No. 1796
8/1989

A Probabilistic Approach to Domains of Partial Attraction *

SÁNDOR CSÖRGŐ

Bolyai Institute, University of Szeged, Szeged, Hungary

1. INTRODUCTION AND RESULTS

A new unified approach to one of the most classical problems of probability theory, the problem of the asymptotic distribution of sums of independent and identically distributed random variables, was recently presented in [6]. This approach, based upon the asymptotic behavior of the uniform empirical distribution function in conjunction with the tail properties of the underlying quantile function, identifies the portions of the sums that contrubute the ingredients of the limiting infinitely divisible law, shows clearly how these ingredients arise, delineates the effect of extreme values on the limiting distribution, and leads to a probabilistic representation for an arbitrary infinitely divisible real random variable. Since in the meantime Sato's bounds for the tail probabilities of infinitely divisible distributions, used in the proof of Corollary 1 in [6], have been derived among other things in [9] from exactly this representation, our approach is now fully self-contained and uses Fourier analysis only to ensure the uniqueness of this probabilistic representation, given in Theorem 3 in [6], through the uniqueness of the Lévy canonical form of the characteristic function of an infinitely divisible law.

Even though the main theorems in [6] were strong enough to derive the basic results on domains of attraction and stochastic compactness in Corallaries 1, 3, 4, 10, 11 and 12, and some results concerning domains of partial attraction in Corollaries 2, 5, 7, 8 and 9, there are a number of important classical or potentially new results on domains of partial attraction which do not follow from them as they stand in [6]. The main reason for this is that the construction of the sequence $\{r_n\}$ in Theorem 1 in [6] appears to be complicated and hence this theorem does not contain an analytic criterion for the choice of the variance σ^2 of the normal component of the limiting law. Also, some classical results of Doeblin and Gnedenko were felt to be "out of the reach" of this approach altogether (cf. the second paragraph on p. 328 in [6]).

The present paper is an organic continuation of [6], completing the theory of our 'probabilistic approach'. We begin with augmenting Theorem 1 in [6] by showing that the

* Work partially supported by the Hungarian National Foundation for Scientific Research, Grants 1808/86 and 457/88.

sequence $\{r_n\}$ can be constructed, or in fact is already constructed in such a way that allows an analytic description of σ^2. Also, starting out from the probabilistic representation of an infinitely divisible random variable obtained in [6], we show that not only all of the mentioned classical results of Doeblin and Gnedenko are well within the present approach, but their present formulations and purely probabilistic proofs shed more light on their essence and the most interesting of them can considerably be improved. A number of seemingly new results are also derived. Considering that the constructional problems mentioned on p. 328 in [6] are also solved in [10], the present paper virtually completes the theory in its essential lines.

This theory has been designed primarily for sums of independent and identically distributed random variables, or, more generally, for various sums of order statistics of such variables (cf. [6], [7], and [8]). However, the formulation and the purely probabilistic proof of Theorem 11 below for the convergence of an arbitrary sequence of infinitely divisible distributions opens the door for the problem of the asymptotic distribution of row sums of row-wise independent but not identically distributed infinitesimal random variables in an arbitrary triangular array to be included into the theory. Indeed, this theorem is a variant of the purely Fourier-analytic Theorem 2 on pp. 88-92 in [14], one of the core results in that book, and using then accompanying infinitely divisible laws (p. 112 and Chapters 4-6 in [14]) this inclusion becomes feasible.

First we review the basic notation from [6] and then state the results. The proofs are in Section 2, and the results and their place in the literutere are discussed in Section 3. Theorem 1* and Corollary 5* below are completed or improved forms of Theorem 1 and Corollary 5 in [6]. Emphasizing that this paper is a continuation of [6], the numbering of the theorems and corollaries here continues that of in [6]. Whenever there is a reference to any one of Theorems 1-5 or Corollaries 1-12 without a reference number, we refer to the corresponding result in [6].

Let $X_1, X_2, \ldots$ be a sequence of independent random variables with a common (right-continuous) non-degenerate distribution function F and quantile function $Q(s) = \inf\{x : F(x) \le s\}$, $0 < s \le 1$, $Q(0) = Q(0+)$, for each integer $n \ge 1$, let $X_{1,n} \le \ldots \le X_{n,n}$ denote the order statistics based on $X_1, \ldots, X_n$, and for $0 < s < 1/2$, introduce

$$\sigma^2(s) = \int_s^{1-s} \int_s^{1-s} (u \wedge v - uv) dQ(u) dQ(v), \tag{1.1}$$

where $u \wedge v = \min(u, v)$. Then for all n large enough the quantity

$$a(n) = n^{1/2} \sigma(1/n) \tag{1.2}$$

is positive. Let $\alpha_n > 0$ be any sequence such that $\alpha_n \downarrow 0$ and $n\alpha_n \to 0$ as $n \to \infty$, and consider the functions

$$\psi_1(n,s) = \begin{cases} Q(s/n+)/a(n), & 0 < s \le n - n\alpha_n, \\ Q((1-\alpha_n)+)/a(n), & n - n\alpha_n < s < \infty, \end{cases}$$

and

$$\psi_2(n,s) = \begin{cases} -Q(1-s/n)/a(n), & 0 < s \le n - n\alpha n, \\ -Q(\alpha_n)/a(n), & n - n\alpha_n < s < \infty. \end{cases}$$

Let $E_1^{(j)}, E_2^{(j)}, \ldots; j = 1,2$, be two independent sequences of independent exponentially distributed random variables with mean 1 and with their partial sums $S_n^{(j)} = E_1^{(j)} + \ldots + E_n^{(j)}$, $n \ge 1$, $j = 1,2$, as jump points, consider the standard left-continuous independent Poisson processes

$$N_j(u) = \sum_{n=1}^{\infty} I(S_n^{(j)} < u), \quad 0 \le u < \infty, \ j = 1,2, \tag{1.3}$$

where $I(\cdot)$ is the indicator function. Considering also two non-decreasing, non-positive, right-continuous functions ψ_1 and ψ_2 on $(0,\infty)$ such that

$$\int_\varepsilon^\infty \psi_j^2(u)du < \infty, \quad \text{for all} \quad \varepsilon > 0, \quad j = 1,2, \tag{1.4}$$

define the random variables

$$V_m^{(1)} = \int_{S_{m+1}^{(1)}}^\infty (u - N_1(u))d\psi_1(u) + \int_1^{S_{m+1}^{(1)}} u d\psi_1(u) - m\psi_1(S_{m+1}^{(1)}) + \int_1^{m+1} \psi_1(u)du + \psi_1(1)$$

and

$$V_k^{(2)} = -\int_{S_{k+1}^{(2)}}^\infty (u - N_2(u))d\psi_2(u) - \int_1^{S_{k+1}^{(2)}} u d\psi_2(u) + k\psi_2(S_{k+1}^{(2)}) - \int_1^{k+1} \psi_2(u)du - \psi_2(1),$$

where $m \ge 0$ and $k \ge 0$ are arbitrary integers. These variables are non-degenerate if $\psi_1 \not\equiv 0$ and $\psi_2 \not\equiv 0$, respectively. Let $N(\mu,\sigma^2)$ denote a normal random variable with mean μ and variance σ^2, understood to be the constant μ if $\sigma = 0$, let $\to_D$ and $\to_P$ denote convergence in distribution and in probability, respectively, and finally, let $\Rightarrow$ denote weak convergence of functions, that is, pointwise convergence on $(0,\infty)$ at each continuity point of the limiting function.

The following is a completed form of Theorem 1, where the additional results appear in (1.6)-(1.10) holding along the original given subsequence $\{n'\}$.

THEOREM 1*. *Assume that there exists a subsequence $\{n'\}$ of the positive integers $\{n\}$ such that for two non-decreasing, non-positive, right-continuous functions ψ_1 and ψ_2 defined on $(0,\infty)$ we have*

$$\psi_j(n',\cdot) \Rightarrow \psi_j(\cdot),\ j=1,2,\ \text{as } n' \to \infty.$$

(i) If $\psi_1 = \psi_2 \equiv 0$, then for all fixed $m \geq 0$ and $k \geq 0$, as $n' \to \infty$,

$$\frac{1}{a(n')}\left\{\sum_{j=m+1}^{n'-k} X_{j,n'} - n'\int_{(m+1)/n'}^{1-(k+1)/n'} Q(u)du\right\} \to_D N(0,1).$$

(ii) If the limits ψ_1 and ψ_2 are arbitrary, they necessarily satisfy (1.4) and there exist two sequences $\{l_{n'}\}$ and $\{r_{n'}\}$ of positive integers such that, as $n' \to \infty$, $l_{n'} \to \infty$, $r_{n'}/n' \to 0$, $l_{n'}/r_{n'} \to 0$, and for any pair of fixed $m \geq 0$ and $k \geq 0$,

$$\begin{aligned}
&\frac{1}{a(n')}\left\{\sum_{j=m+1}^{l_{n'}} X_{j,n'} - n'\int_{(m+1)/n'}^{(l_{n'}+1)/n'} Q(u)du\right\} \to_D V_m^{(1)},\\
&\frac{1}{a(n')}\left\{\sum_{j=l_{n'}+1}^{r_{n'}} X_{j,n'} - n'\int_{(l_{n'}+1)/n'}^{(r_{n'}+1)/n'} Q(u)du\right\} \to_P 0,\\
&\frac{1}{a(n')}\left\{\sum_{j=n'-r_{n'}+1}^{n'-l_{n'}} X_{j,n'} - n'\int_{1-(r_{n'}+1)/n'}^{1-(l_{n'}+1)/n'} Q(u)du\right\} \to_P 0,\\
&\frac{1}{a(n')}\left\{\sum_{j=n'-l_{n'}+1}^{n'-k} X_{j,n'} - n'\int_{1-(l_{n'}+1)/n'}^{1-(k+1)/n'} Q(u)du\right\} \to_D V_k^{(2)},
\end{aligned} \tag{1.5}$$

and

$$\liminf_{n'\to\infty} \frac{\sigma((r_{n'}+1)/n')}{\sigma(1/n')} = \underline{\sigma} \quad \text{and} \quad \limsup_{n'\to\infty} \frac{\sigma((r_{n'}+1)/n')}{\sigma(1/n')} = \overline{\sigma} \tag{1.6}$$

where $0 \leq \underline{\sigma} = \underline{\sigma}_{\{n'\}} \leq \overline{\sigma}_{\{n'\}} = \overline{\sigma} \leq 1$ are defined as

$$\underline{\sigma} = \lim_{h\to\infty} \liminf_{n'\to\infty} \frac{\sigma(h/n')}{\sigma(1/n')} \quad \text{and} \quad \overline{\sigma} = \lim_{h\to\infty} \limsup_{n'\to\infty} \frac{\sigma(h/n)}{\sigma(1/n')}. \tag{1.7}$$

If $\underline{\sigma} = \overline{\sigma} = \sigma$, then, as $n' \to \infty$

$$\frac{1}{a(n')}\left\{\sum_{j=r_{n'}+1}^{n'-r_{n'}} X_{j,n'} - n'\int_{(r_{n'}+1)/n'}^{1-(r_{n'}+1)/n'} Q(u)du\right\} \to_D \sigma Z, \tag{1.8}$$

where Z is a standard normal random variable such that $N_1(\cdot), Z$, and $N_2(\cdot)$ are independent, and hence

$$\frac{1}{a(n')}\left\{\sum_{j=m+1}^{n'-k} X_{j,n'} - n'\int_{(m+1)/n'}^{1-(k+1)/n'} Q(u)du\right\} \to_D V_{m,k}(\psi_1,\psi_2,\sigma) := V_m^{(1)} + \sigma Z + V_k^{(2)}. \tag{1.9}$$

Moreover, whenever $\sigma > 0$, we have, as $n' \to \infty$,

$$\sigma((l_{n'}+1)/n')/\sigma((r_{n'}+1)/n') \to 1. \tag{1.10}$$

If $0 \le \underline{\sigma} < \bar{\sigma} \le 1$, then for some limit point $\sigma \in [\underline{\sigma}, \bar{\sigma}]$ of the sequence $\sigma((r_{n'} + 1)/n')/\sigma(1/n')$ and for some subsequence$\{n''\} \subset \{n'\}$ we have

$$\sigma((r_{n''}+1)/n'')/\sigma(1/n'') \to \sigma \quad \text{as} \quad n'' \to \infty,$$

and the conclusions (1.8), (1,9) and (1.10) holding along $\{n''\}$.

We know from Theorem 5 that if $\{\sum_{j=1}^{n_k} X_j - C_{n_k}\}/A_{n_k}$ converges in distribution along some subsequence $\{n_k\}_{k=1}^{\infty}$ of the positive integers, where $A_{n_k} > 0$ and C_{n_k} are some constants, then what we have is

$$\frac{1}{A_{n_k}}\left\{\sum_{j=1}^{n_k} X_j - C_{n_k}\right\} \to_D \alpha V_{0,0}(\psi_1,\psi_2,\sigma) + \beta \quad \text{as} \quad k \to \infty, \tag{1.11}$$

where $\alpha > 0$, $\beta \in \mathbb{R}$, and $V_{0,0}(\psi_1,\psi_2,\sigma)$, with some ψ_1 and ψ_2 described before and in (1.4) and $0 \le \sigma \le 1$, is defined in (1.9). In this case, we write $F \in D_p(\psi_1,\psi_2,\sigma)$ to denote that F is in the domain of partial attraction of the infinitely divisible law determined by the triple (ψ_1,ψ_2,σ). The next theorem gives a thorough characterization of this relation.

THEOREM 6. *(i) If $\sigma > 0$, then (1.11) holds along some $\{n_k\}_{k=1}^{\infty}$ if and only if there exist a subsequence $\{n'_k\}_{k=1}^{\infty}$ and a $\delta \ge \alpha\sigma > 0$ such that*

$$\psi_j(n'_k,\cdot) \Rightarrow \frac{\alpha}{\delta}\psi_j(\cdot), \quad j = 1,2, \tag{1.12}$$

$$a(n'_k)/A_{n'_k} \to \delta, \tag{1.13}$$

$$\left\{n'_k\int_{1/n'_k}^{1-1/n'_k} Q(u)du - C_{n'_k}\right\}\Big/A_{n'_k} \to \frac{\beta}{\delta} \tag{1.14}$$

as $k \to \infty$, and

$$\lim_{h\to\infty} \liminf_{k\to\infty} \frac{\sigma(h/n'_k)}{\sigma(1/n'_k)} = \frac{\alpha}{\delta}\sigma = \lim_{h\to\infty} \limsup_{k\to\infty} \frac{\sigma(h/n'_k)}{\sigma(1/n'_k)}. \tag{1.15}$$

If (1.11) holds with $\psi_1 = \psi_2 \equiv 0$ (that is, if we talk about the domain of partial attraction of a normal law) then, necessarily, $\delta = \alpha\sigma$ and (1.12)-(1.15) all hold with $\alpha\sigma/\delta = 1$ along the original $\{n_k\}_{k=1}^{\infty}$ for which (1.11) is true. If one of ψ_1 and ψ_2 is not identically zero, then, having (1.11), every subsequence of $\{n_k\}_{k=1}^{\infty}$ contains a further subsequence $\{n'_k\}_{k=1}^{\infty}$ for which (1.12)-(1.15) all hold but δ may in general depend on the chosen subsequence $\{n'_k\}_{k=1}^{\infty}$.

(ii) If $\sigma = 0$ and at least one of ψ_1 and ψ_2 is not identically zero on $[1,\infty)$, then (1.11) holds along some $\{n_k\}_{k=1}^{\infty}$ if and only if there exist a subsequence $\{n'_k\}_{k=1}^{\infty}$ and a $\delta > 0$ such that we have (1.12)-(1.14) and

$$\lim_{h\to\infty} \limsup_{k\to\infty} \sigma(h/n'_k)/\sigma(1/n'_k) = 0. \tag{1.16}$$

If (1.11) holds, then (1.12)-(1.14) and (1.16) generally take place only along subsequences of the given $\{n_k\}_{k=1}^{\infty}$ again, with δ depending on the chosen subsequence.

