

## THERMAL AND MECHANICAL STRESSES IN NUCLEAR REACTOR VESSELS

R.E. HOFFMANN, T. ARIMAN,

*Department of Aerospace and Mechanical Engineering,  
University of Notre Dame, Notre Dame, Indiana, U.S.A.*

### ABSTRACT

This paper is concerned with the stress and deformation analysis of a finite circular cylindrical, thin, elastic shell of constant thickness which represents a nuclear reactor vessel. The shell has an insulated cutout of arbitrary shape on its lateral surface and it is subjected to a stationary heat flow and various mechanical loads.

### 1. INTRODUCTION

In nuclear reactors cylindrical shells are used as portions of containment vessels, fuel sleeves, and even as moderators. In many of these applications such shells are subjected to thermal gradients in both the axial and radial directions. Furthermore when a temperature field is disturbed by the presence of cutouts, there is high elevation of the local temperature gradient accompanied by thermal stress. Thermal disturbances of this kind, in some cases, may cause serious damage to a cylindrical vessel. This phenomenon is particularly important for the nuclear reactor vessels because of the extreme difficulties associated with repair of radioactive components and because of the personnel and systems hazards involved.

At present because of the mathematical difficulties involved, only a limited number of solutions are available for cylindrical shells with cutouts subjected to mechanical [1-12] and/or thermal loads [13-16]. Particularly the availability of an accurate method for determining the stress distribution in the neighborhood of cutout is essential for designing against the various possible modes of failure (plastic instability, incremental collapse, fatigue) at such a cutout [17, 18].

This paper is concerned with the stress and deformation analysis of a finite circular cylindrical, elastic shell of constant thickness, representing a nuclear reactor vessel, with an insulated cutout of arbitrary shape on its lateral surface. The shell is subjected to a stationary heat flow and various mechanical loads. It is assumed that the shell is thin and deformations and strains are infinitesimal. Furthermore, the shell material is isotropic and homogeneous.

The analysis is based on the method of superposition. The actual stresses in the shell are considered as the sum of the following three parts:

- (a) Stresses caused by prescribed stationary heat flow and mechanical loads in a similar shell of infinite length without a hole.
- (b) Stresses in the shell due to the specified boundary conditions at both ends of the shell.
- (c) Stresses in the shell caused by edge loads along the boundary of the hole otherwise free of any other thermal and mechanical loads.

The combination of the first two solutions are referred to as the nominal solution and the last one is usually termed the residual stress due to edge loads. The edge loads around the cutout consist of the stresses, with opposite signs, obtained from the nominal solution and/or prescribed loads on the contour of the hole.

The governing equations of the analysis are the so-called Morley equations [19] which have nearly the same simple form as the well-known Donnell equations but include no other assumptions than those of the linear thin shell theory. Morley equations are considered to be a significant improvement over the often-used Donnell equations. The accuracy of Donnell equations decreases as the wavelength of the circumferential distortion increases. The fact that Donnell equations do not include the ring bending equation is evidence of this limitation. When the cutout is large enough to be of the order of the cylindrical shell radius, Donnell equations will probably not be valid. Furthermore, in the analysis of cylindrical shells under distributed loads, Donnell's equations lead to reliable results only for short shells. Morley's equations should be used for the larger shells [20].

## 2. GEOMETRY OF THE SHELL

The shell geometry under consideration as well as the coordinate systems to be used are presented in figure 1. The primary coordinate system is rectangular in the developed shell surface and the origin lies in the middle surface of the shell. The axes  $\bar{x}$  and  $\bar{y}$  lie along the longitudinal and circumferential axes of the shell respectively, with the  $\bar{z}$  axis mutually perpendicular to both  $\bar{x}$  and  $\bar{y}$  and directed outward. In conjunction with this coordinate system a second non-dimensional system is defined such that

$$x = \bar{x}/a, \quad \theta = \bar{y}/a, \quad z = \bar{z}/a$$

where  $a$  is the radius of the circular cylindrical shell. Finally, a third system in the cylindrical polar coordinates  $r$ ,  $\varphi$  and  $z'$  is superimposed upon the others and is defined by the relationships

$$x = r \cos \varphi, \quad \theta = r \sin \varphi, \quad z = z' . \quad (1)$$

Figure 2a shows a representative shell element with the membrane stress resultants which act on the element and the surface tractions produced by external loads. Figure 2b shows a similar element with the moment resultants and transverse shear forces depicted. All forces and moments are measured per unit length of middle surface and are considered positive in the directions indicated.

## 3. OUTLINE OF THE ANALYTICAL PROCEDURE

The stress and displacement system in a cylindrical shell of finite length having a cutout in its lateral surface and subjected to static mechanical and thermal loads is found

through an application of the principle of superposition. Let  $S$  be the system of stresses and displacements at any point in a cylindrical shell which has no cutout but is loaded with the same mechanical loads and thermal loads as a punctured shell. A second system  $\bar{S}$  may then be found for the case of a shell with no thermal or mechanical surface loads but subjected to the restriction that

$$S(x = \pm l) + \bar{S}(x = \pm l) = \bar{S}_0 \quad (2)$$

Thus the sum of the systems  $S$  and  $\bar{S}$  represents the solution for a cylindrical shell of length  $2l$  with given mechanical and thermal loads and attaining specified boundary values at the shell ends.  $S$  and  $\bar{S}$  taken together shall be referred to as the nominal solution.

If the cutout in the shell surface is represented as a closed contour  $C$  in the  $x, \theta$  plane, a third stress-displacement system may be constructed with the following properties. This system,  $\bar{S}$ , represents the behavior of a shell with no mechanical or thermal loading. It satisfies

$$S(x = \pm l) + \bar{S}(x = \pm l) + \bar{S}(x = \pm l) = \bar{S}_0 \quad (3)$$

and also satisfies

$$S + \bar{S} + \bar{S} = \bar{S}_0 \quad (4)$$

for all points which lie on  $C$ .

If, for example, the term  $\bar{S}_0$  represents a stress free condition on  $C$ ; then, since  $S$  was determined for the thermal load distribution found in a shell with a cutout, the sum of  $S$ ,  $\bar{S}$  and  $\bar{S}$  gives the complete stress displacement system for a shell with thermal and mechanical surface loads, with specified boundary conditions at the shell ends, and possessing a stress-free cutout.

It is also possible to consider cases where the contour  $C$  is other than stress-free through the use of the specified functions  $\bar{S}_0$ .

This last system,  $\bar{S}$ , is termed the residual solution.

