Stability of Thin Beams Containing Cracks

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SUMMARY

Fail safe design concepts have been developed for important structures like nuclear reactors, aeroplanes, etc. There are several modes of failure like general yielding, instability (buckling), excessive deformation, creep, fatigue, brittle fracture & corrosion.

In this work the combination of two modes together is investigated by discussing the stability behavior of structures containing cracks. The concept of minimum potential energy is used in this study to determine the critical loading condition. The presence of a crack increases the potential energy and thus reduces the critical buckling load.

A theoretical and experimental study is presented for the buckling of a thin beam containing an edge crack subjected to pure bending. In the presence of the crack, the change in work done (ΔW) by external loading is balanced by the sum of the changes in strain energies (ΔU) due to bending, torsion, warping, and the reduction of the strain energy resulting from the crack. Thus the minimum potential energy principle at buckling takes the form :

$$\Delta W = \Delta U_{bending} + \Delta U_{torsion} + \Delta U_{warping} - \Delta U_{crack}$$

Assuming the rigidity in the plane of loading to be large in comparison with the rigidity in the lateral direction, the warping term ($\Delta U_{warping}$) may be neglected.

Different crack to width ratios have been investigated theoretically and experimentally. Steel sheet specimens are first notched and then fatigue precracked and loaded by four point bending on a press. The critical load is measured for each crack depth.

The critical buckling load is reduced with increasing crack to width ratio. Moreover, the experimental results show the same trends predicted theoretically. However, some discrepancies appear due to the presence of the plastic zone ahead of the crack tip. Elastic-plastic analysis is expected to show closer results.

1- Introduction

The fail safe design concept has been dictated by the importance and safety of structures like nuclear reactors, aeroplanes, etc. A component may fail in one or more of the following modes :

1- Yielding, 2- Large elastic deformation, 3- Buckling, 4- Crack propagation, 5- Fatigue, 6- Creep.

A lot of literature has been appearing in each failure mode separately and more recently in the effects of coupling these modes. In this work an attempt is made to couple modes 3 & 4 namely buckling and crack propagation.

Several papers have appeared [1] investigating the effects of defects on buckling loads. The word defect in that sense may be any form of defect, whether geometrical, or other.

In this paper an energy approach to the problem of buckling of a thin cracked plate is presented. The results are compared with those obtained experimentally using thin steel sheets. The approach, however, may be easily generalized to other problems where buckling & crack propagation need to be considered simultaneously.

2- Theoretical Analysis

The problem of lateral buckling of thin uncracked beams is readily available in the literature [2]. If a beam is subjected to a bendling moment in its plane of greatest flextural rigidity (Fig.1), it may buckle at a certain critical value. The energy approach to determine the buckling load may be applied by assuming a buckled form for the beam and determining the change in strain energy ΔU and in work ΔW. If $\Delta W < \Delta U$ the beam is stable since the change in the work is not enough to deform it. The buckled form is at the point of instability when :

$$\Delta W = \Delta U \tag{1}$$

Calculating the change in strain energy due to bending and torsion from the buckled form, the following expression for strain energy may be used [2] :

$$\Delta U = \frac{EI_\eta}{2}\int_0^L \left(\frac{d^2u}{dz^2}\right)^2 dz + \frac{C}{2}\int_0^L \left(\frac{d\phi}{dz}\right)^2 dz + \frac{C_1}{2}\int_0^L \left(\frac{d^2\phi}{dz^2}\right) dz \tag{2}$$

where : $C = GJ$ is the torsional rigidity

$C_1 = EC_w$ is the warping rigidity.

Other notations are illustrated in Fig.1.

In equation (2) the three terms represent, respectively, the strain energy due to lateral bending, twisting, and warping of the beam. For a narrow rectangular beam the warping rigidity C can be neglected compared to the benging and twisting terms.

