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ON THE COMPOSITION OF BALANCED INCOMPLETE BLOCK DESIGNS

by

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Some inequalities for the lower bound of the number of constraints in an orthogonal array of strength two are derived. A method of constructing balanced incomplete block designs is given which depends upon the existence of two such designs with the same block size. A large number of these designs with $k=4,5$ and $\lambda=1$ are also given.

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ON THE COMPOSITION OF BALANCED INCOMPLETE BLOCK DESIGNS.[1]

by

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Introduction. The object of the present paper is to develop a method of constructing balanced incomplete block designs. It consists in utilizing the existence of two balanced incomplete block designs to obtain another such design by what may be called the method of composition.

1. Preliminary results on orthogonal arrays and balanced incomplete block designs. Consider a matrix $A = (a_{ij})$ of k rows and N columns, where each a_{ij} represents one of the integers 1, 2, ..., s. Consider all t-rowed submatrices of N columns, which can be formed from this array, $t \leq k$. Each column of any t-rowed submatrix can be regarded as an ordered t-plet. The matrix will be called an orthogonal array [N, k, s, t] of size N, k constraints, s levels, strength t, and index λ if each of the c_t^k t-rowed submatrices that can be formed from the array contains every every one of the s^t possible ordered t-plets exactly λ times. Obviously $N = \lambda s^t$ and each row contains the integers 1, 2, ..., s exactly λs^{t-1} times. The idea of orthogonal array is originally due to Rao [16] who utilized it in the construction of factorial arrangements in the design of experiments.

Denote by $f(\lambda s^t)$ the maximum number of constraints which are possible in an orthogonal array of size λs^t, s levels, strength t and index λ.

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Then from Plackett and Burman [15] we have <u>Theorem</u> A. For any s and λ,

$$f(\lambda s^2) \leq \left[\frac{\lambda s^2 - 1}{s-1}\right]$$

where $[x]$ is the largest integer not exceeding x.

This inequality has been improved by Bose and Bush in [2], where they also give methods of constructing orthogonal arrays of strength two and three when the number of levels s is a prime power.

Let a set of s distinct symbols be arranged in s x s square in such a way that every symbol occurs exactly once in every row and once in every column. Such a square is called a Latin square of order s. Two Latin squares of order s are called orthogonal, if when one of the squares is superimposed on the other, every symbol of the first square occurs with every symbol of the second square once and only once. A set of Latin squares of order s are said to form a set of mutually orthogonal Latin squares (m.o.ℓ.s) if any two of them are orthogonal. It is known [2] that the existence of k-2 m.o.ℓ.s. of order s is equivalent to the existence of an orthogonal array $[s^2, k, s, 2]$. Hence for any s, $f(s^2) \leq s+1$ implies that $N(s) \leq s-1$, where by N(s) we denote the maximum number of m.o.ℓ.s. of order s. If s is a prime power, then it is known [10, 11] that N(s) = s-1. If $v = p_1^{n_1} p_2^{n_2} \ldots p_u^{n_u}$ is the prime power decomposition of an integer v, and we define $n(v) = \min(p_1^{n_1}, p_2^{n_2}, \ldots, p_u^{n_u}) - 1$, then MacNeish [10] and Mann [11] showed that there exists a set of at least n(v) m.o.ℓ.s. of order v, i.e., $N(v) \geq n(v)$. Recently Parker [13] showed how in certain cases one could construct more than n(v) m.o.ℓ.s. of order v. Parker's method has been generalized by the present authors who showed [3,4] in particular that Euler's conjecture about the nonexistence of two orthogonal Latin squares of order 2(mod 4) is false for an infinity of values of $v \geq 22$. By using the method of differences Parker [14] later on showed that $N(v) \geq 2$ for $v = \frac{3q-1}{2}$, where q is a prime power $\equiv 3 \pmod 4$. In a joint paper with Parker [5] the present authors have shown that Euler's conjecture is false for all values of $v \geq 10$.