(iii) If $\sigma = 0$ and both ψ_1 and ψ_2 are identically zero on $[1,\infty)$, then (1.11) holds along some $\{n_k\}_{k=1}^{\infty}$ if and only if either there exists a subsequence $\{n'_k\}_{k=1}^{\infty}$ such that we have (1.12)-(1.14) and (1.16) with some $\delta > 0$, or

$$\frac{a(n'_k)}{A_{n'_k}} \psi_j(n'_k, \cdot) \Rightarrow \alpha\psi_j(\cdot), \quad j = 1,2, \tag{1.17}$$

$$a(n'_k)/A_{n'_k} \to 0, \tag{1.18}$$

and

$$\left\{ n'_k \int_{1/n'_k}^{1-1/n'_k} Q(u)du - C_{n'_k} \right\} \Big/ A_{n'_k} \to \beta. \tag{1.19}$$

If (1.11) holds then again the conclusions take place only along subsequenes of the given $\{n_k\}_{k=1}^{\infty}$.

On the other hand, if any of these sets of conditions is satisfied along some $\{n'_k\}_{k=1}^{\infty}$, then we have (1.11) along the same $\{n'_k\}_{k=1}^{\infty}$ and, besides, in cases (i), (ii) and the first subcase of case (iii) we also have (1.5) and (1.8)-(1.10) along $\{n'_k\}$, while in the second *subcase of case (iii) we also have (1.13)-(1.15) of Theorem 2 in [6], along $\{n'_k\}$, with the appropriate limiting functions taken from (1.12) or (1.17), respectively.*

We note that it is possible to replace limsup by liminf in (1.16), but then we generally have to go down to a subsequence also in the sufficiency direction.

The normalizing and centering constants are fixed in the above theorem. If we do not require this, combine cases (i) and (ii), and are not interested in particular subsequences, the following simplified form is perhaps more transparent.

THEOREM 7. *(a) If $\sigma > 0$, or $\sigma = 0$ but at least one of ψ_1 and ψ_2 is not identically zero on $[1, \infty)$, then $F \in D_p(\psi_1, \psi_2, \sigma)$ if and only if there exist a subsequence $\{n'\}$ of the positive integers and a constant $c > 0$ such that $c\sigma \leq 1$ and*

$$\psi_j(n', \cdot) \Rightarrow c\psi_j(\cdot), \quad j = 1, 2, \tag{1.20}$$

as $n' \to \infty$ and

$$\lim_{h\to\infty} \liminf_{n'\to\infty} \frac{\sigma(h/n')}{\sigma(1/n')} = c\sigma = \lim_{h\to\infty} \limsup_{n'\to\infty} \frac{\sigma(h/n')}{\sigma(1/n')}. \tag{1.21}$$

(b) If both ψ_1 and ψ_2 are identically zero on $[1, \infty)$, then $F \in D_p(\psi_1, \psi_2, 0)$ if and only if there exists an $\{n'\}$ such that either we have (1.20) and (1.21) with some $c > 0$ and $\sigma = 0$, or we have

$$\frac{a(n')}{A_{n'}} \psi_j(n', \cdot) \Rightarrow \psi_j(\cdot), \quad j = 1, 2,$$

and $a(n')/A_{n'} \to 0$ as $n' \to \infty$.

Theorem 6 or 7 readily implies the following improved form of Corollary 5 for the characterization of the domain of partial attraction $D_p(\alpha)$ of a stable distribution with exponent $\alpha \in (0, 2)$.

COROLLARY 5*. *$F \in D_p(\alpha)$ if and only if there exist a subsequence $\{n'\}$ of $\{n\}$ and constants $c_1, c_2 \geq 0$, $c_1 + c_2 > 0$, such that*

$$\psi_j(n', s) \to^{(\alpha)} \psi_j(s) := -c_j s^{-1/\alpha}, \quad 0 < s < \infty, \ j = 1, 2,$$

and

$$\lim_{h\to\infty} \limsup_{n'\to\infty} \sigma(h/n')/\sigma(1/n') = 0. \tag{1.22}$$

In this case, for each fixed $m \geq 0$ and $k \geq 0$ we have

$$\frac{1}{a(n')} \left\{ \sum_{j=m+1}^{n'-k} X_{j,n'} - n' \int_{(m+1)/n'}^{1-(k+1)/n'} Q(u)du \right\} \to_D V_{m,k}({}^{(\alpha)}\psi_1, {}^{(\alpha)}\psi_2, 0)$$

as $n' \to \infty$.

In order to further illustrate Theorems 6 and 7, we consider two more examples. For $\alpha > 0$ and $\lambda > 0$, let $Y_{\alpha,\lambda}$ denote a random variable with the gamma distribution of order α and parameter λ:

$$F_{\alpha,\lambda}(x) = P\{Y_{\alpha,\lambda} \le x\} = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^x e^{-\lambda t} t^{\alpha-1} dt, & x \ge 0, \\ 0, & x < 0, \end{cases}$$

where $\Gamma(\cdot)$ is the usual gamma function. Consider the function $h_{\alpha,\lambda}(s) > 0$, $s > 0$, which is defined to be the unique solution $x > 0$ of the equation

$$\int_{\lambda x}^{\infty} e^{-t} t^{-1} dt = \frac{s}{\alpha},$$

and set $\psi_{\alpha,\lambda}(s) = -h_{\alpha,\lambda}(s)$, $s > 0$. Introduce also the constant

$$C_{\alpha,\lambda} = \alpha \left\{ \int_0^\infty \frac{e^{-\lambda t}}{1+t^2} dt - \int_{-\infty}^{\psi_{\alpha,\lambda}(1)} \frac{e^t}{1+t^2} dt + \int_{\psi_{\alpha,\lambda}(1)}^{0} \frac{t^2 e^t}{1+t^2} dt \right\}.$$

For a different purpose it has been checked in [2] that $V_{m,k}(0, \psi_{\alpha,\lambda}, 0) = V_{0,k}(0, \psi_{\alpha,\lambda}, 0) = Y_{\alpha,\lambda}(k)$ for any integers $m \ge 0$ and $k \ge 0$, where

$$Y_{\alpha,\lambda}(k) = -\sum_{l=k+1}^{\infty} \psi_{\alpha,\lambda}(S_l) - \alpha(1 - e^{\psi_{\alpha,\lambda}(k+1)}) + C_{\alpha,\lambda}, \tag{1.23}$$

where S_l is obtained by dropping the superscript $j = 2$ in (1.3), and we have the distributional equality

$$Y_{\alpha,\lambda}(0) =_D Y_{\alpha,\lambda}. \tag{1.24}$$

(Here we use the occasion to correct a misprint in [2]: in the rate formula for the approximation of $F_{\alpha,\lambda}$ we should have $-h_{\alpha,\lambda}(k+1)$ instead of $-h_{\alpha,\lambda}(k)$ in the exponent.) Using now (1.23) and (1.24), case (ii) of Theorem 6 at once gives the following.

COROLLARY 13. *$F \in D_p\ (F_{\alpha,\lambda})$ if and only if there exists a subsequence $\{n'\} \subset \{n\}$ such that for each $s > 0$,*

$$Q(s/n')/a(n') \to 0 \quad \text{and} \quad Q(1 - s/n')/a(n') \to \beta\psi_{\alpha,\lambda}(s)$$

as $n' \to \infty$, where $\beta > 0$ is some constant, and (1.22) holds. In this case there exists a sequence $\{l_{n'}\}$ of integers such that $l_{n'} \to \infty$, $l_{n'}/n' \to 0$,

$$\frac{1}{a(n')} \left\{ \sum_{j=1}^{n'-l_{n'}} X_{j,n'} - n' \int_{1/n'}^{1-(l_{n'}+1)/n'} Q(u)du \right\} \to_P 0 \tag{1.25}$$

and

$$\frac{1}{a(n')} \left\{ \sum_{j=n'-l_{n'}+1}^{n'-k} X_{j,n'} - n' \int_{1-(l_{n'}+1)/n'}^{1-(k+1)/n'} Q(u)du \right\} \to_D \beta Y_{\alpha,\lambda}(k)$$

for each fixed $k \ge 0$ as $n' \to \infty$.

The second example is the Geometric (p) distribution

$$P\{Y_p = n\} = (1-p)p^n, \quad n = 0,1,2,\dots$$

with parameter $0 < p < 1$. Set

$$\psi_p(s) = \begin{cases} -\sum_{n=1}^{\infty} nI(\log\frac{1}{1-p} - \sum_{j=1}^{n}\frac{p^j}{j} \le s < \log\frac{1}{1-p} - \sum_{j=1}^{n-1}\frac{p^j}{j}) & , 0 < s < \log\frac{1}{1-p}, \\ 0 & , s \ge \log\frac{1}{1-p}. \end{cases}$$

Again, it has been checked in [2] that $V_{m,k}(0,\psi_p,0) = V_{0,k}(0,\psi_p,0) = Y_p(k)$ for any integers $m \ge 0$ and $k \ge 0$, where,

$$Y_p(k) = -\sum_{j=k+1}^{\infty}\psi_p(S_j) \quad \text{and} \quad Y_p(0) =_{\mathcal{D}} Y_p.$$

Using these relations, since $-\log(1-p) > 1$, case (ii) of Theorem 6 again gives the following.

COROLLARY 14. *$F \in D_p$ (Geometric (p)) if and only if there exist a subsequence $\{n'\} \subset \{n\}$ and a constant $\alpha > 0$ such that $Q(s/n')/a(n') \to 0$ for each $s > 0$,*

$$Q(1 - \frac{\cdot}{n'})/a(n') \Rightarrow \alpha\psi_p(\cdot),$$

and (1.22) holds. In this case there exists a sequence $\{l_{n'}\}$ of integers such that $l_{n'} \to \infty$, $l_{n'}/n' \to \infty$, we have (1.25), and

$$\frac{1}{a(n')}\left\{\sum_{j=n'-l_{n'}+1}^{n'-k} X_{j,n'} - n'\int_{1-(l_{n'}+1)/n'}^{1-(k+1)/n'} Q(u)du\right\} \to_{\mathcal{D}} \alpha Y_p(k)$$

for each fixed $k \ge 0$ as $n' \to \infty$.

Perhaps the most interesting example is the domain of partial attraction of a Poisson law with mean $\lambda > 0$. This example presents special intricate problems which are solved in [3]. When $\lambda > 1$, we are in case (ii) of Theorem 6. However, as a complete surprise it turns out that when $\lambda < 1$, we are always in the second subcase of case (iii). The latter situation provides the simplest possible example to show that Theorem 2 is not empty. The situation when $\lambda = 1$ turns out to be a real borderline case in that both alternatives of case (iii) of Theorem 6 may occur. For details we refer to [3].

Now we turn to qualitative and deeper results on domains of partial attraction. All of these will depend upon the probabilistic representation of an infinitely divisible random variable in Theorem 3. For the Poisson processes in (1.3) and a ψ function satisfying the conditions above and in (1.4), introduce

$$W_j(\psi) = \int_1^\infty (N_j(s) - s)d\psi(s) + \int_0^1 N_j(s)d\psi(s) + \Theta(\psi), \quad j = 1,2, \tag{1.26}$$

where

$$\Theta(\psi) = -\psi(1) + \int_0^1 \frac{\psi(s)}{1+\psi^2(s)}ds - \int_1^\infty \frac{\psi^3(s)}{1+\psi^2(s)}ds,$$

and consider

$$V(\psi_1,\psi_2,\sigma) = -W_1(\psi_1) + \sigma Z + W_2(\psi_2), \tag{1.27}$$

where ψ_1, ψ_2 are two such functions, $\sigma \geq 0$, and Z is a standard normal random variable such that $N_1(\cdot), Z$, and $N_2(\cdot)$ are independent. Note that with the notation in (1.9) we have

$$V(\psi_1,\psi_2,\sigma) = V_{0,0}(\psi_1,\psi_2,\sigma) - \Theta(\psi_1) - \psi_1(1) + \Theta(\psi_2) + \psi_2(1) \tag{1.28}$$

by elementary computation, and hence by Theorem 3,

$$Ee^{itV(\psi_1,\psi_2,\sigma)} = \exp\left\{ -\frac{\sigma^2}{2}t^2 + \int_{-\infty}^0 (e^{itx} - 1 - \frac{itx}{1+x^2})dL(x) \right.$$
$$\left. + \int_0^\infty (e^{itx} - 1 - \frac{itx}{1+x^2})dR(x) \right\}$$

for all $t \in \mathbb{R}$, where $L(x) = \inf\{s > 0 : \psi_1(s) \geq x\}$, $x < 0$, and $R(x) = -\inf\{s > 0 : \psi_2(s) \geq -x\}$, $x > 0$, $L(\cdot)$ is left-continuous and non-decreasing on $(-\infty, 0)$ with $L(-\infty) = 0$ and $R(\cdot)$ is right-continuous and non-decreasing on $(0,\infty)$ with $R(\infty) = 0$ and by (1.4),

$$\int_{-\varepsilon}^0 x^2 dL(x) + \int_0^\varepsilon x^2 dR(x) < \infty \quad \text{for any} \quad \varepsilon > 0.$$

Thus, modulo an additive constant, $V(\psi_1,\psi_2,\sigma)$ represents an arbitrary infinitely divisible random variable by p. 84 in [14].

A basic role will be played by the rescaled functions

$$\psi^{(r)}(s) = \psi(\frac{s}{r}), \quad s > 0, \tag{1.29}$$

where r is an arbitrary positive number. This is because if $(N_1^{(1)}(\cdot),\ Z^{(1)},\ N_2^{(1)}(\cdot)),\ldots,$ $(N_1^{(r)}(\cdot),\ Z^{(r)},\ N_2^{(r)}(\cdot))$ are independent copies of $(N_1(\cdot),\ Z,\ N_2(\cdot))$ and $V_1(\psi_1,\psi_2,\sigma),\ldots,$ $V_r(\psi_1,\psi_2,\sigma)$ are the corresponding independent copies of $V(\psi_1,\psi_2,\sigma)$ determined through (1.27), then by Lemma 2.1 below,

$$\sum_{l=1}^r V_l(\psi_1,\psi_2,\sigma) =_D V(\psi_1^{(r)},\psi_2^{(r)},\sqrt{r}\sigma). \tag{1.30}$$

A key result is then the following.

THEOREM 8. *If $F \in D_p(\psi_1, \psi_2, \sigma)$, then $F \in D_p(\psi_1^{(\tau)}, \psi_2^{(\tau)}, \sqrt{\tau}\sigma)$ for any $\tau > 0$, and if, furthermore,*

$$\frac{1}{a_m}\{V(\psi_1^{(\tau_m)}, \psi_2^{(\tau_m)}, \sqrt{\tau_m}\sigma) - c_m\} \to_D V(\psi_1^*, \psi_2^*, \sigma_*) \tag{1.31}$$

for some sequences $\tau_m > 0$, $a_m > 0$, and $c_m \in \mathbb{R}$, as $m \to \infty$, where $\sigma_* \geq 0$ and the functions ψ_1^* and ψ_2^* satisfy the conditions above and in (1.4), then also $F \in D_p(\psi_1^*, \psi_2^*, \sigma_*)$.

Let $F^{\psi_1, \psi_2, \sigma}(\cdot)$ denote the distribution function of $V(\psi_1, \psi_2, \sigma)$. Although the following result is just a simple corollary to Theorem 8, we call it a theorem in view of its special interest.

THEOREM 9. *(Gnedenko's transitivity theorem). If $F \in D_p(\psi_1, \psi_2, \sigma)$ and $F^{\psi_1, \psi_2, \sigma} \in D_p(\psi_1^*, \psi_2^*, \sigma_*)$, then $F \in D_p(\psi_1^*, \psi_2^*, \sigma_*)$.*

Introducing the class

$$D_p^{-1}(F) = \{(\psi_1, \psi_2, \sigma) : F \in D_p(\psi_1, \psi_2, \sigma)\}$$

of infinitely divisible distributions, or types rather, that partially attract F and denoting by F^{*r} the r^{th} convolution power of F, another consequence of Theorem 8 is the following result.