Noting that :

$$\frac{d^2u}{dz^2} = \phi \frac{M_o}{EI_\eta} \quad (3)$$

Then the change in the strain energy due to buckling becomes :

$$U = \frac{M_o^{\ 2}}{2EI_\eta} \int_o^L \phi^2 \, dz + \frac{C}{2} \int_o^L \left(\frac{d\phi}{dz}\right)^2 dz \quad (4)$$

Effect of Crack

Now, if the beam is cracked, the strain energy is reduced by an amount U_c that may be determined through the following analysis :

In any cracked body the energy available to propagate the crack comes from the increase in work done (W) and the decrease in strain energy (-U). Thus the energy release rate G per unit crack area A may be given as [3,4]:

$$G = \frac{\partial (W-U)}{\partial A} \quad (5)$$

Now, if the crack is propagating under fixed grip conditions, there is no change in the work done and hence :

$$G = -\frac{\partial U}{\partial A} \quad (6)$$

Thus to determine the change in strain energy between a cracked and an uncracked body loaded with same load point deformations eq.(6) may be integrated. That is the difference in strain energy between the cracked and uncracked beam may be given as :

$$U_c = -\int_o^a G da \quad \text{per unit beam thickness} \quad (7)$$

in which a is the crack length.
Now, for a plane stress problem :

$$G = \frac{K^2}{E}$$

i.e. $$U_c = -\int_o^a \frac{K^2}{E} da \quad \text{per unit beam thickness} \quad (8)$$

where K is the stress intensity factor and for an edge cracked rectangular beam of depth h subjected to bending, the stress intensity factor K_I may be given as [5] :

$$K_I = Y\sigma \sqrt{a} \quad (9)$$

where Y = calibration factor
and σ = nominal applied stress.

Now, to determine the change in the effect of the crack tip between the buckled and unbuckled forms consider first the change in the stress distribution at the point where the crack tip lies as if the crack is not present. As the beam buckles, the component of M_o about the η axis (M_η) causes a linear stress distribution across the thickness. That is due to M_η :

$$\sigma_z = \frac{M_\eta \cdot \xi}{I_\eta} \tag{10}$$

This additional component is tensile on half the crack edge and compressive on the other half. It is this component that causes the change in strain energy due to the crack.

The effect of M_o is large compared with M_η (small deformations), therefore all the crack edge is under tension. The change in strain energy due to crack in the buckled form may thus be obtained by integrating equation (8) across the thickness of the plate.

i.e. :
$$\Delta U_c = - \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_o^a \frac{K_I^2}{E} \, da \, d\xi \tag{11}$$

Now if the plate is thought of as composing of several layers forming the thickness b, then the factor Y in eq. (9) may be taken as that of an edge cracked plate under uniform tension and the nominal stress σ is taken from eq.(10). Thus from Ref. [5] :

$$Y = 1.99-0.41\lambda+18.7\lambda^2-38.48\lambda^3+53.85\lambda^4 \quad 0 \leqslant \lambda \leqslant 0.6 \tag{12}$$

where : $\lambda = \frac{a}{h}$

Therefore substituting for Y and σ in eq. (9) :

$$K_I = Y \sqrt{a} \, \frac{M_\eta}{I_\eta} \cdot \xi$$

and
$$\Delta U_c = - \frac{M_\eta^2}{E I_\eta^2} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_o^a Y^2 a \, \xi^2 da \, d\xi$$

For $\lambda < 0.6$ (range of experimental work) and using eq. (11) then :

$$\Delta U_c = - \frac{M_\eta^2}{E I_\eta^2} \, B \int_{-\frac{b}{2}}^{\frac{b}{2}} \xi^2 \, d\xi$$

Where : $B = a^2(1.98-0.54\lambda+18.65\lambda^2-33.7\lambda^3+99.26\lambda^4$
$-211.9\lambda^5+436.84\lambda^6$
$-460.48\lambda^7+290\lambda^8$

i.e. :
$$\Delta U_c = - \frac{M_\eta^2}{E I_\eta} \cdot \frac{B}{h} \tag{14}$$

Adding expression (14) to the strain energy terms in (4) and taking the buckled shape of the form :

$$\phi = A \sin \frac{\pi z}{L} \tag{15}$$

At the middle section ($z= \frac{L}{2}$) which is the crack plane, ϕ = A and noting that

