

RADIAL OSCILLATIONS OF HIGHLY STRESSED, NONHOMOGENOUS, THICK-WALLED CYLINDRICAL AND SPHERICAL SHELLS

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SUMMARY

The stability and the vibrational characteristics of thick and thick-walled bodies using the rigorous finite elasticity theory in conjunction with the theory of small deformations superposed on large deformations have been investigated by several authors. Usually the material of the body is assumed to be elastic, isotropic, incompressible, and homogeneous. Recently, the problem of radial oscillations of nonhomogeneous spherical shells of arbitrary wall thickness subjected to uniform pressure has been studied for neo-Hookean materials by introducing material nonhomogeneity in the radial direction as some continuous function of the radial distance. However, the questionable results obtained have led the present investigators to re-examine the problem and also extend it to cylindrical shells.

The present work investigates the infinitesimal breathing motions of spherical and long cylindrical shells of arbitrary wall thickness subjected to a finite, axisymmetric deformation field caused by uniform internal and/or external pressures. A neo-Hookean material with a material constant varying continuously along the radial direction is used. The shell is first subjected to finite, axisymmetric, static deformations and is then exposed to a secondary, axisymmetric, dynamic displacement field. Based on the theory of small deformations superposed on large deformations, closed form expressions are obtained for the frequency of small oscillations about the highly prestressed state. Frequency versus initial deformation parameter curves are given for several nonhomogeneity functions and for various wall thicknesses. The softening or the hardening behaviors of the shells for varying prestress values are observed from these curves. When the frequency of breathing motions cease to be real valued, the superposed secondary motions are no longer periodic. Thus, the critical value of the prestress causing instability is defined as the one which corresponds to zero frequency. It is seen that when the nonhomogeneity parameters are taken zero, the known results of the homogeneous case are obtained.

Although the material is assumed to be incompressible, the theory is general enough to include compressible materials. However, in this case, the governing equations of the problem become more complicated for a closed form solution. An interesting and practical extension of the problem is the vibration analysis of layered shells.

1. Introduction

The theory of small deformations superposed on large elastic deformations {1} has been extensively applied to analyze the stability and/or vibrational characteristics of thick and thick-walled bodies such as hollow spherical and circular cylindrical shells subjected to finite elastic deformations. In most cases {2-7} , the material of the body is assumed to be elastic, isotropic, incompressible and homogeneous. To the authors' knowledge there has not been much progress in the case of nonhomogeneous media. To a certain extent, Nowinski and Shahinpoor {8} considered the effect of nonhomogeneity. They studied the radial oscillations, or the so-called breathing motions of hollow spherical shells of arbitrary wall thickness subjected to finite deformations caused by a uniform internal hydrostatic pressure. In {8} , a continuous nonhomogeneity is introduced ; the material of the shell is of neo-Hookean type with the material constant varying as a function of the radial distance. However, the results obtained indicate, incorrectly, an erratic behavior of the shells.

The erratic behavior in the results of {8} has led the present authors to re-examine the problem, and also extend it to circular cylindrical shells. In the present work, the effect of initial internal and/or external pressures on the free oscillatory motions of hollow cylindrical and spherical shells is studied by applying the theory outlined in {1}. The material of the shells is assumed to be of the neo-Hookean type, with the material constant varying as a function of the radial coordinate. For spherical shells, both the initial and the secondary deformation fields are assumed to be spherically symmetric. Similarly, for cylindrical shells, both fields have axial symmetry.

The field equations governing the small oscillations superposed on large elastic deformations have closed form solutions for both cylindrical and spherical shells. The characteristic equations of both problems are obtained from their respective boundary conditions. The characteristic equations contain the frequencies of small oscillations as the unknown. It is observed that, in the case of cylindrical shells pure radial oscillations are not possible when the material constant contains a term which varies as a linear function of the radial coordinate. A similar situation exists for spherical shells, though it is not included here.