We call an array $[\lambda s^t, k, s, t]$ α-resolvable if the λs^t columns can be separated into $\frac{\lambda s^{t-1}}{\alpha}$ sets of αs each, such that in each set every row contains each of the s integers 1, 2, ..., s exactly α times. A 1-resolvable array is called resolvable. Suppose there exists a set Σ of k-1 m.o.ℓ.s. of order s, then we can take Σ in the standard form in which the first row of each Latin square contains the integers 1, 2, ..., s in that order. We now prefix to the set Σ a $s \times s$ square containing the integer i in each position in the i-th column. If we then write down the elements of each square in a single row such that the integer in the i-th row and the j-th column occupies the n-th position, where $n = s(i-1) + j$; i, $j = 1, 2, \ldots, s$; then we get an orthogonal array $[s^2, k, s, 2]$ which is resolvable. We thus have the following theorem which is essentially contained in [6].

<u>Theorem</u> B. Existence of k-1 m.o.ℓ.s. of order s implies the existence of a resolvable array $[s^2, k, s, 2]$.

A balanced incomplete block design (BIB) [18] with parameters v, b, r, k, λ is an arrangement of v objects or treatments in b sets or blocks such that (i) every block contains $k < v$ different objects and (ii) every pair of treatments occurs in λ blocks. Then it is easy to see that each treatment occurs in exactly r blocks and the parameters satisfy the relations

$$\lambda(v-1) = r(k-1), \; bk = vr, \; b \geq v \qquad (1.1)$$

The last inequality is due to Fisher [7]. A BIB design is called symmetrical if $b = v$ and hence $k = r$. It will be called β-resolvable if the blocks can be separated into sets such that each set contains every treatment exactly β times. A 1-resolvable BIB design is called resolvable. A BIB design with parameters v, k, λ will be denoted by BIB $(v;\ k;\ \lambda)$ and if $\lambda = 1$ by BIB $(v;\ k)$. If the design is resolvable we denote it by RBIB$(v;\ k;\ \lambda)$ and RBIB$(v;\ k)$ respectively.

A BIB design is called separable if its blocks can be divided into sets of type I or II [4].

From Theorem 2 in [4] we have:

Theorem C. If there exists a BIB $(v;k)$ then

$$N(v) \geq N(k) - 1.$$

If further the design is separable then

$$N(v) \geq N(k).$$

From Theorem 2 of [5] and corollary of Theorem 12 of [4] we have

Theorem D. Existence of BIB $(v;\ k)$ implies that

$$N(v-1) \geq \min(N(k),\ 1 + N(k-1)) - 1 .$$

If further the design is resolvable, then

$$N(v-1) \geq \min(N(k\ ),\ N(k-1)) .$$

2. Pairwise balanced designs of index λ. An arrangement of v treatments in b blocks will be called a pairwise balanced design (D) of index λ, if each block contains either k_1, k_2, ... or k_m treatments which are all distinct ($k_i \neq k_j \leq v$), and every pair of treatments occurs in exactly λ blocks. Such a design will be said to be of type $(v;\ k_1,\ \ldots,\ k_m;\ \lambda)$. If the number of blocks containing k_i treatments is b_i, then

$$b = \sum_1^m b_i, \quad \lambda v(v-1) = \sum_1^m b_i k_i (k_i - 1) .$$

The subdesign (D_i) formed by the blocks of size k_i, will be called the i-th equi-block component of (D), $i = 1, 2, \ldots, m$.

A subset of blocks of (D_i) will be said to be of general type I, if every treatment occurs in the subset αk_i times, where α is a divisor of λ. The number of blocks in the subset is clearly αv. As proved in [8, 17] we can arrange the treatments within the blocks of the subset in such a way that every treatment comes in each position exactly α times. If the blocks are written as columns, each treatment occurs α times in every row. When so written out the blocks will be said to be in the standard form.

A subset of blocks of (D_i) will be said to be of general type II, if every treatment occurs in the subset exactly β times, when β is a divisor of λ. The set of blocks will be said to form a β-replicate. The number of blocks in such a subset is clearly $(\beta v)/(k_i)$.

The component (D_i) is said to be separable in the general sense, if the blocks of (D_i) can be divided into subsets of general type I or II. (Both types may occur in (D_i) at the same time.) If $\alpha = \beta = 1$, then (D_i) is called separable [4].

