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ERROR ESTIMATES FOR FINITE ELEMENT METHODS FOR SCALAR CONSERVATION LAWS*

BERNARDO COCKBURN† AND PIERRE-ALAIN GREMAUD‡

Abstract. In this paper, new a posteriori error estimates for the shock-capturing streamline diffusion (SCSD) method and the shock-capturing discontinuous galerkin (SCDG) method for scalar conservation laws are obtained. These estimates are then used to prove that the SCSD method and the SCDG method converge to the entropy solution with a rate of at least $h^{1/8}$ and $h^{1/4}$, respectively, in the $L^\infty(L^1)$-norm. The triangulations are made of general acute simplices and the approximate solution is taken to be piecewise a polynomial of degree k. The result is independent of the dimension of the space.

Key words. error estimates, streamline diffusion method, discontinuous galerkin method, multidimensional conservation laws

AMS subject classifications. 65M60, 65N30, 35L65

1. Introduction. In this paper, we consider the problem of estimating the difference between the entropy solution of the initial value problem, [7],

$$\partial_t u + \nabla \cdot f(u) = 0 \quad \text{in } (0, T_\infty) \times \mathbb{R}^d, \tag{1.1}$$

$$u(0) = u_0 \quad \text{on } \mathbb{R}^d, \tag{1.2}$$

and the approximate solution given by the so-called shock-capturing streamline diffusion (SCSD) method, see [3], [10], [11], and the references therein, or given by the so-called shock-capturing discontinuous galerkin (SCDG) method; see [5] and [4]. More precisely, we obtain new a posteriori error estimates which are then used to prove that the SCSD method and the SCDG method converge to the entropy solution of (1.1), (1.2) in the $L^\infty(L^1)$-norm at least as $h^{1/8}$ and $h^{1/4}$, respectively, as the discretization parameter h goes to zero. We assume the flux function $f : \mathbb{R} \to \mathbb{R}^d$ to be smooth and take the compactly supported initial data u_0 in the space $L^\infty(\mathbb{R}^d) \cap \mathrm{BV}(\mathbb{R}^d)$ of bounded functions of bounded variation in $\mathbb{R}^d$.

Convergence of the SCSD method was first obtained by Szepessy [10]. By using DiPerna's theory [2] of measure-valued solutions for (1.1), (1.2), Szepessy proved that the piecewise-linear approximate solution given by the SCSD method converges in $L^p_{loc}((0, T) \times \mathbb{R}^2)$ to the entropy solution of (1.1), (1.2), for any $p \in [1, \infty)$. Later, Szepessy [11] extended this result to the case of a general scalar conservation law in several space dimensions with boundary conditions and an approximate solution which is piecewise polynomial of degree k.

To obtain convergence results in the framework of measure-valued solutions for (1.1), (1.2), [2], the approximate solution must

(i) be bounded in the $L^\infty((0, T_\infty) \times \mathbb{R}^d)$-norm,
(ii) be weakly consistent with all entropy inequalities,
(iii) be strongly consistent (in time) with the initial condition.

Recently, Jaffré, Johnson, and Szepessy [4] proved that the SCDG method converges to the entropy solution of problem (1.1), (1.2) by using an extension of the measure-value convergence theory of DiPerna [2] obtained by Szepessy [12] which allows them to replace in (i) the $L^\infty((0, T_\infty) \times \mathbb{R}^d)$-norm by the $L^\infty(0, T_\infty; L^2(\mathbb{R}^d))$-norm. In this paper, we consider the

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†School of Mathematics, University of Minnesota, 127 Vincent Hall, Minneapolis, MN 55455 (cockburn@math.umn.edu). This research was partially supported by National Science Foundation grant DMS-91-03997.

‡School of Mathematics, University of Minnesota, 127 Vincent Hall, Minneapolis, MN 55455. Current address: Department of Mathematics, North Carolina State University, Box 8205, Raleigh, NC 27695-8205 (gremaud@dali.math.ncsu.edu). This research was partially supported by a grant from the Minnesota Supercomputer Institute and by the Fonds National Suisse pour la Recherche Scientifique.

case of piecewise polynomial approximations of degree k and show how to obtain not only convergence, but also error estimates with only a suitable version of (ii); the properties (i) and (iii) do not need to be obtained. The basic idea is to combine the estimates of the entropy dissipation, needed in (ii), with a modification of the Kuznetsov approximation theory [8]; see also Cockburn, Coquel, and LeFloch [1].

The paper is organized as follows. In §2, we display the SCSD and SCDG methods and state and briefly discuss our main results. The remainder of the paper is devoted to proving them. In §3, we display a basic approximation inequality, Lemma 3.1. This inequality states that to obtain error estimates, only an estimate of the entropy dissipation is required. Those estimates are obtained in §4 and the a posteriori error estimates are proven. In §5, a very simple key regularity property of the approximate solution is obtained (by a simple L^2-energy argument) which is then used to prove the remaining main results. In §6, we give a proof of Lemma 3.1 and we end in §7 with some concluding remarks.

2. The main results. In this section, we describe the SCSD and SCDG finite element methods and state and briefly discuss our main results.

The methods we have in mind being essentially implicit, we first decompose our domain into "slabs." More precisely, let $0 = t_0 < t_1 < t_2 < \cdots < t_N = T_\infty$ be a sequence of time levels. We set

$$S_n = (t_n, t_{n+1}) \times \mathbb{R}^d, \quad n = 0, \ldots, N-1,$$
$$\mathbb{R}^d_n = \{t_n\} \times \mathbb{R}^d, \quad n = 0, \ldots, N.$$

In each slab S_n, we define a triangulation $\mathcal{T}_{h,n}$ of $(d+1)$-simplices. No compatibility at $t = t_n$ between the meshes of two consecutive slabs S_n and S_{n+1}, $n = 0, \ldots, N-2$, is required. We only assume that the triangulations satisfy the following simple conditions:

$$T \text{ is acute} \quad \text{for all } T \in \mathcal{T}_{h,n}, \text{ and } n = 0, \ldots, N-1, \tag{2.1a}$$
$$\frac{h_T}{\rho_T} \le \sigma \quad \text{for all } T \in \mathcal{T}_{h,n}, \text{ and } n = 0, \ldots, N-1, \tag{2.1b}$$
$$h_T \ge \bar{c}\, h \quad \text{for all } T \in \mathcal{T}_{h,n}, \text{ and } n = 0, \ldots, N-1, \tag{2.1c}$$
$$\Delta t_n \le \bar{\bar{c}}\, h \quad \text{for all } n = 0, \ldots, N-1, \tag{2.1d}$$

where h_T is the diameter of $T \in \mathcal{T}_{h,n}$, ρ_T is the diameter of the biggest ball totally included in T, $h = \max\{h_T, T \in \mathcal{T}_{h,n}, n = 0, \ldots, N-1\}$, $\bar{c}$ and $\bar{\bar{c}}$ are positive constants, and $\Delta t_n = t_{n+1} - t_n$ is the width of the slab S_n. The set of d-simplices corresponding to the edges of $\mathcal{T}_{h,n}$ is denoted by $\partial\mathcal{T}_{h,n}$. Also, throughout this paper, for any $\tau > 0$, we set N_τ for the largest integer such that $t_{N_\tau} \le \tau$.

Taking into account the fact that u_0 and (consequently) $u(t,\cdot)$, $t \in [0, T_\infty]$, have compact support, we introduce the following spaces:

$$V_{h,n} = \{v;\, v|_T \in \mathcal{P}_k(T), \forall T \in \mathcal{T}_{h,n}, v = 0 \text{ for } |\, x\, | \ge M\},$$
$$V_h = \{v : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R},\, v|_{(t_n,t_{n+1})} \in V_{h,n}\},$$

where M is a sufficiently large constant and $\mathcal{P}_k$ stands for the space of polynomials of degree k. We emphasize that no continuity requirements are imposed upon the functions in V_h. Following [11], we partition each $(d+1)$-simplex T into k^{d+1} congruent $(d+1)$-simplices, denoted by T_i, and introduce the spaces

$$\hat{V}_{h,n} = \{v;\, v|_{T_i} \in \mathcal{P}_1(T_i), i = 1, \ldots, k^{d+1}, \forall T \in \mathcal{T}_{h,n}, v = 0 \text{ for } |\, x\, | \ge M\},$$
$$\hat{V}_h = \{v : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R},\, v|_{(t_n,t_{n+1})} \in \hat{V}_{h,n}\}.$$

For each $v \in V_h$, we define $\hat{v}$ to be the element of $\hat{V}_h$ that coincides with v on each of the vertices of the $(d+1)$-simplices T_i, for $i = 1, \dots, k^{d+1}$ and every T of $\mathcal{T}_{h,n}$, for $n = 0, \dots, N-1$. In the case $k = 0$, we set $\hat{V}_{h,n} = V_{h,n}$, $\hat{V}_h = V_h$, and $\hat{v} = v$.

Finally, we define, for all $T \in \mathcal{T}_{h,n}$ and $n = 0, \dots, N-1$,

$$\mathbb{P}_h(v)|_T = \frac{1}{|T|} \int_T v \, dx dt,$$

$$\| v \|_{\mathbb{P}_h}(t,x) = \frac{1}{|T|^{1/2}} \| v \|_{L^2(T)}, \text{ for } (t,x) \in T,$$

and, for all $e \in \partial\mathcal{T}_{h,n}$ and $n = 0, \dots, N-1$,

$$\mathbb{P}_h(v^T)|_e = \frac{1}{|e|} \int_e v^T \, d\lambda,$$

$$\| v^T \|_{\mathbb{P}_h}(t,x) = \frac{1}{|e|^{1/2}} \| v^T \|_{L^2(e)}, \text{ for } (t,x) \in e,$$

where for $p = (t,x) \in e \backslash \partial e$, we have used the notation $v^T(p) = \lim_{s\downarrow 0} v(p - s n_{e,T})$, $n_{e,T}$ being the unit outer normal to T along its face e.