COROLLARY 15. *We have $D_p^{-1}(F) = D_p^{-1}(F^{*r})$ for any $r = 1, 2, \ldots$.*

As usual ([14], p. 40) we say that the distribution functions G_1 and G_2 are of the same type if $G_2(x) = G_1(ax + b), x \in \mathbb{R}$, for some constants $a > 0$ and $b \in \mathbb{R}$. Since the property of being of the same type is symmetric and transitive, the set of all distribution functions (of the cardinality of the continiuum) can be decomposed into mutually disjoint families of distribution functions, each family consisting of distribution functions that are of the same type, that is, into mutually disjoint types. Of course the same can be said about the subset of all infinitely divisible distribution functions, a type being determined by a triple (ψ_1, ψ_2, σ). Since if F belongs to the domain of partial attraction of an infinitely divisible distribution function then it also belongs to the domain of partial attraction of any other distribution function of the same type, the property of partially attracting an F is, as tacitly used throughout above, the property of the corresponding infinitely divisible type. By the remark on p. 270 in [6], the non-normal stable type with exponent $0 < \alpha < 2$ is given by the triple $({}^{(\alpha)}\psi_1, {}^{(\alpha)}\psi_2, 0)$, defined in Corollary 5* above, while the normal type, usually called the stable type with exponent 2, is given by the triple $(0, 0, \sigma), \sigma > 0$.

THEOREM 10. *(i) If F belongs to the domain of partial attraction of only one type, then this type must be a stable type with some exponent $0 < \alpha \leq 2$.*

(ii) If F belongs to the domain of partial attraction of a non-stable type, then it belongs to the domain of partial attraction of continuum many different types.

The only reason to include the constants $\Theta(\psi_1)$ and $\Theta(\psi_2)$, defined after (1.26), into the difinition of $V(\psi_1, \psi_2, \sigma)$ in (1.27) was to have the equality (1.30) in the given nice form. The follwing result, the general convergence theorem for infinitely divisible laws, is more transparent if we leave these constants aside, and hence introduce (cf. (1.28))

$$\begin{aligned} \overline{V}(\psi_1, \psi_2, \sigma) =& V(\psi_1, \psi_2, \sigma) + \Theta(\psi_1) - \Theta(\psi_2) \\ =& V_{0,0}(\psi_1, \psi_2, \sigma) - \psi_1(1) + \psi_2(1) \\ =& -\int_1^\infty (N_1(s) - s) d\psi_1(s) - \int_0^1 N_1(s) d\psi_1(s) \\ &+ \sigma Z \\ &+ \int_1^\infty (N_2(s) - s) d\psi_2(s) + \int_0^1 N_2(s) d\psi_2(s) \end{aligned} \tag{1.32}$$

As so far above, we assume in what follows, unless the contrary is evident from the text, that all occuring ψ functions (with subscripts and/or special marks such as ψ^* or $\overline{\psi}$) satisfy the usual conditions above and in (1.4). Let ψ_1, ψ_2, $\{\psi_{1k}\}_{k=1}^\infty$ and $\{\psi_{2k}\}_{k=1}^\infty$ all be such ψ functions, let $\sigma \geq 0$, $\sigma_k \geq 0$, $c \in \mathbb{R}$, and $c_k \in \mathbb{R}$ be constants, and consider the functions

$$s_k^2(h) = \int_h^\infty \psi_{1k}^2(s) ds + \sigma_k^2 + \int_h^\infty \psi_{2k}^2(s) ds, \quad h > 0; \ k = 1, 2, \ldots. \tag{1.33}$$

Then convergence is characterized as follows.

THEOREM 11. *We have*

$$\overline{V}(\psi_{1k}, \psi_{2k}, \sigma_k) + c_k \to_{\mathcal{D}} \overline{V}(\psi_1, \psi_2, \sigma) + c, \quad k \to \infty, \tag{1.34}$$

if and only if

$$\psi_{jk}(\cdot) \Rightarrow \psi_j(\cdot), \quad j = 1, 2, \tag{1.35}$$

$$c_k \to c, \tag{1.36}$$

as $k \to \infty$, and

$$\underline{s}^2 := \lim_{h \to \infty} \liminf_{k \to \infty} s_k^2(h) = \sigma^2 = \lim_{h \to \infty} \limsup_{k \to \infty} s_k^2(h) =: \overline{s}^2. \tag{1.37}$$

The last theorem, based on Theorem 11, details convergence in distribution of sums of independent and identically distributed infinitely divisible random variables and hence is a counterpart of Theorem 6. Of course, the general theory based on the quantile function is applicable to this problem in principle. However, it is more natural to base here everything on the behavior of the corresponding ψ functions.

Let $\{n'\}$ be any given sequence of positive integers tending to infinity, possibly the whole $\{n\}$, and consider the infinitely divisible random variable $V(\psi_1, \psi_2, \sigma)$ belonging to an arbitrary triple $(\psi_1, \psi_2, \sigma) \not\equiv (0,0,0)$. For a number $h \geq 1$, introduce

$$\begin{aligned} a_1^2(n';h) &= \int_h^\infty \psi_1^2(u/n')du + n'\sigma^2 + \int_h^\infty \psi_2^2(u/n')du \\ &= n'\left\{\int_{h/n'}^\infty \psi_1^2(s)ds + \sigma^2 + \int_{h/n'}^\infty \psi_2^2(s)ds\right\}, \end{aligned} \tag{1.38}$$

which is non-zero for all n' large enough. The role of the "natural" normalizing sequence $a(n')$ in (1.2) will now be taken over either by

$$a_1(n') = a_1(n';1), \tag{1.39}$$

or by

$$\begin{aligned} a_2(n') &= \left(n'\left\{\int_{1/n'}^\infty \int_{1/n'}^\infty (u \wedge v)d\psi_1(u)d\psi_1(v) + \sigma^2 \right.\right. \\ &\quad \left.\left. + \int_{1/n'}^\infty \int_{1/n'}^\infty (u \wedge v)d\psi_2(u)d\psi_2(v)\right\}\right)^{1/2} \\ &= \left(a_1^2(n') + \psi_1^2(1/n') + \psi_2^2(1/n')\right)^{1/2}, \end{aligned} \tag{1.40}$$

where the last equality follows from Lemma 2.6 below. Furthermore, corresponding to the first sequence, define

$$v_{n'}^2(h) = a_1^2(n';h)/a_1^2(n'), \quad h \geq 1. \tag{1.41}$$

Then the quantities

$$\underline{v}^2 = \underline{v}^2_{\{n'\}} = \lim_{h\to\infty} \liminf_{n'\to\infty} v_{n'}^2(h)$$

and

$$\overline{v}^2 = \overline{v}^2_{\{n'\}} = \lim_{h\to\infty} \limsup_{n'\to\infty} v_{n'}^2(h)$$

are well defined and we have $0 \leq \underline{v} \leq \overline{v} \leq 1$. While $a_1(n')$ or $a_2(n')$ in (1.39) and (1.40) will not always be natural, it will turn out that

$$\begin{aligned} \mu_{n'}(\psi_1, \psi_2) &= \int_1^{n'} u d\psi_2(u/n') - \int_1^{n'} u d\psi_1(u/n') \\ &= n'\left\{\int_{1/n'}^1 s d\psi_2(s) - \int_{1/n'}^1 s d\psi_1(s)\right\} \end{aligned}$$

is always a correct centering sequence.

As before (1.30), consider now an infinite sequence

$$\left\{(N_1^{(l)}(\cdot),\ Z^{(l)},\ N_2^{(l)}(\cdot))\right\}_{l=1}^{\infty}$$

of independent copies of $(N_1(\cdot),\ Z,\ N_2(\cdot))$ and let $\{\overline{V}_l(\psi_1,\psi_2,\sigma)\}_{l=1}^{\infty}$ be the corresponding infinite sequence of independent copies of $\overline{V}(\psi_1,\psi_2,\sigma)$ defined through (1.32). The convergence problem for partial sums of these is apparently determined by the behavior of ψ_1 and ψ_2 near zero. Note in this connection that if

$$\int_0^\infty \psi_1^2(s)ds + \int_0^\infty \psi_2^2(s)ds < \infty,$$

then $\underline{v}^2 = \bar{v}^2 = 1$ for any sequence $\{n'\}$.

THEOREM 12. *(i) If, as* $n' \to \infty$,

$$\psi_j(s/n')/a_1(n') \to 0, \quad s > 0,\ j = 1,2,$$

then

$$\frac{1}{a_1(n')}\left\{\sum_{l=1}^{n'}\overline{V}_l(\psi_1,\psi_2,\sigma) - \mu_{n'}(\psi_1,\psi_2)\right\} \to_D N(0,1).$$

(ii) If, as $n' \to \infty$,

$$\psi_j(\cdot/n')/a_1(n') \Rightarrow \psi_j^*(\cdot), \quad j = 1,2, \tag{1.42}$$

for some non-positive, non-decreasing, right-continuous functions ψ_1^* *and* ψ_2^*, *then, necessarily, both* ψ_1^* *and* ψ_2^* *satisfy (1.4) and there exists a subsequence* $\{n''\} \subset \{n'\}$ *such that*

$$\frac{1}{a_1(n'')}\left\{\sum_{l=1}^{n''}\overline{V}_l(\psi_1,\psi_2,\sigma) - \mu_{n''}(\psi_1,\psi_2)\right\} \to_D \overline{V}(\psi_1^*,\psi_2^*,\sigma_*),$$

as $n'' \to \infty$, *where* $\underline{v} \le \sigma_* \le \bar{v}$. *If* $\underline{v} = \bar{v}$, *then this last convergence takes place along the original* $\{n'\}$.

(iii) If, as $n' \to \infty$,

$$\psi_j(\cdot/n')/a_2(n') \Rightarrow \psi_j^*(\cdot), \quad j = 1,2, \tag{1.43}$$

for some non-positive, non-decreasing, right-continuous functions ψ_1^* *and* ψ_2^* *and*

$$a_1(n')/a_2(n') \to 0 \tag{1.44a}$$

or, what is the same by (1.40),

$$a_1^2(n')/\{\psi_1^2(1/n') + \psi_2^2(1/n')\} \to 0, \tag{1.44b}$$

then again ψ_1^* *and* ψ_2^* *satisfy (1.4) and*

$$\frac{1}{a_2(n')}\left\{\sum_{l=1}^{n'} \overline{V}_l(\psi_1,\psi_2,\sigma) - \mu_{n'}(\psi_1,\psi_2)\right\} \to_D \overline{V}(\psi_1^*,\psi_2^*,0).$$

(iv) If there is a sequence of numbers $A_{n'} > 0$ *such that*

$$\psi_j(\cdot/n')/A_{n'} \Rightarrow \psi_j^*(\cdot), \quad j = 1,2, \tag{1.45}$$

for some non-positive, non-decreasing, right-continuous functions ψ_1^* *and* ψ_2^* *and*

$$a_2(n')/A_{n'} \to 0 \tag{1.46}$$

as $n' \to \infty$*, then, necessarily,* $\psi_j^*(s) = 0$ *for all* $s \geq 1$, $j = 1,2$*, and, as* $n' \to \infty$*,*

$$\frac{1}{A_{n'}}\left\{\sum_{l=1}^{n'} \overline{V}_l(\psi_1,\psi_2,\sigma) - \mu_{n'}(\psi_1,\psi_2)\right\} \to_D \overline{V}(\psi_1^*,\psi_2^*,0).$$

(v) Conversely, suppose that for some constants $A_{n'} > 0$ *and* $C_{n'} \in \mathbb{R}$*,*

$$\frac{1}{A_{n'}}\left\{\sum_{l=1}^{n'} \overline{V}_l(\psi_1,\psi_2,\sigma) - C_{n'}\right\} \to_D W, \quad n' \to \infty, \tag{1.47}$$

where W *is a non-degenerate random variable. Then with some constant* c*,* W *is necessarily of the form* $\overline{V}(\psi_1^*,\psi_2^*,\sigma_*) + c$*, and we have*

$$c_{n'} := \{\mu_{n'}(\psi_1,\psi_2) - C_{n'}\}/A_{n'} \to c, \tag{1.48}$$

$$\psi_j(\cdot/n')/A_{n'} \Rightarrow \psi_j^*(\cdot), \quad j = 1,2, \tag{1.49}$$

as $n' \to \infty$, and

$$\lim_{h\to\infty} \liminf_{n'\to\infty} \frac{a_1^2(n';h)}{A_{n'}^2} = \sigma_*^2 = \lim_{h\to\infty} \limsup_{n'\to\infty} \frac{a_1^2(n';h)}{A_{n'}^2}. \tag{1.50}$$

Furthermore, if

$$\limsup_{n'\to\infty} |\psi_j(s/n')|/a_1(n') < \infty, \quad j = 1,2, \tag{1.51}$$

then in the case when $\underline{v} = \overline{v} > 0$, *there exists a finite* $\delta > 0$ *such that* $a_1(n')/A_{n'} \to \delta$ *as* $n' \to \infty$, *and when* $\overline{v} = 0$ *or* $\underline{v} < \overline{v}$, *there exist a subsequence* $\{n''\} \subset \{n'\}$ *and a finite* $\delta = \delta_{\{n''\}} > 0$, *possibly depending on* $\{n''\}$, *such that* $a_1(n'')/A_{n''} \to \delta$ *as* $n'' \to \infty$. *If, on the other hand,*

$$\limsup_{n'\to\infty} |\psi_1(s/n')|/a_1(n') = \infty \quad \text{or} \quad \limsup_{n'\to\infty} |\psi_2(s/n')|/a_1(n') = \infty \tag{1.52}$$

for some $s > 0$, *then*

$$a_1(n')/A_{n'} \to 0 \quad \text{as} \quad n' \to \infty, \tag{1.53}$$

and hence, necessarily, $\sigma_* = 0$. *Moreover, if (1.52) holds for some* $s > 0$, *but*

$$\limsup_{n'\to\infty} |\psi_j(s/n')|/a_2(n') < \infty, \quad j = 1,2, \tag{1.54}$$

then there exist a subsequence $\{n''\} \subset \{n'\}$ *and a* $\delta = \delta_{\{n''\}} > 0$ *such that, besides (1.53),* $a_2(n'')/A_{n''} \to \delta$ *as* $n'' \to \infty$. *Finally, if*

$$\limsup_{n'\to\infty} |\psi_1(s/n')|/a_2(n') = \infty \quad \text{or} \quad \limsup_{n'\to\infty} |\psi_2(s/n')|/a_2(n') = \infty \tag{1.55}$$

for some $s > 0$, *which implies (1.52) for the same* s, *then*

$$a_2(n')/A_{n'} \to 0 \quad \text{as} \quad n' \to \infty. \tag{1.56}$$

We demonstrate Theorem 12 by way of some examples included in the next two corollaries. In the first of these, for the sake of simplicity, we only deal with one-sided examples without a normal component that have the form

$$\overline{V}(0,\psi,0) = \int_1^\infty (N(s) - s)d\psi(s) + \int_0^1 N(s)d\psi(s).$$

In fact, since we always assume that $\psi(s) = 0$ for all $s \geq 1$, we will have

$$\overline{V}_l(0,\psi,0) = \int_0^1 N^{(l)}(s)d\psi(s); \quad l = 1,2,\ldots,$$

for independent copies $N^{(1)}(\cdot)$, $N^{(2)}(\cdot),\ldots$ of the standard Poisson process $N(\cdot)$. Part (a) below illustrates part (i) of Theorem 12. The two cases of part (b) when $\alpha \geq 2$ still illustrate part (i), while the third case when $\alpha < 2$ is an example for part (ii). The case of part (c) when $\lambda > 1$ is an example for part (ii), while the case when $\lambda \leq 1$ is an illustration of part (iv). The case $\alpha < 2$ of part (b), used in conjunction with Theorem 9, has an interesting application to be discussed in Section 3.

