Structural Reanalysis Using Eigenvalue Modification Technique

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Abstract

The structural response of a system due to any arbitrary loading can be computed if the eigenvalues and eigenvectors are known. The proposed method updates these eigenvalues and eigenvectors due to localized changes in stiffness or mass. It is assumed that the unmodified eigenvalues and eigenvectors for the previous analysis are known. The cost of such re-analysis is far less than that of a regular re-analysis in which the eigenvalues and eigenvectors are extracted from stiffness and mass matrices. The application of this method to various types of design changes has been examined, and a series of simple example problems solved to demonstrate its correctness.

1. Introduction

During the process of designing dynamic elastic systems it is often necessary to determine the effect of a change in a particular system component on the dynamic characteristics of the total system. A new analysis is required for any moderate change in the design parameter. To reduce the cost of re-analysis the eigenvalue modification technique (1,2,3) has been widely used by practicing engineers. In this method the eigenfunctions of the modified system are expressed in terms of the eigenfunctions of the unmodified system. The method is cost effective as it utilizes the eigenfunctions of the existing unmodified system. The purpose of this paper is to demonstrate the insight of the method through solving several simple example problems. The practical difficulties that may be encountered by practicing engineers and the accuracy of the method have also been discussed in detail.

2. Method Of Analysis

The homogenous equation of motion describing the unmodified system is:

$$M\ddot{X} + KX = 0 \tag{1}$$

where M, K, $\ddot{X}$ and X are the mass matrix, stiffness matrix, acceleration and displacement vectors of the system, respectively.

Transform the physical coordinate to the modal coordinate as follows:

$$X = \phi \xi \tag{2}$$

From Eq. 1 and 2 we have:

$$\phi^T M \phi \ddot{\xi} + \phi^T K \phi \xi = 0 \tag{3}$$

or: $$I \ddot{\xi} + \omega_1^2 \xi = 0 \tag{4}$$

where I is an identity matrix and ω_1^2, ϕ and ξ are diagonal eigenvalue matrix, modal vector matrix and modal coordinate vector, respectively.

If Δk and Δm are the changes in the stiffness and mass matrices, respectively, then the equation of motion of the modified system is:

$$(M + \Delta m) \ddot{X} + (K + \Delta k) X = 0 \tag{5}$$

By performing the coordinate transformation as given in Eq. 2, we thus have:

$$[\phi^T M \phi + \phi^T \Delta m \phi] \ddot{\xi} + [\phi^T K \phi + \phi^T \Delta k \phi] \xi = 0 \tag{6}$$

or: $$(I + \Delta\overline{m}) \ddot{\xi} + (\omega_1^2 + \Delta\overline{k}) \xi = 0 \tag{7}$$

where, $\Delta\overline{m}$ and $\Delta\overline{k}$ are expressed as follows:

$$\Delta\overline{m} = \phi^T \Delta m \phi , \quad \Delta\overline{k} = \phi^T \Delta k \phi \tag{8}$$

Changes in stiffness and mass matrices Δk and Δm can be expressed in as follows:

$$\Delta k \text{ or } \Delta m = T A T^T = \sum t_i a_i t_i^T \tag{9}$$

The vector t_i is known as the tie vector and a_i is a constant.

2.1 Equations for Stiffness and Mass Tie

The equation of motion for the modified system due to stiffness change can be expressed as:

$$M\ddot{X} + (K + \Delta k) X = 0 \tag{10}$$

Upon transformation to modal coordinate ξ, Eq. 10 can be expressed as:

$$\phi^T M \phi \ddot{\xi} + (\phi^T K \phi + \phi^T \Delta k \phi) \xi = 0 \tag{11}$$

or: $$I\ddot{\xi} + (\omega_1^2 + \Delta\bar{k})\,\xi = 0, \tag{12}$$

where: $$\Delta\bar{k} = \phi^T \Delta k\, \phi \tag{13}$$

The stiffness change Δk can be expressed as:

$$\Delta k = T\,A\,T^T = \sum t_i\, a_i\, t_i^T \tag{14}$$

Assume: $$\xi = P\, e^{i\omega_2 t}, \tag{15}$$
where ω_2 is the eigenvalues of the modified system and P is the amplitude vector. From Eq. 12, 14, and 15 we have:

$$[-\omega_2^2 I + (\omega_1^2 + \phi^T \sum (t_i\, a_i\, t_i^T)\,\phi]\, P = 0 \tag{16}$$

Express V_i in terms of ϕ and t_i, as follows:

$$V_i = \phi^T t_i \tag{17}$$

Eq. 16 can then be expressed as:

$$(\omega_1^2 - \omega_2^2)\, P = -\sum V_i\, a_i\, V_i^T\, P \tag{18}$$

or: $$\sum \frac{V_i^2}{\omega_1^2 - \omega_2^2} = -\frac{1}{a_i} \tag{19}$$

The solution of the above polynomial equation generates new eigenvalues ω_2. Once the new eigenvalues ω_2 are evaluated, the eigenvectors of the modified system can be computed using the transformation equations.

The equation for mass tie can be derived similarly and can be expressed as:

$$\sum \frac{\omega_2^2 V_i^2}{\omega_1^2 - \omega_2^2} = \frac{1}{a_i} \tag{20}$$

The solution of the above polynomial equation generates new eigenvalues ω_2. New eigenvectors can be computed using the transformation equations.

3. Discussion of Results and Conclusions

Two example problem are solved to demonstrate the validity of the proposed methodology.

