ON THE CRACKED ELEMENT APPROACH FOR THE COMPUTATION OF STRESS INTENSITY FACTORS

R. M. E. J. SPIERING, C. de PATER

Department of Mechanical Engineering,
Twente University of Technology, Postbox 217, Enschede, The Netherlands

SUMMARY

The purpose of the work is to evaluate the influence of different matching conditions between a cracked element and the surrounding regular elements upon the stress intensity factors for plane problems. The influence of size and type of the elements has been studied as well.

The cracked element is a polygon with a linear crack. The coordinate system has its origin at the crack tip and the negative x_1-axis along the crack. The work is based upon the complex variable formulations of Muskhelishvili for homogeneous isotropic material. The stresses and displacements are written as series expansions of the complex variable in the $x_1 - x_2$ plane, taking into account the condition of traction free crack surfaces. The stress intensity factors K_I and K_{II} corresponding to the crack edge opening resp. sliding mode can be found directly from the coefficient of the first term of one of the stress functions.

The above representation of the crack tip element satisfies the conditions of internal equilibrium (neglecting body forces), compatibility and Hooke's law.

The strain energy of the element is found by numerical integration (Gauses quadrature) along the element edges.

Matching the edges of the cracked element with those of the adjacent regular elements can be done by:

(a) prescribing compatibility at the nodal points only (Byskov);
(b) enforcing compatibility in an additional number of edge points. These edge points are equidistant points resp. the Gaussian integration points along the edges.
(c) compatibility along the edges is obtained in a least square sense by using the displacement differences at the equidistant resp. Gaussian points;
(d) compatibility is obtained by minimizing the integral of the squared displacement differences along the edges;
(e) the element edge matching is obtained using the hybrid functional as defined by P. Tong *et al.*, Int. J. Num. Meth. Eng. 7 (1973) 297.

The calculations for various configurations show that the methods c, d, and e are to be preferred above a and b.

By varying the element sizes and element types it has been found that a very limited number of surrounding regular elements with a quadratic displacement field is sufficient to compute very accurate stress intensity factors.

1. Introduction

One of the interesting possibilities to determine elastic stress intensity factors for plane cracks by finite elements is the method where a cracked element, including the stress singularity is adopted.

In this paper use is made of the complex variable technique for the cracked element. This implies that the requirements of equilibrium (excluding body forces), compatibility, Hooke's law in this element as well as the condition of traction free crack surfaces are satisfied.

The question now arises how to match the boundaries of the cracked element with those of its surrounding regular elements. The present paper deals with this problem. A number of possibilities are discussed and numerical results are given.

2. Formulation

The cracked element is a polygonon with a linear crack as indicated in Fig. 1. The coordinate system has its origin at the cracktip and the negative x_1-axis along the crack. The work is based upon the complex variable formulations of Muskhelishvili for homogeneous, isotropic, linear elastic material as described in reference [1] . The stresses and displacements are written as series expansions of the complex variable z in the x_1-x_2 plane, i being the imaginary number

$$z = x_1 + ix_2 \qquad (1)$$

Taking into account the condition of traction free crack surfaces, the complex stress functions of Muskhelishvili are

$$\begin{aligned} \Phi(z) &= z^{-\frac{1}{2}} \sum_{p=0}^{n} c_{1p} z^p + \sum_{p=0}^{m} c_{2p} z^p \\ \Omega(z) &= z^{-\frac{1}{2}} \sum_{p=0}^{n} c_{1p} z^p - \sum_{p=0}^{m} c_{2p} z^p \end{aligned} \qquad (2)$$

where c_{1p} and c_{2p} are complex coefficients.

The stresses follow from

$$\begin{aligned} \sigma_{11} + \sigma_{22} &= 2\left[\Phi(z) + \overline{\Phi(z)}\right] = 4\mathrm{Re}\left[\Phi(z)\right] \\ \sigma_{22} - i\sigma_{12} &= \Phi(z) + \Omega(\bar{z}) + (z-\bar{z})\, \overline{\Phi'(z)} \end{aligned} \qquad (3)$$

$( )'$ denotes differentiation and $\overline{( )}$ is the complex conjugate.