2. Formulation of the Problems

2.1. Cylindrical Shells

Consider a long, circular cylindrical shell of arbitrary wall thickness made of a neo-Hookean type material. Let A_1 and A_2 be, respectively, the inner and the outer radii of the undeformed shell. Let

$$a_1 = \mu_1 A_1 \quad , \quad a_2 = \mu_2 A_2 \quad , \quad (1)$$

be the inner and the outer radii of the shell in the finitely deformed state caused by the static pressures q_1 and q_2 applied on the inner and the outer curved surfaces. The differential pressure $q = q_2 - q_1$ required to procedure such a deformation field is given by, (ref. {6} , eq.(3)),

$$q = q_2 - q_1 = \int_{a_1}^{a_2} \Phi (Q^2 - Q^{-2}) \frac{dr}{r}, \quad (2)$$

where

$$\Phi = 2 \frac{\partial W}{\partial I}, \quad Q = R/r. \quad (3)$$

In eq. (3), $W = C_1(I-3)$ denotes the strain energy density function, R and r are the radial coordinates of a material point in the undeformed and the deformed states. If the material of the shell is nonhomogeneous with respect to the radial coordinate, then Φ is a function of R .

We now consider a state of free, infinitesimal radial oscillations superposed onto the finitely deformed state such that the only nonzero incremental displacement component is in the radial direction and it is given by

$$w_1 = w_1(r, t). \quad (4)$$

The secondary displacement field is governed by the equation

$$\Phi(Q^2 - Q^{-2}) w_{1,rr} - 2 \{ \Phi(Q^{-1} - Q)^2 - Q(1 - Q^2) \} \frac{d\Phi}{dQ} \frac{w_{1,r}}{r} + p'_{,r} = \rho \ddot{w}_1, \quad (5)$$

and by the incompressibility condition in the secondary deformation field :

$$w_{1,r} + \frac{1}{r} w_1 = 0. \quad (6)$$

In eq.(5), ρ denotes the mass density and a dot denotes differentiation with respect to time. $p'(r)$ is an unknown hydrostatic pressure.

The boundary conditions, obtained by the requirement that the secondary surface tractions vanish, are

$$2\Phi Q^2 w_{1,r} + p' = 0, \quad \text{on } r = a_1, a_2. \quad (7)$$

If the secondary motions are free harmonic oscillations in the radial direction, then

$$w_1(r, t) = u(r) \cdot \exp(i\omega t),$$

and

$$p'(r, t) = f(r) \cdot \exp(i\omega t). \quad (8)$$

Upon substitution of eqs.(8) into the governing equations, eqs. (5,6), we obtain closed form solutions for $u(r)$ and $f(r)$ containing two integration constants. Substituting $u(r)$ and $f(r)$ into the boundary conditions, eqs.(7), and requiring that the determinant of the coefficient matrix vanishes for a non-trivial solution, we obtain the characteristic equation of the problem, and hence an expression for the frequencies of small oscillations;

$$\bar{\omega}^2 = \frac{8}{\rho A_1^2 \ln \left\{ \frac{(1-\lambda^2 \bar{K})}{\lambda^2 (1-\bar{K})} \right\}} \int_{Q_1}^{Q_2} \phi Q^3 dQ \quad (9)$$

where $\bar{K} = 1 - \mu_1^2$, $\lambda = A_1/A_2$. If the material is homogeneous, then ϕ is constant and the resulting expression corresponds to eq.(29) of ref. {6}. Furthermore, if the primary deformation is infinitesimal, then $\bar{K} \rightarrow 0$, and the expression thus obtained reduces to eq.(32) of ref. {6}.