COROLLARY 16. *(a) As $n \to \infty$,*

$$\frac{1}{\sqrt{n}}\left\{\sum_{l=1}^{n}\int_0^1 N^{(l)}(s)d(-\sqrt{\log 1/s}) - \frac{\sqrt{\pi}}{2}n\right\} \to_D N(0,1).$$

(b) As $n \to \infty$,

$$\sqrt{\frac{\alpha-2}{\alpha}}\frac{1}{\sqrt{n}}\left\{\sum_{l=1}^{n}\int_0^1 N^{(l)}(s)d(-s^{-1/\alpha}) - \frac{n}{\alpha-1}\right\} \to_D N(0,1), \text{ if } \alpha > 2,$$

$$\frac{1}{\sqrt{n\log n}}\left\{\sum_{l=1}^{n}\int_0^1 N^{(l)}(s)d(-s^{-1/2}) - \frac{n}{2}\right\} \to_D N(0,1),$$

and if $0 < \alpha < 2$, then

$$\sqrt{\frac{2-\alpha}{\alpha}}\frac{1}{n^{1/\alpha}}\left\{\sum_{l=1}^{n}\int_0^1 N^{(l)}(s)d(-s^{-1/\alpha}) - \mu_n(\alpha)\right\} \to_D \overline{V}(0,\psi_\alpha^*,0),$$

where $\psi_\alpha^*(s) = -((2-\alpha)/\alpha)^{1/2}s^{-1/\alpha}$, $s > 0$, and

$$\mu_n(\alpha) = \begin{cases} \frac{n}{\alpha-1}, & 1 < \alpha < 2, \\ n\log n, & \alpha = 1, \\ \frac{n^{1/\alpha}}{1-\alpha}, & 0 < \alpha < 1. \end{cases}$$

(c) Let $t_j = 2^{-2^j}$, $j = 1,2,\ldots$, and

$$b_j = (t_{j-1} - t_j)^{-1} = 2^{2^{j-1}}/(1 - 2^{-2^{j-1}}), \quad j = 1,2,\ldots,$$

and define

$$\psi(s) = \begin{cases} -b_k, & t_k \le s < t_{k-1},\ k = 1,2,\ldots, \\ 0, & s \ge t_0 = 1/2. \end{cases}$$

Then

$$\overline{V}(0,\psi,0) = \int_0^{1/2} N(s)d\psi(s) = \sum_{k=0}^{\infty} N(t_k)(b_{k+1} - b_k),$$

where $b_0 = 0$, and if $\lceil x \rceil$ denotes the smallest integer not less than x, then for each $\lambda > 0$,

$$\frac{1}{2^{2^k}}\left\{\sum_{l=1}^{\lceil \lambda 2^{2^k} \rceil}\int_0^{1/2} N^{(l)}(s)ds - \mu_k(\lambda)\right\} \to_D N(\lambda) - \lambda I(\lambda > 1) \tag{1.57}$$

as $k \to \infty$, where

$$\mu_k(\lambda) = \lceil \lambda 2^{2^k} \rceil \left\{ \sum_{j=0}^{\nu(k,\lambda)} (b_{j+1} - b_j) t_j + \left(t_{\nu(k,\lambda)} - \frac{1}{\lceil \lambda 2^{2^k} \rceil} \right) b_{\nu(k,\lambda)+1} \right\}, \tag{1.58}$$

and where

$$\nu(k,\lambda) = \begin{cases} k-1, & \text{if } \lambda < 1, \\ k, & \text{if } \lambda \geq 1. \end{cases}$$

Furthermore,

$$\frac{a_1^2(\lceil \lambda 2^{2^k} \rceil)}{a_2^2(\lceil \lambda 2^{2^k} \rceil)} = \frac{\lceil \lambda 2^{2^k} \rceil \int_{1/\lceil \lambda 2^{2^k} \rceil}^{1/2} \psi^2(s)ds}{\lceil \lambda 2^{2^k} \rceil \int_{1/\lceil \lambda 2^{2^k} \rceil}^{1/2} \psi^2(s)ds + \psi^2(1/\lceil \lambda 2^{2^k} \rceil)} \to \begin{cases} 1 - \frac{1}{\lambda}, & \text{if } \lambda > 1, \\ 1, & \text{if } \lambda \leq 1, \end{cases} \tag{1.59}$$

and if $\lambda > 1$, then

$$\frac{a_2^2(\lceil \lambda 2^{2^k} \rceil)}{A_k^2} = \frac{\lceil \lambda 2^{2^k} \rceil \int_{1/\lceil \lambda 2^{2^k} \rceil}^{1/2} \psi^2(s)ds + \psi^2(1/\lceil \lambda 2^{2^k} \rceil)}{(2^{2^k})^2} \to \lambda \tag{1.60}$$

while if $\lambda \leq 1$, then

$$a_2^2(\lceil \lambda 2^{2^k} \rceil)/A_k^2 \to 0 \tag{1.61}$$

as $k \to \infty$.

Let $\overline{F}^{\psi_1,\psi_2,\sigma}$ denote the distribution function of $\overline{V}(\psi_1,\psi_2,\sigma)$. Part (v) of Theorem 12 can be used to construct negative examples.

COROLLARY 17. *If at least one of ψ_1 and ψ_2 decreases to $-\infty$ so fast that there exist $0 < s < t < \infty$ such that $\psi_j(s/n)/\psi_j(t/n) \to \infty$, as $n \to \infty$, for the corresponding $j = 1$ or $j = 2$ or both, then $\overline{F}^{\psi_1,\psi_2,\sigma}$ does not belong to the domain of partial attraction of any law. In particular, this is the case for any of the distrubution functions $\overline{F}^{0,{}_\alpha\psi,0}$, where*

$$_\alpha\psi(s) = \begin{cases} -\exp((1/s)^\alpha), & 0 < s < 1, \\ 0, & s \geq 1, \end{cases}$$

and α is an arbitrary fixed positive number.

Let $\overline{Q}^{\psi_1,\psi_2,\sigma}$ be the quantile function pertaining to $\overline{F}^{\psi_1,\psi_2,\sigma}$. Many authors worked on the problem of comparing the tail behavior of $\overline{F}^{\psi_1,\psi_2,\sigma}$ and those of the Lévy functions $L(\cdot)$ and $R(\cdot)$ belonging to ψ_1 and ψ_2 as determined below (1.28). (Cf. the references in [9].) Our last corollary, following from a joint application of Theorems 5 and 12, gives a result of such a flavor.

COROLLARY 18. *Suppose that* $\overline{F}^{\psi_1,\psi_2,\sigma} \in D_p(\psi_1^*, \psi_2^*, \sigma_*)$ *for some* ψ_1^*, ψ_2^*, *and* $\sigma_* \geq 0$, *where at least one of* ψ_1^* *and* ψ_2^* *is not identically zero, that is, the attracting infinitely divisible law has a non-degenerate non-normal component. Then there exist a subsequence* $\{n'\} \subset \{n\}$ *and finite constants* $c_1, c_2 > 0$ *such that if* $\psi_1^* \not\equiv 0$, *then*

$$\overline{Q}^{\psi_1,\psi_2,\sigma}\left(\frac{s}{n'}+\right) \Big/ \psi_1\left(\frac{s}{n'}\right) \to c_1, \quad 0 < s < s_1,$$

and if $\psi_2^* \not\equiv 0$, *then*

$$-\overline{Q}^{\psi_1,\psi_2,\sigma}\left(1 - \frac{s}{n'}\right) \Big/ \psi_2\left(\frac{s}{n'}\right) \to c_2, \quad 0 < s < s_2,$$

as $n' \to \infty$, *where*

$$0 < s_j = \inf\{s : \psi_j^*(s) = 0\} \leq \infty, \quad j = 1, 2.$$

We close this section by using the opportunity to correct a few misprints and an oversight in [6]. In lines -10 and -12 on p. 263, the reference should be to Corollary 11 instead of Corollary 10. On p. 270, the minus sign should be deleted from before $Q(1-s)$ in (1.32). In line -3 on p. 292, $R(s_{n_1}, l_{n_1}, n_1)$ should be $R(s_{n_1} l_{n_1}, n_1)$. The summation index k in the definition of N' in line 8 on p. 297 should start with $k = 1$ rather than $k = 0$. Finally, the bottom-line inequality on p. 314 is wrong, it holds the opposite way. However, leaving this line out and referring to (1.7) in the present paper instead, the proof is correct as it stands.

2. PROOFS

Proof of Theorem 1*. We need only to show (1.6), the validity of all the statements following this relation follows from the original proof of Theorem 1.

For any real $h \geq 1$, define $\underline{\sigma}(h) = \underline{\sigma}_{\{n'\}}$ and $\overline{\sigma}(h) = \overline{\sigma}_{\{n'\}}$ by setting

$$\underline{\sigma}(h) = \liminf_{n' \to \infty} \frac{\sigma(h/n')}{\sigma(1/n')} \quad \text{and} \quad \overline{\sigma}(h) = \limsup_{n' \to \infty} \frac{\sigma(h/n')}{\sigma(1/n')},$$

where $\sigma^2(\cdot)$ is given in (1.1). Since $\sigma(s)$ is a monotone non-incereasing function of s, we see that both $\underline{\sigma}(h)$ and $\overline{\sigma}(h)$ are monotone non-increasing functions of h, $0 \leq \underline{\sigma}(h) \leq \overline{\sigma}(1) = 1$ and $\overline{\sigma}(h) \leq \overline{\sigma}(1) = 1$ for any $h \geq 1$. Hence the limits in (1.7) are well defined. Also, whatever is the sequence $\{r_{n'}\}$ to be constructed such that $r_{n'} \to \infty$ as $n' \to \infty$, the monotonicity of $\sigma(\cdot)$ implies that

$$\liminf_{n' \to \infty} \sigma((r_{n'} + 1)/n')/\sigma(1/n') \leq \underline{\sigma} \tag{2.1}$$

and

$$\limsup_{n'\to\infty} \sigma((r_{n'}+1)/n')/\sigma(1/n') \leq \bar{\sigma}. \tag{2.2}$$

In order to prove the opposite inequalities, we have to go into the construction of $\{r_{n'}\}$ at the end of the proof of case (ii) of Theorem 1. Taking up the line in the middle of p.292 and using the notation developped there with n_1 replaced by n' everywhere one can construct a strictly increasing sequence $\{\bar{l}_s\}$ of positive integers such that for all $s \geq 1$ we have

$$\rho(\max_{0\leq h\leq \bar{l}_s} |V_h^{(j)}(\bar{l}_s, n') - V_h^{(j)}(\bar{l}_s)|) \leq s^{-1},$$

$$\rho(\max_{0\leq h\leq \bar{l}_s} |V_h^{(j)}(\bar{l}_s) - V_h^{(j)}|) \leq s^{-1},$$

$$\rho(\triangle_j(\bar{l}_s, s\bar{l}_s, n') - \triangle_j(s, \bar{l}_s)) \leq s^{-1},$$

$$\rho(\triangle_j(s, \bar{l}_s)) \leq s^{-1}, \; s\bar{l}_s\psi_j^2(\bar{l}_s) \leq s^{-1}, \quad j = 1, 2,$$

and

$$\rho(R(s\bar{l}_s, n')) \leq s^{-1}, \quad s\bar{l}_s/n' \leq s^{-1},$$

for all $n' \geq n(s)$, where the positive integer $n(s)$ is chosen so large that, together with the above twelve inequalities, we also have

$$\underline{\sigma} - \frac{1}{s} \leq \underline{\sigma}(s\bar{l}_s + 1) - \frac{1}{s} \leq \sigma(\frac{s\bar{l}_s + 1}{n'})/\sigma(\frac{1}{n'}), \quad n' \geq n(s),$$

and

$$\bar{\sigma}(s\bar{l}_s + 1) - \frac{1}{s} \leq \sigma(\frac{s\bar{l}_s + 1}{n'})/\sigma(\frac{1}{n'}) \quad \text{for infinitely many} \quad n'.$$

Clearly, we can choose these threshold numbers $n(s)$, $s \geq 1$, so that $n(1) < n(2) < \ldots$. For each $n' \geq n(1)$ from the sequence $\{n'\}$ there exists an integer $s = s(n')$ such that $n(s) \leq n' < n(s+1)$. Now we simply set $s_{n'} = s(n')$, $l_{n'} = \bar{l}_{s(n')}$, and $\varepsilon_{n'} = 1/s(n')$ for any $n' \geq n(1)$, and for the finitely many $n' < n(1)$ we define these three sequences in an appropriate but otherwise arbitrary fashion. Obviously, $s_{n'} \to \infty$, $l_{n'} \to \infty$, and $\varepsilon_{n'} \to 0$ as $n' \to \infty$, and together with the twelve inequalities in the last seven lines of p. 292 (with the misprint corrected in the third line from below as noted above and with n' standing everywhere in place of n_1) we also have

$$\underline{\sigma} - \varepsilon_{n'} \leq \underline{\sigma}(s_{n'}l_{n'} + 1) - \varepsilon_{n'} \leq \sigma((s_{n'}l_{n'} + 1)/n')/\sigma(1/n') \tag{2.3}$$

for each n' in $\{n'\}$ and

$$\underline{\sigma}(s_{n'}l_{n'} + 1) - \varepsilon_{n'} \leq \sigma((s_{n'}l_{n'} + 1)/n')/\sigma(1/n') \tag{2.4}$$

for infinitely many n' in $\{n'\}$. Hence, with $r_{n'} = s_{n'} l_{n'}$, we have all the conclusions of page 293 and, moreover, from (2.3),

$$\underline{\sigma} \leq \liminf_{n' \to \infty} \sigma((r_{n'}+1)/n')/\sigma(1/n'),$$

and, since $\overline{\sigma}(s_{n'} l_{n'}+1) = \overline{\sigma}(r_{n'}+1)$ is just a sequence in the set of the values of the function $\overline{\sigma}(h), h \geq 1$, and $\overline{\sigma}(h) \downarrow \overline{\sigma}$ as $h \to \infty$, from (2.4),

$$\overline{\sigma} \leq \limsup_{n' \to \infty} \sigma((r_{n'}+1)/n')/\sigma(1/n').$$

These inequalities, together with those in (2.1) and (2.2), prove (1.6). ■

Proof of Theroem 6. In all theree case (i)-(iii), the results in the sufficiency direction as stated at the end of the theorem follow directly from respective applications of Theorem 1* and Theorem 2, and hence we only have to deal with necessity.

Assume, therefore, (1.11) and that we are either in case (i) or in (ii). Then by Theorem 5 there exists an $\{n''\} \subset \{n_k\}_{k=1}^{\infty}$ such that

$$\frac{a(n'')}{A_{n''}} \psi_j(n'', \cdot) \Rightarrow \psi_j^*(\cdot), \quad j = 1, 2, \tag{2.5}$$

and

$$a(n'')/A_{n''} \to \delta, \quad 0 \leq \delta < \infty, \tag{2.6}$$

as $n'' \to \infty$. If δ were zero, then by Theorem 2 we would have

$$\frac{1}{A_{n''}} \left\{ \sum_{j=1}^{n''} X_j - n'' \mu_{0,0}(n'') \right\} \to_D V_{0,0}(\psi_1^*, \psi_2^*, 0) \quad \text{as} \quad n'' \to \infty,$$

where

$$\mu_{0,0}(n) = \int_{1/n}^{1-1/n} Q(s)ds, \quad n = 1, 2 \ldots, \tag{2.7}$$

and $V_{0,0}(\cdot,\cdot,\cdot)$ is defined in (1.9), and we would know that $\psi_j^*(s) = 0$ for $1 \leq s < \infty$, $j = 1, 2$. However, (1.11) also holds along $\{n''\}$ and hence the convergence of types theorem ([14], pp. 40-42) implies that, as $n'' \to \infty$,

$$\{n'' \mu_{0,0}(n'') - C_{n''}\}/A_{n''} \to \gamma$$

with some $\gamma \in \mathbb{R}$ and, therefore,

$$\frac{1}{A_{n''}} \left\{ \sum_{j=1}^{n''} X_j - n'' \mu_{0,0}(n'') \right\} \to_D \alpha V_{0,0}(\psi_1, \psi_2, \sigma) + \beta - \gamma.$$

Thus, if δ were zero, we would have

$$V_{0,0}(\psi_1^*, \psi_2^*, 0) = \alpha V_{0,0}(\psi_1, \psi_2, \sigma) + \beta - \gamma = V_{0,0}(\alpha\psi_1, \alpha\psi_2, \alpha\sigma) + \beta - \gamma,$$

which is impossible in either case in view of the uniqueness of the representation of an infinitely divisible random variable in Theorem 3(ii), already mentioned in the introduction in Section 1. Whence $\delta > 0$ in (2.6).