The displacements u_1 and u_2 in the directions x_1 resp. x_2 are given by

$$2G(u_1 + iu_2) = \kappa\phi(z) - z\overline{\Phi(z)} - \overline{\psi(z)}$$

with $\overline{\psi(z)} = \omega(\bar{z}) - \bar{z}\overline{\Phi(z)}$ (4)

and $\phi(z) = \int_0^z \Phi(\eta)d\eta \;, \qquad \omega(z) = \int_0^z \Omega(\eta)d\eta$

The shear modulus $G = \frac{E}{2(1+\nu)}$, E being Young's modulus and ν Poisson's ratio.

For plane stress problems $\kappa = \frac{3-\nu}{1+\nu}$ and for plane strain $\kappa = 3-4\nu$.

The analytic functions $\phi(z)$ and $\psi(z)$ are written as series expansions

$$\phi(z) = \sum_{j=0}^{N} A_j z^{\frac{1}{2}j}$$
$$\psi(z) = \sum_{j=0}^{N} B_j z^{\frac{1}{2}j} \tag{5}$$

with $A_j(=a_j+ib_j)$ and B_j being complex coefficients. The condition of traction free crack surfaces is fulfilled by putting

$$B_j = -\tfrac{1}{2}jA_j - (-1)^j\overline{A_j} \tag{6}$$

The stress intensity factors K_I and K_{II} corresponding with the crack opening mode resp. the crack edge-sliding mode follow from

$$K_I + iK_{II} = \sqrt{2\pi}\lim_{\substack{x_2=0\\ x_1\to 0+}} \left[\sqrt{x_1}(\sigma_{22} + i\sigma_{12})\right] \tag{7}$$

Substitution of the eqs. (5), (6) and (3) in eq. (7) leads to the well known result

$$K_I + iK_{II} = \sqrt{2\pi}\,\bar{A}_1 \tag{8}$$

With the above representation of the cracked element the conditions of internal equilibrium (neglecting volume forces), compatibility and Hooke's law in this element, as well as the condition of zero tractions along the crack surfaces are satisfied.

3. Strain energy and stiffness matrix of the cracked element

The elastic strain energy U_e of the cracked element in index notation (using the summation convention with respect to repeated indices) is

$$U_e = \tfrac{1}{2}t\iint_A \sigma_{ij}\varepsilon_{ij}\,dA \tag{9}$$

where t is the element thickness, A the element area and the strain components ε_{ij} are defined by

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{10}$$

with $(\ )_{,j}$ denoting the partial derivative with respect to x_j.

As the stress field in the element satisfies the equilibrium conditions

$$\sigma_{ij,j} = 0 \tag{11}$$

the strain energy can also be written as an integral along the contour "c" of the cracked element

$$U_e = \tfrac{1}{2}t\int_c T_i u_i\,ds \tag{12}$$

where the components T_i of the boundary traction are given by

$$T_i = \sigma_{ij} n_j \qquad (13)$$

with n_j representing the direction cosine of the outward normal to the contour of the cracked element with respect to $x_j (j=1,2)$.

Using the series expansions eq. (5) the displacements eq. (4) can be written as

$$u_i = D_{ij}(x_1,x_2)\alpha_j^* \qquad (14)$$

The 2x2(N+1) matrix D_{ij} contains functions of the coordinates x_1 and x_2. Along the contour the matrix contains functions of the parametrized coordinate s. The vector α_i^* contains 2(N+1) components being the parameters $a_1, a_2 \ldots, a_N, b_1, b_3, \ldots, b_N, a_0, b_0, b_2$ in this order.
The parameters a_0, b_0 and b_2 are related to the translation and rotation of the element as a rigid body. Likewise the stresses $\sigma_{ij} (i,j=1,2)$ from eq. (3) can be expressed as

$$\sigma_{ij} = S_{ijk}(x_1,x_2)\alpha_k \qquad (15)$$

where the vector α_i contains the same parameters as α_i^*, excluding a_0, b_0 and b_2. Substitution of eq. (15) in eq. (13) gives the equation

$$T_i = B_{ij}\alpha_j \qquad (16)$$

where the matrix B_{ij} contains functions of the parametrized coordinate s.