The approximate solution u_h given by the SCDG method is defined to be the element of V_h such that

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_{h,n}} \int_T A(u_h)(v + \delta \tilde{A}(u_h, v)) \, dxdt + \sum_{T\in\mathcal{T}_{h,n}} \sum_{e\in\partial T} \int_e \Big(f^{LF}_{e,T}(u_h^T, u_h^{T_e}) - \tilde{f}(u_h^T)\cdot n_{e,T} \Big) v^T d\lambda \\
&+ \sum_{T\in\mathcal{T}_{h,n}} \int_T \varepsilon_1(u_h) \mathbb{P}_h(\nabla \hat{u}_h \cdot \nabla \hat{v}) \, dxdt + \sum_{T\in\mathcal{T}_{h,n}} \sum_{e\in\partial T} \int_e \varepsilon_2(u_h^T) \mathbb{P}_h(\nabla_e \hat{u}_h^T \cdot \nabla_e \hat{v}^T) d\lambda = 0, \\
&\qquad \forall v \in V_{h,n}, \quad n = 0, 1, \dots, N-1,
\end{aligned} \tag{2.2}$$

where $\tilde{f}(u_h) = (u_h, f(u_h))$,

$$A(v) = \partial_t v + \sum_{i=1}^d \partial_i f_i(v), \quad \tilde{A}(w,v) = \partial_t v + \sum_{i=1}^d f_i'(w) \partial_i v,$$

$$\nabla = (\partial_t, \partial_1, \partial_2, \dots, \partial_d), \qquad \nabla_x = (\partial_1, \partial_2, \dots, \partial_d), \qquad \nabla_e v^T = \nabla(v^T|_e),$$

and where $v^{T_e}(p) = \lim_{s\downarrow 0} v(p + s n_{e,T})$, for $p = (t,x) \in e\backslash\partial e$ and $e \in \partial T$. The initial condition is defined as follows:

$$u_h^{T_e}(0,\cdot) = u_{0h}, \text{ for all } e \in \partial\mathcal{T}_{h,0} \cap \mathbb{R}^d_0, \tag{2.3}$$

u_{0h} being the standard L^2-projection of u_0 into the space of functions which are polynomials of degree k on $e \in \partial\mathcal{T}_{h,0} \cap \mathbb{R}^d_0$.

We use throughout the paper the so-called local Lax-Friedrichs flux

$$f^{LF}_{e,T}(u_h^T, u_h^{T_e}) = \frac{1}{2}(\tilde{f}(u_h^T) + \tilde{f}(u_h^{T_e})) \cdot n_{e,T} + C_e^{LF}(u_h^T - u_h^{T_e}), \tag{2.4a}$$

where $C_e^{LF} = \frac{1}{2}$ if $e \subset \mathbb{R}^d_n$ for some $n = 0, 1, \dots, N$. We assume that

$$C_e^{LF} - \frac{1}{2}|\tilde{f}'|_{e,\infty} \geq c_\star > 0, \qquad \text{for } e \in \mathcal{E}_n, \tag{2.4b}$$

with $|\tilde{f}'|_{e,\infty} = \max\{|\tilde{f}'(\xi)\cdot n_{e,T}|, \xi \in [\min_{y\in e}\{u_h^T(y), u_h^{T_e}(y)\}, \max_{y\in e}\{u_h^T(y), u_h^{T_e}(y)\}]\}$ and where $\mathcal{E}_n$ denotes the subset of the edges $\partial\mathcal{T}_{h,n}$ whose intersection with $\mathbb{R}_n^d \cup \mathbb{R}_{n+1}^d$ is of measure zero. The condition (2.4b) ensures that the flux $f_{e,T}^{LF}(a,b)$ is a strictly monotone flux, that is, it is an increasing function of a and a decreasing function of b.

Finally, for $T \in \mathcal{T}_{h,n}$, $n = 0, 1, \dots, N-1$, the shock-capturing terms ε_1 and ε_2 are defined as follows:

$$\varepsilon_1(v)|_T = \delta_1 \frac{|A(v)|}{\|\nabla\hat{v}\|_{\mathbb{P}_h} + \delta_3}$$

and

$$\varepsilon_2(v^T)|_e = \delta_2 \frac{|f_{e,T}^{LF}(v^T, v^{T_e}) - \tilde{f}(v^T)\cdot n_{e,T}|}{\|\nabla_e \hat{v}^T\|_{\mathbb{P}_h} + \delta_4},$$

for any $v \in V_h$. The positive parameters δ, δ_1, δ_2, δ_3, and δ_4 will be defined later.

On the other hand, the SCSD method is the exact analog of SCDG but with *continuous* approximations inside each slab S_n, $n = 0, 1, \dots, N-1$. Namely, instead of $V_{h,n}$, one considers the space $\mathcal{V}_{h,n}$, where

$$\mathcal{V}_{h,n} = \{v \in \mathcal{C}^0(S_n);\, v|_T \in \mathcal{P}_k(T), \forall T \in \mathcal{T}_{h,n}, v = 0 \text{ for } |x| \geq M\}.$$

The spaces $\mathcal{V}_h$, $\hat{\mathcal{V}}_{h,n}$, $\hat{\mathcal{V}}_h$ are also defined accordingly. Thus, the approximate solution u_h given by the SCSD method is the element of $\mathcal{V}_h$ such that

$$\int_{S_n} A(u_h)(v + \delta\tilde{A}(u_h, v))dxdt + \int_{\mathbb{R}_n^d}(u_{h,+} - u_{h,-})v_+ dx + \int_{S_n} \varepsilon_1(u_h)\mathbb{P}_h(\nabla\hat{u}_h \cdot \nabla\hat{v})dxdt$$
$$+ \int_{\mathbb{R}_n^d} \varepsilon_2^{SD}(u_{h,+})\mathbb{P}_h(\nabla_x\hat{u}_{h,+}\cdot\nabla_x\hat{v}_+)dx = 0, \qquad \forall v \in \mathcal{V}_{h,n}, \quad n = 0, 1, \dots, N-1,$$

(2.5)
where

$$\varepsilon_2^{SD}(v(t_n,\cdot))|_K = \delta_2 \frac{|v_+(t_n,\cdot) - v_-(t_n,\cdot)|}{\|\nabla_x\hat{v}_+(t_n)\|_{\mathbb{P}_h} + \delta_4}$$

for any $v \in \mathcal{V}_h$, and where $u_{h,\pm} = \lim_{s\to\pm 0} v(t+s, x)$. Instead of (2.3), the initial condition is now given by

$$u_{h,-}(0,\cdot) = u_{0h}. \tag{2.6}$$

Notice that for both methods, $u_h|_{S_n}$ is not coupled to $u_h|_{S_{n+1}}$, $n = 0, \dots, N-2$, as a consequence of the "slab structure" and of the upwinding in time used in both numerical schemes.

For the sake of completeness, we include in the Appendix a proof of the existence of a solution to (2.2), (2.3) (or (2.5), (2.6)). The proof uses a fixed point argument. See [10], [11] and the references quoted therein for a similar application of this argument.

Since the only relevant values of the nonlinear flux f are those in the range of the entropy solution u, $[a = \inf u_0, b = \sup u_0]$, we extend each of the components of f smoothly in such a way that the extension is affine linear outside a fixed compact including $[a, b]$. We use that extension, which we still call f, to define the above schemes. Thus, we have

$$\| f' \| = \sup_{|n|=1} | f' \cdot n |_{L^\infty(\mathbb{R})} < \infty. \tag{2.7a}$$

In what follows, we restrict the choice of local viscosity coefficients C_e^{LF} to those satisfying

$$C_e^{LF} \leq c_\infty, \tag{2.4c}$$

where c_∞ is an arbitrary constant such that $c_\infty \geq \frac{1}{2} | f' |_{e,\infty} + c_\star$ (see (2.4b)). We also assume that each of the components of the flux function f has a Lipschitz continuous derivative:

$$\| f'' \| = \sup_{|n|=1} | f'' \cdot n |_{L^\infty(\mathbb{R})} < \infty. \tag{2.7b}$$

Next, we state and briefly discuss our main results.

2.1. Error estimates for the SCDG with $k = 0$. We start by considering the SCDG method with a piecewise-constant approximate solution. In this case, the only term of the left-hand side of (2.2) that survives is the second term and the resulting scheme is nothing but an implicit monotone scheme.

THEOREM 2.1 (a posteriori error estimate). *Let u be the unique entropy satisfying solution of* (1.1), (1.2), *and let u_h be the solution given by the SCDG method with $k = 0$. Then, for any $n = 1, 2, \ldots, N_{T_\infty} - 1$, we have*

$$\begin{aligned}\|u_h(t_n, \cdot) - u(t_n, \cdot)\|_{L^1(\mathbb{R}^d)} &\leq 2 \, \|u_{0h} - u_0\|_{L^1(\mathbb{R}^d)} \\ &\quad + C_{1a}\big\{| u_0 |_{TV(\mathbb{R}^d)} \, \Theta_0(u_h) \big\}^{1/2} h^{1/2},\end{aligned}$$

where

$$\begin{aligned}\Theta_0(u_h) = &\sum_{n=0}^{N_\tau - 1} \sum_{T\in\mathcal{T}_{h,n}} \frac{h_T}{h} \sum_{e\in\partial T} \int_e | f_{e,T}^{LF}(u_h^T, u_h^{T_e}) - \tilde{f}(u_h^T) \cdot n_{e,T} | \, d\lambda' \\ &+ \sum_{n=0}^{N_\tau - 1} \frac{\Delta t_n}{h} \bigg(\sum_{e\in\mathcal{E}_n} \int_e |u_h^T - u_h^{T_e}| d\lambda' + \int_{\mathbb{R}^d} |u_{h,+} - u_{h,-}| dx' \bigg),\end{aligned}$$

$C_{1a} = c(d, k) \, (1 + \| f' \|)^{1/2}$, *and* $\Delta t_n = t_{n+1} - t_n$.