If each (D_i) is separable in the general sense with α and β independent of i, then (D) is called separable in the general sense. If each (D_i) is separable then (D) will be called separable.

The set of equi-block components $(D_1), (D_2), \ldots, (D_\ell)$ will be said to form a clear set if $\sum_1^\ell b_i$ blocks comprising $(D_1), (D_2), \ldots, (D_\ell)$ are such that no two blocks of the set have a treatment in common. Clearly a necessary condition for this is

$$\sum_1^\ell b_i k_i \leq v \quad .$$

3. Use of pairwise balanced design in the construction of orthogonal arrays.

Theorem 1. Let there exist a pairwise balanced design (D) of type $(v;\, k_1, \ldots, k_m;\lambda)$ and suppose that there exist q_i-1 m.o.l.s. of order k_i, $i=1,2,\ldots,m$. Put

$$q = \min\,(q_1,\, q_2,\, \ldots,\, q_m).$$

Then

$$f(\lambda v^2) \geq q.$$

If the set of equiblock components (D_1), (D_2), ..., (D_ℓ) form a clear set, and

$$q^* = \min\,(q_1+1,\, q_2+1,\, \ldots,\, q_\ell+1,\, q_{\ell+1},\, \ldots,\, q_m)$$

then $f(\lambda v^2) \geq q^*$.

If the design (D) is separable in the general sense, then

$$f(\lambda v^2) \geq q+1$$

and we can construct $A = [\lambda v^2,\, q, v, 2]$ which is λ-resolvable. If in particular (D) is separable then A is resolvable.

Proof. Proof follows the general lines of Theorem 1 in [4,5]. Let the treatments of the design be $t_1, t_2, \ldots, t_v$ and let the blocks of the design (written out as columns) belonging to the equiblock component (D_i) be $\delta_{i1}, \delta_{i2}, \ldots, \delta_{ib_i}$ $(i=1,2,\ldots,m)$. Define the $k_i \times b_i$ matrix D_i by

$$D_i = [\delta_{i1},\, \delta_{i2},\, \ldots,\, \delta_{ib_i}]$$

Let P_i be the matrix of order $q_i \times k_i\,(k_i-1)$ defined in Lemma 2 of [4], the elements of P_i being the symbols 1, 2, ..., k_i. Let P_{ic}, $c=1,2,\ldots,k_i-1$

be the submatrices of P_i, such that each row of P_{ic} contains the symbols $1,2,\ldots,k_i$, exactly once. Define $P_i(\delta_{iu})$ in the same manner as in Lemma 2 of [4] and let

$$P_i(D_i) = [P_i(\delta_{i1}), P_i(\delta_{i2}, \ldots, P_i(\delta_{ib_i})]$$

Then $P_i(D_i)$ is of order $q_i \times b_i k_i(k_i-1)$. If t_a and t_b are any two treatments occuring in the same block (δ_{iu}), then the ordered pair $\binom{t_a}{t_b}$ occurs exactly once as a column in any two rowed submatrix of $P_i(\delta_{iu})$. Let Δ_i be the matrix obtained from $P_i(D_i)$ by retaining only the first q rows, and let

$$\Delta = (\Delta_1, \Delta_2, \ldots, \Delta_m).$$

Then from (2.1), Δ is of order $q \times \lambda v(v-1)$ and since any two treatments occur exactly in λ blocks of (D), any two rowed submatrix of Δ contains exactly λ columns of ordered pairs of any two distinct treatments chosen from $t_1, t_2, \ldots, t_v$. Let Δ_o be a $q \times \lambda v$ matrix containing t_i in all positions in columns numbered from $(i-1)\lambda + 1$ to $i\lambda$, $i=1,2,\ldots,v$. Then the matrix (Δ_o, Δ) obviously gives $A=[\lambda v^2, q, v, 2]$.

The second part of the theorem can be proved along the same lines as Theorem 1 in [5].