Substitution of eqs. (14) and (16) in eq. (12) gives

$$U_e = \tfrac{1}{2}\alpha_i \{t \int_c B_{ki} D_{kj} ds\} \alpha_j^* \qquad (17)$$

Changing to vector-matrix notation with $\overline{B} = (B_{ij})$, $\overline{D} = (D_{ij})$, $\vec{\alpha} = (\alpha_i)$ and $\vec{\alpha}^* = (\alpha_i^*)$, eq. (17) reads

$$U_e = \tfrac{1}{2}\vec{\alpha}^T \{t \int_c \overline{B}^T \overline{D} \, ds\} \vec{\alpha}^* \qquad (18)$$

where the superscript T denotes the transpose of the column vector or matrix. Omitting the last three columns which contain zero's as a result of the equilibrium of the cracked element, the elements of the (2N-1)x(2N-1) matrix $\overline{K^{**}}$

$$\overline{K^{**}} = t \int_c \overline{B}^T \overline{D} \, ds \qquad (19)$$

are obtained by numerical integration along the boundary of the cracked element using Gausz quadrature.
As the matrix $\overline{K^{**}}$ generally will not be symmetric a symmetric matrix $\overline{K^*}$ is obtained by defining

$$\overline{K^*} = \tfrac{1}{2}(\overline{K^{**}} + \overline{K^{**}}^T) \qquad (20)$$

The strain energy then becomes finally

$$U_e = \tfrac{1}{2}\vec{\alpha}^T \overline{K^*} \vec{\alpha} \qquad (21)$$

where the stiffness matrix $\overline{K^*}$ is related to the parameters in the series expansions of the functions $\phi(z)$ and $\psi(z)$, omitting the parameters a_0, b_0 and b_2.

4. Matching procedures

There are several possibilities to match the boundary of the cracked element with those of the adjacent regular elements.

(a). In his paper Byskov [2] requires compatibility at the L nodal points of the cracked element only. In this case N = L - 1.

Writing for the displacements of the k^{th} nodal point $u_1^k = \delta_{2k-1}$ and $u_2^k = \delta_{2k}$, eq. (14) can be written as

$$\begin{aligned} D_{1j}^k \alpha_j^* &= \delta_{2k-1} \\ D_{2j}^k \alpha_j^* &= \delta_{2k} \end{aligned} \qquad (22)$$

in which $D_{ij}^k = D_{ij}(x_1^k, x_2^k)$; $x_1{}^k$ and $x_2{}^k$ being the coordinates of the k^{th} point.

Eq. (22) can be written as

$$\bar{A}\vec{\alpha}^* = \bar{H}\vec{\delta} \qquad (23)$$

where $\vec{\delta}$ contains the 2L nodal displacements, $\vec{\alpha}^*$ contains 2L parameters and the matrix $\bar{A}$ and the unit matrix $\bar{H}$ are 2L x 2L matrices each.

Solving for $\vec{\alpha}^*$ one gets from eq. (23)

$$\vec{\alpha}^* = \bar{A}^{-1}\bar{H}\vec{\delta} \qquad (24)$$

Now a new matrix $\bar{C}$ is defined as being equal to $(\bar{A}^{-1}\bar{H})$ omitting however the last three rows.

Clearly one can write now

$$\vec{\alpha} = \bar{C}\vec{\delta} \qquad (25)$$

The matrix $\bar{C}$ is a (2L-3) x 2L matrix.

Substitution of eq. (25) in eq. (21) gives the expression for the elastic strain energy of the cracked element

$$U_e = \tfrac{1}{2}\vec{\delta}^T\{\bar{C}^T\bar{K}^*\bar{C}\}\vec{\delta} = \tfrac{1}{2}\vec{\delta}^T\bar{K}\vec{\delta} \qquad (26)$$

where the element stiffness matrix $\bar{K}$, now related to the nodal point displacements, is defined by

$$\bar{K} = \bar{C}^T\bar{K}^*\bar{C} \qquad (27)$$

(b). As the displacements of the edges of the regular elements are known functions of the 2L nodal point displacements one can enforce compatibility in a number of M points at the boundary of the cracked element. Now N = M-1. Eq. (14) again leads to eqs. (23), (24) and (25), where now $\bar{A}$ is a 2Mx2M matrix and $\bar{H}$ is a 2M x 2L matrix.