We point out that this result does not require either the hypotheses (2.1) on the triangulations or the conditions (2.4) on the numerical flux to be satisfied. It is a new general result for implicit schemes which could be used as the basis for an adaptivity strategy; however, we do not pursue this avenue of research in this paper.

Thus, to obtain an error estimate we only have to obtain an upper bound for the quantity $\Theta_0(u_h)$ which, ideally, we expect to be of the following form:

$$\Theta_0(u_h) \leq C \, T_\infty \, | u_0 |_{TV(\mathbb{R}^d)}.$$

In this case, Theorem 2.1 gives the classical rate of $h^{1/2}$ for the error. If the standard L^2-energy technique associated with the SCDG (and SCSD) methods is used, we can obtain the following estimate.

PROPOSITION 2.2 (estimate of $\Theta_0(u_h)$). *Suppose that the hypotheses* (2.1b), (2.1c), *and* (2.1d) *on the triangulations are satisfied. Suppose that the conditions* (2.4) *on the numerical flux are also satisfied. Let u_h be the solution given by the SCDG method with $k = 0$. Then,*

$$\Theta_0(u_h) \leq C_{2a} \left\{ \frac{\sigma}{\bar{c}} \right\}^{1/2} \| u_0 \|_{L^2(\mathbb{R}^d)} \, h^{-1/2},$$

where $C_{2a} = c(d)\big(\|f'\|/2 + c_\infty + \bar{c}\big)\{(2M)^d \, T_\infty / \min\{1, c_\star\}\}^{1/2}$.

Thus, we can see that the use of the simple L^2-energy technique gives an upper bound that does not depend on $|\, u_0 \,|_{TV(\mathbb{R}^d)}$ but only on $\|\, u_0 \,\|_{L^2(\mathbb{R}^d)}$. As a consequence, the upper bound is not independent of h, but blows up like $h^{-1/2}$. This fact is reflected in the loss of one-fourth of the expected optimal order of convergence of one-half, as we can see in the corollary below. This same phenomenon occurs in the treatment of explicit monotone schemes defined in general triangulations; see [1].

COROLLARY 2.3. *Suppose that the hypotheses* (2.1b), (2.1c), *and* (2.1d) *on the triangulations are satisfied. Suppose that the conditions* (2.4) *on the numerical flux are also satisfied. Let u be the unique entropy satisfying the solution of* (1.1), (1.2), *and let u_h be the solution given by the SCDG method with $k = 0$. Then, for any $n = 1, 2, \dots, N_{T_\infty} - 1$, we have*

$$\|u_h(t_n, \cdot) - u(t_n, \cdot)\|_{L^1(\mathbb{R}^d)} \le 2\, \|u_{0h} - u_0\|_{L^1(\mathbb{R}^d)} + C_{1a}\, C_{2a}^{1/2} \left\{ |\, u_0 \,|_{TV(\mathbb{R}^d)}\, \|\, u_0 \,\|_{L^2(\mathbb{R}^d)} \right\}^{1/2} \left\{ \frac{\sigma\, h}{\bar{c}} \right\}^{1/4}.$$

Notice that, since the hypothesis (2.1a) on the triangulations is not required to be satisfied, the time-space tetraedra T could be very flat. In this case, σ can blow up and $\bar{c}$ can go to zero as h goes to zero. The above corollary states that when the triangulations become more irregular as h goes to zero, the error is of order $\{h\,\sigma/\bar{c}\}^{1/4}$.

2.2. Error estimates for the SCDG method with arbitrary k. We start with the following result.

THEOREM 2.4 (a posteriori error estimate). *Let u be the unique entropy satisfying solution of* (1.1), (1.2), *and let u_h be the solution given by the SCDG method. Then, for any $n = 1, 2, \dots, N_{T_\infty} - 1$, we have*

$$\begin{aligned} \|u_h(t_n, \cdot) - u(t_n, \cdot)\|_{L^1(\mathbb{R}^d)} &\le 2\, \|u_{0h} - u_0\|_{L^1(\mathbb{R}^d)} \\ &\quad + C_{1b} \left\{ |\, u_0 \,|_{TV(\mathbb{R}^d)}\, \Theta_1(u_h) \right\}^{1/2} h^{1/2} \\ &\quad + C_{2b} \left\{ \max\{1, |\, u_0 \,|_{TV(\mathbb{R}^d)}\}\, \Theta_2(u_h) \right\}^{1/2} h^{1/2}, \end{aligned}$$

where

$$\Theta_1(u_h) = \Theta_0(u_h) + \sum_{n=0}^{N_\tau - 1} \frac{\Delta t_n}{h} \sum_{T \in \mathcal{T}_{h,n}} \int_T |A(u_h)|\, dx'dt' + \sum_{n=0}^{N_\tau - 1} \sum_{T \in \mathcal{T}_{h,n}} \frac{h_T}{h} \int_T |\, A(u_h) \,|\, dx'dt',$$

$$\begin{aligned} \Theta_2(u_h) = &\sum_{n=0}^{N_\tau - 1} \sum_{T \in \mathcal{T}_{h,n}} \sum_{e \in \partial T} \int_e |\, f_{e,T}^{LF}(u_h^T, u_h^{T_e}) - \tilde{f}(u_h^T) \cdot n_{e,T} \,| \left(\frac{h_T}{h} \right) \|\, \nabla_e \hat{u}_h^T \,\|_{\mathbb{P}_h}\, d\lambda' \\ &+ \sum_{n=0}^{N_\tau - 1} \sum_{T \in \mathcal{T}_{h,n}} \int_T |\, A(u_h) \,| \left(\frac{h_T}{h} \right) \|\, \nabla' \hat{u}_h \,\|_{\mathbb{P}_h}\, dx'dt', \end{aligned}$$

and

$$\begin{aligned} C_{1b} &= c(d, k)\, (1 + \|\, f' \,\|)^{1/2}, \\ C_{2b} &= c(d, k)\, (c_0 (2M)^d + c_1 T_\infty \|\, f'' \,\|)^{1/2}. \end{aligned}$$

(The numerical constants c_0 and c_1 are defined in Lemma 3.1.*)*

Notice that this new a posteriori estimate reduces to the one in Theorem 2.1 for the case $k = 0$, as expected.

To estimate the quantities $\Theta_1(u_h)$ and $\Theta_2(u_h)$ we use once more the standard L^2-energy technique. We obtain the following result.

PROPOSITION 2.5 (estimates of $\Theta_1(u_h)$ and $\Theta_2(u_h)$). *Suppose that the hypotheses* (2.1b), (2.1c), *and* (2.1d) *on the triangulations are satisfied. Suppose that the conditions* (2.4) *on the numerical flux are also satisfied. Let u_h be the solution given by the SCDG method. Then,*

$$\Theta_1(u_h) \le C_{2a}\left\{\frac{\sigma}{\bar{c}}\right\}^{1/2} \| u_0 \|_{L^2(\mathbb{R}^d)}\, h^{-1/2} + (1+\bar{\bar{c}})\, C_{3b} \, \| u_0 \|_{L^2(\mathbb{R}^d)}\, \delta^{-1/2},$$

$$\Theta_2(u_h) \le \frac{1}{2}\,(\delta_1^{-1}+\delta_2^{-1})\, \| u_0 \|^2_{L^2(\mathbb{R}^d)} + C_{2a}\left\{\frac{\sigma}{\bar{c}}\right\}^{1/2} \|u_0\|_{L^2(\mathbb{R}^d)} \frac{\delta_4}{h^{1/2}} + C_{3b}\, \|u_0\|_{L^2(\mathbb{R}^d)} \frac{\delta_3}{\delta^{1/2}},$$

where $C_{3b} = \{(2M)^d\, T_\infty/2\}^{1/2}$.

From the above results, we obtain the following error estimate.

COROLLARY 2.6. *Suppose that the hypotheses* (2.1) *on the triangulations are satisfied. Suppose that the conditions* (2.4) *on the numerical flux are also satisfied. Let u be the unique entropy satisfying solution of* (1.1), (1.2), *and let u_h be the solution given by the SCDG. Then, for any $n = 1, 2, \dots, N_{T_\infty} - 1$, we have*

$$\begin{aligned}\|u_h(t_n,\cdot) - u(t_n,\cdot)\|_{L^1(\mathbb{R}^d)} &\le 2\,\|u_{0h} - u_0\|_{L^1(\mathbb{R}^d)} \\ &+ C_{4b}\, C_{2a}^{1/2} \left\{ \max\{1, | u_0 |_{TV(\mathbb{R}^d)}\}\, \| u_0 \|_{L^2(\mathbb{R}^d)}\right\}^{1/2} \left\{\frac{\sigma\, h}{\bar{c}}\right\}^{1/4} \\ &+ C_{4b}\, C_{3b}^{1/2} \left\{ \max\{1, | u_0 |_{TV(\mathbb{R}^d)}\}\, \| u_0 \|_{L^2(\mathbb{R}^d)}\right\}^{1/2} \frac{h^{1/2}}{\delta^{1/4}} \\ &+ \frac{1}{\sqrt{2}}\, C_{2b}\, (\max\{1, | u_0 |_{TV(\mathbb{R}^d)}\})^{1/2}\, \| u_0 \|_{L^2(\mathbb{R}^d)} \left(\frac{h^{1/2}}{\delta_1^{1/2}} + \frac{h^{1/2}}{\delta_2^{1/2}}\right),\end{aligned}$$

where $C_{4b} = C_{1b}(1+\bar{\bar{c}})^{1/2} + C_{2b}\, \max\{\delta_3^{1/2}, \delta_4^{1/2}\}$.