#### 4. THE GOVERNING EQUATIONS

To insure static equilibrium of any shell element the following equations must be satisfied.

$$\begin{aligned} \frac{\partial N_x}{\partial x} + \frac{\partial N_{\theta x}}{\partial \theta} + a q_x &= 0 \\ \frac{\partial N_{x\theta}}{\partial x} + \frac{\partial N_\theta}{\partial \theta} + Q_\theta + a q_\theta &= 0 \\ \frac{\partial Q_x}{\partial x} + \frac{\partial Q_\theta}{\partial \theta} - N_\theta + a q &= 0 \\ \frac{\partial M_x}{\partial x} + \frac{\partial M_{\theta x}}{\partial \theta} - a Q_x &= 0 \end{aligned} \quad (5)$$

$$\frac{\partial M_{x\theta}}{\partial x} + \frac{\partial M_\theta}{\partial \theta} - a Q_\theta = 0 \quad (5)$$

The stress resultants and moments may be expressed in terms of the displacement of the middle surface of the shell  $u$ ,  $v$  and  $w$  which are in the directions of  $x$ ,  $\theta$  and  $z$  respectively. Converting to the non-dimensional displacements  $u = \underline{u}/a$ ,  $v = \underline{v}/a$  and  $w = \underline{w}/a$  the relationships are

$$\begin{aligned} N_x &= \frac{Eh}{(1-\nu^2)} \left[ \frac{\partial u}{\partial x} + \nu \left( \frac{\partial v}{\partial \theta} + w \right) - \frac{1}{12} \left( \frac{h}{a} \right)^2 \frac{\partial^2 w}{\partial x^2} \right. \\ &\quad \left. - (1+\nu) \alpha T_o \right] \\ N_\theta &= \frac{Eh}{(1-\nu^2)} \left[ \frac{\partial v}{\partial \theta} + w + \nu \frac{\partial u}{\partial x} + \frac{1}{12} \left( \frac{h}{a} \right)^2 \left( \frac{\partial^2 w}{\partial \theta^2} \right. \right. \\ &\quad \left. \left. + w \right) - (1+\nu) \alpha T_o \right] \end{aligned} \quad (6)$$

$$\begin{aligned} N_{x\theta} &= \frac{Eh}{2(1+\nu)} \left[ \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x} + \frac{1}{12} \left( \frac{h}{a} \right)^2 \left( -\frac{\partial^2 w}{\partial x \partial \theta} + \frac{\partial v}{\partial x} \right) \right] \\ N_{\theta x} &= \frac{Eh}{2(1+\nu)} \left[ \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x} + \frac{1}{12} \left( \frac{h}{a} \right)^2 \left( \frac{\partial^2 w}{\partial x \partial \theta} + \frac{\partial u}{\partial \theta} \right) \right] \\ M_x &= -\frac{D}{a} \left[ \frac{\partial^2 w}{\partial x^2} + \nu \left( \frac{\partial^2 w}{\partial \theta^2} - \frac{\partial v}{\partial \theta} \right) - \frac{\partial u}{\partial x} \right. \\ &\quad \left. + (1+\nu) \alpha a T_1 \right] \\ M_\theta &= -\frac{D}{a} \left[ \frac{\partial^2 w}{\partial \theta^2} + w + \nu \frac{\partial^2 w}{\partial x^2} + (1+\nu) \alpha a T_1 \right] \\ M_{x\theta} &= -\frac{D}{a} (1-\nu) \left[ \frac{\partial^2 w}{\partial x \partial \theta} - \frac{\partial v}{\partial x} \right] \\ M_{\theta x} &= \frac{1}{2} \frac{D}{a} (1-\nu) \left[ 2 \frac{\partial^2 w}{\partial x \partial \theta} - \frac{\partial v}{\partial x} + \frac{\partial u}{\partial \theta} \right] \\ Q_x &= -\frac{D}{a^2} \left[ \frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial \theta^2} + \frac{\partial^2 u}{\partial x^2} - \frac{1}{2} (1-\nu) \frac{\partial^2 u}{\partial \theta^2} \right. \\ &\quad \left. + \frac{1}{2} (1+\nu) \frac{\partial^2 v}{\partial x \partial \theta} + (1+\nu) \alpha a \frac{\partial T_1}{\partial x} \right] \\ Q_\theta &= -\frac{D}{a^2} \left[ \frac{\partial^3 w}{\partial \theta^3} + \frac{\partial w}{\partial \theta} + \frac{\partial^3 w}{\partial x^2 \partial \theta} + (1-\nu) \frac{\partial^2 v}{\partial x^2} \right. \\ &\quad \left. + (1+\nu) \alpha a \frac{\partial T_1}{\partial \theta} \right] \end{aligned} \quad (7)$$

Where

- E = Young's modulus
- h = shell thickness =  $\bar{h}/a$
- $\nu$  = Poisson's ratio
- $\alpha$  = coefficient of linear thermal expansion

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

$$T_0 = \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} T dz \quad (8)$$

$$T_1 = \frac{12}{h^3} \int_{-\frac{h}{2}}^{\frac{h}{2}} T z dz$$

where  $T(x, \theta, z)$  represents the temperature field in the shell.

Substituting these stress-displacement relations into the equilibrium equations (5), a set of three partial differential equations in the displacements  $u$ ,  $v$  and  $w$  may be derived. These equations, the so-called Morley equations, with their attendant loading terms are:

$$\nabla^4 (\nabla^2 + 1)^2 w + 4K^4 \frac{\partial^4 w}{\partial x^4} = P_w \quad (9)$$

$$\nabla^4 u = \frac{\partial^3 w}{\partial x \partial \theta^2} - \nu \frac{\partial^3 w}{\partial x^3} + O\left[\left(\frac{h}{a}\right)^2\right] + P_u \quad (10)$$

$$\nabla^4 v = -(2+\nu) \frac{\partial^3 w}{\partial x^2 \partial \theta} - \frac{\partial^3 w}{\partial \theta^3} + O\left[\left(\frac{h}{a}\right)^2\right] + P_v \quad (11)$$

where

$$\begin{aligned} K^4 &= 3(1-\nu^2) \left(\frac{a}{h}\right)^2 \\ \nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \theta^2} \\ \nabla^4 &= \nabla^2 \nabla^2 \end{aligned} \quad (12)$$

$$O\left[\left(\frac{h}{a}\right)^2\right] = \text{terms of order } \left(\frac{h}{a}\right)^2$$

