A Three-Class Association Scheme on the Flags of a Finite Projective Plane and a (PBIB) Design Defined by the Incidence of the Flags and the Baer Subplanes in $PG(N,q^2)$*

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0. Summary.

First we define relations between the $v = (s^2+s+1)(s+1)$ flags (point-line incident pairs) of a finite projective plane of order s. Two flags $a \equiv (p,\ell)$ and $b \equiv (p',\ell')$, where p and p' are two points and ℓ and ℓ' are two lines of the projective plane, are defined to be *first* associates if either $p = p'$ or $\ell = \ell'$; *second associates* if $p \neq p'$, $\ell \neq \ell'$ but either p is incident also with ℓ' or p' is incident also with ℓ; *third associates*, otherwise.

We show that these relations define a three-class association scheme on $v = (s^2+s+1)(s+1)$ flags with $n_1 = 2s$, $n_2 = 2s^2$ and $n_3 = s^3$ (n_i denotes the number of i-th associates of a given flag, $i = 1,2,3$) and the association matrices are

$$P_1 = (p^1_{ij}) = \begin{bmatrix} s-1 & s & 0 \\ s & s(s-1) & s^2 \\ 0 & s^2 & (s-1)s^2 \end{bmatrix},$$

$$P_2 = (p^2_{ij}) = \begin{bmatrix} 1 & s-1 & s \\ s-1 & s & 2s(s-1) \\ s & 2s(s-1) & s(s-1)^2 \end{bmatrix},$$

$$P_3 = (p^3_{ij}) = \begin{bmatrix} 0 & 2 & 2(s-1) \\ 2 & 4(s-1) & 2(s-1)^2 \\ 2(s-1) & 2(s-1)^2 & (s-1)(s^2-s+1) \end{bmatrix}.$$

If a finite projective plane of order s admits a subplane of order q, then it is known (Bruck, 1955) that either $s = q^2$ or $s \geq q^2+q$. If $s = q^2$, then the subplane is called a *Baer subplane*. In a Desarguesian finite projective plane of order q^2, $PG(2,q^2)$, all subplanes are Baer subplanes of order q and there

are $b = q^3(q^3+1)(q^2+1)$ Baer subplanes in $PG(2,q^2)$.

Next, we consider the incidence of the flags and the Baer subplanes (blocks) of $PG(2,q^2)$. We show that every flag occurs in $r = q^3(q+1)^2$ Baer subplanes; a pair of flags which are *first associates* occur together in $\lambda_1 = q^2(q+1)^2$ blocks; a pair of flags which are *second associates* occur together in $\lambda_2 = q(q+1)^2$ blocks; any two flags which are *third associates* occur together in $\lambda_3 = (q+1)^2$ blocks. Each Baer subplane is incident with $k = (q^2+q+1)(q+1)$ flags. Thus the incidence of the flags and the Baer subplanes of $PG(2,q^2)$ defines an incomplete block design (called a partially balanced incomplete block design) with parameter, $v = (q^4+q^2+1)(q^2+1)$, $b = q^3(q^3+1)(q^2+1)$, $r = q^3(q+1)^2$, $k = (q^2+q+1)(q+1)$, $\lambda_1 = q^2(q+1)^2$ $\lambda_2 = q(q+1)^2$ and $\lambda_3 = (q+1)^2$. The parameters of the association matrices are obtained by replacing s by q^2 in the second paragraph.

I. Introduction. Definitions.

A finite projective plane is defined by a set of points $\mathcal{P}$ and a set of lines $\mathcal{L}$ together with a relation of incidence between a point $p \in \mathcal{P}$ and a line $\ell \in \mathcal{L}$, which satisfies the following axioms.

(a) To any two distinct points, there exists a unique line incident with both of them.

(b) To any two distinct lines, there exists a unique point incident with both of them.

(c) There exist four points of which no three are incident with the same line. Such a set of four points is called a quadrangle or a quadrilateral.

(d) There exists one line which is incident with a finite of number of points, say, $s + 1$.

Then it is known, that each line will be incident with exactly (s + 1) points, and each point is incident with (s + 1) lines. There will be exactly s^2+s+1 points and s^2+s+1 lines in the plane. s is called the order of the plane.

A point-line pair (p,ℓ) $p \in \mathcal{P}$, $\ell \in \mathcal{L}$ such that p is incident with ℓ is called a flag. There are $(s^2+s+1)(s+1)$ flags in the plane.

A *subplane* of a finite projective plane Π is a subset $\mathcal{C}$ of points and lines, which itself is a projective plane.