In particular, when $\delta = \mathcal{O}(h\,\bar{c}/\sigma)$, $\delta_1 = \delta_2 = \mathcal{O}(\| u_0 \|_{L^2(\mathbb{R}^d)}\, \{h\, \bar{c}/\sigma\}^{1/2})$, *and* $\delta_3 = \delta_4 = \mathcal{O}(1)$, *we have*

$$\begin{aligned}\|u_h(t_n,\cdot) - u(t_n,\cdot)\|_{L^1(\mathbb{R}^d)} &\le 2\,\|u_{0h} - u_0\|_{L^1(\mathbb{R}^d)} \\ &+ C \left\{ \max\{1, | u_0 |_{TV(\mathbb{R}^d)}\}\, \| u_0 \|_{L^2(\mathbb{R}^d)}\right\}^{1/2} \left\{\frac{\sigma\, h}{\bar{c}}\right\}^{1/4},\end{aligned}$$

where the constant C does not depend on u_0.

In practice, δ_1 and δ_2 are taken to be of order h and so the previous choice can be considered a choice that introduces too much artificial viscosity. The coefficients δ_1 and δ_2 can be taken of order arbitrarily close to $\mathcal{O}(h)$ by setting $\delta_1 = \delta_2 = \mathcal{O}(\| u_0 \|^{2-4\nu}_{L^2(\mathbb{R}^d)}\, h^{1-2\nu}\, (\bar{c}/\sigma)^{2\nu})$. If δ is as before, then for any $\nu \in (0, \frac{1}{2})$, the error estimate given by the corollary above reads as follows:

$$\begin{aligned}\|u_h(t_n,\cdot) - u(t_n,\cdot)\|_{L^1(\mathbb{R}^d)} &\le 2\,\|u_{0h} - u_0\|_{L^1(\mathbb{R}^d)} \\ &+ C\, (\max\{1, | u_0 |_{TV(\mathbb{R}^d)}\})^{1/2} \left(\| u_0 \|^{1/2}_{L^2(\mathbb{R}^d)} \left\{\frac{h\,\sigma}{\bar{c}}\right\}^{1/4} + \| u_0 \|^{2\nu}_{L^2(\mathbb{R}^d)} \left\{\frac{h\,\sigma}{\bar{c}}\right\}^{\nu}\right).\end{aligned}$$

2.3. Error estimates for the SCSD method. We start with the following result.

THEOREM 2.7 (a posteriori error estimate). *Let u be the unique entropy satisfying solution of* (1.1), (1.2), *and let u_h be the solution given by the SCSD method. Then, for any $n = 1, 2, \dots, N_{T_\infty} - 1$, we have*

$$\|u_h(t_n,\cdot) - u(t_n,\cdot)\|_{L^1(\mathbb{R}^d)} \le 2\,\|u_{0h} - u_0\|_{L^1(\mathbb{R}^d)} + C_{1c}\big\{ |\, u_0 \,|_{TV(\mathbb{R}^d)}\, \Theta_3(u_h) \big\}^{1/2} h^{1/2} + C_{2c}\big\{ \max\{1, |\, u_0\,|_{TV(\mathbb{R}^d)}\}\, \Theta_4(u_h) \big\}^{1/2} h^{1/2},$$

where

$$\Theta_3(u_h) = \sum_{n=0}^{N_\tau-1} \int_{\mathbb{R}^d_n} |u_{h,+} - u_{h,-}| \left(\frac{h_T}{h} + \frac{\Delta t_n}{h} + \frac{\sigma \delta_2}{h} \right) d\lambda' + \sum_{n=0}^{N_\tau-1} \sum_{T\in\mathcal{T}_{h,n}} \int_T |A(u_h)| \left(\frac{h_T}{h} + \frac{\sigma\delta_1}{h} + (1+\|f'\|)\frac{\sigma\delta}{h} + \frac{\Delta t_n}{h} \right) dx'dt',$$

$$\Theta_4(u_h) = \sum_{n=0}^{N_\tau-1} \int_{\mathbb{R}^d_n} |u_{h,+} - u_{h,-}| \left(\frac{h_T}{h} \right) \|\nabla_{x'} \hat{u}_{h,-}\|_{\mathbb{P}_h}\, d\lambda' + \sum_{n=0}^{N_\tau-1} \sum_{T\in\mathcal{T}_{h,n}} \int_T |A(u_h)| \left(\frac{h_T}{h} + (1+\|f'\|)\frac{\sigma\delta}{h} \right) \|\nabla' \hat{u}_h\|_{\mathbb{P}_h} dx'dt',$$

$C_{1c} = c(d,k)\, C_{1b}$, *and* $C_{2c} = c(d,k)\, C_{2b}$.

Notice that this new a posteriori estimate is almost identical to the one in Theorem 2.4. The term with the factor δ appears as a reflection of the presence of the streamline diffusion term of the SCSD method, and the terms with the factors δ_1 and δ_2 appear as a reflection of the presence of the corresponding shock-capturing terms of the SCSD method. The fact that the SCDG uses discontinuous approximations allowed us to discard those terms, as we can see in Theorem 2.4; however, we cannot do this for the SCSD method since it uses continuous approximations.

PROPOSITION 2.8 (estimate of $\Theta_3(u_h)$ and $\Theta_4(u_h)$). *Suppose that the hypotheses* (2.1b), (2.1c), *and* (2.1d) *on the triangulations are satisfied. Suppose that the conditions* (2.4) *on the numerical flux are also satisfied. Let u_h be the solution given by the SCSD method. Then,*

$$\Theta_3(u_h) \le C_{2a} \left\{ \frac{\sigma}{\bar{c}} \right\}^{1/2} \|\, u_0 \,\|_{L^2(\mathbb{R}^d)} \left(\frac{1+\bar{\bar{c}}}{h^{1/2}} + \frac{\sigma\,\delta_2}{h^{3/2}} \right) + C_{3b}\, \|\, u_0\,\|_{L^2(\mathbb{R}^d)} \left(\frac{1+\bar{\bar{c}}}{\delta^{1/2}} + \frac{\sigma\,\delta_1}{h\,\delta^{1/2}} + (1+\|f'\|)\frac{\sigma\delta^{1/2}}{h} \right),$$

$$\Theta_4(u_h) \le \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{R}^d)} \left(\frac{1}{\delta_1}\left(1 + (1+\|f'\|)\frac{\sigma\delta}{h} \right) + \frac{1}{\delta_2} \right) + \max\{\delta_3, \delta_4\}\|u_0\|_{L^2(\mathbb{R}^d)} \left(C_{2a} h^{-1/2} + C_{3b}\delta^{-1/2} \right).$$

From the above results, we obtain the following error estimate.

COROLLARY 2.9. *Suppose that the hypotheses* (2.1) *on the triangulations are satisfied. Suppose that the conditions* (2.4) *on the numerical flux are also satisfied. Let u be the unique entropy satisfying solution of* (1.1), (1.2), *and let u_h be the solution given by the SCSD. Then,* *for any* $n = 1, 2, \dots, N_{T_\infty} - 1$, *we have*

$$\|u_h(t_n,\cdot) - u(t_n,\cdot)\|_{L^1(\mathbb{R}^d)} \le 2\,\|u_{0h} - u_0\|_{L^1(\mathbb{R}^d)}$$
$$+ C_{3c}|u_0|^{1/2}_{TV(\mathbb{R}^d)}\|u_0\|^{1/2}_{L^2(\mathbb{R}^d)} \left(1 + \frac{\sigma^{1/2}\delta_2^{1/2}}{h^{1/2}} + \bar{c}^{\,1/4}\frac{\sigma^{1/4}\delta_1^{1/2}}{h^{1/4}\delta^{1/4}}\right)\left(\frac{\sigma h}{\bar{c}}\right)^{1/4}$$
$$+ C_{3c}|u_0|^{1/2}_{TV(\mathbb{R}^d)}\|u_0\|^{1/2}_{L^2(\mathbb{R}^d)} \left(\frac{h^{1/2}}{\delta^{1/4}} + \sigma^{1/2}\delta^{1/4}\right)$$
$$+ C_{2c}C_{3c}\max\{1, |u_0|^{1/2}_{TV(\mathbb{R}^d)}\}\|u_0\|^{1/2}_{L^2(\mathbb{R}^d)}\max\{\delta_3,\delta_4\}\left(h^{1/4} + \frac{h^{1/2}}{\delta^{1/4}}\right)$$
$$+ \frac{1}{\sqrt{2}}C_{4c}\max\{1, |u_0|^{1/2}_{TV(\mathbb{R}^d)}\}\|u_0\|_{L^2(\mathbb{R}^d)}\left(\frac{h^{1/2}}{\delta_1^{1/2}} + \frac{h^{1/2}}{\delta_2^{1/2}}\right),$$

where

$$C_{3c} = \max\{(1+\bar{c})^{1/2}, (1+\|f'\|)^{1/2}\}\max\{C_{2a}^{1/2}, C_{3b}^{1/2}\},$$
$$C_{4c} = C_{2c}\left(1 + (1+\|f'\|)^{1/2}\frac{\sigma^{1/2}\delta^{1/2}}{h^{1/2}}\right).$$

In particular, if $\delta_1 = \delta_2 = \mathcal{O}(\delta^{3/4})$, $\delta_3 = \delta_4 = \mathcal{O}(1)$, *and* $\delta = \mathcal{O}(h\,\bar{c}/\sigma)$, *we have*

$$\|u_h(t_n,\cdot) - u(t_n,\cdot)\|_{L^1(\mathbb{R}^d)} \le 2\,\|u_{0h} - u_0\|_{L^1(\mathbb{R}^d)}$$
$$+ C\Big\{\max\{1, |\,u_0\,|_{TV(\mathbb{R}^d)}\}\|\,u_0\,\|_{L^2(\mathbb{R}^d)}\Big\}^{1/2}\left\{\frac{\sigma h}{\bar{c}}\right\}^{1/8},$$

where C *does not depend on* u_0.