and where

$$\begin{aligned} P_w &= \nabla^4 P + \nu \frac{\partial P_u}{\partial x} + 12 \left(\frac{a}{h}\right)^2 \frac{\partial P_v}{\partial \theta} - \frac{1}{2} (3-\nu) \frac{\partial^3 P_v}{\partial x^2 \partial \theta} \\ &\quad - \frac{\partial^3 P_u}{\partial x^3} + \frac{1}{2} (1-\nu) \frac{\partial^3 P_u}{\partial x \partial \theta^2} \\ P &= \frac{a^3}{D} q - (1+\nu) \alpha a \nabla^2 T_1 - 12 \left(\frac{a}{h}\right)^2 (1+\nu) \alpha a T_0 \quad (13) \\ P_u &= \frac{\partial^2 P_x}{\partial x^2} + \frac{2}{1-\nu} \frac{\partial^2 P_x}{\partial \theta^2} - \frac{1+\nu}{1-\nu} \frac{\partial^2 P_\theta}{\partial x \partial \theta} + \frac{3}{12} \left(\frac{h}{a}\right)^2 \frac{\partial^2 P_x}{\partial x^2} \\ P_v &= \frac{2}{1-\nu} \frac{\partial^2 P_\theta}{\partial x^2} + \frac{\partial^2 P_\theta}{\partial \theta^2} - \frac{1+\nu}{1-\nu} \frac{\partial^2 P_x}{\partial x \partial \theta} + \frac{1}{12} \left(\frac{h}{a}\right)^2 \frac{\partial^2 P_\theta}{\partial \theta^2} \end{aligned}$$

and in turn where

$$P_x = (1+\nu) \alpha \frac{\partial T_0}{\partial x} - \frac{(1-\nu^2) a}{Eh} q_x \quad (14)$$

$$P_{\theta} = (1+\nu) \alpha \frac{\partial T_o}{\partial \theta} - \frac{1}{12} \left(\frac{h}{a}\right)^2 (1+\nu) \alpha a \frac{\partial T_1}{\partial \theta} - \frac{(1-\nu^2)a}{Eh} q_{\theta} \quad (14)$$

and  $q_x$ ,  $q_{\theta}$  and  $q$  are the distributed surface loads in the  $x$ ,  $\theta$  and  $z$  directions respectively.

Equations (9), (10), and (11) are the governing equations which must be solved to determine the displacements for the problem at hand. Then the stress resultants and moments may be found from equations (6) and (7).

In the construction of that part of nominal solution designated  $S$ , the mechanical loads on the surface may be specified at will; however, the thermal loads represented by the temperature field are dependent upon the geometry of the cutout. Therefore, an expression for the temperature must be developed which takes into account the shape and size of the cutout and the desired thermal boundary conditions on its boundary.

### 5. THE TEMPERATURE FIELD

As this analysis deals with the behavior of a thin shell, the assumption of a stationary temperature field which varies with the shell thickness in a linear form would seem warranted. Thus we write

$$T(x, \theta, z) = \bar{T}_1(x, \theta) + z \bar{T}_2(x, \theta) \quad (15)$$

where  $\bar{T}_1$  is then the middle surface temperature and  $\bar{T}_2$  represents the difference between the temperatures of the outer and inner shell surfaces at any point  $(x, \theta)$ .

If the shell is considered to contain no heat sources, then the temperature must satisfy the Laplace equation i. e.

$$\nabla^2 T = 0 \quad (16)$$

In particular, at the middle surface, where  $z = 0$

$$\nabla^2 \bar{T}_1 = 0 \quad (17)$$

If the length of the shell is greater than approximately three times the shell radius and if it is desired to specify temperatures at the shell ends then  $\bar{T}_1$  can be written

$$\bar{T}_1 = \bar{T}_1 + \bar{T}_2 \quad (18)$$

where  $\bar{T}_1(x = l) = T_l$ ,  $\bar{T}_2(x = l) = 0$  (19)

and  $\bar{T}_1(x = -l) = T_{-l}$ ,  $\bar{T}_2(x = -l) = 0$

and where  $T_l$  and  $T_{-l}$  are the specified temperatures at  $x = l$  and  $x = -l$  respectively. The term  $\bar{T}_1$  can be thought of as the temperature distribution in an unpunctured shell with specified temperatures on the shell end boundaries. Likewise the term  $\bar{T}_2$  represents a distortion of the heat flow due to the cutout in the shell surface and, as is experimentally known, such a distortion has decreasing effects on the total temperature field with increasing distance from the point of disturbance.

Then a suitable solution to equation (17) in polar coordinates can be written as

$$\bar{T}_1 = T_A + \tau r \cos \varphi + \sum_{n=1}^{\infty} [(C_n \cos n\varphi + D_n \sin n\varphi) r^{-n}] \quad (20)$$

where

$$T_A = \frac{(T_{\ell} + T_{-\ell})}{2}$$

and (21)

$$\tau = \frac{(T_{\ell} - T_{-\ell})}{2\ell}$$

The coefficients  $C_n$  and  $D_n$  are determined from the boundary condition on the contour of the cutout. For this analysis the condition is that the hole possesses an insulated boundary. Therefore eq. (20) is subject to the requirement

$$\nabla \bar{T}_1 \cdot \underline{n} = 0 \quad (22)$$

where  $\underline{n}$  is the normal to the contour defining the cutout boundary. Only for the case of a circular hole is it possible to obtain analytically all  $2^{\infty}$  coefficients  $C_n$  and  $D_n$ . Therefore the infinite series in  $n$  is truncated and the least squares point matching (App.A) method is employed to determine an approximate solution for the temperature distribution.

## 6. THE NOMINAL SOLUTION

### 6.1 The Nominal Solution: Part 1 (S)

That part of the nominal solution which has been defined as the stress displacement system  $S$  must represent the behavior of a shell with specified surface loads and with the temperature distribution determined in the previous section. Since the stress resultants and moments may be found once we have determined the deflections, from the stress-displacement relations, the solution of the three Morley equations for  $u$ ,  $v$  and  $w$  will suffice to determine  $S$ .

In order to deal with a large number of different loading situations, it is desirable to have a representation for the surface mechanical and thermal loads which is quite general. The multiple Fourier series offers itself as a solution to the problem of generality. Thus, as an example, the internal pressure can be written as

$$q = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left\{ q_{nm}^1 \sin \alpha_n x \sin m \theta \right. \\ \left. + q_{nm}^2 \sin \alpha_n x \cos m \theta + q_{nm}^3 \cos \alpha_n x \sin m \theta \right. \\ \left. + q_{nm}^4 \cos \alpha_n x \cos m \theta \right\} \quad (23)$$

where  $\alpha_n = \frac{n\pi}{\ell}$ , (24)

where  $\ell' > \ell$  to insure convergence of the series at  $x = \pm \ell$ . The other mechanical loads can be expanded in similar series' as can the polar coordinate representation for the temperature given earlier, although this latter operation requires, in practice, extensive numerical integration.

To determine the displacements from these load expansions, a similar Fourier series is written for each displacement with the coefficients in the series as unknown quantities. Then by substituting the displacements and loads into the field equations and

equating the coefficients of the independent sine and cosine terms the displacement coefficients are found.