A closed subset $\mathcal{C}$ of Π is called a *Baer subset* or a Baer subplane in case it is a subplane, if it satisfies the following conditions:

(1) Every point of Π is incident with a line of $\mathcal{C}$.

(2) Every line of Π is incident with a point of $\mathcal{C}$.

Thus a Baer subplane if it exists is a maximal subplane of Π.

Let Π be a projective plane of order s and $\mathcal{C}$ a proper subplane of Π, of order q. Then

(i) $q^2 = s$ if and only if $\mathcal{C}$ is a Baer subplane.

and (ii) $q^2 + q \leq s$ if $\mathcal{C}$ is not a Baer subplane.

If Π is a finite Desarguesian projective plane PG(2,s) of square order that is, $s = q^2$, every subplane of Π is a Baer subplane and every quadrangle can be completed to a unique Baer subplane. The number of Baer subplanes in Π is then, $q^3(q^3+1)(q^2+1)$. The definitions and the results given above can be found in Dembowski (1968) and Cofman (1972).

An *association scheme* with m classes (or relations) consists of a finite set X of v points together with m+1 relations $R_0, R_1, \ldots, R_m$ defined on X which satisfy

(i) Each R_i is symmetric, that is, $(x,y) \in R_i \Rightarrow (y,x) \in R_i$.

(ii) For every x,y in X, $(x,y) \in R_i$ for exactly one i. Then x and y are called i-th associates.

(iii) $R_0 = \{(x,x) : x \in X\}$ is the identity relation.

(iv) If $(x,y) \in R_k$ the number of $z \in X$ such $(x,z) \in R_i$ and $(y,z) \in R_j$ is a constant p_{ij}^k depending on i,j,k but not on the particular choice of x and y.

For further properties of and results in association schemes, see Bose and Mesner (1959) and MaWilliams and Sloane (1977, Chapter 21).

Given an m class association scheme with its parameters, a partially balanced incomplete blocks (pbib) design is an arrangement of the v points in b sets (blocks) of k points each such that each point is contained in r sets and a pair of points which are i-th associates occur together in λ_i sets $i=1,\dots,m$.

2. A Three-class Association Scheme Defined on the Flags of a Finite Projective Plane of Order s

We define two flags $a = (p,\ell)$ and $b = (p',\ell')$ where p,p' are two points and ℓ,ℓ' are two lines of the plane, to be *first associates* if either $p = p'$ or $\ell = \ell'$; *second associates* if $p \neq p'$, $\ell \neq \ell'$ but either p is incident also with ℓ or p' is incident also with ℓ; *third associates*, otherwise. The number of flags in the plane is $v = (s^2+s+1)(s+1)$.

Given a flag $a = (p,\ell)$, its first associates are the flags of the types, (p,ℓ') and (p',ℓ). Since there are s lines other than ℓ, incident with p and s points other than p incident with ℓ, the number of first associates of (p,ℓ) is $n_1 = 2s$.

The second associates of the flag $a = (p,\ell)$ are the flags (p',ℓ'), $p \neq p'$, $\ell \neq \ell'$ but either p' is incident with ℓ or p is incident with ℓ'. Now there are s points other than p, on ℓ and each such point is incident with s lines other than ℓ. Again there are s lines other than ℓ incident with p and each line is incident with s points other than p. Hence the number of second associates of (p,ℓ) is $2s^2$.

The third associates of the flag $a = (p,\ell)$ are the flags (p',ℓ'), $p \neq p'$ and $\ell \neq \ell'$ and neither p is incident with ℓ' nor p' is incident with ℓ. Since there are s points other than p, incident with ℓ and s lines other than ℓ incident with each one of these points, the number of third associates of $a = (p,\ell)$ is $n_3 = s^3$.

Consider two flags $a = (p,\ell)$ and $b = (p,\ell')$, $\ell \neq \ell'$ which are first associates. Then the flags which are first associates of both a and b, are of the type (p,ℓ'') and hence $p^1_{11}(a,b) = s-1$ which is independent of a and b.

Let us calculate $p^1_{12}(a,b)$. It is clear that (p',ℓ) is a first associate of $a = (p,\ell)$ and a second associate of $b = (p,\ell')$ and there are s such flags. Hence $p^1_{12}(a,b) = s$, which is independent of a and b.