The matrix $\bar{C}$ has the dimensions (2M-3) x 2L. The expression for the elastic strain energy is equal to eq. (26) and the element stiffness matrix is defined by eq. (27).

The M boundary points can be chosen arbitrarily. In the computerprogram the possibilities of taking equidistant points or the Gauszian integration points have been realised.

(c). In the two cases described above, the number of points where compatibility is required is directly coupled to the number of functions that can be used. If the boundary points are greater in number than corresponds with the number of functions the result will be that there are more equations than unknown parameters and a solution is then possible by the least square's method. If the displacements of the k^{th} boundary point where compatibility is desired are, as a point of the cracked element resp. the adjacent regular element, defined by u_i^k resp. $\tilde{u}_i^k$ (i=1,2) then the function to be minimized (with a total number of M points) is

$$F = \sum_{k=1}^{M} \{(u_1^k - \tilde{u}_1^k)^2 + (u_2^k - \tilde{u}_2^k)^2\} \qquad (28)$$

With eq. (14) one can write for eq. (28)

$$F = \sum_{k=1}^{M} \{(D_{1j}^k \alpha_j^* - \tilde{u}_1^k)^2 + (D_{2j}^k \alpha_j^* - \tilde{u}_2^k)^2\} \qquad (29)$$

in which $D_{ij}^k = D_{ij}(x_1^k, x_2^k)$; x_1^k and x_2^k being the coordinates of the k^{th} boundary point.

The displacements $\tilde{u}_1^k$ and $\tilde{u}_2^k$ can be expressed in the displacements δ_i of the L nodal points as follows

$$\tilde{u}_i^k = M_{ij}^k \delta_j \qquad (i=1,2;\ j=1,2...,2L;\ k=1,2,...,M) \qquad (30)$$

The matrix M_{ij} contains functions of the parametrized coordinate s. M_{ij}^k is this matrix after substitution of the coordinate of the k^{th} point.

Minimizing F as a function of the (2N+2) parameters α_i^*

$$\frac{\partial F}{\partial \alpha_i^*} = 0 \qquad (i=1,2,...,2N+2) \qquad (31)$$

leads to the set of equations

$$A_{ij}\alpha_j^* = H_{il}\delta_l$$

in which

$$A_{ij} = D_{1i}^k D_{1j}^k + D_{2i}^k D_{2j}^k \quad \text{(sum on k)} \qquad (32)$$

and

$$H_{il} = D_{1i}^k M_{1l}^k + D_{2i}^k M_{2l}^k \quad \text{(sum on k)}$$

The boundary points used in the least square technique can again be chosen arbitrarily. The choice of equidistant points and the Gauszian integration points has been realised in the program.

The eq. (32) in vector-matrix notation with $\overline{A} = (A_{ij})$ and $\overline{H} = (H_{ij})$ again leads to eqs. (23), (24) and (25).

$\overline{A}$ now is a (2N+2) x (2N+2) matrix and $\overline{H}$ has the dimensions (2N+2) x 2L.

The strain energy and the element stiffness matrix are again defined by eqs. (26) and (27).

(d). Instead of minimizing the sum of the squared displacement differences in a number of boundary points one can also minimize the following integral

$$F = \int_c \{(u_1 - \tilde{u}_1)^2 + (u_2 - \tilde{u}_2)^2\} ds \qquad (33)$$

taken along the contour of the cracked element. In this equation u_i resp. $\tilde{u}_i$ (i=1,2) are the displacements of the edges of the cracked element resp. the adjacent regular elements.
With eq. (14) one can write for eq. (33)

$$F = \int_c \{(D_{1j}\alpha_j^* - \tilde{u}_1)^2 + (D_{2j}\alpha_j^* - \tilde{u}_2)^2\} ds \qquad (34)$$

Minimizing F with respect to the (2N+2) parameters α_i^* gives

$$\int_c (D_{1i}D_{1j} + D_{2i}D_{2j})\alpha_j^* ds = \int_c (D_{1i}\tilde{u}_1 + D_{2i}\tilde{u}_2) ds \qquad (35)$$