For simplicity, we assume a quadratic form for ϕ . If ϕ_i , ϕ_m , and ϕ_o denote, respectively, the material constants on the inner, middle, and the outer surfaces, then we can write

$$\phi = \phi_i \left\{ C_1 \left(\frac{R}{A_1} \right)^2 + C_2 \left(\frac{R}{A_1} \right) + C_3 \right\} \quad (10)$$

where

$$C_1 = \frac{\lambda^2}{(1-\lambda)^2} \{ 2(\phi_o/\phi_i) + 2-4(\phi_m/\phi_i) \} \quad (11.a)$$

$$C_2 = \frac{\lambda}{1-\lambda} (\phi_o/\phi_i - 1) - \frac{1+\lambda}{\lambda} C_1 \quad (11.b)$$

$$C_3 = 1 - C_1 - C_2$$

By substituting eq.(10) into eq.(9) and integrating, for $\bar{K} \neq 0$ the nondimensionalized frequency expression is given by

$$\begin{aligned} \bar{\omega}^2 = & \frac{2}{\ln \frac{(1-\lambda^2 \bar{K})}{\lambda^2 (1-\bar{K})}} \left\{ C_1 \left[\frac{3-2\bar{K}}{(1-\bar{K})^2} - \frac{3-2\lambda^2 \bar{K}}{(1-\lambda^2 \bar{K})^2} + 2 \ln \frac{(1-\lambda^2 \bar{K})}{\lambda^2 (1-\bar{K})} \right] \right. \\ & + C_2 \left[\frac{5-3\bar{K}}{2(1-\bar{K})^2} - \frac{5\lambda-3\lambda^3 \bar{K}}{2(1-\lambda^2 \bar{K})^2} + \frac{3}{2\sqrt{\bar{K}}} \ln \frac{(1+\sqrt{\bar{K}}) \sqrt{1-\lambda^2 \bar{K}}}{(1+\lambda^2 \sqrt{\bar{K}}) \sqrt{1-\bar{K}}} \right] \\ & \left. + C_3 \left[\frac{\lambda^4 \bar{K} - 2\lambda^2 + 2 - \bar{K}}{(1-\bar{K})^2 \cdot (1-\lambda^2 \bar{K})^2} \right] \right\} \quad (12.a) \end{aligned}$$

and for $\bar{K} = 0$,

$$\bar{\omega}^2 = \frac{2}{\ln \frac{1}{\lambda^2}} \left[2C_1 \ln \frac{1}{\lambda} + C_2 \left(4 - \frac{5}{2}\lambda - \frac{3}{2}\lambda^2 \right) + 2C_3 (1-\lambda^2) \right] \quad (12.b)$$

where

$$\bar{\omega}^2 = \rho A_1^2 \omega^2 / \phi_i \quad (13)$$

Figures (1-4) show $\bar{\omega}^2$ versus the initial deformation parameter \bar{K} . For a general quadratic form of the material constant Φ , fig.1. indicates that pure radial oscillations can exist only for $\bar{K} > 0^*$. In fact, a close examination of eq.(12) shows that $\bar{\omega}^2$ becomes a complex number for $\bar{K} < 0$ due to the presence of \bar{K} associated with the linear term, i.e., the last term associated with C_2 . When C_2 is taken to be zero (i.e., no linear term), the shell shows a hardening behavior in the compression region and a softening behavior in the tension region, (fig.2). It is also observed that when the material constant increases with increasing R, the frequencies of small oscillation are larger compared with the homogeneous case and vice versa. An interesting behavior is observed when only the quadratic term is retained in eq.(10) (i.e. $C_2 = C_3 = 0$, $C_1 = 1$), the frequencies of small oscillations about the undeformed state ($\bar{K} \rightarrow 0$) are given by

$$\bar{\omega}^2 = 4, \text{ or } \omega^2 = \frac{4\Phi_1}{\rho A_1^2} \quad (14)$$

which is independent of wall thickness (fig.3).

2.2. Spherical Shells**

The general expression for the frequency of small radial oscillations about the prestressed state is given by

$$\omega^2 = \frac{2(1-\bar{K})^{1/3}(1-\lambda^3\bar{K})^{1/3}}{\rho A_1^2 \{ (1-\lambda^3\bar{K}) - \lambda(1-\bar{K})^{1/3} \}} \cdot \frac{1}{\bar{K}} \int_{Q_2}^{Q_1} \Phi (7Q^6 - 1) dQ, \quad (15)$$

where

$$\bar{K} = 1 - \left(\frac{a_1}{A_1} \right)^3.$$

In order to compare the results with those given in {8}, we now choose a cubic variation for the material constant in the following form :