To prove the last part of the theorem we note that each (D_i) can be broken up into x_i sets of αv blocks of general type I and y_i sets of $\frac{\beta v}{k_i}$ blocks of general type II, where α and β are divisors of λ and are the same for each (D_i). Thus each treatment occurs $\alpha k_i x_i + \beta y_i = r_i$ times in (D_i), $i=1,2,\ldots,m$. Following the proof of the latter part of Theorem 1 in [4], it is easily seen that the columns of Δ can be divided into

$\sum_1^m x_ik_i(k_i-1)$ sets of αv and $\sum_1^m y_i(k_i-1)$ sets of βv columns respectively, where in each set every row contains all the treatments exactly α and β times respectively.

If $\alpha = \beta = 1$, then (Δ_0,Δ) is a resolvable array $[\lambda v^2,q,v,2]$. We can now add an additional row by putting t_i in the q+1-th position under λ sets of v columns of (Δ_0,Δ), $i=1,2,\ldots,v$. This gives an array $[\lambda v^2,q+1,v,2]$.

Now consider the case where both α and β are not equal to 1. Since

$$\begin{aligned}\alpha v\sum_1^m x_ik_i(k_i-1)+\beta v\sum_1^m y_i(k_i-1) &= v\sum_1^m (k_i-1)\ [\alpha k_ix_i+\beta y_i] \\ &= v\sum_1^m (k_i-1)r_i \\ &= \sum_1^m (k_i-1)\ k_ib_i \\ &= \lambda v(v-1),\end{aligned}$$

$$(3.1)\quad \alpha\sum_1^m x_ik_i(k_i-1)+\beta\sum_1^m y_i(k_i-1) = \lambda(v-1)$$

Let $\lambda = p\alpha = p'\beta$, say, and let

$$(3.2)\qquad \sum_1^m x_ik_i(k_i-1) = pc+d,\ c \geq 0,\ 0 \leq d < p$$

$$(3.3)\text{ and}\qquad \sum_1^m y_i(k_i-1) = p'c' + d',\ c' \geq 0,\ 0 \leq d' < p'.$$

Then (3.1) gives

$$\alpha\ [pc+d] + \beta\ [p'c'+d'] = \lambda(v-1)$$

or

$$(3.4)\qquad \lambda(c+c')+d\alpha+d'\beta = \lambda(v-1).$$

Since $d\alpha < p\alpha = \lambda$, $d'\beta < p'\beta = \lambda$, (3.4) implies that

(3.5) $$d\alpha + d'\beta = \lambda.$$

From (3.2), it is clear that $\Sigma x_i k_i(k_i-1)$ sets of αv columns of Δ can be separated into c sets of λv columns, each set containing everyone of the v treatments exactly λ times in any row and another set containing $d\alpha v$ columns in which every treatment occurs exactly $d\alpha$ times in every row. Similarly from (3.4) we get c' sets of λv columns containing each treatment λ times in every row and a set of $d'\beta v$ columns contains each treatment exactly $d'\beta$ times in a row. Combining the sets of $d\alpha v$ and $d'\beta v$ columns we get λv columns containing each treatment λ times in every row. Thus the columns of Δ are divisible into (v-1) sets each of λv columns, such that in each set every row contains all the v treatments exactly λ times. It is now obvious that (Δ_o,Δ) is an array $[\lambda v^2, q, v, 2]$ which is λ-resolvable. We can now add an additional row by placing t_i in the (q+1)-th position under the i-th set i=1,2,...,v, thus giving $[\lambda v^2, q+1, v, 2]$.

Corollary 1.1. Existence of BIB $(v;k;\lambda)$ and the existence of q-1 m.o.l.s. of order k implies that

$$f(\lambda v^2) \geq q.$$

Further if the design is separable in the general sense, then

$$f(\lambda v^2) \geq q+1$$

and we can construct $A = [\lambda v^2, q, v, 2]$ which is λ-resolvable. If the design is separable, then A is resolvable.

4. Composition of balanced incomplete block designs.

Theorem 2A. If BIB $(v_1; k; \lambda_1)$ and BIB $(v_2; k; \lambda_2)$ exist and if $f(\lambda_2 v_2^2) \geq k$, then BIB $(v_1 v_2; k; \lambda_1\lambda_2)$ exists.