### 6.2 The Nominal Solution: Part 2 ( $\bar{S}$ )

The second system of solutions  $\bar{S}$  must yield the behavior of a shell with no surface loading and possess the property that when added to  $S$  the sum equals specified values of stresses and/or displacements at the shell ends. It is a well known restriction of the theory of thin shells that at most eight boundary conditions may be satisfied on a coordinate line. Subject to this restriction the following conditions are chosen at  $x = l$  and at  $x = -l$

$$\begin{aligned} u &= u^0 \\ v &= v^0 \\ w &= w^0 \end{aligned} \quad (25)$$

and either one of

$$\frac{\partial w}{\partial x} = 0 \quad (26)$$

or

$$M_x = M_x^0 \quad (27)$$

where  $u$ ,  $v$ ,  $w$  and  $M_x$  are obtained from the sum of the systems  $S$  and  $\bar{S}$ . However, a judicious choice of  $u^0$  and  $v^0$  allow one to deal with the cases of uniform axial tension or a uniform torque at  $x = \pm l$ .

$\bar{S}$  must once again be solutions to the Morley equations but this time in their homogeneous form. The general term in the solution for  $w$  is obtained by using the fact that the displacements must be periodic in the coordinate  $\theta$  and thus can be written as

$$\bar{w}_m = w^1(x) \sin m\theta + w^2(x) \cos m\theta \quad (28)$$

Substitution into the homogeneous Morley equation for  $w$  yields, for  $m > 1$

$$\bar{w}_m = \sum_{j=1}^8 \bar{w}_{mj}^1 e^{P_j x} \sin m\theta + \sum_{j=1}^8 \bar{w}_{mj}^2 e^{P_j x} \cos m\theta \quad (29)$$

where

$$P_j = P_j(m) \text{ and}$$

$$P_j^2 \quad (j = 1, 3, 5, 7) = m^2 - \frac{1}{2} + iK^4 \left\{ 1 \pm \left[ 1 - i \left( \frac{2m^2 - 1}{K^2} - \frac{1}{4K^4} \right) \right]^{\frac{1}{2}} \right\} \quad (30)$$

$$P_j^2 \quad (j = 2, 4, 6, 8) = m^2 - \frac{1}{2} - iK^4 \left\{ 1 \pm \left[ 1 + i \left( \frac{2m^2 - 1}{K^2} - \frac{1}{4K^4} \right) \right]^{\frac{1}{2}} \right\}$$

However, since  $w$  must be a real quantity, and due to the complex conjugate nature of the  $P_j$ 's the expression (29) can be rewritten as

$$\begin{aligned} \bar{w}_m &= \sum_{i=1}^2 \left\{ \left( e^{a_{1m} x} \left[ \bar{w}_{1m}^{-i} \cos \beta_{1m} x + \bar{w}_{2m}^{-i} \sin \beta_{1m} x \right] \right. \right. \\ &\quad \left. \left. + e^{a_{2m} x} \left[ \bar{w}_{3m}^{-i} \cos \beta_{2m} x + \bar{w}_{4m}^{-i} \sin \beta_{2m} x \right] \right) \right\} \end{aligned} \quad (31)$$

$$\begin{aligned}
 & + e^{-a_{1m} x} \left[ \bar{w}_{5m}^i \cos \beta_{1m} x + \bar{w}_{6m}^i \sin \beta_{1m} x \right] \\
 & + e^{-a_{2m} x} \left[ \bar{w}_{7m}^i \cos \beta_{2m} x + \bar{w}_{8m}^i \sin \beta_{2m} x \right] \\
 & \cdot \left( \begin{matrix} \sin m \theta \\ \cos m \theta \end{matrix} \right)_i \} \}
 \end{aligned} \tag{31}$$

where, for  $i = 1$  the above expression is the  $\sin m \theta$  series, etc., and where

- $a_{1m}$  = real part ( $P_1$ )
- $b_{1m}$  = imaginary part ( $P_1$ )
- $a_{2m}$  = real part ( $P_2$ )
- $b_{2m}$  = imaginary part ( $P_2$ )

For the cases  $m = 0$  and  $m = 1$  four repeated roots,  $P_j$  with value zero arise and consequently the terms in the general expression for  $\bar{w}_m$  which contain  $\cos \beta_2 x$  must be replaced by a polynomial up to the third order.

The solutions for  $\bar{u}$  and  $\bar{v}$  are found from the homogeneous form of the Morley equations (10) and (11), to have the same form as  $\bar{w}$  and are found through these equations by equating terms of the  $\theta$  Fourier series for common values of  $m$ .

Therefore, the  $48 m$  unknown  $\bar{u}$ ,  $\bar{v}$  and  $\bar{w}$  coefficients giving the  $\bar{S}$  system are found from the  $16 m$  (8 sin series, 8 cos series) boundary conditions that are necessary to satisfy the requirements at  $x = \pm l$  and the  $32 m$  equations resulting from the last two Morley equations.

## 6.2 The Residual Solution, ( $\bar{S}$ )

The residual solution  $\bar{S}$  must also satisfy the homogeneous Morley equations since the surface loads play no part in its determination. It must however, satisfy boundary conditions on two specified regions of the shell. Given that the boundary of the cutout is loaded so that the net resultant of the stresses applied on that boundary, (namely the system  $\bar{S}_0$ ), has a net resultant of zero stress, the principle of St. Venant shows that the effects of this system will be only local to the region near the cutout. Thus if the length of the shell is restricted to being greater than approximately three multiples of the radius, the effects of the residual solution will very nearly vanish at  $x = \pm l$ , and therefore, the conditions that

$$S + \bar{S} + \bar{S} = \bar{S}_0 \tag{32}$$

and

$$S + \bar{S} = \bar{S}_0 \tag{33}$$

can be mutually satisfied.

Since  $\bar{S}$  is the same solution to the homogeneous Morley equations as  $\bar{S}$ , this condition can be easily enforced by dropping all positive exponential terms in  $\bar{u}$ ,  $\bar{v}$  and  $\bar{w}$  for  $x > 0$  and likewise the negative exponential terms for  $x < 0$ . The polynomial terms which arise for  $m = 0$  and  $m = 1$  must also be deleted.

Figure 3 shows a part of the boundary of the cutout along with a local coordinate system of  $\underline{t}$ , a unit tangent vector and  $\underline{n}$ , a unit normal vector pointing into the cutout. On the contour  $C$  defining the boundary of the cutout the following conditions are to be

enforced

$$\begin{aligned}
 N_n &= N_n^o \\
 N_{nt} &= N_{nt}^o \\
 M_n &= M_n^o \\
 Q_n^* &= Q_n^{*o}
 \end{aligned}
 \tag{34}$$

where  $Q_n^*$  is the Kirchoff resultant shear composed of the transverse shear plus the rate of change of the twisting moment caused by shear stress. The resultants  $N_n, N_{nt}$ , etc., consist of the sum of the stress resultants due to both parts of the nominal solution as well as the residual solution. (See Appendix B).

