions eq. (9) and eq. (10), one has

$$\phi_h(x_o,y_o) = \int_C [\Delta G\phi_{p,n} - \Delta G,_n\phi_p]ds$$

$$- \int_{C_\infty} [\Delta G\phi_{h,n} - \Delta G,_n\phi_h + G\Delta\phi_{h,n} - G,_n\Delta\phi_h]ds \tag{14}$$

where C_∞ denotes the contour at infinity and (x_o,y_o) a point in R. The condition at infinity eq. (7), requires that the stresses due to ϕ_h vanish at infinity. Thus, if the general form of ϕ_h corresponding to no traction at infinity is substituted into the second integral of eq. (14), one can show that it reduces to a polynomial of degree one, as was noted in reference [2]. Eq. (14) thus becomes

$$\phi_h(x_o,y_o) = \int_C [\Delta G\phi_{p,n} - \Delta G,_n\phi_p]ds$$

$$+ a_o + a_1x_o + a_2y_o \tag{15}$$

(a) Uniform Tension Problem

For an infinite plane with a circular hole under uniform tension $\sigma_x = T$ at infinity, the stress state in a plane without a hole is given by stress function

$$\phi_p = (Tr^2/4)(1-\cos 2\theta) \tag{16}$$

where r,θ are the usual polar coordinates.

Substituting eq. (16) into eq. (15) and using the Green's function eq. (12) in polar coordinates, one obtains, after carrying out the integration along the hole contour,

$$\int_C \Delta G \phi_{p,r} ds = -(T/4)[2 + 2\ln r_0 + (1/r_0 - 1)\cos 2\theta_0]$$

$$\int_C \Delta G_{,r} \phi_p ds = -(T/4)\cos 2\theta_0$$

Note here r_0 has been nondimensionalized with respect to the radius of the hole. Thus, apart from a polynomial of degree one which produces no stresses,

$$\phi_h = -(T/2)\ln r_0 + (T/4)(2-1/r_0^2)\cos 2\theta_0 \tag{17}$$

Combining eq. (16) and eq. (17), one obtains the stress function for the solution of the problem

$$\begin{aligned} \phi &= \phi_h + \phi_p \\ &= (T/4)[(r_0^2 - 2\ln r_0) + (2 - r_0^2 - r_0^{-2})\cos 2\theta_0] \end{aligned} \tag{18}$$

which yields,

$$\begin{aligned} \sigma_r &= (T/2)[1 - r_0^{-2} + (1 + 3r_0^{-4} - 4r_0^{-2})\cos 2\theta_0] \\ \sigma_\theta &= (T/2)[1 + r_0^{-2} - (1 + 3r_0^{-4})\cos 2\theta_0] \\ \sigma_{r\theta} &= -(T/2)(1 + 2r_0^{-2} - 3r_0^{-4})\sin 2\theta_0 \end{aligned} \tag{19}$$

These coincide with the results of the classical Kirsch's solution [5]

(b) Uniform Heat Flow Problem

The problem of an insulated circular hole of radius a in an infinite plane under uniform heat flow of constant temperature gradient q in the direction of negative y axis is next considered.

As shown by Parkus [6], the solution of the problem is given by

$$\phi' = \phi'_p + \phi'_h \tag{20}$$

where ϕ'_p satisfies

$$\Delta \phi'_p = (1+\nu)\alpha\Theta \tag{21}$$

and ϕ'_h satisfies the biharmonic equation, eq. (1). In eq. (21) α is the coefficient of linear thermal expansion, Θ is the temperature distribution, and ν the Poisson's ratio.

It can be easily shown that the temperature distribution Θ satisfying the condition of no heat flow at the hole is

$$\Theta = qy + (qa^2/r')\sin\theta \tag{22}$$

Only the second term in eq. (22) produces thermal stresses. Using this term, the particular solution of eq. (21) in nondimensional form is given by

$$\phi_p = \frac{\phi'_p}{(1+\nu)\alpha q a^3} = (1/2) r \ln r \sin\theta \tag{23}$$

where $r = r'/a$.

Substituting eq. (23) into eq. (15) and carrying out the integration around the unit circle, one obtains

$$\phi_h = \frac{\phi_h'}{(1+\nu)\alpha q a^3} = (1/4)(r+r^{-1})\sin\theta \tag{24}$$

where the polynomial of degree one is again neglected. Eqs. (23) and (24) then yields

$$\phi' = (1+\nu)\alpha q a^3[(1/2)r\ell n r + (r+r^{-1})/4] \tag{25}$$

which gives stresses

$$\begin{aligned}\sigma_r &= -(E\alpha q a/2)(r^{-1}-r^{-3})\sin\theta\\ \sigma_\theta &= -(E\alpha q a/2)(r^{-1}+r^{-3})\sin\theta\\ \sigma_{r\theta} &= (E\alpha q a/2)(r^{-1}-r^{-3})\cos\theta\end{aligned} \tag{26}$$

These coincide with those given in [6].