Corollary 2.9 does not allow us to take δ_1 and δ_2 of order h. However, we can take them arbitrarily close to such a choice. More precisely, if we take δ as before and $\delta_1 = \delta_2 = \mathcal{O}(h^{1-2\nu})$, for any $\nu \in (0, 1/4)$, we get

$$\|u_h(t_n,\cdot) - u(t_n,\cdot)\|_{L^1(\mathbb{R}^d)} \le 2\,\|u_{0h} - u_0\|_{L^1(\mathbb{R}^d)} + \mathcal{O}(h^{\min\{\nu,1/4-\nu\}}).$$

3. The approximation inequality. In this section, we obtain an approximation inequality which we then use in §4 to obtain our error estimates. Following Kuznetsov [8], we let $w:\mathbb{R}\to\mathbb{R}$ be a smooth function such that

(3.1a) $$w(t) \ge 0, \ \text{for } t > 0,$$

(3.1b) $$w(t) = w(-t), \ \text{for } t > 0,$$

(3.1c) the support of w is $[-1, 1]$,

(3.1d) $$\int_{\mathbb{R}} w(r)\,dr = 1,$$

(3.1e) $$\int_{\mathbb{R}} |\,w'(r)\,|\,dr \le 2,$$

and set

(3.2) $$\varphi = w_{\varepsilon_t}(t-t')\prod_{i=1}^{d} w_{\varepsilon_x}(x_i - x_i'), \qquad (x,t), (x',t') \in \mathbb{R}^d\times\mathbb{R}^+,$$

where ε_t and ε_x are two arbitrary positive numbers and $w_\lambda(s) = w(s/\lambda)/\lambda$ for any $s \in \mathbb{R}$, $\lambda = \varepsilon_t, \varepsilon_x$. Finally, we define $W(t) = \int_0^t w_{\varepsilon_t}(s)\,ds$. We point out that the hypothesis (3.1e) is used only to estimate the entropy dissipation form in Lemmas 4.8–4.12; it is also used to prove the approximation result of Lemma 4.2, which in turn is used to prove the above-mentioned lemmas.

In what follows, we will set $w = w^\ell$, where $\{w^\ell\}_{\ell\in\mathbb{N}}$ is a sequence of functions satisfying conditions (3.1) that converges pointwisely to one-half of the characteristic function of $(-1, 1)$ which we denote by w^∞. When relevant, the dependence with respect to w^ℓ is emphasized by a superscript ℓ.

For u and v right-continuous functions from $(0, T_\infty)$ to $L^1(\mathbb{R}^d)$ for which the left limits, $u_-(t)$ and $v_-(t)$, respectively, exist for $t \in (0, T_\infty)$, we define

$$E^{\varepsilon_t,\varepsilon_x}(v, u; \tau) = \int_0^\tau \int_{\mathbb{R}^d} \Theta_\tau^{\varepsilon_t,\varepsilon_x}(v, u; t, x)\,dxdt$$

with

$$\begin{aligned}
\Theta_\tau^{\varepsilon_t,\varepsilon_x}(v, c; t, x) = &- \int_0^\tau \int_{\mathbb{R}^d} U(v(t', x') - c)\partial_{t'} \varphi\, dx'dt' \\
&- \int_0^\tau \int_{\mathbb{R}^d} F(v(t', x'), c) \cdot \nabla_{x'} \varphi\, dx'dt' \\
&- \int_{\mathbb{R}^d} U(v_-(0, x') - c)\, \varphi(t, x, 0, x')\, dx' \\
&+ \int_{\mathbb{R}^d} U(v_-(\tau, x') - c)\, \varphi(t, x, \tau, x')\, dx',
\end{aligned}$$

where U is an arbitrary even entropy and F is its associated flux, i.e., $\partial_u F(u, c) = U'(u) f'(u)$, and $v_-(0, \cdot)$ is the exterior trace of v at $t = 0$. We recall that if u is the entropy solution of (1.1), (1.2), and if we set $u_-(0, x) = u_0(x)$, then $E^{\varepsilon_t,\varepsilon_x}(u, v; \tau) \leq 0$ for any v.

Since $\Theta_\tau^{\varepsilon,\varepsilon_x}$ is without any slab structure, necessary to the definition of our finite element approximation, we rearrange it as follows:

$$\begin{aligned}
\Theta_\tau^{\varepsilon_t,\varepsilon_x}(v, u; t, x) = &- \int_0^{t_{N_\tau}} \int_{\mathbb{R}^d} \tilde{F}(v, u) \cdot \nabla' \varphi\, dx'dt' \\
&- \int_{t_{N_\tau}}^\tau \int_{\mathbb{R}^d} \tilde{F}(v, u) \cdot \nabla' \varphi\, dx'dt' \\
&- \int_{\mathbb{R}^d} U(v_-(0, x') - u)\, \varphi(t, x, 0, x')\, dx' \\
&+ \int_{\mathbb{R}^d} U(v_-(\tau, x') - u)\, \varphi(t, x, \tau, x')\, dx',
\end{aligned}$$

where $\tilde{F}(v, u) = (U(v - u), F(v, u))$ is the "extended" entropy flux and where N_τ denotes the largest integer such that $t_{N_\tau} \leq \tau$. In the above expression, the explicit dependence of each function is stated only when confusion is possible. Now, for any $v \in V_h$, we have after integration by parts

$$\Theta_\tau^{\varepsilon_t,\varepsilon_x}(v, u; t, x) = \tilde{\Theta}_\tau^{\varepsilon_t,\varepsilon_x}(v, u; t, x) + D_\tau(v, u; t, x),$$

where

$$\begin{aligned}\tilde{\Theta}_\tau^{\varepsilon_t,\varepsilon_x}(v,u;t,x) = &\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_h}\int_T A(v)U'(v-u)\varphi dx'dt'\\ &-\sum_{n=0}^{N_\tau-1}\sum_{e\in\mathcal{E}_n}\int_e\Big(\tilde{F}(v^T,u)-\tilde{F}(v^{T_e},u)\Big)\cdot n_{e,T}\varphi d\lambda'\\ &+\sum_{n=0}^{N_\tau-1}\int_{\mathbb{R}^d_n}\Big(U(v_+-u)-U(v_--u)\Big)\varphi dx',\end{aligned}$$

and

$$\begin{aligned}D_\tau(v,u;t,x) = &\sum_{T\in\mathcal{T}_{h,N_\tau}}\int_T A(v)U'(v-u)\chi_{(t_{N_\tau},\tau]}(t')\varphi dx'dt'\\ &-\sum_{e\in\mathcal{E}_{N_\tau}}\int_e\Big(\tilde{F}(v^T,u)-\tilde{F}(v^{T_e},u)\Big)\cdot n_{e,T}\chi_{(t_{N_\tau},\tau]}(t')\varphi d\lambda'\\ &+\int_{\mathbb{R}^d_{N_\tau}}\Big(U(v_+-u)-U(v_--u)\Big)\chi_{(t_{N_\tau},\tau]}(\tau)\varphi dx',\end{aligned}$$

where $\chi_{(t_{N_\tau},\tau]}(t') = 1$ if $t_{N_\tau} < t' \leq \tau$ and is zero otherwise; $\chi_{(t_{N_\tau},\tau]} \equiv 0$ if $t_{N_\tau} = \tau$. The presence of the term $D_\tau(v,u;t,x)$ is a consequence of the piecewise-constant character of the finite element formulation. Note that if τ corresponds to the boundary of a slab, i.e., if $\tau = t_i$ for some $i = 0, 1, \ldots, N$, then $D_\tau(v,u;t,x) = 0$.

For future use, we also define

$$\begin{aligned}\tilde{E}^{\varepsilon_t,\varepsilon_x}(v,u;\tau) &= \int_0^\tau\int_{\mathbb{R}^d}\tilde{\Theta}_\tau^{\varepsilon_t,\varepsilon_x}(v,u;t,x)dxdt,\\ D_\tau(v,u) &= \int_0^\tau\int_{\mathbb{R}^d}D_\tau(v,u;t,x)dxdt.\end{aligned}$$

We restrict our attention to one particular family of entropies. Let $G:\mathbb{R}\to\mathbb{R}^+$ be a smooth even function such that

$$G(0)=0,\quad G'(x)=\begin{cases}1 & \text{if } x\geq 1,\\ -1 & \text{if } x\leq -1,\end{cases}\quad G''\geq 0. \tag{3.3a}$$

For any $c\in\mathbb{R}$ and any $\varepsilon>0$, we define

$$U(u)=\varepsilon G\left(\frac{u}{\varepsilon}\right)\quad\text{and}\quad F_i(u,c)=\int_c^u U'(\lambda-c)f_i'(\lambda)d\lambda,\quad i=1,\ldots,d. \tag{3.3b}$$

The tool for proving our main results is the following approximation inequality; see also [1], which we prove in §6.