The above conditions suffice to specify the loading conditions on the hole boundary for thin shell theory. Since the solution  $\xi$  is represented in two different forms (for  $x > 0$  and  $x < 0$ ), certain conditions of continuity must be insured on the curve  $x = 0$  with the exception of that part of the curve encompassed by the cutout. Therefore, for all points where  $x = 0$  that are not inside the cutout the following conditions are to be satisfied

$$\begin{aligned}
 \lim_{x \rightarrow 0^+} \tilde{w} &= \lim_{x \rightarrow 0^-} \tilde{w} \\
 \lim_{x \rightarrow 0^+} \tilde{u} &= \lim_{x \rightarrow 0^-} \tilde{u} \\
 \lim_{x \rightarrow 0^+} \tilde{v} &= \lim_{x \rightarrow 0^-} \tilde{v} \\
 \lim_{x \rightarrow 0^+} \frac{\partial w}{\partial x} &= \lim_{x \rightarrow 0^-} \frac{\partial w}{\partial x} \\
 \lim_{x \rightarrow 0^+} N_x &= \lim_{x \rightarrow 0^-} N_x \\
 \lim_{x \rightarrow 0^+} N_{x\theta} &= \lim_{x \rightarrow 0^-} N_{x\theta} \\
 \lim_{x \rightarrow 0^+} M_x &= \lim_{x \rightarrow 0^-} M_x \\
 \lim_{x \rightarrow 0^+} Q_x^* &= \lim_{x \rightarrow 0^-} Q_x^*
 \end{aligned}
 \tag{35}$$

Since neither the contour of the cutout nor the locus of points outside the cutout save on  $x = 0$  are coordinate lines for the system  $x, \theta$ , the boundary point matching technique is again brought to bear for the determination of the residual solution coefficients by truncating the infinite series after a sufficient number of terms.

## 7. NUMERICAL RESULTS

At the time of the preparation of this paper, a general computer program, written in the FORTRAN IV language, for the UNIVAC 1107 digital computer of the University of Notre Dame is being completed. This program is expected to handle a large number of cases such as arbitrary normal and tangential surface loads as well as thermal gradients in the longitudinal and radial directions. It is also expected it will be capable of dealing with a variety of cutout configurations. These configurations are prescribed either in terms of a polynomial in  $x$  and  $\theta$  or since the point matching method is used, from input data specifying points on the cutout.

The ability to handle the cases in which edge loads and/or moments are applied on the boundary of the cutout, and also outer loads and/or moments at the shell ends is being included in the computer program.

The following graphs are presented as preliminary results of the computer program. Figure 1 represents a plot of the angular variation of the shell middle surface temperature distribution for a shell having an insulated circular cutout of radius  $r_0$ . Here  $T_A$  and  $\tau$  represent the average temperature and temperature gradient respectively (Eq. 21). A similar plot for a shell having an insulated elliptical cutout is given in Figure 2. In this figure  $r_0$  and  $r_1$  are the half length of the major and minor axes of the elliptical hole respectively. The curves are given for the case of  $r_0/r_1 = 2$ .

The extensive numerical results for the stress and displacement fields with a wide variety of mechanical and thermal loads as well as cutout geometries are not yet available at the present time and they will be presented in a forthcoming publication.

### Acknowledgement

The work reported here was supported by the National Science Foundation under Grant No. GK-11194 to the University of Notre Dame. The content of this paper is a part of a Ph.D. dissertation submitted by the first author, R. E. Hoffman to the University of Notre Dame.

REFERENCES

- [1] LUR'E, A. E., "Statics of thin-walled elastic shells," State Publishing House of Technical and Theoretical Literature, Moscow, Chapter 4, Art. 9, (1947). (In Russian).
- [2] WITHUM, D. "Die Kreiszylinderschale mit kreisförmigem Ausschnitt unter Schubbeanspruchung, Ing.-Arc., Vol. 26, 435 (1958).
- [3] ERINGEN, A. C., NAGHDI, A. K., THIEL, C. C., "State of stress in a circular cylindrical shell with a circular hole," Welding Research Council Bulletin No. 102, January (1965).
- [4] VAN DYKE, P., "Stresses about a circular hole in a cylindrical shell," Harvard U. Div. of Eng. and Appl. Phy., Tech. Rep. No. 21, (1964).
- [5] LEKKERKERKER, J. G., "On the stress distribution in cylindrical shells weakened by a circular hole," Ph.D. diss., Tech. Univ. of Delft, Holland, (1964).
- [6] LEKKERKERKER, J. G., "Stress concentration around circular holes in cylindrical shells," Proc., the Eleventh Int. Cong. of Appl. Mech., (1964).
- [7] REIDELBACH, W., "Der Spannungszustand in Übergangsbereich einer rechtwinkligen Rohrabzweigung, Ing.-Arch., Vol. 30, 293 (1961).
- [8] MYINT, T., RADOK, J. R. M., WOLFSON, M., "Numerical solution of shell intersection problems," Öst. Ing.-Arch., Vol. 16, 253 (1962).
- [9] ERINGEN, A. C., SUHUBI, E. S., "Stress distribution at two normally intersecting cylindrical shells," General Technology Corporation Technical Report No. 3-7, April (1965).
- [10] ERINGEN, A. C., NAGHDI, A. K., MAHMOOD, S. S., THIEL, C. C., ARIMAN, T., "Analysis of stress and deformation in two normally intersecting cylindrical shells subject to internal pressure," Gen. Tech. Corp., Tech. Rep. No. 3-8, Dec. (1965).
- [11] ERINGEN, A. C., NAGHDI, A. K., MAHMOOD, S. S., THIEL, C. C., ARIMAN, T., "Stress concentrations in two normally intersecting cylindrical shells subject to internal pressure," Welding Research Council Bulletin No. 139, April (1969).
- [12] FLEISHMAN, N. P., "Effect of reinforcing ring on the stresses in a cylindrical shell with a circular hole," DTMB Trans. No. 319, March (1965).
- [13] VAN CAMPEN, D. H., "Mechanical and thermal stresses in cylinder-to-cylinder intersections of equal or nearly equal diameters," Proceedings of the Int. Conf. on Pressure Vessel Technology, 293, (1969).
- [14] HOFFMAN, R. E., ARIMAN, T., "Thermal bending of plates with circular holes," Nuclear Engineering and Design, Vol. 14, 23, (1970).
- [15] LEE, L. H. N., ARIMAN, T., HOFFMAN, R. E., "Thermal stresses in open cylindrical shells with circular holes," Tech. Rep. THEMIS-UND-68-2, December (1968).