$M_\eta = \phi M_o$

the total change in the strain energy in the buckled form becomes :

$$\Delta U_t = \frac{A^2 M_o^2 L}{4E I_\eta} + \frac{A^2 C}{4} \frac{\pi^2}{L} - \frac{A^2 M^2_o}{EI_\eta} \frac{B}{h} \qquad (16)$$

The change in external work due to buckling may be given by :

$$\Delta W = \frac{\frac{1}{2} A^2 M_o^2 L}{EI_\eta} \qquad (17)$$

Substituting from (16), (17), in (1), then the critical buckling moment M_{ocr} is given by :

$$M_{ocr} = \frac{\pi}{2} \sqrt{\frac{EI_\eta G J}{L(\frac{L}{4} + \frac{B}{h})}} \qquad (18)$$

Noting that the modulus of section in torsion J of a thin rectangular strip may be given as :

$$J = 4 I_\eta = \frac{1}{3} h b^3 \qquad (19)$$

and that $G = \frac{E}{2(1+\nu)}$ (20)

where ν is Poisson's ratio = 0.3 for steel, then eq.(18) may be rewritten as :

$$M_{ocr} = \frac{1.95 EI_\eta}{\sqrt{L(\frac{L}{4} + \frac{B}{h})}} \qquad (21)$$

3- Test Rig

The test rig used is shwon in (Fig.2). The rig is composed of a horizontal rigid base (1) bolted to a rigid floor (2). The two supporting columns (3) are bolted at their ends to the base (1), while their upper ends are supporting two rigid cylinders (4) used for the simulation of simply supported end conditions. The test specimen (5), Prepared as shown in (Fig.3), rests on the cylinders (4) to simulate the condition of two-point loading. The test specimen is loaded through two loading arms connected to the load tackle (7). The loading is made by the use ofacalibrated loading cylinder (8).

The positions of the vertical columns (3) and the horizontal load tackle (7) could be adjusted on the test rig to suit the length of the different test specimen. A hydro-electric pump (8) of maximum pressure 250 Kg/cm^2 was used for loading. The hydraulic jack used has a maximum capacity of 2.2 tons.

4- Test Specimens

Test specimens were made from commercial structural steel sheets (Fig.3). The length of each specimen was always chosen to be more than five times its height. This length is believed to be sufficient for uniform stress distribution across the middle part of the specimen. The shape of both ends is adjusted to suit the loading conditions. Specimens with different crack length/height ratios were tested and two specimens were tested for each result point.

5- Results & Discussion

Fig.4 shows the theoretical and experimental results for the variation of the critical moment M_{ocr} with different crack to height ratios. The specimen length used in the eq. (18) is taken to be 55 cms. and 49 cms. The second length may be considered as an effective length to remove the effect of the end constraints. It is obtained by equating the experimental results to the theoretical ones at zero crack using the equation [2] :

$$M_{ocr} = \frac{\pi}{L} \sqrt{EI_{\eta} \, G \, J} \tag{22}$$

which may also be obtained from eq.(18) by putting the crack length a=0.

Though the theoretical and experimental results have the same trends, there is a maximum difference of about 10% in the predicted M_{ocr} value. This may be attributed to the plastic zone ahead of the crack tip since it dissipates some of the stored strain energy. This is a significant factor since the beam is thin and the material is free to yield.

If an effective crack length accounting for the plastic zone using Dugdale's [5] model or any other suitable one, closer results are expected. However, the integration of the term Y^2 in eq.(13) will be cumbersome and could only be performed through an itterative technique.

Specimens with large crack to depth ratios failed by tearing and were excluded. For such cases ductile rather than brittle fracture analysis would be more useful.

6- Conclusion

An energy approach has been developed to couple the effects of cracks and buckling. The effect of edge cracks in reducing the critical buckling moment of thin sheets gave close results to the experimental. This approach may be easily extended to other problems.