LEMMA 3.1 (the approximation inequality). *Let u be the entropy solution of* (1.1), (1.2), *and let v be any right-continuous function (in time) coinciding with a function of $V_{h,n}$ on each interval (t_n, t_{n+1}), $n = 0, 1, \ldots, N-1$. We have for any t_n, $0\leq t_n < T_\infty$,*

$$\begin{aligned}\| v(t_n)-u(t_n)\|_{L^1(\mathbb{R}^d)} \leq\ &2\,\| v_-(0)-u_0\|_{L^1(\mathbb{R}^d)}\\ &+8\,L(\varepsilon_x+\varepsilon_t\|f'\|)\,|u_0|_{TV(\mathbb{R}^d)})\\ &+\big(c_0(2M)^d+2\,c_1\,\|f''\|\,T_\infty\,|u_0|_{TV(\mathbb{R}^d)}\big)\,\varepsilon\end{aligned}$$

$$+2\lim_{\ell\to\infty}\left\{\sup_{0\le t\le T_\infty}\frac{\tilde{E}^{\varepsilon_t,\varepsilon_x;\ell}(v,u;t)}{W^\ell(t)}\right\}$$
$$+\left\{\chi_{(\varepsilon_t,\infty)}(t_n)\,\frac{1}{\varepsilon_t}\int_0^{\varepsilon_t}\frac{D_t^\infty(v,u)}{W^\infty(t)}dt\right\},$$

where $c_0=\sup_{|r|\le 1}\big|\,|r|-G(r)\,\big|$, $c_1=\sup_{|r|\le 1}G''(r)$, $L=\sup_{r\in\mathbb{R}}|U'(r)|$.

If we take $v_-(0,x)=u_{0h}$ and set v equal to the right-continuous function that coincides with u_h on each interval of the form (t_n,t_{n+1}), we see that to obtain our error estimates, we only have to estimate the entropy dissipation $\tilde{E}^{\varepsilon_t,\varepsilon_x}(v_h,u;\tau)$, as well as the term involving $D_t(v,u)$. From now on, we will not distinguish between v_h and u_h.

Hereafter, we will put a prime as a superscript on the operators Π_h, ∇, and ∇_e to emphasize the fact that they are considered to act on the "primed" variables only. Following [10], we define the operator Π_h' (which acts only on the variables t' and x') as the classical interpolation nodal operator L^2-projection when we deal with the SCSD method. When we deal with the SCDG method, we take Π_h' to be the classical L^2-projection into the space of piecewise-constant functions.

We can now rewrite $\tilde{\Theta}_\tau^{\varepsilon_t,\varepsilon_x}$ as follows:

$$\begin{aligned}
\tilde{\Theta}_\tau^{\varepsilon_t,\varepsilon_x}(u_h,u;t,x)=&\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T A(u_h)\,\Pi_h'(U'(u_h-u)\,\varphi)\,dx'dt'\\
&+\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T A(u_h)\Big(U'(u_h-u)\,\varphi-\Pi_h'(U'(u_h-u)\,\varphi)\Big)\,dx'dt'\\
&-\sum_{n=0}^{N_\tau-1}\sum_{e\in\mathcal{E}_n}\int_e\Big(\tilde{F}(u_h^T,u)-\tilde{F}(u_h^{T_e},u)\Big)\cdot n_{e,T}\;\varphi\,d\lambda'\\
&+\sum_{n=0}^{N_\tau-1}\int_{\mathbb{R}_n^d}\Big(U(u_{h,+}-u)-U(u_{h,-}-u)\Big)\,\varphi\,dx'.
\end{aligned}$$

By inserting the definition of u_h and setting $v=\Pi_h'(U'(u_h-u)\;\varphi)$ in (2.2), we get

$$\begin{aligned}
\tilde{\Theta}_\tau^{\varepsilon_t,\varepsilon_x}\;(u_h,u;t,x)=&-\delta\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T A(u_h)\,\tilde{A}\Big(u_h,\Pi_h'[U'(u_h-u)\,\varphi]\Big)\,dx'dt'\\
&-\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e\Big(f_{e,T}^{LF}(u_h^T,u_h^{T_e})-\tilde{f}(u_h^T)\cdot n_{e,T}\Big)\Pi_h'[U'(u_h^T-u)\,\varphi]d\lambda'\\
&-\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T\varepsilon_1(u_h)\mathbb{P}_h'(\nabla'\hat{u}_h\cdot\nabla'\hat{\Pi}_h'[U'(u_h-u)\,\varphi])\,dx'dt'\\
&-\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e\varepsilon_2(u_h^T)\mathbb{P}_h'(\nabla_e'\hat{u}_h^T\cdot\nabla_e'\hat{\Pi}_h'[U'(u_h^T-u)\,\varphi])\,d\lambda'\\
&+\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T A(u_h)\Big(U'(u_h-u)\,\varphi-\Pi_h'[U'(u_h-u)\,\varphi]\Big)\,dx'dt'\\
&-\sum_{n=0}^{N_\tau-1}\sum_{e\in\mathcal{E}_n}\int_e\Big(\tilde{F}(u_h^T,u)-\tilde{F}(u_h^{T_e},u)\Big)\cdot n_{e,T}\;\varphi\,d\lambda'\\
&+\sum_{n=0}^{N_\tau-1}\int_{\mathbb{R}_n^d}\Big(U(u_{h,+}-u)-U(u_{h,-}-u)\Big)\,\varphi\,dx',
\end{aligned}$$

where if Π_h' is the projection onto the constants, we have set $\hat{\Pi}_h'=\Pi_h'$. A few simple rearrangements yield

$$
\begin{aligned}
\tilde{\Theta}_{\tau}^{\varepsilon_t,\varepsilon_x}(u_h,u;t,x) = &-\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e\Big(f_{e,T}^{LF}(u_h^T,u_h^{T_e})-\tilde{f}(u_h^T)\cdot n_{e,T}\Big)U'(u_h^T-u)\,\varphi\,d\lambda'\\
&-\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T\varepsilon_1(u_h)\mathbb{P}_h'(\nabla'\hat{u}_h\cdot\nabla'\hat{\Pi}_h'[U'(u_h-u)\,\mathbb{P}_h'\varphi])\,dx'dt'\\
&-\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e\varepsilon_2(u_h^T)\mathbb{P}_h'(\nabla_e'\hat{u}_h^T\cdot\nabla_e'\hat{\Pi}_h'[U'(u_h^T-u)\,\mathbb{P}_h'\varphi])\,d\lambda'\\
&-\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T\varepsilon_1(u_h)\mathbb{P}_h'(\nabla'\hat{u}_h\cdot\nabla'\hat{\Pi}_h'[U'(u_h-u)(\varphi-\mathbb{P}_h'\varphi)])\,dx'dt'\\
&-\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e\varepsilon_2(u_h^T)\mathbb{P}_h'(\nabla_e'\hat{u}_h^T\cdot\nabla_e'\hat{\Pi}_h'[U'(u_h^T-u)(\varphi-\mathbb{P}_h'\varphi)])\,d\lambda'\\
&+\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T A(u_h)\Big(U'(u_h-u)\,\varphi-\Pi_h'[U'(u_h-u)\,\varphi]\Big)\,dx'dt'\\
&-\sum_{n=0}^{N_\tau-1}\sum_{e\in\mathcal{E}_n}\int_e\Big(\tilde{F}(u_h^T,u)-\tilde{F}(u_h^{T_e},u)\Big)\cdot n_{e,T}\;\varphi\,d\lambda'\\
&+\sum_{n=0}^{N_\tau-1}\int_{\mathbb{R}_n^d}\Big(U(u_{h,+}-u)-U(u_{h,-}-u)\Big)\,\varphi\,dx'\\
&+\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e\Big(f_{e,T}^{LF}(u_h^T,u_h^{T_e})-\tilde{f}(u_h^T)\cdot n_{e,T}\Big)\\
&\qquad\qquad\cdot\Big(U'(u_h^T-u)\,\varphi-\Pi_h'[U'(u_h^T-u)\,\varphi]\Big)d\lambda'\\
&-\delta\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T A(u_h)\,\tilde{A}\Big(u_h,\Pi_h'[U'(u_h-u)\,\varphi]\Big)\,dx'dt'.
\end{aligned}
$$

Finally, noting that, by (2.4a),

$$
\begin{aligned}
-\sum_{n=0}^{N_\tau-1}&\sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e\Big(f_{e,T}^{LF}(u_h^T,u_h^{T_e})-\tilde{f}(u_h^T)\cdot n_{e,T}\Big)U'(u_h^T-u)\,\varphi d\lambda'\,d\lambda'\\
&=\sum_{n=0}^{N_\tau-1}\sum_{e\in\mathcal{E}_n}\int_e\Big(-f_{e,T}^{LF}(u_h^T,u_h^{T_e})(U'(u_h^T-u)-U'(u_h^{T_e}-u))\\
&\qquad\qquad+\tilde{f}(u_h^T)\cdot n_{e,T}U'(u_h^T-u)-\tilde{f}(u_h^{T_e})\cdot n_{e,T}U'(u_h^{T_e}-u)\Big)\,\varphi d\lambda'\\
&\quad-\sum_{n=0}^{N_\tau-1}\int_{\mathbb{R}_n^d}(u_{h,+}-u_{h,-})U'(u_{h,+}-u)\;\varphi dx',
\end{aligned}
$$