- [16] ARIMAN, T., RAO, K.S., RAO, B.M.N., "Thermal stresses in plates with circular holes," appears in the Nuclear Engineering and Design.
- [17] Reactor Handbook, Vol. IV, Engineering, Editors S. McLain and J. H. Martens, Second Edition, Interscience Publishers, New York (1964), (under the auspices of the USAEC).
- [18] Fast Reactor Technology: Plant Design, Editors J. G. Yevick and A. Amorosi, The M.I.T. Press (1966), (under the auspices of the USAEC).
- [19] MORLEY, L.S.D., "An improvement on Donnell's approximation for thin-walled circular cylinders," Quart. Journ. Appl. Math., Vol. 12, 89, (1969).
- [20] KRAUS, H., Thin Elastic Shells, John Wiley (1967).

APPENDIX A

The Least Squares Boundary Point Matching Method

Given a linear function of the variables  $g_i$  defined by

$$f(x_k) = \sum_{i=1}^N a_i g_i(x_k) \quad (36)$$

where the  $g_i$  may be functions of any finite number of coordinates  $(x_k)$ .

It is desired to determine the coefficients  $a_i$  such that  $f(x_i)$  satisfies some specified conditions on some contour  $L$ . If  $L$  is not a coordinate line of the coordinate system in which the  $x_k$  are defined, it is presently impossible to do this exactly. However, if at some finite number  $\bar{M}$  of points on  $L$  the specified conditions are satisfied, the following system of equations results.

$$F_m[f(x_k)] = F_m\left[\sum_{i=1}^N a_i g_i(x_k)\right] = b_m \quad (37)$$

where  $(m = 1, \dots, \bar{M})$  and  $F_i =$  or  $\neq F_j$ . If  $F$  is a linear operator

$$b_m = \sum_{i=1}^N a_i F_m(g_i(x_k)) \quad (m = 1, \dots, \bar{M}) \quad (38)$$

Then if  $\bar{M} = N$  the system is determined. However if  $\bar{M} > N$  consider a solution to

$$\sum_{m=1}^{\bar{M}} \left[ b_m - \sum_{i=1}^N a_i F_m(g_i(x_k)) \right]^2 = d \quad (39)$$

where  $d$  is minimized with respect to the desired solution vector  $a_i$ .

That is

$$\min_{a_i} \sum_{m=1}^{\bar{M}} \left[ b_m - \sum_{i=1}^N a_i F_m(g_i(x_k)) \right]^2 \quad (40)$$

It can be shown that if

$$a_i = \underline{a} \quad (41)$$

$$F_m(g_i(x_k)) = [C]$$

and  $b_m = \underline{b}$  then for  $\underline{a}$  to minimize equation (39) for  $\bar{M} > N$  it is necessary that  $\underline{a}$  be a solution of

$$[C]^T [C] \underline{a} = [C]^T \underline{b} \quad (42)$$

APPENDIX B

If  $u$ ,  $v$  and  $w$  in the following expressions represent the sum of those displacements due to both nominal solutions and the residual solution then

$$\begin{aligned} N_n = & Eh \left\{ \frac{1}{(1-\nu^2)} \left[ \frac{\partial u}{\partial x} + \nu \left( \frac{\partial v}{\partial \theta} + w \right) + \frac{1}{12} \left( \frac{h}{a} \right)^2 \frac{\partial^2 w}{\partial x^2} - (1+\nu) \alpha T_o \right] \cos^2 \alpha \right. \\ & + \frac{1}{(1+\nu)} \left[ \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x} + \frac{1}{2} \frac{1}{12} \left( \frac{h}{a} \right)^2 \left( \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x} \right) \right] \sin \alpha \cos \alpha \\ & \left. + \frac{1}{(1-\nu^2)} \left[ \frac{\partial v}{\partial \theta} + w + \nu \frac{\partial u}{\partial x} - \frac{1}{12} \left( \frac{h}{a} \right)^2 \left( \frac{\partial^2 w}{\partial \theta^2} - w \right) - (1+\nu) \alpha T_o \right] \sin^2 \alpha \right\} \quad (43) \end{aligned}$$

where in this and subsequent equations  $\alpha$  is the angle between  $-\underline{n}$  and the  $x$  axis.

$$\begin{aligned}
 N_{nt} = & Eh \left\{ \frac{1}{(1-\nu)^2} \left[ (1-\nu) \frac{\partial u}{\partial x} - (1-\nu) \frac{\partial v}{\partial \theta} \right. \right. \\
 & - (1-\nu) w + \frac{1}{12} \left( \frac{h}{a} \right)^2 \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial \theta^2} - w \right) \left. \right] \sin \alpha \cos \alpha \\
 & - \frac{1}{2(1+\nu)} \left[ \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x} + \frac{1}{12} \left( \frac{h}{a} \right)^2 \left( -\frac{\partial^2 w}{\partial x \partial \theta} + \frac{\partial v}{\partial x} \right) \right] \\
 & \cdot \cos^2 \alpha + \frac{1}{2(1+\nu)} \left[ \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x} + \frac{1}{12} \left( \frac{h}{a} \right)^2 \left( \frac{\partial^2 w}{\partial x \partial \theta} + \frac{\partial u}{\partial \theta} \right) \right] \cdot \sin^2 \alpha \left. \right\} \quad (44)
 \end{aligned}$$

$$\begin{aligned}
 M_n = & -\frac{D}{a} \left\{ \left[ \frac{\partial^2 w}{\partial x^2} + \nu \left( \frac{\partial^2 w}{\partial \theta^2} - \frac{\partial v}{\partial \theta} \right) - \frac{\partial u}{\partial x} + (1+\nu) \alpha a T_1 \right] \cos^2 \alpha \right. \\
 & + \frac{(1-\nu)}{2} \left[ 4 \frac{\partial^2 w}{\partial x \partial \theta} - 3 \frac{\partial v}{\partial x} + \frac{\partial u}{\partial \theta} \right] \sin \alpha \cos \alpha \\
 & \left. + \left[ \frac{\partial^2 w}{\partial \theta^2} + w + \nu \frac{\partial^2 w}{\partial x^2} + (1+\nu) \alpha a T_1 \right] \sin^2 \alpha \right\} \quad (45)
 \end{aligned}$$