7- References

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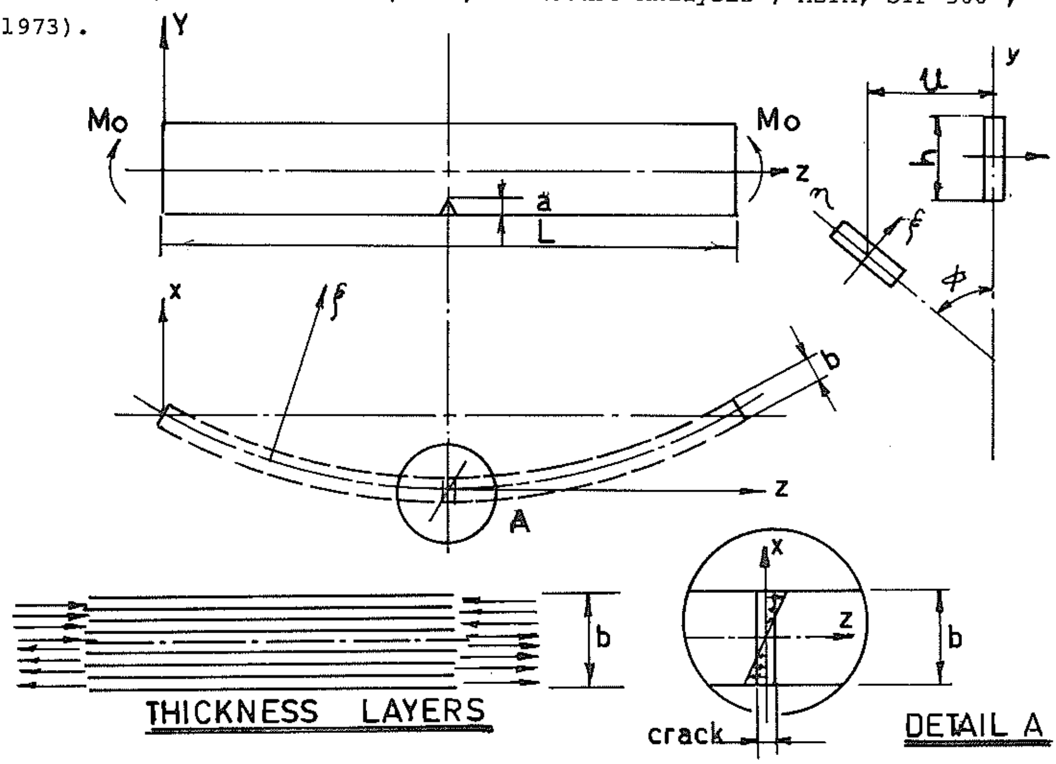
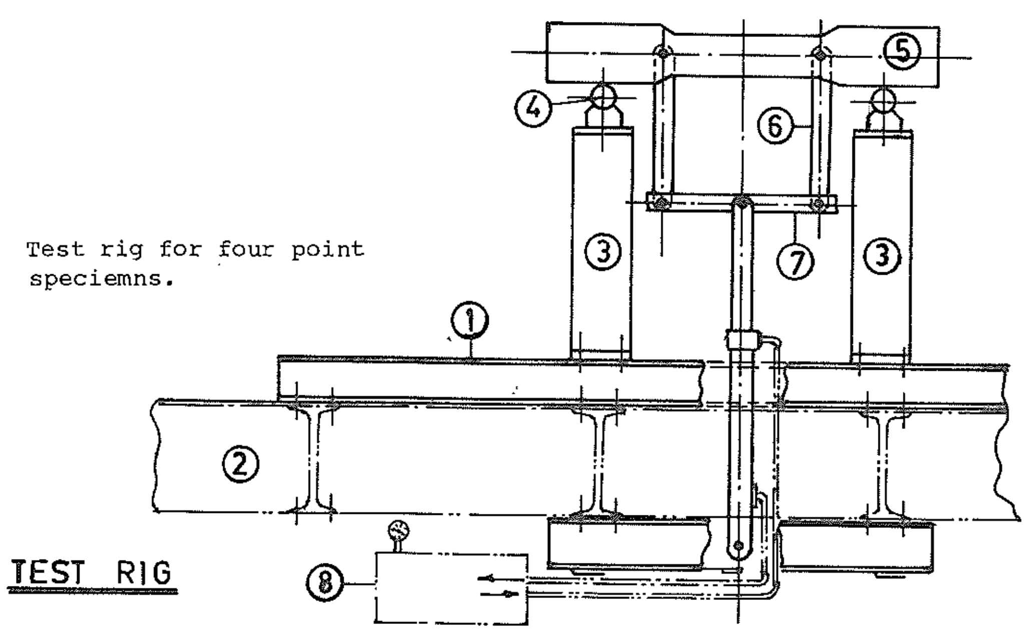