$$\Phi = B_0 + B_3 \left(\frac{R}{A_1} \right)^3. \quad (16)$$

The corresponding frequency expression for $\bar{K} \neq 0$ is given by

$$\bar{\omega}^2 = \frac{\rho A_1^2 \omega^2}{B_0} = \frac{2(1-\bar{K})^{1/3}(1-\lambda^3\bar{K})^{1/3}}{(1-\lambda^3\bar{K})^{1/3} - \lambda(1-\bar{K})^{1/3}} \left\{ \frac{(2-\bar{K})}{(1-\bar{K})^{7/3}} - \frac{\lambda^3(2-\lambda^3\bar{K})}{(1-\lambda^3\bar{K})^{7/3}} \right\} + \frac{B_3}{B_0} \left[\frac{24(1-\bar{K})^2 + 7(1-\bar{K}) + 4}{4(1-\bar{K})^{7/3}} - \frac{24(1-\lambda^3\bar{K})^2 + 7(1-\lambda^3\bar{K}) + 4}{4(1-\lambda^3\bar{K})^{7/3}} \right] \quad (\text{continued})$$

* This behavior is independent of A_1/A_2 ratio. $\bar{K} > 0$ means compression while $\bar{K} < 0$ means tension. We also note that \bar{K} cannot be larger than 1.

** Since the detailed formulation of the problem is given in {8}, here we only give the final relations between $\bar{\omega}$ and \bar{K} .

$$\left. \begin{aligned}
 &+ 3\lambda n \frac{\lambda\{1-(1-\bar{K})^{-1/3}\}}{\{1-(1-\lambda^3\bar{K})^{-1/3}\}} + 2\sqrt{3} \tan^{-1} \left(\frac{2}{\sqrt{3}(1-\lambda^3\bar{K})^{1/3}} + \frac{1}{\sqrt{3}} \right) \\
 &- 2\sqrt{3} \tan^{-1} \left(\frac{2}{\sqrt{3}(1-\bar{K})^{1/3}} + \frac{1}{\sqrt{3}} \right) \left. \right\}, \quad (17)
 \end{aligned}$$

and for $\bar{K} = 0$ is given by

$$\bar{\omega}^2 = \frac{4}{(1-\lambda)} \left\{ (1-\lambda^3) + 3 \frac{B_3}{B_0} \lambda n \frac{1}{\lambda} \right\} \quad (18)$$

Figure 5 shows $\bar{\omega}^2$ versus \bar{K} curves for $\frac{A_1}{A_2} = 0.8$ for different values of B_3/B_0 . When $\bar{K} = 0$ eq.(16) becomes indeterminate. However, using L'Hospital rule eq.(17) is obtained. As in the case of cylindrical shells, the shell hardens in the compression region ($\bar{K} > 0$) and softens in the tension region. The curves do not show the erratic behavior stated in {8}.

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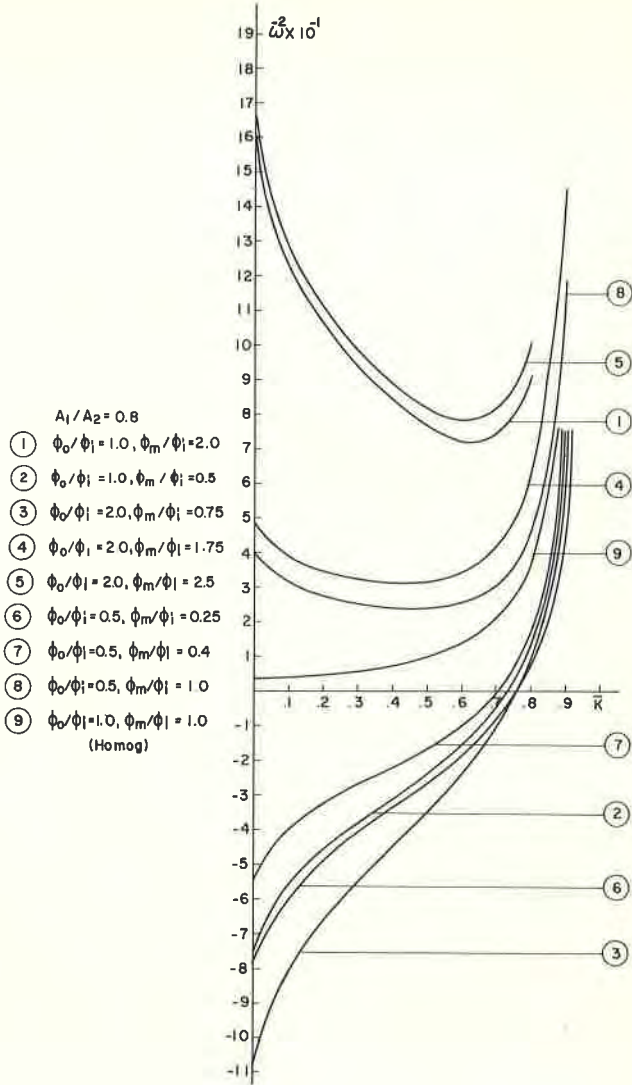