$$\begin{aligned}
 Q_n^* = & \frac{D}{a} \left\{ -\frac{1}{a} \left[ \frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial \theta^2} + \frac{\partial^2 u}{\partial x^2} \right. \right. \\
 & - \frac{1}{2} (1-\nu) \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{2} (1+\nu) \frac{\partial^2 v}{\partial x \partial \theta} + (1+\nu) \alpha a \frac{\partial T_1}{\partial x} \left. \right] \cos \alpha \\
 & - \frac{1}{a} \left[ \frac{\partial^3 w}{\partial \theta^3} + \frac{\partial w}{\partial \theta} + \frac{\partial^3 w}{\partial x^2 \partial \theta} + (1-\nu) \frac{\partial^2 v}{\partial x^2} + (1+\nu) \alpha a \frac{\partial T_1}{\partial \theta} \right] \sin \alpha \\
 & + \left[ \left[ \frac{\partial^3 w}{\partial x^3} + \nu \left( \frac{\partial^3 w}{\partial x \partial \theta^2} - \frac{\partial^2 v}{\partial x \partial \theta} \right) - \frac{\partial^2 u}{\partial x^2} \right. \right. \\
 & - \frac{\partial^3 w}{\partial x \partial \theta^2} - \frac{\partial w}{\partial x} - \nu \frac{\partial^3 w}{\partial x^3} \left. \right] \sin \alpha \cos \alpha - (1-\nu) \left[ \frac{\partial^3 w}{\partial x^2 \partial \theta} - \frac{\partial^2 v}{\partial x^2} \right] \cos^2 \alpha \\
 & + \frac{(1-\nu)}{2} \left[ 2 \frac{\partial^3 w}{\partial x^2 \partial \theta} - \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial \theta} \right] \sin^2 \alpha \left. \right] (-\sin \alpha) \\
 & + \left[ \left[ \frac{\partial^3 w}{\partial x^2 \partial \theta} + \nu \left( \frac{\partial^3 w}{\partial \theta^3} - \frac{\partial^2 v}{\partial \theta^2} \right) - \frac{\partial^2 u}{\partial x \partial \theta} \right. \right. \\
 & - \frac{\partial^3 w}{\partial \theta^3} - \frac{\partial w}{\partial \theta} - \nu \frac{\partial^3 w}{\partial x^2 \partial \theta} \left. \right] \sin \alpha \cos \alpha - (1-\nu) \left[ \frac{\partial^3 w}{\partial x \partial \theta^2} - \frac{\partial^2 v}{\partial x \partial \theta} \right] \cos^2 \alpha \\
 & + \frac{(1-\nu)}{2} \left[ 2 \frac{\partial^3 w}{\partial x \partial \theta^2} - \frac{\partial^2 v}{\partial x \partial \theta} + \frac{\partial^2 u}{\partial \theta^2} \right] \sin^2 \alpha \left. \right] \cos \alpha \\
 & + \left[ \left[ \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial \theta^2} - w + \nu \left( \frac{\partial^2 w}{\partial \theta^2} - \frac{\partial^2 w}{\partial x^2} \right) - \nu \frac{\partial v}{\partial \theta} - \frac{\partial u}{\partial x} \right] (\cos^2 \alpha - \sin^2 \alpha) \right. \\
 & + (1-\nu) \left[ \frac{\partial^2 w}{\partial x \partial \theta} - \frac{\partial v}{\partial x} \right] 2 \cos \alpha \sin \alpha + \frac{(1-\nu)}{2} \left[ 2 \frac{\partial^2 w}{\partial x \partial \theta} - \frac{\partial v}{\partial x} + \frac{\partial u}{\partial \theta} \right] \\
 & \left. \cdot 2 \cos \alpha \sin \alpha \right] \cdot \left[ \frac{\partial \alpha}{\partial \theta} \cos \alpha - \frac{\partial \alpha}{\partial x} \sin \alpha \right] \left. \right\} \quad (46)
 \end{aligned}$$

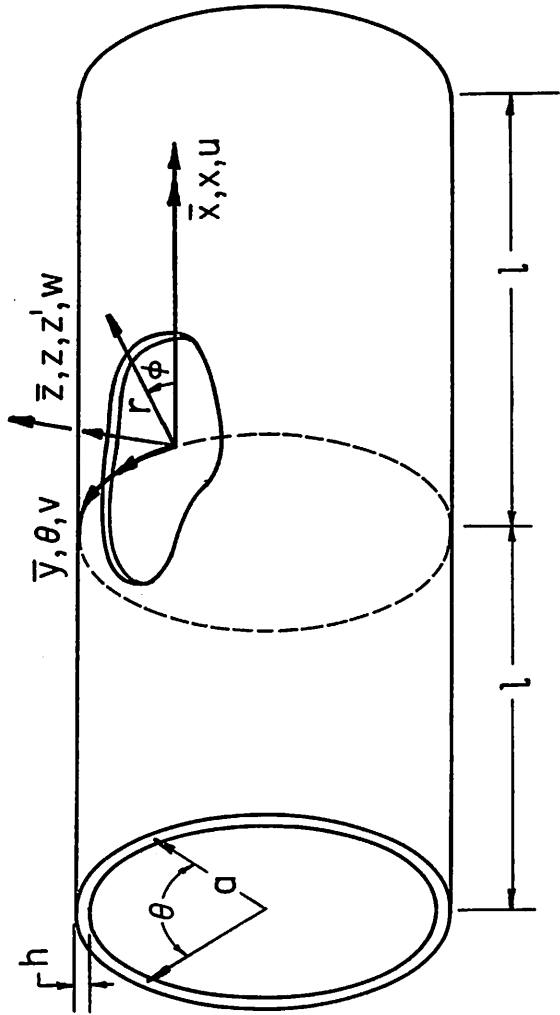


Figure 1: Shell geometry and the coordinate system.

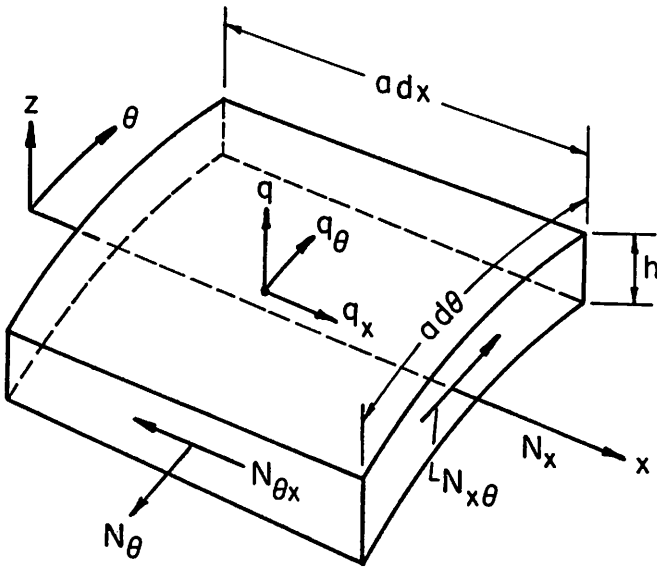


Figure 2a: Shell element with the membrane stress resultants and external loads.