Suppose now that $a = (p,\ell)$ and $b = (p',\ell')$, $p \neq p'$, $\ell \neq \ell'$ but p is incident with ℓ', so that a and b are second associates. Let us calculate, say, $p^2_{23}(a,b)$. A flag (p'',ℓ'') where p" is incident with ℓ but $p'' \neq p$ is a second associate of $a = (p,\ell)$ and a third associate of $b = (p',\ell')$ provided ℓ'' is not the line joining p' and p". There are s(s-1) such flags. Again, a flag (p^*,ℓ^*) where ℓ^* is one of the lines through p other than ℓ and ℓ', is a second associate of a and a third associate of b. And there are s(s-1) such flags. Hence $p^2_{23}(a,b) = 2s(s-1)$. In this manner, we have calculated all the p^i_{jk}'s, i,j,k = 1,2,3, which are given below:

$$P_1 = (p^1_{ij}) = \begin{bmatrix} s-1 & s & 0 \\ s & s(s-1) & s^2 \\ 0 & s^2 & (s-1)s^2 \end{bmatrix},$$

$$P_2 = (p_{ij}^2) = \begin{bmatrix} 1 & s-1 & s \\ s-1 & s & 2s(s-1) \\ s & 2s(s-1) & s(s-1)^2 \end{bmatrix},$$

$$P_3 = (p_{ij}^3) = \begin{bmatrix} 0 & 2 & 2(s-1) \\ 2 & 4(s-1) & 2(s-1)^2 \\ 2(s-1) & 2(s-1)^2 & (s-1)(s^2-s+1) \end{bmatrix}.$$

3. Incidence of Flags and Baer Subplanes in $PG(2,q^2)$. A pbib Design.

There are $q^3(q^3+1)(q^2+1)$ Baer subplanes, each of order q, in $PG(2,q^2)$. We also know that each quadrangle can be completed to a unique Baer subplane.

Each Baer subplane consists of $(q^2+q+1)(q+1)$ flags. Now given a flag (p,ℓ) the number of quadrangles in $PG(2,q^2)$, which include (p,ℓ) is $q^2 \cdot q^4(q^2-1)^2 = q^6(q^2-1)^2$. But each Baer subplane of order q, which includes (p,ℓ), has $q \cdot q^2(q-1)^2 = q^3(q-1)^2$ distinct quadrangles. Hence the number of Baer subplanes incident with a given flag (p,ℓ) is $q^6(q^2-1)^2/q^3(q-1)^2 = q^3(q+1)^2 = r$ (say).

Now, consider two flags a and b which are first associates, say,

$$a = (p,\ell) \quad \text{and} \quad b = (p',\ell), \quad p \neq p'.$$

The number of quadrangles in $PG(2,q^2)$ which include the flags a and b is $q^4(q^2-1)^2$ and the number of quadrangles in a Baer subplane which includes a and b is $q^2(q-1)^2$. Hence the number of Baer subplanes which are incident with both the flags $a = (p,\ell)$ and $b = (p',\ell)$ is $\lambda_1 = q^4(q^2-1)^2/q^2(q-1)^2 = q^2(q+1)^2$.

Let $a = (p,\ell)$ and $c = (p',\ell')$, $p \neq p'$, $k \neq \ell'$ be two flags which are second associates and, say, p is incident with ℓ'. Then the number of quadrangles which include both a and c is $q^2(q^2-1)^2$ and the number of quadrangles in a Baer subplane which includes both a and c is $q(q-1)^2$. Hence a and c occur together in $\lambda_2 = q^2(q^2-1)^2/q(q-1)^2 = q(q+1)^2$.

Now consider two flags $a = (p, \ell)$ and $d = (p', \ell')$, $p \neq p'$, $\ell \neq \ell'$, p not incident with ℓ' and p' not incident with ℓ, so that a and d are third associates. Suppose ℓ and ℓ' meet at the point $p'' \neq p$, $p'' \neq p'$. Any quadrangle which includes the flags (p, ℓ) and (p', ℓ') must also include p" the point of intersection of ℓ and ℓ'. Hence the number of quadrangles in $PG(2, s^2)$ which include both a and d is $(q^2-1)^2$ and the number of quadrangles in a Baer subplane which include both a and d is $(q-1)^2$. Hence a and d which are third associates, occur together in $\lambda_3 = (q^2-1)^2/(q-1)^2 = (q+1)^2$.

Thus the incidence of the flags and the Baer subplanes in $PG(2, q^2)$ defines a partially balanced incomplete block (pbib) design with $v = (q^4+q^2+1)(q^2+1)$, $b = q^3(q^3+1)(q^2+1)$, $r = q^3(q+1)^2$, $k = (q^2+q+1)(q+1)$, $\lambda_1 = q^2(q+1)^2$, $\lambda_2 = q(q+1)^2$, $\lambda_3 = (q+1)^2$ with the same association matrices as in Section 2, where s is to be replaced by q^2.

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