Proof. Let the two designs be denoted by (D_1) and (D_2) respectively. Write the blocks of each design as columns. Then (D_1) can be written as a matrix of k rows and b_1 columns, where b_1 is the number of blocks in (D_1). Let the treatments of (D_1) be $t_1, t_2, \ldots, t_v$; and let A be an orthogonal array $A = [\lambda_2 v_2^2, k, v_2, 2]$ in the integers 1, 2, ..., v_2. Let (β_1) be any block of (D_1) containing treatments $t_{i_1}, t_{i_2}, \ldots, t_{i_k}$ in positions 1, 2, ..., k respectively. Whenever j (j = 1, 2, ..., v_2) occurs in row p of A, replace it by $t_{i_p,j}$, p = 1, 2, ..., k. We thus get a matrix $A(\beta_1)$ of k rows and $\lambda_2 v_2^2$ columns. Define

$$A(D_1) = (A(\beta_1), \ldots, A(\beta_{b_1}) \quad .$$

Then $A(D_1)$ is a matrix of k rows and $b_1\lambda_2 v_2^2$ columns in which the entries are the $v_1 v_2$ symbols $t_{i,j}$, i = 1, 2, ..., v_1; j = 1, 2, ..., v_2. If $c \neq c'$, the treatments t_c and $t_{c'}$ occur together in exactly λ_1 blocks of (D_1). Suppose (β) is a block of (D_1) containing t_c and $t_{c'}$ in positions i and i' respectively. In A integers j,j' occur in positions i and i' respectively in exactly λ_2 columns. Hence treatments $t_{c,j}$ and $t_{c',j'}$ occur together in the corresponding λ_2 columns of $A(\beta)$. Obviously then these treatments will occur in $\lambda_1\lambda_2$ columns of $A(D_1)$. We note that this is true whether or not j and j' are equal. Thus the $v_1 v_2$ treatments $t_{i,j}$ can be divided into v_1 sets $(t_{i,1}, t_{i,2}, \ldots, t_{i,v_2})$, i = 1, 2, ..., v_1, such that any two treatments coming from different sets occur exactly $\lambda_1\lambda_2$ times in the blocks (columns) of $A(D_1)$. We now take λ_1 repetitions of the design (D_2) for each of these v_1 sets of v_2 treatments. The totality of blocks thus obtained obviously provide BIB $(v_1 v_2; k; \lambda_1\lambda_2)$.

Corollary 2A.1. If BIB $(v_1;k)$ and BIB$(v_2;k)$ exist and $N(v_2) \geq k-2$, then BIB $(v_1v_2; k)$ exists.

Using the above corollary and Theorem C, we have the following result due to Skolem given in the notes to Netto's book [12].

Corollary 2A.2. If k is a prime power and BIB $(v_1; k)$ and BIB$(v_2;k)$ exist, the BIB $(v_1v_2; k)$ exists.

Theorem 2B. If separable designs BIB $(v_1; k; \lambda_1)$ and BIB $(v_2;k;\lambda_2)$ exist and if a resolvable array $A = [\lambda_2v_2^2, k, v_2, 2]$ exists, then a separable design BIB $(v_1v_2; k; \lambda_1\lambda_2)$ exists. If in particular the original designs are resolvable so is the obtained design.

Proof. Suppose that the first design (D_1) can be separated into sets $S_1, S_2, \ldots, S_x$ of type I and $S_1^*, S_2^*, \ldots, S_{x'}^*$ of type II. Then obviously $xk + x' = r_1$, the number of replications of any treatment in (D_1). The sets S_q each contain v_1 blocks and the sets $S_{q'}^*$ each contain v_1/k blocks. Without loss of generality assume that each set S_q is put in the standard form, i.e., in each row of S_q every treatment of (D_1) occurs exactly once.