3.1 Problem 1:

Verification of the mass modification option is performed by considering a spring mass system with two dof, as shown in Figure 1. A concentrated mass Δm (=4) is added to the unmodified system. The stiffness and mass matrices of the modified system are as follows:

$$K = \begin{bmatrix} 16 & -8 \\ -8 & 20 \end{bmatrix} \quad M = \begin{bmatrix} 8 & 0 \\ 0 & 4 \end{bmatrix} \tag{21}$$

The eigenvalues (ω^2) and eigenvectors (ϕ) of this system are as follows:

$$\omega^2 = \begin{bmatrix} 1.4384472, & 5.5615528 \end{bmatrix}, \quad \phi = \begin{bmatrix} 0.32859615 & 0.130529 \\ 0.18452409 & -0.464879 \end{bmatrix} \tag{22}$$

The eigenvalues (ω^2) and eigenvectors (ϕ) obtained using the program compare exactly with those computed manually in Eq. 22.

3.2 Problem 2:

The proposed method utilizes the eigenvalues and eigenvectors of the Δk or Δm. In many instances it is possible to have singular Δk or Δm. The computer program written for the computation will encounter numerical instability in such cases. To avoid this, the singular matrix Δk or Δm can be made nonsingular by adding a nonsingular stiffness or mass element. In the next step, we subtract the nonsingular stiffness or mass element which was added earlier. To demonstrate, a spring mass system (Figure 2) is considered for verification of the method. The stiffness (K) and mass (M) matrices of the modified system are as follows:

$$K = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 12 & -8 \\ 0 & -8 & 12 \end{bmatrix} \quad M = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \tag{23}$$

The eigenvalues (ω^2) and eigenvectors (ϕ) of the system are shown below:

$$\omega^2 = \begin{bmatrix} 0.6086, & 2.22707, & 5.1643 \end{bmatrix}, \quad \phi = \begin{bmatrix} 0.241 & -0.423 & -.113 \\ 0.336 & 0.097 & 0.358 \\ 0.281 & 0.249 & -0.336 \end{bmatrix} \tag{24}$$

The stiffness (k) and mass (M) matrices of the unmodified system are as follows:

$$K = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix}, \quad M = \begin{bmatrix} 4 & & \\ & 4 & \\ & & 4 \end{bmatrix} \tag{25}$$

The eigenvalues (ω^2) and eigenvectors (ϕ) of the system are as follows:

$$\omega^2 = \begin{bmatrix} 3.414, & 0.586, & 2.0 \end{bmatrix}, \quad \phi = \begin{bmatrix} 0.2500 & 0.2500 & -0.3536 \\ -0.3536 & 0.3536 & 0.6000 \\ 0.2500 & 0.2500 & 0.3536 \end{bmatrix} \tag{26}$$

The stiffness change Δk is as follows:

$$\Delta k = \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} \qquad (27)$$

The stiffness matrix Δk given in Eq. 27 is singular. Therefore, Δk is reconstructed by adding two nonsingular matrices as follows:

$$\Delta k = \Delta k_1 - \Delta k_2 \qquad (28)$$

$$= \begin{bmatrix} 6 & -4 \\ -4 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

The modification was done in two steps in the computation: the first with Δk_1 and the second with Δk_2. The results obtained using the proposed method compare exactly with those (Eq. 24) obtained from hand computations.

It can be concluded from the results of the analyses considered that the proposed method is capable of computing the dynamic behavior of the modified system by utilizing the eigenfunctions of the unmodified system.

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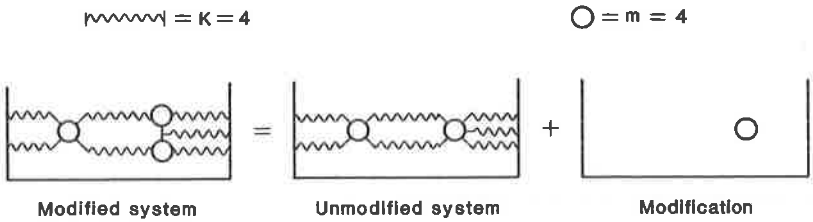

Figure 1. Mass addition in a 2 DOF system.

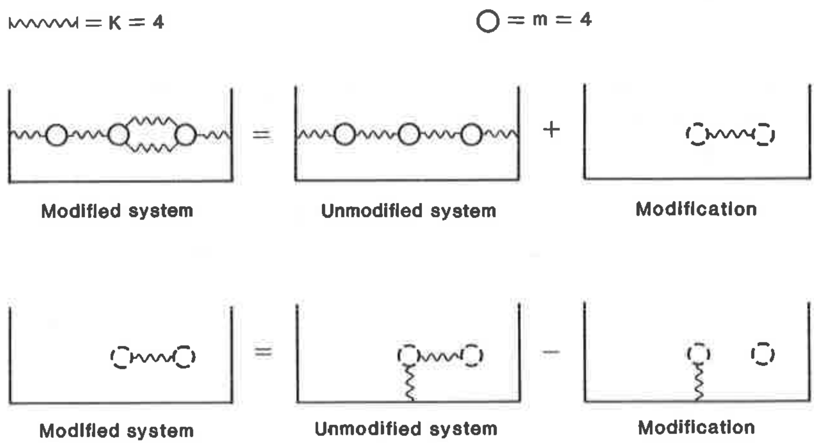

Figure 2 Singular stiffness addition in a 3 DOF system.