From (2.5) and (2.6), with $\delta > 0$, we obtain

$$\psi_j(n'', \cdot) \Rightarrow \frac{1}{\delta}\psi_j^*(\cdot), \quad j = 1, 2, \tag{2.8}$$

and from (1.11), holding along $\{n''\}$, and (2.6),

$$\frac{1}{a(n'')}\left\{\sum_{j=1}^{n''} X_j - C_{n''}\right\} \to_D \frac{\alpha}{\delta} V_{0,0}(\psi_1, \psi_2, \sigma) + \frac{\beta}{\delta} \tag{2.9}$$

as $n'' \to \infty$. Using (2.8), for any subsequence $\{n_3\} \subset \{n''\}$ there exist by Theorem 1* a further subsequence $\{n_4\} \subset \{n_3\}$ and some $0 \le \sigma_* \le 1$ such that

$$\frac{1}{a(n_4)}\left\{\sum_{j=1}^{n_4} X_j - n_4\mu_{0,0}(n_4)\right\} \to_D V_{0,0}\left(\frac{1}{\delta}\psi_1^*, \frac{1}{\delta}\psi_2^*, \sigma_*\right) \tag{2.10}$$

as $n_4 \to \infty$. Now (2.9) and (2.10) together imply, again by convergence of types, the existence of some $\gamma \in \mathbb{R}$ such that

$$\{n_4\mu_{0,0}(n_4) - C_{n_4}\}/a(n_4) \to \gamma \tag{2.11}$$

and hence

$$\frac{1}{a(n_4)}\left\{\sum_{j=1}^{n_4} X_j - n_4\mu_{0,0}(n_4)\right\} \to_D \frac{\alpha}{\delta} V_{0,0}(\psi_1, \psi_2, \sigma) + \frac{\beta}{\delta} - \gamma \tag{2.12}$$

as $n_4 \to \infty$. Form (2.10) and (2.12), by uniqueness,

$$\psi_j^*/\delta = \alpha\psi_j/\delta, \quad j = 1, 2, \quad \text{and} \quad \sigma_* = \alpha\sigma/\delta, \ \gamma = \beta/\delta.$$

Since $\{n_3\} \subset \{n''\}$ was arbitrary, by (2.6), (2.8), and (2.11) this means that (1.12), (1.13) and (1.14) all hold along $\{n''\} = \{n_k''\}_{k=1}^{\infty}$, and as $n'' \to \infty$, we have

$$\frac{1}{a(n'')}\left\{\sum_{j=1}^{n''} X_j - n''\mu_{0,0}(n'')\right\} \to_D V_{0,0}\left(\frac{\alpha}{\delta}\psi_1, \frac{\alpha}{\delta}\psi_2, \frac{\alpha}{\delta}\sigma\right). \tag{2.13}$$

If $\psi_1 = \psi_2 \equiv 0$, then by part (i) of Theorem 1* the limit in (2.13) must be standard normal and hence in this special case we necessarily have $\delta = \alpha\sigma$. Also, $\{n''\}$ can be chosen as a subsequence of an arbitrary subsequence $\{n'\}$ of the original $\{n_k\}_{k=1}^{\infty}$ along which (1.11) holds and, since the limits in (1.12), (1.13), and (1.14) are the same for all such $\{n''\}$ in this special case, we conclude that (1.12), (1.13), and (1.14) hold true along the original $\{n_k\}_{k=1}^{\infty}$ for which (1.11) is valid.

Let $\{n_3\}$ be again any subsequence of $\{n''\}$. Then by one more application of Theorem 1* one can find a sequence $\{n_5\} \subset \{n_3\}$ and a $\tilde{\sigma}$, $0 \le \tilde{\sigma} \le 1$, such that

$$\frac{1}{a(n_5)}\left\{\sum_{j=1}^{n_5} X_j - n_5\mu_{0,0}(n_5)\right\} \to_D V_{0,0}\left(\frac{\alpha}{\delta}\psi_1, \frac{\alpha}{\delta}\psi_2, \tilde{\sigma}\right)$$

and

$$\sigma((r_{n_5}+1)/n_5)/\sigma(1/n_5) \to \tilde{\sigma}$$

as $n_5 \to \infty$. By (2.13) it follows that $\tilde{\sigma} = \alpha\sigma/\delta$, and hence we see that

$$\rho_{n''} := \sigma((r_{n''}+1)/n'') \to \alpha\sigma/\delta, \quad \text{as} \quad n'' \to \infty,$$

since any subsequence of $\{\rho_{n''}\}$ contains a further subsequence with the same limit. Thus by (1.6) in Theorem 1* we obtain (1.15) with $\{n'_k\}_{k=1}^{\infty}$ replaced by $\{n''\} = \{n''_k\}_{k=1}^{\infty}$. Hence we have (1.12), (1.13), (1.14), and (1.15) along $\{n'_k\}_{k=1}^{\infty} := \{n''_k\}_{k=1}^{\infty}$ in both cases (i) and (ii).

Suppose now (1.11) and that we are in case (iii). Then we have (2.5) and (2.6). If $\delta > 0$, then we can follow the above proof beginning from (2.8) and conclude that (1.12)-(1.15) all hold along the same $\{n''\}$ chosen at (2.5) and (2.6).

If we have (2.5) and (2.6) with $\delta = 0$, then by Theorem 2, for any subsequence $\{n_3\} \subset \{n''\}$ we can choose a further subsequence $\{n_6\} \subset \{n_3\}$ such that

$$\frac{1}{A_{n_6}}\left\{\sum_{j=1}^{n_6} X_j - n_6\mu_{0,0}(n_6)\right\} \to_D V_{0,0}(\psi_1^*, \psi_2^*, 0), \quad \text{as} \quad n_6 \to \infty,$$

and of course we have (1.11) along $\{n_6\}$. By convergence of types, as $n_6 \to \infty$,

$$\{n_6\mu_{0,0}(n_6) - C_{n_6}\}/A_{n_6} \to \gamma$$

and

$$\frac{1}{A_{n_6}}\left\{\sum_{j=1}^{n_6} X_j - n_6\mu_{0,0}(n_6)\right\} \to_D V_{0,0}(\alpha\psi_1, \alpha\psi_2, 0) + \beta - \gamma.$$

Whence $\psi_j^* = \alpha\psi_j$, $j = 1,2$, and $\gamma = \beta$. Therefore, along $\{n'_k\}_{k=1}^{\infty} := \{n''_k\}_{k=1}^{\infty}$ we have (1.17), (1.18), and (1.19), and the theorem is completely proved. ∎

The proof of Theorem 8 requires four lemmas. Let $N(\cdot)$ denote any of the two standard Poisson processes $N_1(\cdot)$ or $N_2(\cdot)$ in (1.26) and consider $W(\psi)$ defined in (1.26) by accordingly dropping the subscript j. Let $N^{(1)}(\cdot), \ldots, N^{(r)}(\cdot)$ be independent copies of $N(\cdot)$, consider the corresponding independent copies $W^{(1)}(\psi), \ldots, W^{(r)}(\psi)$ of $W(\psi)$, where r is an arbitrary integer, and recall the notation in (1.29).

LEMMA 2.1. *We have* $\sum_{l=1}^{r} W^{(l)}(\psi) =_D W(\psi^{(r)})$.

Proof. Using the fact that the Poisson process has independent increments,

$$\begin{aligned}
\sum_{l=1}^{r} W^{(l)}(\psi) &= \int_1^{\infty} \left(\sum_{l=1}^{r} (N^{(l)}(s) - s) \right) d\psi(s) + \int_0^1 \left(\sum_{l=1}^{r} N^{(l)}(s) \right) d\psi(s) + r\Theta(\psi) \\
&=_D \int_1^{\infty} (N(rs) - rs) d\psi(s) + \int_0^1 N(rs) d\psi(s) + r\Theta(\psi) \\
&= \int_r^{\infty} (N(u) - u) d\psi^{(r)}(u) + \int_0^r N(u) d\psi^{(r)}(u) + r\Theta(\psi) \\
&= \int_1^{\infty} (N(u) - u) d\psi^{(r)}(u) + \int_0^1 N(u) d\psi^{(r)}(u) \\
&\qquad + \int_1^r u\, d\psi^{(r)}(u) + r\Theta(\psi).
\end{aligned} \tag{2.14}$$

But a very elementary calculation, involving integration by parts, shows that

$$\int_1^r u\, d\psi^{(r)}(u) + r\Theta(\psi) = \Theta(\psi^{(r)})$$

and hence the lemma. ∎

Recalling now the notation in (1.27), we see that Lemma 2.1 implies (1.30) indeed.

LEMMA 2.2. *If for some subsequence* $\{n'\} \subset \{n\}$, *constants* $A_{n'} > 0$ *and* $C_{n'} \in \mathbb{R}$,

$$\frac{1}{A_{n'}} \left\{ \sum_{j=1}^{n'} X_j - C_{n'} \right\} \to_D V(\psi_1, \psi_2, \sigma) \quad \text{as} \quad n' \to \infty, \tag{2.15}$$

then for any integer $r \geq 1$, *we have*

$$\frac{1}{A_{n'}} \left\{ \sum_{j=1}^{rn'} X_j - rC_{n'} \right\} \to_D V(\psi_1^{(r)}, \psi_2^{(r)}, \sqrt{r}\sigma), \quad \text{as} \quad n' \to \infty. \tag{2.16}$$

Proof. Let $\{X_j^{(m)}\}_{j=1}^{\infty}$ be independent copies of $\{X_j\}_{j=1}^{\infty}$, $m = 1, \ldots, r$. Then by (2.15) and (1.30) we obtain

$$\frac{1}{A_{n'}}\left\{\sum_{j=1}^{n'}\left(\sum_{m=1}^{r} X_j^{(m)}\right) - rC_{n'}\right\} \to_D V(\psi_1^{(r)}, \psi_2^{(r)}, \sqrt{r}\sigma) \tag{2.17}$$

as $n' \to \infty$. Writing $Y_1, Y_2, \ldots$ for the sequence $X_1^{(1)}, \ldots, X_1^{(r)}, X_2^{(1)}, \ldots, X_2^{(r)}, \ldots$, that is, writing

$$Y_{(j-1)r+m} = X_j^{(m)}, \quad 1 \le m \le r, \quad j = 1, 2, \ldots, \tag{2.18}$$

we have

$$\sum_{j=1}^{n'}\left(\sum_{m=1}^{r} X_j^{(m)}\right) = \sum_{j=1}^{rn'} Y_j,$$

and since $Y_1, Y_2, \ldots$ are independent with the common distribution function F, (2.17) is nothing but (2.16). ■

Let $[\cdot]$ denote the usual integer part function.

LEMMA 2.3. *If (2.15) holds, then for any integer $r \ge 1$,*

$$\frac{1}{A_{n'}}\left\{\sum_{j=1}^{[n'/r]} X_j - \frac{1}{r}C_{n'}\right\} \to_D V(\psi_1^{(1/r)}, \psi_2^{(1/r)}, \sigma/\sqrt{r}) \quad \text{as} \quad n' \to \infty. \tag{2.19}$$

Proof. If $Y_1, Y_2, \ldots$ are independent with the same distribution function F, then (2.15) can be written as

$$\frac{1}{A_{n'}}\left\{\sum_{j=1}^{r[n'/r]} Y_j - C_n'\right\} + \frac{1}{A_{n'}}\sum_{j=r[n'/r]+1}^{n'} Y_j \to_D V(\psi_1, \psi_2, \sigma), \tag{2.20}$$

and Theorem 5 clearly implies that $A_{n'} \to \infty$ as $n' \to \infty$. Therefore, since the second sum here has at most $r - 1$ terms, the second term on the left side converges to zero in probability as $n' \to \infty$. Thus, breaking up the sequence $\{Y_j\}_{j=1}^{\infty}$ into the union of r independent sequences $\{X_j^{(m)}\}_{j=1}^{\infty}$ of independent variables, $m = 1, \ldots, r$, according to the rule in (2.18), (2.20) can in fact be written as

$$\sum_{l=1}^{r}\left(\frac{1}{A_{n'}}\left\{\sum_{j=1}^{[n'/r]} X_j^{(l)} - \frac{1}{r}C_{n'}\right\}\right) \to_D \sum_{l=1}^{r} V_l(\psi_1^{(1/r)}, \psi_2^{(1/r)}, \sigma/\sqrt{r}) \tag{2.21}$$

as $n' \to \infty$, where, with appropriate independent copies of $V(\psi_1^{(1/r)}, \psi_2^{(1/r)}, \sigma/\sqrt{r})$, we also used (1.30) on the right side in conjunction with the trivial fact that $\psi^{(1/r)(r)} = \psi$. Now this convergence obviously implies that in (2.19). ■

LEMMA 2.4. *If (2.15) holds, then for any two integers $r \geq 1$ and $l \geq 1$,*

$$\frac{1}{A_{n'}}\left\{\sum_{j=1}^{l[n'/r]} X_j - \frac{l}{r}C_{n'}\right\} \to_D V\left(\psi_1^{(l/r)}, \psi_2^{(l/r)}, \sqrt{\frac{l}{r}}\sigma\right) \quad \text{as} \quad n' \to \infty.$$

Proof. This follows by applying first Lemma 2.3 and then Lemma 2.2, upon noting that $\psi_j^{(1/r)(l)} = \psi_j^{(l/r)}, j = 1,2$. ■

Proof of Theorem 8. For two distribution functions G and H let

$$L(G,H) = \inf\{\varepsilon > 0 : G(x-\varepsilon) - \varepsilon \leq H(x) \leq G(x+\varepsilon) + \varepsilon \quad \text{for all} \quad x \in \mathbb{R}\}$$

denote their Lévy distance. It is well known that this distance metrizes the weak convergence of distribution functions on the line.

To prove the first statement, let τ_m be rational numbers such that $0 < \tau_m < \tau$ and $\tau_m \uparrow \tau$ as $m \to \infty$, where $\tau > 0$ is any given number. Then, as $m \to \infty$,

$$\psi_j^{(\tau_m)}(s) \to \psi_j^{(\tau)}(s) \quad \text{in each} \ s > 0, \ j = 1,2.$$

Consequently,

$$V(\psi_1^{(\tau_m)}, \psi_2^{(\tau_m)}, \sqrt{\tau_m}\sigma) \to V(\psi_1^{(\tau)}, \psi_2^{(\tau)}, \sqrt{\tau}\sigma) \quad \text{almost surely,}$$

and, *a fortiori*, with G_m and G denoting the distribution functions of the two sides respectively, $L(G_m, G) \to 0$ as $m \to \infty$. By Lemma 2.4, $F \in D_p(\psi_1^{(\tau_m)}, \psi_2^{(\tau_m)}, \sqrt{\tau_m}\sigma)$ for each $m \geq 1$, and hence we can pick a subsequence $\{n_k(m)\}_{k=1}^{\infty}$ and constants $A_k(m) > 0$ and $C_k(m) \in \mathbb{R}$ such that for

$$F_{n_k(m)}(x) = P\left\{\frac{1}{A_k(m)}\left\{\sum_{j=1}^{n_k(m)} X_j - C_k(m)\right\} \leq x\right\}, \ x \in \mathbb{R},$$

we have

$$\lim_{k\to\infty} L(F_{n_k(m)}, G_m) = 0, \quad m = 1,2,\ldots. \tag{2.22}$$

Hence we can choose a subsequence

$$n_{k_1}(1) < n_{k_2}(2) < \ldots < n_{k_m}(m) < \ldots$$

such that

$$L(F_{n_{k_m}(m)}, G_m) \leq \frac{1}{m}, \quad m = 1,2,\ldots.$$

Using now the triangle inequality for the Lévy distance ([14], p.33), we get

$$L(F_{n_{k_m}(m)}, G) \le \frac{1}{m} + L(G_m, G), \quad m = 1, 2, \ldots,$$

and hence the first statement.