Since A is resolvable, we can put

$$A = (A_1, A_2, \ldots, A_{\lambda_2v_2})$$

where each row of A_i contains all the integers 1, 2, ..., v_2 exactly once. As in the previous theorem, let

$$A(D_1) = (\ \ldots, A_p(S_q), \ldots, A_p(S_{q'}^*), \ldots\ )$$

where $p = 1, 2, \ldots, \lambda_2v_2$; $q = 1, 2, \ldots, x$; $q' = 1, 2, \ldots, x'$. Then it is easily seen that with respect to the v_1v_2 treatments $t_{i,j}$ defined in the previous theorem, each set $A_p(S_q)$ gives a set of v_1v_2 blocks of type I and each set $A_p(S_{q'}^*)$ gives a set of $(v_1v_2)/k$ blocks of type II. Taking the additional blocks obtained from λ_1 repetitions of the separable design (D_2) for each of the v_1 sets of v_2 treatments as in the previous theorem, we get a separable design BIB $(v_1v_2;\ k;\ \lambda_1\lambda_2)$. It is obvious that if in particular the original designs are resolvable so is the new design for v_1v_2 treatments.

Taking $\lambda_1 = \lambda_2 = 1$ and the particular case resolvability of separability we get from Theorem B,

Corollary 2B.1 If RBIB $(v_1;\ k)$ and RBIB $(v_2;\ k)$ exist and $N(v_2) \geq k - 1$, then RBIB $(v_1v_2;\ k)$ exists.

Using Theorem C, the above gives

Corollary 2B.2. If k is a prime power and RBIB $(v_1;\ k)$ and RBIB $(v_2;\ k)$ exist, then RBIB $(v_1v_2;\ k)$ exists.

<u>Theorem</u> 2C. If BIB $(v_1;\ k;\ \lambda_1)$ BIB $(v_2;\ k;\ \lambda_2)$ exist and $f(\lambda_2(v_2\mu 1)^2] \geq k$, then BIB $(v_1(v_2-1) + 1;\ k;\ \lambda_1\lambda_2)$ exists.

Proof. Let (D_1) be the design with v_1 treatments and let $A_1 = [\lambda_2(v_2-1)^2,\ k,\ v_2-1,\ 2]$. Then as in Theorem 2A, the matrix $A_1(D_1)$ gives blocks of size k in which any two treatments coming from different sets $(t_{i,1}, \ldots, t_{i,v_2-1})$, $i = 1, 2, \ldots, v_1$, occur together in exactly $\lambda_1\lambda_2$ blocks. Take a new treatment say Θ, and consider the v_1 sets of v_2 treatments $(\Theta, t_{i,1}, \ldots, t_{i,v_2-1})$, $i = 1, 2, \ldots, v_1$. The λ_1 repetitions of the design (D_2) with each of the v_1 sets of v_2 treatments above together with the blocks $A_1(D_1)$ give the BIB $(v_1(v_2-1)+1;\ k;\ \lambda_1\lambda_2)$.

Corollary 2C.1. If BIB $(v_1;\ k)$ and BIB $(V_2;\ k)$ exist and $N(v_2-1) \geq k - 2$, then BIB $(v_1(v_2-1)+1;\ k)$ exists.

The particular case of this corollary when $v_2 = k$ is given in the notes added by Skolem in [12].

Using Theorem D and the above corollary we have

Corollary 2C.2. If k and k-1 are both prime powers, then the existence of BIB $(v_1;\ k)$ and BIB$(v_2;\ k)$ implies the existence of BIB$(v_1(v_2-1)+1;\ k)$.

Theorem 2D. If BIB $(v; k; \lambda)$ exists and k-1 is a prime power, then $BIB((k-2)v + 1; k-1; \lambda)$ exists which is λ-resolvable.

Proof. Let the remaining parameters of (D) the BIB $(v; k; \lambda)$ be b and r. Since k-1 is a prime power, there exists an orthogonal arry $A_1 = [(k-1)^2, k, k-1, 2]$ in k-1 integers 1, 2, ..., k-1. Without loss of generality assume that the first column of A_1 consists entirely of 1's. Using Theorem 2C with $v_1 = v$, $\lambda_1 = \lambda$, $v_2 = k$, $\lambda_2 = 1$, we obtain a BIB design with parameters $v^* = v(k-1) + 1$, $b^* = b(k-1)^2 + \lambda v$, $r^* = \lambda v$, $k^* = k$, $\lambda^* = \lambda$. From this design omit the b blocks of $A_1(D)$ which arise from the first column of A_1. Obviously these blocks form a $BIB(v; k; \lambda)$ for the treatments $t_{1,1}, \ldots, t_{v,1}$, where $t_1, t_2, \ldots, t_v$ are the treatments of (D). In the design (D_1) formed of the remaining $b(k-1)^2 + \lambda v - b$ blocks, each of the treatments $t_{1,1}, \ldots, t_{v,1}$ occurs $\lambda v - r$ times. Further no two of these treatments can occur in the same block of (D_1). Since from (1.1)