Numerical integration of the integrals by Gausz quadrature gives the set of equations

$$A_{ij}\alpha_j^* = H_{il}\delta_l$$

in which

$$A_{ij} = \sum_{k=1}^{NG} W^k (D_{1i}^k D_{1j}^k + D_{2i}^k D_{2j}^k) \qquad (36)$$

and

$$H_{il} = \sum_{k=1}^{NG} W^k (D_{1i}^k M_{1l}^k + D_{2i}^k M_{2l}^k)$$

The superscript k denotes the k^{th} Gausz point with W^k being its associated weight. NG is the total number of Gausz points.
The strain energy and the stiffness matrix of the cracked element is given by eqs. (24), (25), (26) and (27) with the matrices $\bar{A} = (A_{ij})$ and $\bar{H} = (H_{ij})$ now as given by eq. (36). It is noted that the eqs. (32) and (36) are identical for weight factors W^k equal to one.

(e). Pin Tong et. al. [3] matches the boundary of the cracked element with those of the adjacent regular elements by using the hybrid-element concept. Following this line of thought we add to the expression for the elastic energy eq. (12) of the cracked element the following integral along the contour of the element

$$t \int_c T_i(\tilde{u}_i - u_i) ds \qquad (37)$$

The sum of the elastic strain energy and expression (37) now reads

$$U = t \int_c T_i(\tilde{u}_i - \tfrac{1}{2}u_i) ds = t \int_c T_i\tilde{u}_i ds - \tfrac{1}{2}t \int_c T_i u_i ds \qquad (38)$$

Substitution of (see eq. (30))

$$\tilde{u}_i = M_{ij}\delta_j \qquad (39)$$

and eq. (16) in the integral $t \int_c T_i\tilde{u}_i ds$ gives

$$t \int_c T_i\tilde{u}_i ds = \alpha_j\{t \int_c B_{ij}M_{ik} ds\}\delta_k \qquad (40)$$

or in vector-matrix notation

$$t \int_c T_i \tilde{u}_i ds = \vec{\alpha}^T \bar{R} \vec{\delta} \quad \text{with} \tag{41}$$

$$\bar{R} = t \int_c \bar{B}^T \bar{M} ds$$

The expression (38) after substitution of eqs. (21) and (41) becomes

$$U = \vec{\alpha}^T \bar{R} \vec{\delta} - \tfrac{1}{2} \vec{\alpha}^T \overline{K^*} \vec{\alpha} \tag{42}$$

By putting the first variation of U with respect to the parameters in $\vec{\alpha}$ equal to zero, one gets

$$\overline{K^*} \vec{\alpha} = \bar{R} \vec{\delta} \tag{43}$$

which again is a set of equations of the form as given in eq. (23).

Solving eq. (43) for $\vec{\alpha}$ and substitution of $\vec{\alpha}$ in eq. (21) gives for the strain energy of the cracked element

$$U_e = \tfrac{1}{2} \vec{\delta}^T \{ \bar{R}^T \overline{K^*}^{-1} \bar{R} \} \vec{\delta} = \tfrac{1}{2} \vec{\delta}^T \bar{K} \vec{\delta} \tag{44}$$

with the element stiffness matrix defined by

$$\bar{K} = \bar{R}^T \overline{K^*}^{-1} \bar{R} \tag{45}$$

5. Computerprogram

To compute the stiffness matrix of the cracked element four subroutines has been incorporated in the program.
The first one computes the matrix in accordance with the matching procedure described in Ch.4(c) for equidistant points.
The second one does the computation for the same matching procedure but for the Gausz points.
The third subroutine computes the matrix for the matching procedure 4(d) and the fourth one for the hybrid element approach as discussed in 4(e). The stiffness matrices for the procedures 4(a) and 4(b) can be computed with the above mentioned subroutines 1 and 2. Besides the stiffness matrix these subroutines compute the elements of the two vectors, needed to calculate the stress intensity factors. The number of Gausz points can be chosen as 8, 10, 12 or 16 along each side of the cracked element. To compute the stiffness matrix of the cracked element it is necessary to solve the set of equations as given by eq. (23) for $\overline{\alpha^*}$ (resp. eq. (43) for $\vec{\alpha}$). This is done by Choleski-decomposition of the matrix $\bar{A}$ (resp. $\overline{K^*}$) and overwriting the columns of $\bar{H}$ (resp. $\bar{R}$) one by one.
After computation of the stiffness matrix of the cracked element, it is incorporated in the structure stiffness matrix. The regular elements surrounding the cracked element can be of the linear or quadratic displacement type.
After solution of the set of equations for the structure nodal point displacements, the stress intensity factor(s) is (are) calculated by the scalar product of the above mentioned vector(s) with the vector of displacements of the nodal

points of the cracked element. The program is written in FORTRAN. The computations have been done on a DEC-system 10 computer of the Twente University of Technology.