we obtain,

$$
\begin{aligned}
\tilde{\Theta}_{\tau}^{\varepsilon_t,\varepsilon_x}&(u_h,u;t,x)\\
&=\sum_{n=0}^{N_\tau-1}\sum_{e\in\mathcal{E}_n}\int_e\Big(-f_{e,T}^{LF}(u_h^T,u_h^{T_e})(U'(u_h^T-u)-U'(u_h^{T_e}-u))+\tilde{f}(u_h^T)\cdot n_{e,T}U'(u_h^T-u)\\
&\qquad\qquad-\tilde{f}(u_h^{T_e})\cdot n_{e,T}U'(u_h^{T_e}-u)-(\tilde{F}(u_h^T,u)-\tilde{F}(u_h^{T_e},u))\cdot n_{e,T}\Big)\,\varphi d\lambda'
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{n=0}^{N_\tau-1}\int_{\mathbb{R}^d_n}\Big(-(u_{h,+}-u_{h,-})U'(u_{h,+}-u)+U(u_{h,+}-u)-U(u_{h,-}-u)\Big)\varphi dx' \\
&-\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T \varepsilon_1(u_h)\mathbb{P}'_h(\nabla'\hat{u}_h\cdot\nabla'\hat{\Pi}'_h[U'(u_h-u)\,\mathbb{P}'_h\varphi])\,dx'dt' \\
&-\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e \varepsilon_2(u_h^T)\mathbb{P}'_h(\nabla'_e\hat{u}_h^T\cdot\nabla'_e\hat{\Pi}'_h[U'(u_h^T-u)\,\mathbb{P}'_h\varphi])\,d\lambda' \\
&-\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T \varepsilon_1(u_h)\mathbb{P}'_h(\nabla'\hat{u}_h\cdot\nabla'\hat{\Pi}'_h[U'(u_h-u)(\varphi-\mathbb{P}'_h\varphi)])\,dx'dt' \\
&-\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e \varepsilon_2(u_h^T)\mathbb{P}'_h(\nabla'_e\hat{u}_h^T\cdot\nabla'_e\hat{\Pi}'_h[U'(u_h^T-u)(\varphi-\mathbb{P}'_h\varphi)])\;,d\lambda' \\
&+\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T A(u_h)\Big(U'(u_h-u)\,\varphi-\Pi'_h[U'(u_h-u)\,\varphi]\Big)\,dx'dt' \\
&+\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e\Big(f_{e,T}^{LF}(u_h^T,u_h^{T_e})-\tilde{f}(u_h^T)\cdot n_{e,T}\Big) \\
&\qquad\qquad\cdot\Big(U'(u_h^T-u)\,\varphi-\Pi'_h[U'(u_h^T-u)\,\varphi]\Big)d\lambda' \\
&-\delta\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T A(u_h)\,\tilde{A}\Big(u_h,\,\Pi'_h[U'(u_h-u)\,\varphi]\Big)\,dx'dt' \\
&=\sum_{i=1}^{9}\Theta_i^{\varepsilon_t,\varepsilon_x}(u_h,u;t,x).
\end{aligned}
$$

By definition of $E^{\varepsilon_t,\varepsilon_x}(u_h,u;\tau)$, we can write

$$E^{\varepsilon_t,\varepsilon_x}(u_h,u;\tau)=\tilde{E}^{\varepsilon_t,\varepsilon_x}(u_h,u;\tau)+D_\tau(u_h,u),$$

where

$$\tilde{E}^{\varepsilon_t,\varepsilon_x}(u_h,u;\tau)=\sum_{i=1}^{9}E_i^{\varepsilon_t,\varepsilon_x}(u_h,u;\tau)$$

and $E_i^{\varepsilon_t,\varepsilon_x}(u_h,u;\tau)=\int_0^\tau\int_{\mathbb{R}^d}\Theta_i^{\varepsilon_t,\varepsilon_x}(u_h,u;t,x)dxdt$, $i=1,\dots,9$. This is the desired expression of the entropy dissipation form.

4. Proof of the a posteriori error estimates. In this section, we obtain upper bounds for the entropy dissipation form and we prove the a posteriori error estimates of Theorems 2.1, 2.4, and 2.7.

4.1. Preliminary results. We start with three simple auxiliary lemmas.

The following result is a simple consequence of the fact that all norms are equivalent in finite-dimensional spaces.

LEMMA 4.1. *We have*

$$\|\nabla' u_h\|_{L^\infty(T)}\le c(d,k)\,\|\nabla'\hat{u}_h\|_{\mathbb{P}_h} in\ T,\forall T\in\mathcal{T}_{h,n},n=0,\dots,N-1,$$

$$\|\nabla'_e u_h^T\|_{L^\infty(e)}\le c(d,k)\,\|\nabla'_e\hat{u}_h^T\|_{\mathbb{P}_h}\ on\ e,\forall e\in\partial\mathcal{T}_{h,n},n=0,\dots,N-1.$$

LEMMA 4.2 (approximation properties of the operator Π_h). *Let $s\in L^1((0,T_\infty)\times\mathbb{R}^d)$ be a piecewise-continuous function. We have*

$$\begin{aligned}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T s(t',x')\int_0^{t_{N_\tau}}\int_{\mathbb{R}^d}\Big(U'(u_h-u)\,\varphi-\Pi_h'[U'(u_h-u)\,\varphi]\Big)dxdtdx'dt'\\ \le W(\tau)\,\frac{c(d,k)}{\varepsilon}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T |s(t',x')|\,\|\nabla'\hat{u}_h\|_{\mathbb{P}_h}\,h_T\,dx'dt'\\ +W(\tau)\,\frac{c(d,k)}{\varepsilon_x+\varepsilon_t}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T |s(t',x')|\,h_T\,dx'dt',\end{aligned}$$

$$\begin{aligned}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e s^T\int_0^{t_{N_\tau}}\int_{\mathbb{R}^d}\Big(U'(u_h^T-u)\,\varphi-\Pi_h'[U'(u_h^T-u)\,\varphi]\Big)dxdtd\lambda'\\ \le W(\tau)\,\frac{c(d,k)}{\varepsilon}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e |s^T|\,\|\nabla_e\hat{u}_h^T\|_{\mathbb{P}_h}\,h_e\,d\lambda'\\ +W(\tau)\,\frac{c(d,k)}{\varepsilon_x+\varepsilon_t}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e |s^T|\,h_T\,d\lambda',\end{aligned}$$

where h_e is the diameter of e and where C depends on k, d, and σ only.

Proof. Since the proofs of these inequalities are almost identical, we prove only the first one; let us denote by $\mathcal{L}$ its left-hand side. Moreover, we will only prove the result in the case in which the operator Π_h is the standard nodal interpolator; the case in which Π_h is the L^2-projection into the space of piecewise-constant functions can be treated in a similar way.

Let $\{(t_j,p_j)\}_{j=1}^{N_{d+1,k}}$ be the nodal points associated with the canonical basis $\{\Psi_j^T\}_{j=1}^{N_{d+1,k}}$ of the space $\mathcal{P}^k(T)$ where, of course, $N_{d+1,k}=\binom{d+k+1}{k}$. With this notation, we can express $\mathcal{L}$ as follows:

$$\mathcal{L}=\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T s(t',x')\int_0^{t_{N_\tau}}\int_{\mathbb{R}^d}\gamma\,dxdtdt'dx',$$

where $\gamma=\gamma(t,x,t',x')$ is defined by

$$\begin{aligned}\gamma&=U'(u_h-u)\,\varphi-\sum_{j=1}^{N_{d+1,k}}[U'(u_{h,j}-u)\,\varphi_j]\Psi_j^T\\ &=\sum_{j=1}^{N_{d+1,k}}\Big(U'(u_h-u)\,\varphi-U'(u_{h,j}-u)\,\varphi_j\Big)\Psi_j^T\end{aligned}$$

(since $\sum_{j=1}^{N_{d+1,k}}\Psi_j^T\equiv 1$ on T) where $u_{h,j}=u_h(t_j,p_j)$ and $\varphi_j=\varphi(t_j,p_j)$. Moreover, since

$$\begin{aligned}U'(u_h-u)\,\varphi-U'(u_{h,j}-u)\,\varphi_j&=(U'(u_h-u)-U'(u_{h,j}-u))\,\varphi\\ &\quad-U'(u_{h,j}-u)(\varphi_j-\varphi),\end{aligned}$$

we obtain, after simple manipulations,

$$\begin{aligned}\mathcal{L}\le W(\tau)\,\frac{c_0}{\varepsilon}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T |s(t',x')|\sum_{j=1}^{N_{d+1,k}}|u_h-u_{h,j}||\Psi_j^T(t',x')|dx'\\ +\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T |s(t',x')|\sum_{j=1}^{N_{d+1,k}}\int_0^{t_{N_\tau}}\int_{\mathbb{R}^d}|\varphi-\varphi_j|\,|\Psi_j^T(t',x')|dtdxdt'dx',\end{aligned}$$

where we have used the definition of W and the fact that $|U'|\le 1$ and $U''\le c_1/\varepsilon<\infty$.

From this inequality, we immediately get

$$\mathcal{L} \leq W(\tau)\,\frac{c(d,k)}{\varepsilon}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T |s(t',x')|\,\|\nabla' u_h\|_{L^\infty(T)}\,h_T\,dt'dx'$$

$$+\,W(\tau)\,\frac{c(d,k)\,h}{\varepsilon_x+\varepsilon_t}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T |s(t',x')|\,dt'dx',$$

since, by (3.1) and the specific properties of our choice for w, $\int_0^{t_{N_\tau}}\!\int_{\mathbb{R}^d} |\varphi-\varphi_j| \leq c\,W(\tau)/(\varepsilon_t+\varepsilon_x)$. The result follows from Lemma 4.1, completing the proof. □

The following nonnegativity result reflects the nature of the so-called shock-capturing terms used in the definition of the schemes under consideration.