Fig.1 - Frequency versus initial deformation parameter for a cylindrical shell, for various nonhomogeneity factors (general quadratic variation).

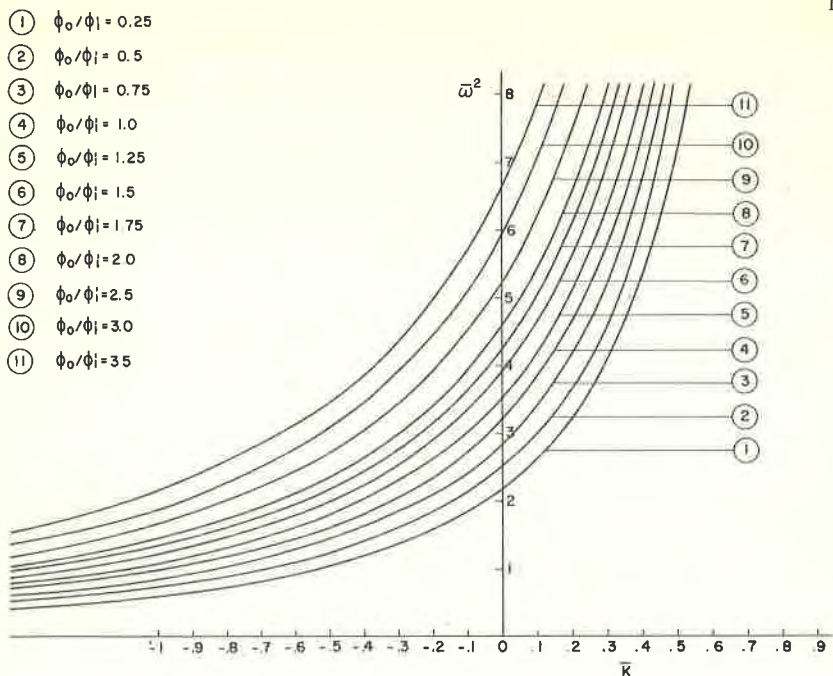


Fig.2 - Frequency versus initial deformation parameter for a cylindrical shell, for various nonhomogeneity factors (no linear term).