$$\begin{aligned} b(k-1)^2 + \lambda v - b &= bk^2 - 2bk + \lambda v \\ &= v(rk - 2r + \lambda) \\ &= v(\lambda v - r) \end{aligned}$$

the blocks of (D_1) can be separated into v sets of $(\lambda v - r)$ blocks, such that each block of the i-th set contains the treatment $t_{i,1}$, i = 1, 2, ..., v. In this set $t_{i,1}$ obviously occurs λ times with each of the remaining $v(k-2) + 1$ treatments excepting $t_{j,1}$, $j \neq i = 1, 2, \ldots, v$. Omitting the treatment $t_{i,1}$ from the blocks of the i-th set i = 1, 2, ..., v, we get BIB $((k-2)v+ 1; k-1; \lambda)$ which is obviously λ-resolvable.

Corollary 2D.1. If BIB $(v; k)$ exists and k-1 is a prime power then RBIB $((k-2)v + 1; k-1)$ exists.

We note that in actual applications of the above theorems the trivial existence of RBIB (k; k) is very useful.

5. <u>BIB designs with $k = 5$, $\lambda = 1$.</u> A BIB design with $k = 5$, $\lambda = 1$ belongs to one of the two series [1]

(G_1) $\quad v = 20t + 1,\ b = t(20t + 1),\ \ r = 5t$

(G_2) $\quad v = 20t + 5,\ b = (5t + 1)(4t + 1),\ 4 = 5t + 1$.

If $v = 20t + 1$, we denote the corresponding design by $G_1(v)$. Similarly if $v = 20t + 5$, the corresponding design is denoted by $G_2(v)$. When v is of the form $20t + 1$ or $20t + 5$ the corresponding design will be denoted by $G(v)$. Using the results in [1] and on p. 118 of [11] and the corollaries 2A.2, 2B.2, 2C.2, we can state the following theorem.

<u>Theorem</u> 3. (a) If $v = 20t + 1$ is a prime power and x is a primitive element of $GF(v)$ and if $x^{4t} + 1 = x^q$, q-odd then $G_1(v)$ exists.

(b) If $v = 20t + 5$ and $4t + 1$ is a prime power, then $G_2(v)$ exists.

(c) Existence of $G(v)$ implies the existence of $G(5v)$, and if $G(v)$ is resolvable so is $G(5v)$.

(d) If $G(v_1)$ and $G(v_2)$ exist, then $G(v_1v_2)$ exists, and if $G(v_1)$ and $G(v_2)$ are both resolvable so is $G(v_1v_2)$.

(e) Existence of $G(v)$ implies the existence of $G(4v + 1)$ and $G(5v - 4)$.

(f) Existence of $G(v_1)$ and $G(v_2)$ implies the existence of $G(v_1(v_2 - 1) + 1)$.

From (a) using the powers of primitive roots of primes given in [9] solutions for v = 41, 61, 241, 281, 641, 701, 881 can be obtained. From (b) we get solutions for v = 25, 45, 65, 85, 125, 145, 185, 205, 245, 265, 305, 365, 405, 445, 485, 505, 545, 565, 605, 625, 685, 745, 785, 845, 865, 905, 965, 985. Similarly (c), (d), (e) and (f) provide solutions for large number of values of v. Using these methods, solutions for all v (≤ 1000) of the form $20t + 1$ or $20t + 5$ can be obtained excepting for 81, 141, 161, 285, 345, 361, 381, 385, 461, 465, 541, 561, 585, 645, 665, 681, 705, 761, 765, 781, 801, 941, 961, 981. We note that resolvable solutions for $v = s^q$, $q \geq 2$ can always be constructed.