4. Plate Bending Problems

For the bending problem of an infinite plate with a circular hole, the Green's function satisfies

$$\Delta\Delta G = \delta(r-r_o,\theta-\theta_o) \tag{27}$$

and boundary conditions

$$M_n(G) = V_n(G) = 0 \text{ on } C \tag{28}$$

Here M_n and V_n denote the bending moment and effective transverse shear normal to the hole contour. The solution to the boundary value problem eq. (27) and eq. (28) is

$$\begin{aligned}G(r,\theta;r_o,\theta_o) = (1/16\pi)\{&[r^2+r_o^{\,2}-2rr_o\cos\theta^*]\\ &\ell n[(rr_o)^{1/\kappa}(r^2+r_o^{\,2}-2rr_o\cos\theta^*)/(1+r^2r_o^{\,2}-2rr_o\cos\theta^*)^{1/\kappa}]\\ &-2[1+\kappa+(1+\kappa)\ell n r_o]\ell n r-(2r_o/r\kappa)\cos\theta^*\\ &+2(1/\kappa-\kappa)\sum_{n=1}^{\infty}[1/n^2(n+1)]r_o^{-n}r^{-n}\cos n\theta^*\end{aligned} \tag{29}$$

where $\theta^*=\theta-\theta_o$ and $\kappa = -(3+\nu)/(1-\nu)$.

Let the transverse deflection w of the plate be given by

$$w = w_p+w_h \tag{30}$$

where w_p is the deflection due to the external load in the plate without a hole, and w_h represents the perturbed deflection due to the presence of a hole. Following the procedures in arriving at eq. (15) for plane stress problem, one obtains

$$\begin{aligned}w_h = -\int_C&[M_n(w_p)G_{,n} + V_n(w_p)G]ds\\ &+ a_o + a_1x_o + a_2y_o\end{aligned} \tag{31}$$

As an example, an infinite plate with a unit circular hole under constant twisting moment $\bar{M}_{xy}$ at infinity is considered. It is easily shown that for this case

$$w_p = (\bar{M}_{xy}/2)r^2\sin 2\theta \tag{32}$$

Substituting eq. (32) into eq. (31) and using the Green's function eq. (29) yields

$$w_h = (1-\nu)/(3+\nu)\bar{M}_{xy}(1-r^{-2}/2)\sin 2\theta \tag{33}$$

which, with eq. (32), then gives

$$w = \bar{M}_{xy}[r^2/2 + (1-\nu)/(3+\nu)(1-r^{-2}/2)]\sin 2\theta \tag{34}$$

This coincides with that given by Savin [7].

5. Numerical Calculations

Using the Green's function eq. (12), eq. (15) can be written in the following form:

$$\phi_h(r_o,\theta_o) = -(1/2\pi)(I_1+I_2+I_3+I_4+I_5) \tag{35}$$

where

$$I_1 = (\ln r_o+2)\int_0^{2\pi}\phi_{p,r}(1,\theta)d\theta - \int_0^{2\pi}\phi_p(1,\theta)d\theta - r_o\int_0^{2\pi}\cos(\theta-\theta_o)[\phi_p(1,\theta)+\phi_{p,r}(1,\theta)]d\theta \tag{36}$$

$$I_2 = -\int_0^{2\pi}\{[2r_o^2 - r_o(1+r_o^2)\cos(\theta-\theta_o)]/[1+r_o^2-2r_o\cos(\theta-\theta_o)]\}\cdot[\phi_{p,r}(1,\theta)-\phi_{p,r}(1,\theta_o)]d\theta \tag{37}$$

$$I_3 = \int_0^{2\pi}\{[(2r_o^2-r_o^4-1)+(r_o^5-2r_o^3+r_o)\cos(\theta-\theta_o)]/[1+r_o^2-2r_o\cos(\theta-\theta_o)]^2\}\cdot[\phi_p(1,\theta)-\phi_p(1,\theta_o)]d\theta \tag{38}$$

$$I_4 = -2\pi\phi_{p,r}(1,\theta_o) \tag{39}$$

$$I_5 = 2\pi\phi_p(1,\theta_o) \tag{40}$$

Note that eq. (36) has no singularities and can be easily integrated numerically. I_2 given by eq. (37) and I_3 given by eq. (38) can be numerically integrated if the integrands are defined to be zero at $\theta=\theta_o$.

A computer program has been written to carry out the numerical integration of eq. (35). Note that in carrying out the integration of eq. (37) and eq. (38), the limits of integration used were $-\theta_o$ to $2\pi-\theta_o$. Finite differences were used to determine the stresses from the stress function ϕ. Numerical results obtained are in excellent agreement with the analytical results.

6. Conclusion

In this paper, a direct method of determining the Airy stress function for the plane stress problems, or the transverse deflection of the plate bending problems is discussed. The method depends on finding a Green's function which satisfies the boundary conditions at the hole contour. Once the Green's function is found, it can be used in the solution of the infinite plate with a hole under different loadings by carrying out relatively simple analytic or numerical integration around the hole contour.

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