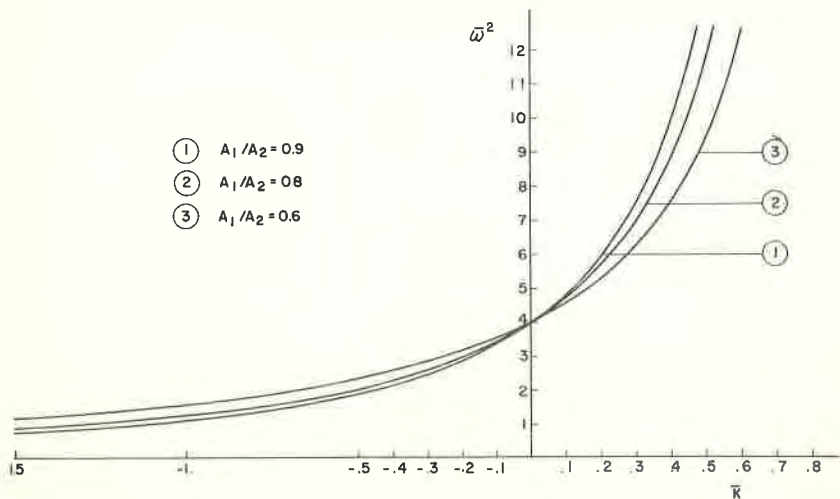


Fig.3 - Frequency versus initial deformation parameter for a cylindrical shells of various wall thicknesses (only the quadratic term is retained).

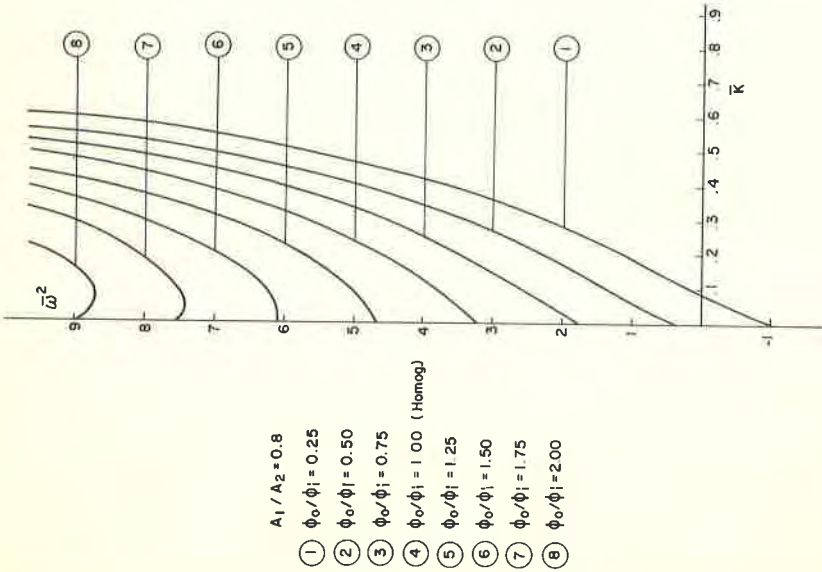


Fig. 4 - Frequency versus initial deformation parameter for a cylindrical shell, for various nonhomogeneity factors (general linear variation).

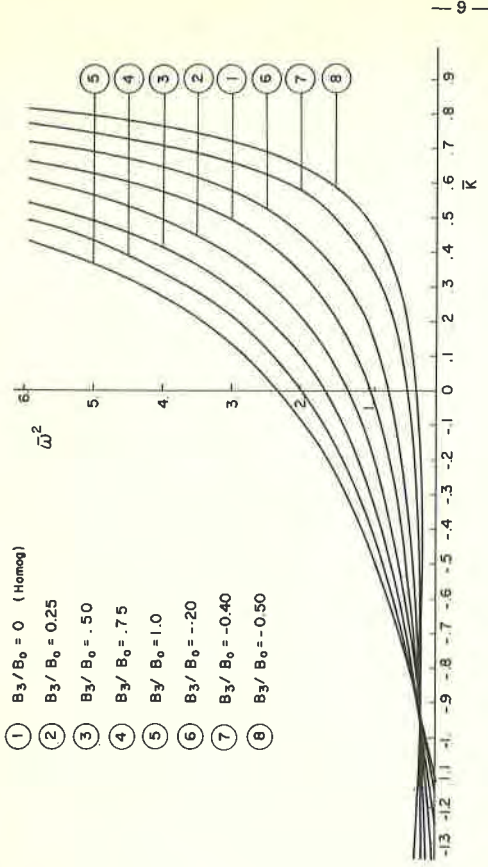


Fig. 5 - Frequency versus initial deformation parameter for a spherical shell, for various nonhomogeneity factors ($\phi = B_0 + B_3 (R/A_1)^3$).