6. BIB designs with $k = 4$, $\lambda = 1$. BIB designs with $k = 4$, $\lambda = 1$ can be classified in two series:

(F_1): $v = 12t + 1$, $b = t(12t + 1)$, $r = 4t$

(F_2): $v = 12t + 4$, $b = (4t + 1)(3t + 1)$, $r = 4t + 1$.

If $v = 12t + 1$, we denote the corresponding solution by $F_1(v)$ and if $v = 12t + 4$ we denote it by $F_2(v)$. $F(v)$ will denote the design when v is of the form $12t + 1$ or $12t + 4$. Using the results in [1] and on page 118 of [11] and the corollaries 2A.2, 2B.2, 2C.2, 2D.1, we have the following theorem.

Theorem 4.

(a) If $v = 12t + 1$ is a prime power and x is a primitive element of $GF(v)$ and $x^{4t} - 1 = x^q$, q odd, then $F_1(v)$ exists.

(b) If $v = 12t + 4$ and $4t + 1$ is a prime power, then a resolvable solution for $F_2(v)$ exists.

(c) Existence of $F(v)$ implies the existence of $F(4v)$, and if $F(v)$ is resolvalbe so is $F(4v)$.

(d) If $F(v_1)$ and $F(v_2)$ exist, then $F(v_1v_2)$ exists, and if $F(v_1)$ and $F(v_2)$ are resolvable so is $F(v_1v_2)$.

(e) Existence of $F(v)$ implies the existence of $F(3v + 1)$ and $F(4v - 3)$.

(f) Existence of $F(v_1)$ and $F(v_2)$ implies the existence of $F(v_1(v_2 - 1) + 1)$.

(g) Existence of G(v) implies the existence of a resolvable solution for F(3v + 1).

From (a) above with the help of [9] we get solutions for v = 13, 25, 73, 181, 277, 409, 457, 541, 709; from (b) resolvable solutions for v = 16, 28, 40, 52, 76, 88, 112, 124, 148, 160, 184, 220, 244, 268, 292, 304, 328, 340, 364, 376, 412, 448, 472, 508, 520, 544, 580, 592, 688, 700, 724, 772, 808, 832, 844, 868, 880, 940, 952, are obtained. From (g) we get resolvable solutions for v = 136, 196, 256, 316, 436, 556, 604, 616, 664, 676, 796, 904, 916, 964. Resolvable solutions for v = 64, 208, 352, 496, 640, 736, 784, 976 are provided by (c). Similarly (d), (e), (f) give solutions for a large number of values of v. It has not bees possible to obtain solutions for $v \leq 1000$ for the following values: 37, 133, 145, 172, 217, 232, 280, 361, 424, 460, 469, 505, 517, 529, 532, 565, 568, 577, 613, 649, 652, 685, 697, 712, 745, 748, 841, 853, 856, 865, 889, 892, 901, 925, 928, 997.

7. Concluding remarks. The BIB designs with $k = 5$, $\lambda = 1$ and RBIB designs with $k = 4$, $\lambda = 1$ are especially interesting from the point of view of constructing orthogonal Latin squares. From Theorem 3 [5] it follows that existence of G(v) with v = 20t + 1 and 20t + 5 implies the existence of at least two orthogonal Latin squares of order 20t - 2 and 20t + 2, respectively, which are of the form 2(mod. 4). Similarly from Theorem 4 of [5] existence of resolvable solution $F_2(12t + 4)$, $t \geq 5$, coupled with the fact that $N(v) \geq 2$, for v = 10, 14, and 18. [14, 5] gives the result that there exist at least two orthogonal Latin squares of orders 12t + 14, 12t + 18 and 12t + 22. Since Euler's conjecture is false [5] for all numbers of the form 4t + 2 which are ≤ 74,

excepting for 2 and 6, another proof for the falsity of the conjecture for all numbers ≥ 10 could be given if it could be shown that a resolvable solution $F_2(12t + 4)$ exists for all $t \geq 5$.

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