In order to prove the second statement, we only have to redifine G_m and G to denote the distribution functions of the left and right sides of (1.31), respectively, and note that by Lemma 2.4 one can pick now $\{n_k(m)\}_{k=1}^{\infty}$, $A_k(m) > 0$, and $C_k(m) \in \mathbb{R}$, $m = 1, 2, \ldots$, such that (2.22) holds again. Hence the same proof works. ■

Proof of Theorem 9. By (1.30), the second condition means that for a subsequence $\{n_m\}_{m=1}^{\infty}$ of the positive integers and some constants $a_m > 0$ and $c_m \in \mathbb{R}$,

$$\frac{1}{a_m}\{V(\psi_1^{(n_m)}, \psi_2^{(n_m)}, \sqrt{n_m}\sigma) - c_m\} \to_D V(\psi_1^*, \psi_2^*, \sigma_*)$$

as $m \to \infty$. Hence the statement is a special case of Theorem 8. ■

Proof of Corollary 15. Let $r \ge 1$ be any integer. Then by Lemma 2.2 or Theorem 8, $F^{*r} \in D_p(\psi_1, \psi_2, \sigma)$ implies that $F^{*r} \in D_p(\psi_1^{(r)}, \psi_2^{(r)}, \sqrt{r}\sigma)$, which in turn, as in the proof of Lemma 2.3 (cf. (2.21)), implies that $F \in D_p(\psi_1, \psi_2, \sigma)$. This means that $D_p^{-1}(F^{*r}) \subset D_p^{-1}(F)$. Conversely, by Lemma 2.3 or Theorem 8, $F \in D_p(\psi_1, \psi_2, \sigma)$ implies that $F \in D_p(\psi_1^{(1/r)}, \psi_2^{(1/r)}, \sigma/\sqrt{r})$, which, as in the proof of Lemma 2.4 (cf. (2.17)), implies $F^{*r} \in D_p(\psi_1, \psi_2, \sigma)$, that is, we have $D_p^{-1}(F) \subset D_p^{-1}(F^{*r})$. ■

Proof of Theorem 10 (i). Let the triple (ψ_1, ψ_2, σ) represent the attracting type. If $\psi_1 = \psi_2 \equiv 0$, and hence, necessarily, $\sigma > 0$, then there is nothing to prove for this means that the attracting type is the normal type.

Suppose that at least one of ψ_1 and ψ_2 is not identically zero. By Theorem 8, $(\psi_1^{(\tau)}, \psi_2^{(\tau)}, \sqrt{\tau}\sigma)$ also attracts F for any $\tau > 0$. But by assumption all these belong to the same given attracting type (ψ_1, ψ_2, σ). This implies that for any $\tau > 0$ there is a constant $c_\tau > 0$ such that

$$c_\tau \psi_j(s/\tau) = \psi_j(s), \quad s > 0, \quad j = 1, 2, \quad \text{and} \quad c_\tau \sqrt{\tau}\sigma = \sigma,$$

or what is the same by setting $\tau = 1/u$ and $a_u = 1/c_{1/u}$,

$$\psi_j(us) = a_u \psi_j(s), \ s > 0; \ a_u > 0; \ j = 1, 2, \quad \text{and} \quad \sigma/(a_u\sqrt{u}) = \sigma, \ u > 0. \tag{2.23}$$

Thus $\psi_j(uvs) = a_{uv}\psi_j(s)$ and $\psi_j(uvs) = a_u\psi_j(vs) = a_u a_v \psi_j(s)$, $s > 0$, $j = 1, 2$, and hence $a_{uv} = a_u a_v$ for any $u, v > 0$. This is the multiplicative form of the Cauchy functional equation ([1], p.17) and from (2.23) we also see that a_u is a non-increasing function of $u > 0$. Hence $a_u = u^{-\rho}, u > 0$, for some constant $\rho \geq 0$. Setting $d_j = -\psi_j(s_0) \geq 0$, $j = 1,2$, where $s_0 > 0$ is chosen so small that $d_1 + d_2 > 0$, we obtain from (2.23) that $\psi_j(s_0 u) = -d_j u^{-\rho}, u > 0, j = 1,2$, and putting finally $u = s/s_0$, we obtain

$$\psi_j(s) = -c_j s^{-\rho}, \ s > 0; \ \rho \geq 0, \ c_j \geq 0, \ j = 1,2, \ c_1 + c_2 > 0, \tag{2.24}$$

where $c_j = d_j s_0^{-\rho}$. So far the side condition concerning the σ in (2.23) did not play any role, nor did the square-integrability condition (1.4) for the ψ_j' s.

Now if σ were positive, then from (2.23) we would get $\rho = 1/2$ which would contradict (1.4). Hence $\sigma = 0$. Also, condition (1.4) implies that $\rho = 1/\alpha$ for some $0 < \alpha < 2$. Hence the attracting type is the triple $({}^{(\alpha)}\psi_1, {}^{(\alpha)}\psi_2, 0)$, which, as noted before the formulation of Theorem 10, represents the stable type with exponent α. ■

The proof of part (ii) of Theorem 10 requires a lemma. Let ψ_1 and ψ_2 be two functions on $(0,\infty)$ satisfying the regularity conditions above (1.4), but not necessarily (1.4) itself, such that at least one of them is not identically zero and consider the following condition:

There exist a set I with cardinality less than continuum and positive numbers $\{\tau_k : k \in I\}$ such that for each $\tau > 0$ one can find a τ_k, $k \in I$, and a constant $c(\tau,k)$ such that $c(\tau,k)\psi_j(\tau s) = \psi_j(\tau_k s)$ for all $s > 0$, $j = 1,2$. (2.25)

If for a given $\tau > 0$ it is $\tau_m \in \{\tau_k : k \in I\}$ that is given by this condition, then we write $\tau \succ \tau_m$. Setting $H_k = \{\tau > 0 : \tau \succ \tau_k\}$, condition (2.25) implies

$$\bigcup_{k \in I} H_k = (0,\infty), \tag{2.26}$$

and the required lemma is the following.

LEMMA 2.5. *If condition (2.25) holds, then for each $u > 0$ there exists a constant $a_u > 0$ such that*

$$\psi_j(us) = a_u \psi_j(s) \quad \text{for all} \quad s > 0, \ j = 1,2. \tag{2.27}$$

Proof. Condition (2.25) implies via (2.26) that there exists a $k_0 \in I$ such that H_{k_0} is uncountable. (Otherwise the cardinality of the half line $(0,\infty)$ would be less than continuum.) Now τ_{k_0} may or may not be in H_{k_0}, but we never the less can fix a $t_0 \in H_{k_0}$ such that $t_0 \neq \tau_{k_0}$. Since the set

$$\{(\log t - \log \tau_{k_0})/(\log t_0 - \log \tau_{k_0}) : t \in H_{k_0}\}$$

is uncountable, it cannot be a subset of or coincide with the set of rational numbers. Hence we can find a $t_1 \in H_{k_0}$ such that with $v_l = t_l/\tau_{k_0}$, $l = 1, 2$,

$$\log v_1 / \log v_0 = (\log t_1 - \log \tau_{k_0})/(\log t_0 - \log \tau_{k_0}) \quad \text{is irrational.} \tag{2.28}$$

Now with the fixed constant $c = c(t_1, k_0) > 0$,

$$\psi_j(v_1 s) = \psi_j(t_1 \frac{s}{\tau_{k_0}}) = \frac{1}{c}\psi_j(\tau_{k_0}\frac{s}{\tau_{k_0}}) = \frac{1}{c}\psi_j(s), \quad s > 0;\ j = 1, 2,$$

from which by induction

$$\psi_j(v_1^k s) = \frac{1}{c^k}\psi_j(s), \quad s > 0; \quad j = 1, 2; \quad k = 1, 2, \ldots.$$

Similarly, with the fixed constant $d = c(t_0, k_0)$,

$$\psi_j(v_0^l s) = \frac{1}{d^l}\psi_j(s), \quad s > 0; \quad j = 1, 2; \quad l = 1, 2, \ldots.$$

Hence for any $k, l = 1, 2, \ldots$,

$$\psi_j\left(\frac{v_1^k}{v_0^l} s\right) = \frac{1}{c^k}\psi_j(\frac{s}{v_0^l}) = \frac{d^l}{c^k}\psi_j(s), \quad s > 0; \quad j = 1, 2.$$

Using now Kronecker's well-known theorem, it follows form (2.28) that the set $\{k \log v_1 - l \log v_0 : k, l \text{ positive integers}\}$ is dense in $\mathbb{R}$, and hence the set $\{v_1^k / v_0^l : k, l \text{ positive integers}\}$ is dense in $(0, \infty)$.

Let $u > 0$ be arbitrary. Then we can find two sequences of integers $k_m, l_m \geq 1$ such that

$$v_1^{k_m}/v_0^{l_m} \downarrow u \quad \text{as} \quad m \to \infty.$$

Choosing now a sufficintly small $s_0 > 0$ for which $\psi_j(\min(s_0, us_0)) \neq 0$ for at least one of $j = 1$ or $j = 2$, for that j we have by right continuity that

$$\psi_j(us_0) = \lim_{m\to\infty} \psi_j\left(\frac{v_1^{k_m}}{v_0^{l_m}} s_0\right) = \lim_{m\to\infty} \frac{d^{l_m}}{c^{k_m}}\psi_j(s_0).$$

Hence the limit

$$0 < a_u := \lim_{m\to\infty} d^{l_m}/c^{k_m} < \infty$$

necessarily exists, and we have (2.27). Since $u > 0$ was arbitrary, the lemma is proved. ■

We point out that the monotonicity of ψ_1 and ψ_2 is not used in the above proof.

Proof of Theorem 10 (ii). The condition means that $F \in D_p(\psi_1, \psi_2, \sigma)$ for some $\sigma \geq 0$ and functions ψ_1 and ψ_2 satisfying the usual conditions above and in (1.4) such that at least one of ψ_1 and ψ_2 is not identically zero but (2.24) is not true for any $\rho = 1/\alpha$, $0 < \alpha < 2$, if $\sigma = 0$.

Suppose the conclusion is not true. This implies by Theorem 8 that, in particular, the set of types given by $\{(\psi_1^{(1/\tau)}, \psi_2^{(1/\tau)}, \sigma/\sqrt{\tau}) : \tau > 0\}$ has cardinality less then continuum, which in turn implies that condition (2.25) holds in such a way that for the constant $c(\tau, k)$ found for $\tau > 0$ also satisfies $c(\tau,k)\tau/\sqrt{\tau} = \sigma$ for each $k \in I$. Lemma 2.5 then implies that

$$\psi_j(us) = a_u\psi_j(s), \quad s > 0; \quad a_u > 0; \quad j = 1,2, \quad \text{and} \quad \frac{a\sigma}{a_u\sqrt{u}} = \sigma, \quad u > 0,$$

for some constant $a > 0$, which is a version of (2.23). Hence the argument following (2.23) in the proof of part (i) implies that $\sigma = 0$ and that (2.24) holds with $\rho = 1/\alpha$ for some $0 < \alpha < 2$, and we have a contradiction. ■

Aiming at the proof of Theorem 11, the following simple formula is very important.

LEMMA 2.6. *If ψ is a function on $(0, \infty)$ satisfying the conditions above and in (1.4), then for any $\varepsilon > 0$,*

$$\int_\varepsilon^\infty \int_\varepsilon^\infty (u \wedge v)d\psi(u)d\psi(v) = \int_\varepsilon^\infty \psi^2(u)du + \varepsilon\psi^2(\varepsilon).$$

Proof. By a somewhat lengthy but elementary computation we obtain that for any $\varepsilon < t < \infty$,

$$\begin{aligned}\int_\varepsilon^t \int_\varepsilon^t (u \wedge v)d\psi(u)d\psi(v) &= \int_\varepsilon^t \psi^2(u)du + \varepsilon\psi^2(\varepsilon) + t\psi^2(t) \\ &\quad - 2\varepsilon\psi(\varepsilon)\psi(t) - 2\psi(t)\int_\varepsilon^t \psi(u)du.\end{aligned} \tag{2.29}$$

By (1.4), $\psi(t) \to 0$ and $t\psi^2(t) \to 0$ as $t \to \infty$, and also

$$\begin{aligned}0 \leq \psi(t)\int_\varepsilon^t \psi(u)du &\leq |\psi(t)|\left(\int_\varepsilon^t du \int_\varepsilon^t \psi^2(u)du\right)^{1/2} \\ &\leq \left(t\psi^2(t)\int_\varepsilon^t \psi^2(u)du\right)^{1/2} \to 0\end{aligned}$$

as $t \to \infty$. Hence, letting $t \to \infty$ in (2.29), the lemma follows. ■

Proof of Theorem 11. First we consider sufficiency. Let $h > 1$ be any common continuity point of ψ_1 and ψ_2 and introduce the decomposition

$$\overline{V}(\psi_{1k}, \psi_{2k}, \sigma_k) + c_k = \overline{V}_k(h) + \overline{Z}_k(h), \tag{2.30}$$

where

$$\begin{aligned}\overline{V}_k(h) = &- \int_1^h (N_1(s) - s)d\psi_{1k}(s) - \int_0^1 N_1(s)d\psi_{1k}(s) \\ &+ \int_1^h (N_2(s) - s)d\psi_{2k}(s) + \int_0^1 N_2(s)d\psi_{2k}(s) \\ &+ (N_1(h) - h)\psi_{1k}(h) - (N_2(h) - h)\psi_{2k}(h)\end{aligned}$$

and

$$\begin{aligned}\overline{Z}_k(h) = &- \int_h^\infty \{(N_1(s) - s) - (N_1(h) - h)\}d\psi_{1k}(s) \\ &+ \int_h^\infty \{(N_2(s) - s) - (N_2(h) - h)\}d\psi_{2k}(s) \\ &+ \sigma_k Z + c_k.\end{aligned}$$

The basic motivation for such a decomposition is that, due to the fact that a Poisson process has independent increments, $\overline{V}_k(h)$ and $\overline{Z}_k(h)$ are *independent* for each k.