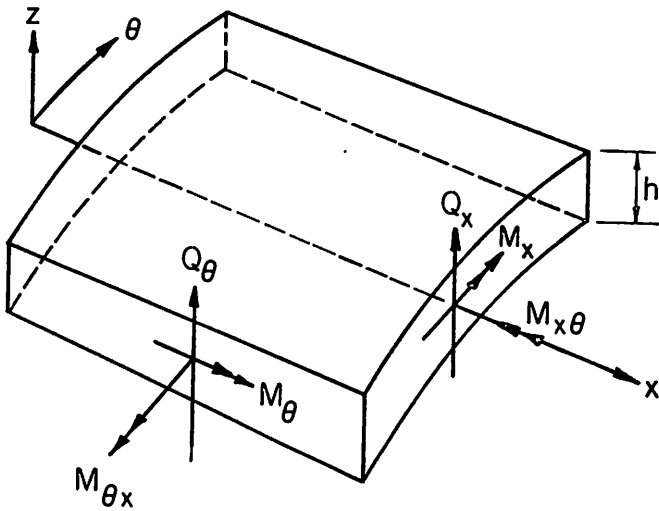


Figure 2b: Shell element with the moment resultants and transverse shear forces.

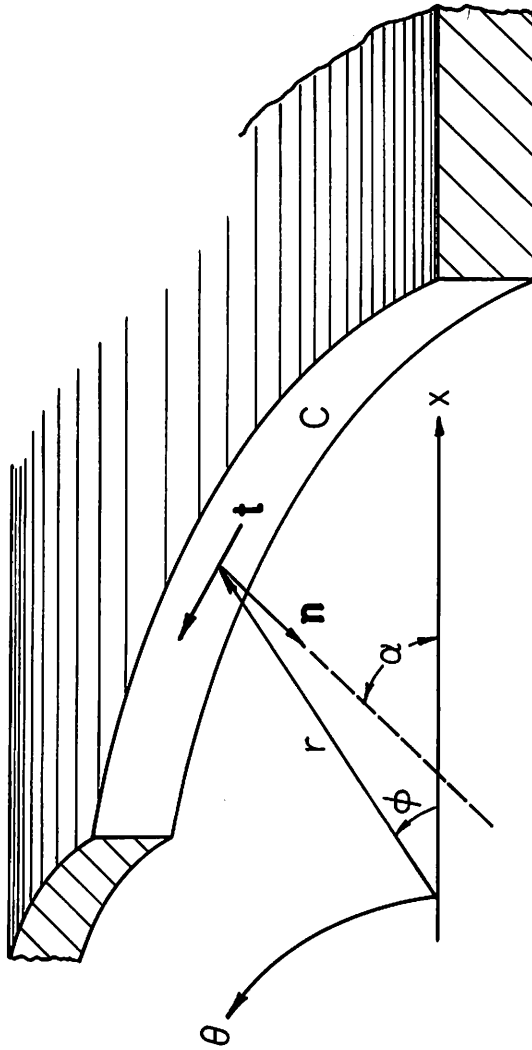


Figure 3: A part of the cutout with the local coordinate system.

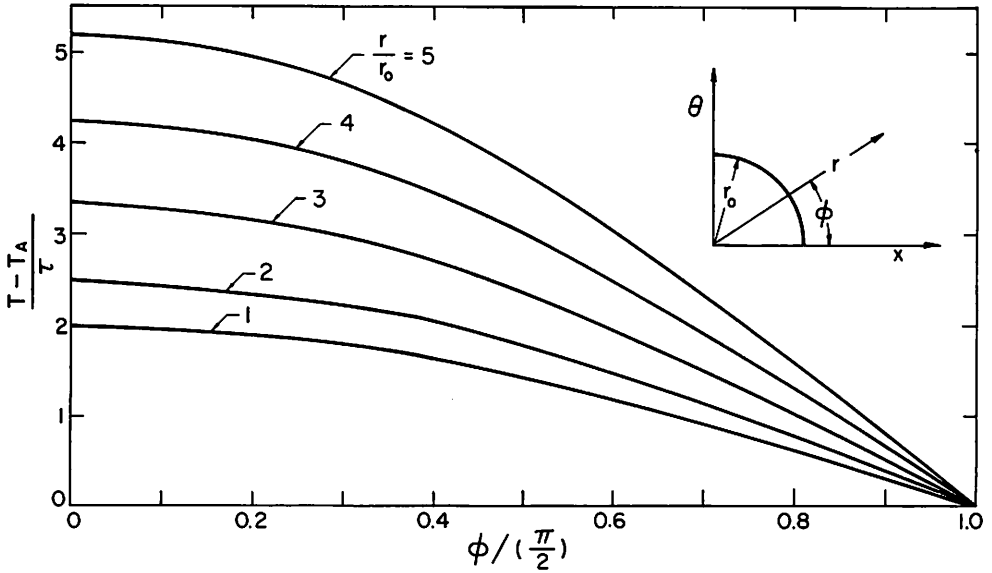


Figure 4: The angular variation of the temperature distribution for the case of a circular cutout.

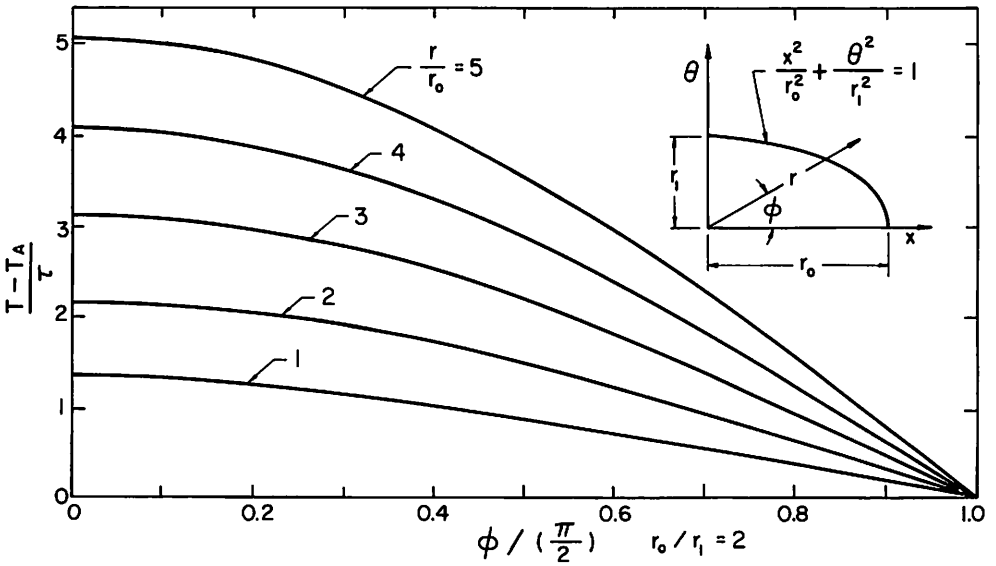


Figure 5: The angular variation of the temperature distribution for the case of an elliptical cutout.