6. Numerical examples

The problem of a finite plate with central crack, subjected to a uniformly distributed end load has been studied. Because of symmetry of plate and loading one can suffice taking a quarter of the plate with the appropriate boundary conditions. The cracked element is a triangle, which is symmetric with respect to the x_1-axis. The number of Gausz points that has been used in the computations is 10 for each side of the cracked element. Three different sizes of the cracked element with surrounding regular linear elements have been examined. The meshes are given in Figs. 2, 3 and 4. The number of unconstrained degrees of freedom is resp. 48, 110, 179.

The height to width ratio and cracklength to width ratio is resp. 2.5 and 0.5.

Isida [4] gives for a strip of infinite length and a cracklength to width ratio of 0.5 a stress intensity magnification factor - based upon the K_I-factor for an infinite plate - of 1.187. He calculates a stress intensity magnification factor of 1.190 for this strip with a height of 1.8 times the width.

In Table I numerical results for the magnification factor are given for the different matching procedures as described in Ch. 4 for the mesh of Fig. 2. The numbers in the 2nd column apply for N = 3 to the procedure described in Ch. 4(a), for N>3 to the procedure 4(b) for equidistant points.

In the third column the magnification factor is given for the procedure 4(c) for equidistant points, where the total number of boundary points has been taken about 1.5N, thus resulting in about 50% more equations than unknown parameters.

In the fourth column the factor is given by taking the Gauszian points in procedure 4(c). The numbers in the fifth and sixth column have been computed in accordance with procedures 4(d) resp. 4(e).

For the mesh of Fig. 2 the magnification factor stabilizes for $N \geqslant 18$ on 1.143 for procedures (c), (d) and (e), no stabilization occurs for the procedure (b), as can be seen in Table I.

For the mesh of Fig. 3 the magnification factor stabilizes on 1.174 and for the mesh of Fig. 4 on 1.187 in both cases for procedures (c), (d) and (e) and for $N \geqslant 18$.

By using regular elements with quadratic displacement fields (again for the mentioned procedures and for $N \geqslant 18$) a stress intensity magnification factor of 1.165 for the mesh of Fig. 2 and a factor of 1.151 for the coarse mesh of Fig. 5 has been computed. The unconstrained degrees of freedom are resp. 188 and 58 for these cases. Computations which have been carried out for a large triangular cracked element, with an increased number of nodal points, surrounded by quadratic regular elements show that the stress intensity magnification factor deviates less than 1% of the values given by Isida.

7. Conclusions

The concept of the cracked element, including the stress singularity at the crack tip and satisfying the conditions of equilibrium, compatibility and Hooke's law in the element as well as the requirement of traction free crack surfaces offers an efficient possibility to compute the stress intensity factors K_I and K_{II} for plane cracks.

Highly accurate results are obtained by matching the boundaries of the cracked element with those of the surrounding linear or quadratic regular elements in a least square sense by using displacement differences in equidistant or Gauszian points,
or by minimizing the integral of the squared displacement differences along the boundaries,
or by basing upon the hybrid functional approach of Pin Tong et. al.

It has been found that a large cracked element with increased number of nodal points surrounded by a limited number of regular quadratic elements gives already very accurate stress intensity factors.

References

[1] N.I. Muskhelishvili, "Some Basic Problems of Mathematical Theory of Elasticity", Noordhoff, Groningen, Holland, 1953.

[2] E. Byskov, "The calculation of stress intensity factors using the finite element method with cracked elements", Int. J. Fract. Mech. 6, 159-167, 1970.

[3] P. Tong, T.H.H. Pian and S. Lasry, "A hybrid-element approach to crack problems in plane elasticity", Int. J. Num. Meth. Eng., 7, 297-308, 1973.