A very elementary probabilistic reasoning, based on the fact that a Poisson process has a fixed discontinuity at the point 1 with probability zero, one can show that by (1.35), as $k \to \infty$,

$$\begin{aligned}\overline{V}_k(h) \to_P \overline{V}(h) := &- \int_1^h (N_1(s) - s)d\psi_1(s) - \int_0^1 N_1(s)d\psi_1(s) \\ &+ \int_1^h (N_2(s) - s)d\psi_2(s) + \int_0^1 N_2(s)d\psi_2(s) \\ &+ (N_1(h) - h)\psi_1(h) - (N_2(h) - h)\psi_2(h).\end{aligned} \tag{2.31}$$

Furthermore, since $(N_j(h) - h)/\sqrt{h}$ has a limiting (standard normal) distrubution and $\sqrt{h}\psi_j(h) \to 0$ as $h \to \infty$, $j = 1, 2$, by (1.4), we see that

$$\overline{V}(h) \to_P \overline{V}(\psi_1, \psi_2, 0) \quad \text{as} \quad h \to \infty. \tag{2.32}$$

On the other hand, since a Poisson process has stationary increments,

$$\begin{aligned}\overline{Z}_k(h) =_D \tilde{Z}_k(h) := &- \int_h^\infty \{\tilde{N}_1(s - h) - (s - h)\}d\psi_{1k}(s) + \sigma_k \tilde{Z} \\ &+ \int_h^\infty \{\tilde{N}_2(s - h) - (s - h)\}d\psi_{2k}(s) + c_k,\end{aligned} \tag{2.33}$$

where

$$(\tilde N_1(\cdot),\ \tilde Z,\ \tilde N_2(\cdot) =_D (N_1(\cdot), Z, N_2(\cdot)).$$

Furthermore, if $(\tilde N_1^{(j)}(\cdot), \tilde Z^{(j)}, \tilde N_2^{(j)}(\cdot))$, $j = 1,2,\ldots,k$, are independent copies of $(\tilde N_1(\cdot), \tilde Z, \tilde N_2(\cdot))$, then, substituting first $s = ku$,

$$\begin{aligned}
\tilde Z_k(h) &= -\int_{h/k}^{\infty}\{\tilde N_1(ku-h)-(ku-h)\}d\psi_{1k}(ku)+\sigma_k\tilde Z\\
&\quad+\int_{h/k}^{\infty}\{\tilde N_2(ku-h)-(ku-h)\}d\psi_{2k}(ku)+c_k\\
&=_D\sum_{j=1}^{k}\left\{\int_{h/k}^{\infty}\left\{\tilde N_1^{(j)}(u-\frac{h}{k})-(u-\frac{h}{k})\right\}d\psi_{1k}(ku)+\frac{\sigma_k}{k}\tilde Z^{(j)}\right.\\
&\quad\left.+\int_{h/k}^{\infty}\left\{\tilde N_2^{(j)}(u-\frac{h}{k})-(u-\frac{h}{k})\right\}d\psi_{2k}(ku)\right\}+c_k,
\end{aligned}\tag{2.34}$$

that is, $\overline{Z}_k(h)$ is distributed as a sum of k independent and identically distributed random variables for each k. Also, $E\overline{Z}_k(h) = c_k \to c$, as $k \to \infty$, by (1.36) for each $h > 1$, and using (2.34) and Lemma 2.6,

$$\begin{aligned}
\mathrm{Var}(\overline{Z}_k(h)) &= \sigma_k^2+\sum_{j=1}^{2}\int_h^{\infty}\int_h^{\infty}\{(s-h)\wedge(t-h)\}d\psi_{jk}(s)\psi_{jk}(t)\\
&=\sigma_k^2+\sum_{j=1}^{2}\left\{\int_h^{\infty}\int_h^{\infty}(s\wedge t)d\psi_{jk}(s)d\psi_{jk}(t)-h\psi_{jk}^2(h)\right\}\\
&=\int_h^{\infty}\psi_{1k}^2(s)ds+\sigma_k^2+\int_h^{\infty}\psi_{2k}^2(s)ds\\
&=s_k^2(h).
\end{aligned}\tag{2.35}$$

Now exacly as in the proof of Theorem 1' above, using (2.31) and (2.32), we can consruct a sequence $\{h_k\}$ of positive numbers such that $h_k \to \infty$, $h_k/k \to 0$, and

$$\overline{V}_k(h_k) \to_P \overline{V}(\psi_1,\psi_2,0) \quad \text{as} \quad k\to\infty, \tag{2.36}$$

and, with $\underline{s}$ and $\bar s$ given in (1.37) but otherwise not using condition (1.37),

$$\liminf_{k\to\infty} s_k^2(h_k) = \underline{s}^2 \quad \text{and} \quad \limsup_{k\to\infty} s_k^2(h_k) = \bar s^2. \tag{2.37}$$

Since by (2.30) we also have

$$\overline{V}(\psi_{1k},\psi_{2k},\sigma_k)+c_k = \overline{V}_k(h_k)+\overline{Z}_k(h_k) \tag{2.38}$$

with the two terms on the right side being independent for each $k = 1, 2, \ldots$, and

$$E\overline{Z}_k(h_k) = c_k \quad \text{and} \quad \mathrm{Var}(\overline{Z}_k(h_k)) = s_k^2(h_k), \; k = 1, 2, \ldots, \tag{2.39}$$

the above considerations, (2.37), and condition (1.37) now applied yield

$$\overline{Z}_k(h_k) \to_D \sigma Z + c \quad \text{as} \quad k \to \infty,$$

where the standard normal variable Z is independent of the two independent Poisson processes through which the limiting random variable in (2.36) is defined. The last convergence, (2.36), and (2.38) now give

$$\overline{V}(\psi_{1k}, \psi_{2k}, \sigma_k) + c_k \to_D \overline{V}(\psi_1, \psi_2, 0) + \sigma Z + c = \overline{V}(\psi_1, \psi_2, \sigma) + c$$

as $k \to \infty$ and hence the sufficiency half of the theorem.

Now we turn to necessity, starting out again from (2.30) and knowing (2.33), (2.34), and (2.35), where presently $h \geq 1$ is arbitrary.

Suppose first that the sequence $\{s_k^2(h)\}$ is unbounded for some $h \geq 1$ and/or $\{c_k\}$ is unbounded. Then, since $\overline{Z}_k(h) = Y_k(h)s_k(h) + c_k$, where

$$Y_k(h) = (\overline{Z}_k(h) - c_k)/s_k(h) \to_D N(0,1) \quad \text{as} \quad k \to \infty,$$

it follows by elementary probabilistic considerations that $\{\overline{Z}_{k'}(h)\}$ is stochastically unbounded for this h along some subsequence $\{k'\} \subset \{k\}$. By assumption $\overline{V}_{k'}(h) + \overline{Z}_{k'}(h)$ has a limiting distribution as $k' \to \infty$, as it follows from condition (1.34) and (2.30), and hence it is stochastically bounded. As another set of elementary probabilistic considerations shows, this can only happen if $\{\overline{V}_{k'}(h)\}$ is also stochastically unbounded. Hence both sequences $\{\overline{V}_{k'}(h)\}$ and $\{\overline{Z}_{k'}(h)\}$ are stochastically unbounded and $\overline{V}_{k'}(h)$ and $\overline{Z}_{k'}(h)$ are independent for each k'. However, this easily implies that the sum sequence $\{\overline{V}_{k'}(h) + \overline{Z}_{k'}(h)\}$ is stochastically unbounded and this, via (2.30), contradicts the assumption in (1.34). Thus we conclude that both the sequence $\{s_k^2(h), h \geq 1\}$ of functions and the sequence $\{c_k\}$ of constants are bounded.

Since $s_k^2(\cdot)$ is a non-increasing continuous function on $[1, \infty)$ for each k, for each subsequence $\{k'\} \subset \{k\}$ we can choose a further subsequence $\{k''\} \subset \{k'\}$ such that

$$s_{k''}^2(\cdot) \Rightarrow s_*^2(\cdot) \quad \text{and} \quad c_{k''} \to c^*, \quad \text{as} \quad k \to \infty, \tag{2.40}$$

for some non-increasing function $s_*^2(\cdot)$ on $[1, \infty)$ and a constant $c^* \in \mathbb{R}$. Then by (2.33), (2.34), and (2.35),

$$\overline{Z}_{k''}(h) \to_D N(c^*, s_*^2(h)), \quad \text{as} \quad k'' \to \infty,$$

for each continuity point $h \geq 1$ of $s_*^2(\cdot)$. This, the independence of the summands in (2.30), and (1.34) now imply that

$$\overline{V}_{k''}(h) \to_D \overline{W}(h), \quad \text{as} \quad k'' \to \infty,$$

where $\overline{W}(h)$ is a proper, possibly degenerate random variable for each continuity point $h \geq 1$ of $s_*^2(\cdot)$. This could not happen if we had $\psi_{jk''}(s) \to -\infty$ for some $s = s_0 > 0$ and $j = 1$ or $j = 2$, because this limit would then be $-\infty$ for all $0 < s \leq s_0$ and then, since again the sum of the three terms in $\overline{V}_{k''}(h)$ that involve $\psi_{1k''}$ and the sum of the other three terms that involue $\psi_{2k''}$ are independent, the sequence $\{\overline{V}_{k''}(h)\}$ could not even be stochastically bounded if h is large enough. Therefore,

$$\limsup_{k'' \to \infty} |\psi_{jk''}(s)| < \infty, \quad s > 0, \quad j = 1, 2.$$

Then along a further subsequence $\{k'''\} \subset \{k''\}$,

$$\psi_{jk'''}(\cdot) \Rightarrow \psi_j^*(\cdot), \quad j = 1, 2, \tag{2.41}$$

for some ψ_1^* and ψ_2^* on $(0, \infty)$. Since we can clearly assume without loss of generality that $\{k''\}$ above has been chosen in such a way that all three terms in $s_{k''}^2(\cdot)$, given in (1.33) or (2.35) converge separately, that is, $\sigma_{k''}^2 \to \tilde{\sigma}^2$,

$$\int_\cdot^\infty \psi_{1k''}^2(s)ds \Rightarrow \varphi_1(\cdot) \quad \text{and} \quad \int_\cdot^\infty \psi_{2k''}^2(s)ds \Rightarrow \varphi_2(\cdot)$$

for some non-increasing functions φ_1 and φ_2 on $(0, \infty)$ as $k'' \to \infty$, it follows by Fatou's lemma that both ψ_1^* and ψ_2^* satisfy (1.4).

Now, by (2.40), (2.41), and the already proved sufficiency part of the theorem we have

$$\overline{V}(\psi_{1k'''}, \psi_{2k'''}, \sigma_{k'''}) + c_{k'''} \to_D \overline{V}(\psi_1^*, \psi_2^*, \sigma^*) + c^*, \quad \text{as} \quad k''' \to \infty,$$

where

$$\sigma^* = \lim_{h \to \infty} s_*(h) < \infty.$$

Then by (1.34) and uniqueness, $\psi_j^* = \psi_j$, $j = 1, 2$, and $c^* = c$, and since $\{k'\} \subset \{k\}$ was arbitrary, we conclude that (1.35) and (1.36) hold true.

Starting out finally from these conditions (1.35) and (1.36), the sufficiency proof provides a sequence $\{h_k\}$ such that $h_k \to \infty$ and $h_k/k \to 0$ as $k \to \infty$, and (2.36)

(2.37), and (2.38) are all in force. Now if we had $\underline{s}^2 < \bar{s}^2$, then along one subsequence $\{k'\}$ for which $s^2_{k'}(h_{k'}) \to \underline{s}^2$ we would have

$$\overline{V}(\psi_{1k'}, \psi_{2k'}, \sigma_{k'}) + c_{k'} \to_D \overline{V}(\psi_1, \psi_2, \underline{s}) + c,$$

and along another subsequence $\{k''\}$ for which $s^2_{k''}(h_{k''}) \to \bar{s}^2$ we would have

$$\overline{V}(\psi_{1k''}, \psi_{2k''}, \sigma_{k''}) + c_{k''} \to_D \overline{V}(\psi_1, \psi_2, \bar{s}) + c,$$

which, in view of condition (1.36), is impossible. Hence $\underline{s}^2 = \bar{s}^2$, and by uniqueness this common value must be σ^2, that is, (1.37) is also satisfied. ■

Proof of Theorem 12. Everything is based on the following distributional equality, following from (1.30) and (2.14) in the proof of Lemma 2.1:

$$\sum_{l=1}^{n'} \overline{V}_l(\psi_1, \psi_2, \sigma) =_D V(\psi_1^{(n')}, \psi_2^{(n')}, \sqrt{n'}\sigma) + \mu_{n'}(\psi_1, \psi_2). \tag{2.42}$$

(i) Setting $\psi_{jk}(\cdot) = \psi_j^{(n'_k)}(\cdot)/a_1(n'_k)$ and $c_k = 0$, $k = 1, 2 \ldots$, $j = 1, 2$, where we obviously denote the k-th element of the sequence $\{n'\}$ by n'_k, the statement follows from Theorem 11 if we note that, with the present ψ_{jk},

$$\lim_{k\to\infty} \int_1^h \psi_{jk}^2(s)ds = 0 \quad \text{for each} \quad h \geq 1, \quad j = 1, 2,$$

the constant $\sup\{\psi_{jk}^2(1) : k \geq 1\} < \infty$ being the common integrable majorant, and hence

$$\lim_{k\to\infty} \left\{ \int_h^\infty \psi_{1k}^2(s)ds + \frac{n'_k \sigma^2}{a_1^2(n'_k)} + \int_h^\infty \psi_{2k}^2(s)ds \right\} = 1, \quad h \geq 1,$$

since h can be replaced by 1 in the latter sequence in the curly braces and then we have 1 for each k. ■

(ii) First we have to show that ψ_1^* and ψ_2^* satisfy (1.4). Let $1 < s < t < \infty$ be arbitrary and choose s' and t' to be continuity points of both ψ_1^* and ψ_2^* such that $1 < s' < s < t < t' < \infty$. Then for a standard left-continuous Poisson process $N(\cdot)$, using (1.42),

$$\int_{s'}^{t'} (N(u) - u) d\psi_j(u/n')/a_1(n') \to \int_{s'}^{t'} (N(u) - u) d\psi_j^*(u)$$

almost surely as $n' \to \infty$. Hence by Fatou's lemma and Lemma 2.6,

$$\begin{aligned}
\int_s^t \int_s^t (u \wedge v) d\psi_j^*(u) d\psi_j^*(v) &\le \int_{s'}^{t'} \int_{s'}^{t'} (u \wedge v) d\psi_j^*(u) d\psi_j^*(v) \\
&= E\left(\int_{s'}^{t'} (N(u) - u) d\psi_j^*(u)\right)^2 \\
&\le \liminf_{n' \to \infty} E\left(\int_{s'}^{t'} (N(u) - u) d\psi_j(u/n')/a_1(n')\right)^2 \\
&= \liminf_{n' \to \infty} \frac{\int_{s'/n'}^{t'/n'} \int_{s'/n'}^{t'/n'} (u \wedge v) d\psi_j(u) d\psi_j(v)}{\int_{1/n'}^{\infty} \psi_1^2(u) du + \sigma^2 + \int_{1/n'}^{\infty} \psi_2^2(u) du} \\
&\le \liminf_{n' \to \infty} \frac{\int_{1/n'}^{\infty} \psi_j^2(u) du + \psi_j^2(1/n')/n'}{\int_{1/n'}^{\infty} \psi_1^2(u) du + \sigma^2 + \int_{1/n'}^{\infty} \psi_2^2(u) du} \\
&\le 1 + (\psi_j^*(1-))^2
\end{aligned}$$

for $j = 1, 2$, where in the last step we again used (1.42). Thus for any $1 < s < \infty$,

$$\int_s^{\infty} \int_s^{\infty} (u \wedge v) d\psi_j^*(u) d\psi_j^*(v) \le 1 + (\psi_j^*(-1))^2, \quad j = 1, 2. \tag{2.43}$$

Also, again by (1.42) and Lemma 2.6, for any continuity point $s \ge 1$ of ψ_j^*,

$$\begin{aligned}
s(\psi_j^*(s))^2 &= \lim_{n' \to \infty} s\psi_j^2(s/n')/a_1(n') \\
&= \lim_{n' \to \infty} \frac{\int_{s/n'}^{\infty} \int_{s/n'}^{\infty} (u \wedge v) d\psi_j(u) d\psi_j(v) - \int_{s/n'}^{\infty} \psi_j^2(u) du}{\int_{1/n'}^{\infty} \psi_1^2(u) du + \sigma^2 + \int_{1/n'}^{\infty} \psi_2^2(u) du} \\
&\le \limsup_{n' \to \infty} \frac{\int_{1/n'}^{\infty} \psi_j^2(u) du + \psi_j^2(1/n')/n'}{\int_{1/n'}^{\infty} \psi_1^2(u) du + \sigma^2 + \int_{1/n'}^{\infty} \psi_2^2(u) du} \\
&\le 1 + (\psi_j^*(1-))^2
\end{aligned}$$

and hence

$$\psi_j^*(s) \to 0 \quad \text{as} \quad s \to \infty, \quad j = 1, 2.$$

By routine manipulation based on (2.29), this and (2.43) together imply that both ψ_1^* and ψ_2^* satisfy (1.4).