Fig.1. Buckling of a general thin edge cracked rectangular strip with illustration of the effect of the crack.

Fig.2. Test rig for four point speciemns.



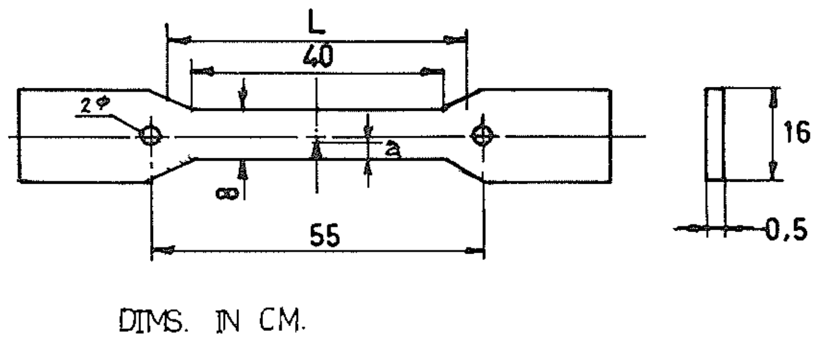


Fig.3. Geometry of specimens used.

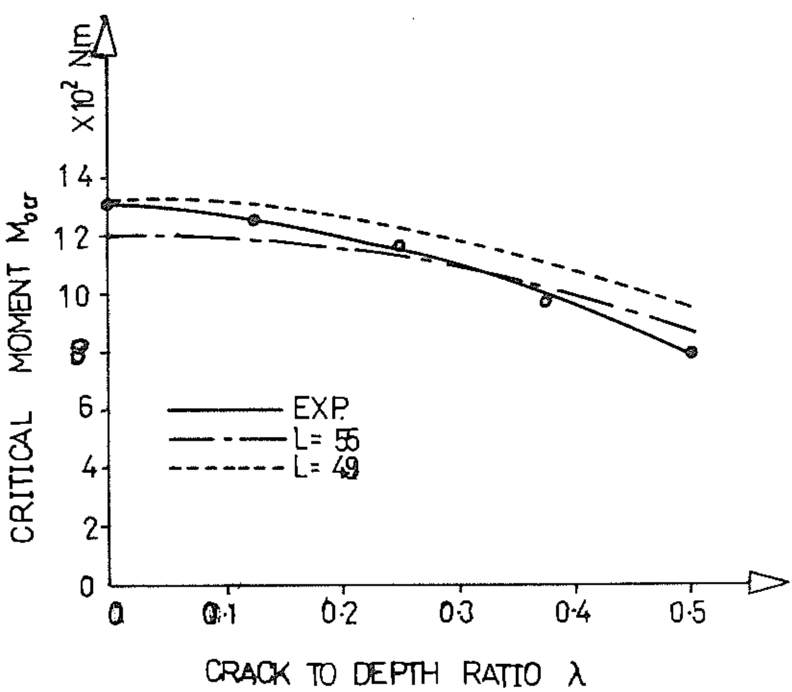


Fig.4. Theoretical and experimental results for the variation of the critical moment (M) with crack to depth ratio (λ).