[4] M. Isida, "Effect of width and length on stress intensity factors of internally cracked plates under various boundary conditions", Int. J. Fract. Mech. 7, 301-316, 1971.

TABLE I

STRESS INTENSITY MAGNIFICATION FACTORS FOR THE MESH OF FIGURE 2.

N	MATCHING PROCEDURE				
	$4(a),(b)_{Eq.}$	$4(c)_{Eq.}$	$4(c)_{Gz.}$	4(d)	4(e)
3	1,217	1,219	1,217	1,204	1,058
6	1,186	1,207	1,196	1,181	1,158
9	1,064	1,122	1,131	1,131	1,164
12	1,143	1,139	1,143	1,143	1,146
15	1,146	1,145	1,144	1,145	1,147
18	1,058	1,143	1,143	1,143	1,144
21	1,135	1,143	1,143	1,143	1,143
24	1,139	1,143	1,143	1,143	1,143

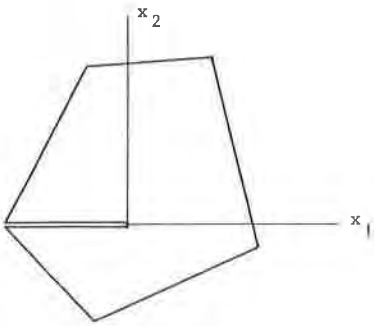

Fig. 1

Cracked element polygonon

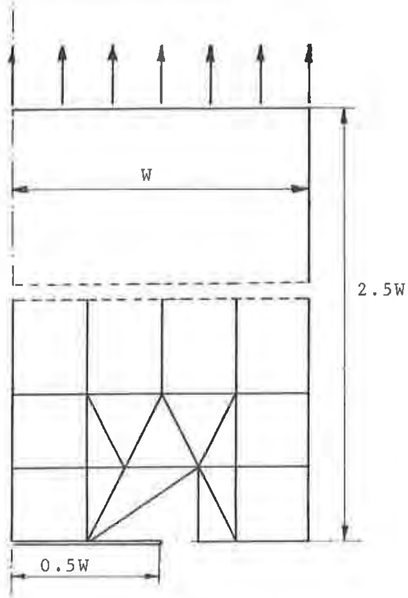

Fig. 2

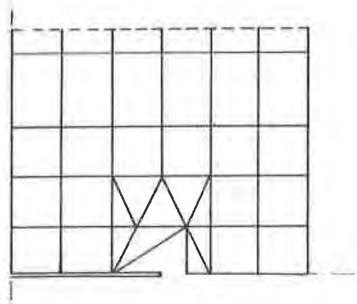

Fig. 3

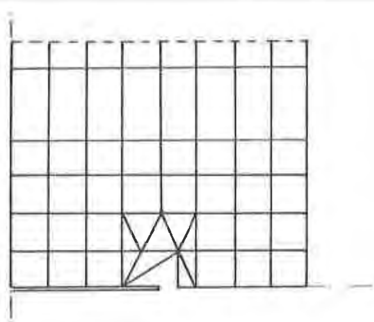

Fig. 4

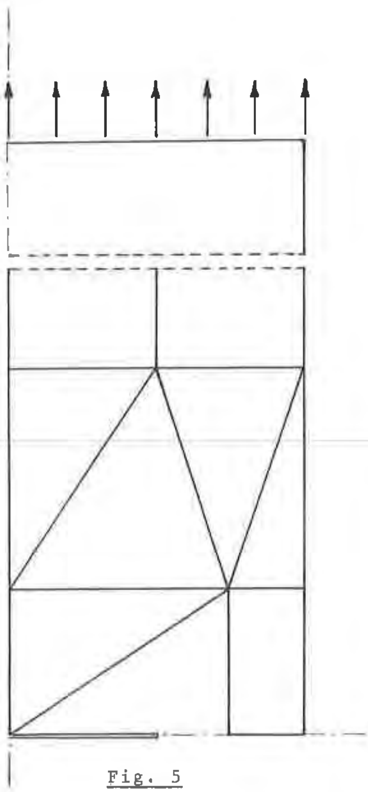

Fig. 5

Finite element meshes