Using now (2.42), (1.42), and the fact that the $s_k^2(h)$ of Theorem 11 belonging to the present $\psi_{jk}(\cdot) = \psi_j^{(n_k')}(\cdot)/a_1(n_k')$, $j = 1, 2$, is the same as $v_{n_k'}^2(h)$ of the present theorem, the second and main statement follows from Theorem 11. ■

(iii) If $s > 1$, then by (1.43) and (1.44) we obtain similarly as in case (ii) above that

$$\int_s^\infty \int_s^\infty (u \wedge v) d\psi_j^*(u) d\psi_j^*(v) \le (\psi_j^*(1-))^2 < \infty$$

and

$$s(\psi_j^*(s))^2 \le (\psi_j^*(1-))^2 < \infty, \quad j = 1,2.$$

Hence (1.4) follows again for both ψ_1^* and ψ_2^*. Using (2.42), (1.43), and the fact that

$$s_k^2(h) = \frac{a_1^2(n_k')}{a_2^2(n_k')} v_{n_k'}^2(h), \quad h \ge 1,$$

the main statement follows from Theorem 11 on account of (1.44). ■

(iv) For any $s \ge 1$, using the monotonicity of the ψ_j, the formula (1.40), and condition (1.46),

$$\begin{aligned}\frac{|\psi_j(s/n')|}{A_{n'}} &= \frac{|\psi_j(s/n')|}{a_2(n')} \frac{a_2(n')}{A_{n'}} \\ &\le \frac{|\psi_j(s/n')|}{|\psi_j(1/n')|} \frac{a_2(n')}{A_{n'}} \\ &\le a_2(n')/A_{n'} \to 0, \quad j = 1,2,\end{aligned}$$

as $n' \to \infty$, so that the first statement follows. Using (2.42), (1.45), the realation

$$s_k^2(h) = \frac{a_1^2(n_k')}{A_{n_k'}} v_{n_k'}^2(h), \quad h \ge 1,$$

and the fact that (1.46) clearly implies that $a_1(n_k')/A_{n_k'} \to 0$ as $k \to \infty$, the second statement follows again from Theorem 11. ■

(v) That the limit is of the stated form follows from Theorem 5, and then (1.48), (1.49), and (1.50) all follow from Theorem 11 via (2.42).

Suppose now (1.51). Then for any subsequence $\{n_3\} \subset \{n'\}$ there is a further subsequence $\{n_4\} \subset \{n_3\}$ such that

$$\psi_j(\cdot/n_4)/a_1(n_4) \Rightarrow \overline{\psi}_j(\cdot), \quad j = 1,2, \tag{2.44}$$

as $n_4 \to \infty$, where $\overline{\psi}_1$ and $\overline{\psi}_2$ are some functions with the usual properties above (1.4). Using now part (i) or part (ii), we see that there exists a further subsequence $\{n''\} \subset \{n_4\}$ such that

$$\frac{1}{a_1(n'')} \left\{ \sum_{l=1}^{n''} \overline{V}_l(\psi_1,\psi_2,\sigma) - \mu_{n''}(\psi_1,\psi_2) \right\}$$

has a non-degenerate limit as $n'' \to \infty$. Hence by the convergence of types theorem ([14], pp.40-42) there is a $\delta = \delta_{\{n''\}} > 0$ such that $a_1(n'')/A_{n''} \to \delta$. In the special case when $\underline{v} = \bar{v} = v > 0$, it follows from (1.50) that $\sigma_* = \delta v > 0$. Since σ_* and v are determined by the whole original sequence $\{n'\}$, this implies that δ must be the same for all such subsequences $\{n''\}$. Since $\{n_3\}$ was an arbitrary subsequence of $\{n'\}$, $a_1(n')/A_{n'} \to \delta > 0$, as $n' \to \infty$, in this special case.

If we have (1.47) and (1.52), then the same (1.52) must be true for all $0 < t < s$, and hence (1.53) follows from (1.49) easily. Also, (1.47) and (1.55) imply (1.56) in the same way.

If (1.47), (1.52), and (1.54) hold, then we have (2.44) with $a_1(n_4)$ replaced by $a_2(n_4)$, and our statement concerning the convergence of $a_2(n'')/A_{n''}$ follows by repeating the argument below (2.44). The theorem is completely proved. ■

Proof of Corollary 16. All the asymptotic equalities below are either obvious or obtained by elementary calculations.

(a) Clearly,

$$a_1^2(n) = -n \int_{1/n}^1 \log s\, ds \sim n,$$

so that

$$\psi\left(\frac{s}{n}\right)/a_1(n) = \begin{cases} (-\frac{1}{n} \log \frac{s}{n})^{1/2}, & 0 < s < n, \\ 0, & s \geq n, \end{cases}$$

and we have the condition of part (i) of Theorem 12 as $n \to \infty$, and the statement follows with the centering sequence

$$\mu_n(0, \psi) = \frac{n}{2} \int_{1/n}^1 (\log \frac{1}{s})^{-1/2} ds = \frac{n}{2} \int_0^{\log n} x^{\frac{1}{2}-1} e^{-x} dx.$$

But

$$\frac{1}{\sqrt{n}} \left(\mu_n(0, \psi) - \frac{n}{2}\sqrt{\pi}\right) = -\frac{\sqrt{n}}{2} \int_{\log n}^\infty x^{\frac{1}{2}-1} e^{-x} dx \to 0$$

as $n \to \infty$, and hence the statement. ■

(b) Presently, for any $h \geq 1$ and $\alpha > 0$,

$$a_1^2(n; h) = n \int_{h/n}^1 s^{-2/\alpha} ds \sim \begin{cases} \frac{\alpha}{\alpha-2} n, & \alpha > 2, \\ n \log n, & \alpha = 2, \\ \frac{\alpha}{2-\alpha} (\frac{n}{h})^{2/\alpha}, & \alpha < 2, \end{cases}$$

and

$$\mu_n(0,\psi) = \frac{n}{\alpha}\int_{1/n}^{1} s^{-1/\alpha}ds \sim \begin{cases} \frac{n}{\alpha-1}, & \alpha > 1, \\ n\log n, & \alpha = 1, \\ \frac{n^{1/\alpha}}{1-\alpha}, & \alpha < 1. \end{cases}$$

Hence, as $n \to \infty$,

$$\psi\left(\frac{s}{n}\right)/a_1(n) = \begin{cases} -n^{1/\alpha}s^{-1/\alpha}/a_1(n), & 0 < s < n, \\ 0, & s \geq n, \end{cases}$$

converges to zero for each $s > 0$ if $\alpha \geq 2$, and we have (1.42) for $j = 2$ if $\alpha < 2$, with limiting function ψ_α^*. Finally, note that if $\alpha < 2$,

$$v_n(h) = a_1(n;h)/a_1(n) \sim h^{-1/\alpha}, \quad h \geq 1,$$

and this implies that $\underline{v} = \bar{v} = 0$. ■

(c) Using the fact that $t_k/t_{k-1} \to 0$ as $k \to \infty$, and setting

$$n_k = \left\lceil \frac{\lambda}{t_k} \right\rceil = \lceil \lambda 2^{2^k} \rceil$$

for each $\lambda > 0$, by elementary manipulations we see that for all k large enough,

$$\psi\left(\frac{s}{n_k}\right) = \begin{cases} -b_{k+1}, & s < \lambda, \\ -b_k, & s \geq \lambda. \end{cases}$$

Since $b_k/b_{k+1} \to 0$, upon setting $A_k = b_{k+1}$ we obtain

$$\psi(s/n_k)/A_k \to \psi_\lambda^*(s) = \begin{cases} -1, & s < \lambda, \\ 0, & s \geq \lambda, \end{cases}$$

as $k \to \infty$. The formula for

$$\mu_k(\lambda) = n_k \int_{1/n_k}^{1/2} s\,d\psi(s)$$

given in (1.58) is valid for all k large enough and follows by simple computation just as the relations

$$a_1^2(n_k) \sim \begin{cases} (\lambda-1)2^{2^{k+1}}, & \text{if } \lambda > 1, \\ \lambda 2^{2^{k-1}} 2^{2^k}, & \text{if } \lambda < 1, \\ 2^{2^k}\sum_{j=1}^{k} 2^{2^{j-1}}/(1-2^{-2^{j-1}}), & \text{if } \lambda = 1, \end{cases}$$

and

$$a_2^2(n_k) \sim \begin{cases} \lambda 2^{2^{k+1}}, & \text{if } \lambda > 1, \\ a_1^2(n_k), & \text{if } \lambda \le 1. \end{cases}$$

Hence (1.59), (1.60), and (1.61) all follow, and we obtain (1.58) from part (ii) or part (iv) of Theorem 12, according to $\lambda > 1$ or $\lambda \le 1$, in view of the fact that by (1.32),

$$\overline{V}(0,\psi_\lambda^*,0) = V_{0,0}(0,\psi_\lambda^*,0) + \psi_\lambda^*(1) = N(\lambda) - \lambda I(\lambda > 1),$$

where the second equality follows from equation (2.2) in [3], also obtained by simple computation.

Proof of Corollary 17. For any subsequence $\{n'\} \subset \{n\}$ we still have $\psi_j(s/n')/\psi_j(t/n') \to \infty$ as $n' \to \infty$, and hence it is impossible to find a numerical sequence $\{A_{n'}\}$ such that (1.49) could hold with finite limiting functions.

3. DISCUSSION

As said in the introduction, the present paper is an "organic" continuation of [6], the results of which have been discussed in detail in Section 4 of [6]. Accordingly, Theorem 1* here is new. It leads to Theorems 6 and 7, which in their full generality and detail may be considered to be new as well, together with Corollaries 5*, 13 and 14. Many similar corollaries can be worked out routinly for the characterization of the domain of partial attraction of given concrete infinitely divisible distributions. As mentioned in Section 1, this problem for the Poisson distribution turns out to be intricate, and the surprising solution is given in [3].

Another application is the following. In a very interesting recent paper [15], A. Martin-Löf proves a limit theorem along the special subsequnce $\{2^k\}_{k=1}^{\infty}$, which theorem "clarifies the Petersburg paradox". By Theorem 10 (ii), however, there might exist continuum many very different clarifications. In [5], we investigate the problem how unique is the one given by Martin-Löf. We construct all possible subsequences and describe all possible limiting laws, and finally conclude that Martin-Löf's clarification is unique modulo a constant factor. The inclusion of this factor has the practical consequence that the game can be played arbitrarily and not just in blocks of size 2^k as suggested by Martin-Löf's particular clarification. This paper [5] has the additional didactic aspect that it also illustrates all the results from [7] and [8] on the Petersburg game.

A characteristic-function version of Theorem 8 was first proved by Gnedenko [13] in 1940 by purely Fourier-analytic methods. In contrast, our proof is purely probabilistic, based on the representation of an infinitely divisible random variable. From his theorem Gnedenko [13] also deduced what is Corollary 15 here and his transitivity result in Theorem 9 (not called by him as such), the latter of which is also cited without proof in [14], p. 189.

In the proof of Lemma 5 in [6], concrete constructions are given to show that the domain of partial attraction $D_p(\alpha)$ of a stable law with exponent $0 < \alpha < 2$ is wider than its domain of attraction $D(\alpha)$. This was first proved independently by Doeblin [11], published also in 1940, and Gnedenko [13]. Both proofs are non-constructive. Doeblin uses characteristic functions, while Gnedenko's interesting proof is based on the transitivity theorem. In our setting, Gnedenko's proof is as follows. Consider any F in the domain of partial attraction of the distribution $\overline{F}^{0,\overline{\psi}_\alpha,0}$ of $\overline{V}(0,\overline{\psi}_\alpha,0)$, where, for $0 < \alpha < 2$,

$$\overline{\psi}_\alpha(s) = \begin{cases} -s^{-1/\alpha}, & 0 < s < 1, \\ 0, & s \geq 1. \end{cases}$$

By Khinchin's theorem ([14], p.184, or [10]) such an F exists. Then by Corollary 16 (b), $\overline{F}^{0,\overline{\psi}_\alpha,0}$ is in the domain of partial attraction $D_p(\alpha)$ of the distribution of $\overline{V}(0,\psi_\alpha,0)$, and hence by transitivity (Theorem 9), $F \in D_p(\alpha)$. However, F cannot be in the domain of attraction of this distribution exactly because it is partially attracted to $\overline{F}^{0,\overline{\psi}_\alpha,0}$. While this proof is not constructive, it is certainly much simpler than the one given by K. L. Chung in his footnote on p. 189 in [14], also based on Gnedenko's transitivity.

Theorem 10 (i) was proved by Gnedenko [13] with the characteristic-function method just as that weaker version of Theorem 10 (ii) where "uncountable" stands in place of our "continuum". This weaker version of Theorem 10 implies what Gnedenko and Kolmogorov [14] write on p. 189: "Each distribution law F belongs to the domain of partial attraction of one or a nondenumerable set of types or else does not belong to any domain of partial attraction at all." This conclusion was also achieved by Doeblin [11]. (Note that distributions not in the domain of partial attraction of any law do exist by Corollary 17 here.) At this point an historical remark is perhaps tolerable. Doeblin and Gnedenko appear to be competing on these results at the time. Doeblin announced his results, proved in [11], without proof in the Paris Comptes Rendus in two communications already in 1938 (Vol. 206, p.306 and p.718). However, Gnedenko [13] states that he obtained his results in the spring of 1937 and they were part of his dissertation that he has defended in that year. This "competition", if there was one, is of course not surprising with Lévy standing behind Doeblin and with Khinchin and Kolmogorov behind Gnedenko.

To the best of our knowledge, the present unimprovable version of Theorem 10 with the "continuum" is new. The original weaker or "uncountable" version would require

Lemma 2.5 under the stronger assumption that condition (2.25) holds with a countable set I. It is perhaps interesting to note that the first proof that we had for this weaker version of Lemma 2.5 started out from the Baire category theorem applied to (2.26). The extension of that proof under the present weaker condition (2.25) is impossible under the usual ZF axioms of set theory since the corresponding extension of the Baire category theorem is known to be an independent axiom just as the continuum hypothesis. Hence the problem of extending the original Doeblin-Gnedenko result appeared to be 'one of those set-theoretic problems' at first sight. However, the present proof of Lemma 2.5 completely bypasses all these problems and is in fact much shorter than the first one was.

The essence of Theorem 11 was already remarked upon in the introduction. We believe that the present formulation is cleaner than the original characteristic-function version, first proved by Gnedenko [12], and also that the present proof, based on the fact that a standard Poisson process has independent and stationary increments, really uncovers the ultimate reason behind it.

Theorem 12 is new with all its details. The construction in Corollary 16(c) is a version of a construction, a different one with different but similar purposes, given in [3]. A modification of the present construction will be given in [4], where again the intricacies of a Poisson limit are treated in more detail and a case also illustrates part (iii) of Theorem 12.

Corollary 17 is new in its generality. The special case of the concrete example when $\alpha = 1$ corresponds to a half-sided version of the example of Gnedenko and Kolmogorov [14], pp. 186-189, showing by a rather complicated characteristic-function proof the existence of a distribution not in the domain of partial attraction of any law at all. Examples similar to theirs were constructed independently by Lévy, Khinchin and Gnedenko in the years 1937-1939.

Corollary 18 is also new.

ACKNOWLEDGEMENTS

This paper has grown out of joint work with Rossitza Dodunekova of the University of Sofia on [3]. I thank her also for her comments on the manuscript of this paper. I am also grateful to Vilmos Totik of the University of Szeged for his help in the proof of Lemma 2.5 and to V. V. Vinogradov of Moscow State University for sending me a copy of [13].

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