LEMMA 4.3 (nonnegativity of the shock-capturing terms). *Suppose that the condition* (2.1a) *on the triangulations is satisfied. Then*

$$\nabla'\hat{u}_h\cdot\nabla'\hat{\Pi}'_h[U'(u_h-u)]\geq 0,$$

$$\nabla'_e\hat{u}_h^T\cdot\nabla'_e\hat{\Pi}'_h[U'(u_h^T-u)]\geq 0.$$

Proof. We only prove the first inequality since the proof for the second is similar. By definition, we have that, on each $(d+1)$-simplex T_ℓ,

$$\hat{u}_h(t',x')=\sum_{i=1}^{d+2}u_{h,\ell,i}\Psi_i^{T_\ell}(t',x'),$$

where $\{\,\Psi_i^{T_\ell}\,\}_{i=1}^{d+2}$ is the canonical basis of the space $\mathcal{P}_1(T_\ell)$, $\{(t_{\ell,i},p_{\ell,i})\}_{i=1}^{d+2}$ are the vertices of T_ℓ, and $u_{h,\ell,i}=u_h(t_{\ell,i},p_{\ell,i})$.

Thus, for $(t',x')\in T_\ell$,

$$\theta=\nabla'\hat{u}_h\cdot\nabla'(\hat{\Pi}'_h[U'(u_h-u)])=\sum_{i,j=1}^{d+2}u_{h,\ell,i}U'(u_{h,\ell,j}-u)\,\Lambda_{i,j},$$

where $\Lambda_{i,j}=\nabla'\Psi_i^{T\ell}(t',x')\cdot\nabla'\Psi_j^{T_\ell}(t',x')$. Since $\sum_{i=1}^{d+2}\Psi_i^{T\ell}(t',x')\equiv 1$ on T_ℓ, we have

$$\theta=\sum_{i,j=1}^{d+2}(-u_{h,\ell,j}+u_{h,\ell,i})U'(u_{h,\ell,j}-u)\Lambda_{i,j}.$$

Interchanging i and j in the previous relation and averaging the two expressions for θ, we get

$$\theta=\frac{1}{2}\sum_{i,j=1}^{d+2}\big(-u_{h,\ell,j}+u_{h,\ell,i}\big)\big(-U'(u_{h,\ell,j}-u)+U'(u_{h,\ell,i}-u)\big)\,\big(-\Lambda_{i,j}\big).$$

Since by (2.1a) the $(d+1)$-simplex T_ℓ is acute, we have $-\Lambda_{i,j}\geq 0$ on T_ℓ. Hence by convexity of U, $\theta\geq 0$ on T_ℓ. This completes the proof. □

4.2. Estimate of the entropy dissipation form. Next, we turn to estimate the entropy dissipation form $E^{\varepsilon_x,\varepsilon_t}(u_h,u;\tau)=\sum_{i=1}^{9}E_i^{\varepsilon_x,\varepsilon_t}(u_h,u;\tau)+D_\tau(u_h,u)$. In what follows, we analyze each of the above terms.

LEMMA 4.4. *Suppose that C_e^{LF} satisfies the condition* (2.4b). *Then we have*

$$E_1^{\varepsilon_x,\varepsilon_t}(u_h,u;\tau)\leq 0.$$

Proof. We have

$$\tilde{F}(u_h^T, u) - \tilde{F}(u_h^{T_e}, u) = \int_{u_h^{T_e}}^{u_h^T} \frac{d}{ds}\tilde{F}(s, u)ds = \int_{u_h^{T_e}}^{u_h^T} \tilde{f}'(s)U'(s-u)\,ds$$
$$= -\int_{u_h^{T_e}}^{u_h^T} \tilde{f}(s)U''(s-u)ds + \tilde{f}(u_h^T)U'(u_h^T - u) - \tilde{f}(u_h^{T_e})U'(u_h^{T_e} - u),$$

and thus

$$E_1^{\varepsilon_x,\varepsilon_t}(u_h, u; \tau) = \int_0^{t_{N_\tau}} \int_{\mathbb{R}^d} \sum_{n=0}^{N_\tau - 1} \sum_{e\in\mathcal{E}_n} \int_e \Bigg(-f_{e,T}^{LF}(u_h^T, u_h^{T_e})(U'(u_h^T - u) - U'(u_h^{T_e} - u))$$
$$+ \int_{u_h^{T_e}}^{u_h^T} \tilde{f}(s)U''(s-u)ds \Bigg)\varphi\, dx dt d\lambda'$$
$$= \int_0^{t_{N_\tau}} \int_{\mathbb{R}^d} \sum_{n=0}^{N_\tau - 1} \sum_{e\in\mathcal{E}_n} \int_e \int_{u_h^{T_e}}^{u_h^T} \left(\tilde{f}(s)\cdot n_{e,T} - f_{e,T}^{LF}(u_h^T, u_h^{T_e}) \right) U''(s-u)\,\varphi\, dx dt ds d\lambda'.$$

Since U is convex and the numerical flux is monotone, by (2.4b), the above quantity is nonnegative. This proves the lemma. □

LEMMA 4.5. *We have*

$$E_2^{\varepsilon_x,\varepsilon_t}(u_h, u; \tau) \le 0.$$

Proof. The result immediately follows from the nonnegativity of the function φ and the convexity of the entropy U. □

LEMMA 4.6. *Suppose that the condition* (2.1a) *on the triangulations is satisfied. Then we have*

$$E_3^{\varepsilon_x,\varepsilon_t}(u_h, u; \tau) \le 0.$$

Proof. The result directly follows from Lemma 4.3 and the fact that $\mathbb{P}_h'\,\varphi$ is a nonnegative piecewise-constant function. □

LEMMA 4.7. *Suppose that the condition* (2.1a) *on the triangulations is satisfied. Then we have*

$$E_4^{\varepsilon_x,\varepsilon_t}(u_h, u; \tau) \le 0.$$

The proof of this result is similar to that of the preceding lemma.

LEMMA 4.8. *If Π_h' is the L^2-projection onto the space of piecewise-constant functions, then*

$$E_5^{\varepsilon_x,\varepsilon_t}(u_h, u; \tau) = 0.$$

If Π_h' is the classical nodal interpolation operator, then, if the condition (2.1b) *on the triangulations is satisfied, we have*

$$\frac{E_5^{\varepsilon_x,\varepsilon_t}(u_h, u; \tau)}{W(\tau)} \le \frac{c(d,k)\sigma}{\varepsilon_x + \varepsilon_t}\delta_1 \sum_{n=0}^{N_\tau - 1} \sum_{T\in\mathcal{T}_{h,n}} \int_T |A(u_h)| dx' dt'.$$

Proof. The first equality can be trivially verified. Let us consider the case in which Π_h is the classical nodal interpolation operator. We have, by definition,

$$
\begin{aligned}
&E_5^{\varepsilon_x,\varepsilon_t}(u_h, u; \tau)\\
&= -\int_0^{t_{N_\tau}}\int_{\mathbb{R}^d}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T \varepsilon_1(u_h)\mathbb{P}'_h(\nabla'\hat{u}_h\cdot\nabla'\hat{\Pi}'_h[U'(u_h-u)(\varphi-\mathbb{P}'_h\varphi)])\,dt'dx'dtdx\\
&\le \int_0^{t_{N_\tau}}\int_{\mathbb{R}^d}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T \varepsilon_1(u_h)\|\,\nabla'\hat{u}_h\,\|_{\mathbb{P}_h}\,\|\,\Gamma\,\|_{L^\infty(T)}\,dt'dx'dtdx,
\end{aligned}
$$

where $\Gamma=\nabla'\hat{\Pi}'_h[U'(u_h-u)(\varphi-\mathbb{P}'_h\varphi)]$. Since, on each $(d+1)$-simplex T_ℓ,

$$
\begin{aligned}
\Gamma &= \nabla'\big(\hat{\Pi}'_h[U'(u_h-u)(\varphi-\mathbb{P}'_h\varphi)]\big)\\
&= \sum_{i=1}^{d+2} U'(u_{h,\ell,i}-u)(\varphi_{\ell,i}-\mathbb{P}'_h\varphi)\nabla'\Psi_i^{T_\ell}(t',x'),
\end{aligned}
$$

and since $|\,\nabla'\Psi_i^{T_\ell}\,|\le c(d,k)/\rho_T$ on T and $|\,U'\,|\le 1$, by (3.3), we have

$$
|\,\Gamma\,| \le \frac{c(d,k)}{\rho_T}\sum_{i=1}^{d+2}|\,\varphi_{\ell,i}-\mathbb{P}'_h\varphi\,|,
$$

which yields, by (2.1b) and (3.1),

$$
\int_0^{t_{N_\tau}}\int_{\mathbb{R}^d}|\,\Gamma\,| \le W(t_{N_\tau})\,c(d,k)\frac{h_T}{\rho_T}\frac{1}{\varepsilon_x+\varepsilon_t} \le W(\tau)\,c(d,k)\,\frac{\sigma}{\varepsilon_x+\varepsilon_t}.
$$

Finally, using the definition of ε_1, we get

$$
\begin{aligned}
\frac{E_5^{\varepsilon_x,\varepsilon_t}(u_h,u;\tau)}{W(\tau)} &\le \frac{c(d,k)\,\sigma}{\varepsilon_x+\varepsilon_t}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T \varepsilon_1(u_h)\|\,\nabla'\hat{u}_h\,\|_{\mathbb{P}_h}\,dx'dt'\\
&\le \frac{c(d,k)\,\sigma}{\varepsilon_x+\varepsilon_t}\delta_1\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T |A(u_h)|dx'dt'.
\end{aligned}
$$

This completes the proof. □

LEMMA 4.9. *Suppose that the conditions* (2.1b), (2.1c) *on the triangulations and* (2.4b) *on the coefficients C_e^{LF} are satisfied. If Π'_h is the L^2-projection onto the space of piecewise-constant functions, then*

$$
E_6^{\varepsilon_x,\varepsilon_t}(u_h,u;\tau)=0.
$$

If Π'_h is the classical nodal interpolation operator, then, if the condition (2.1b) *on the triangulations is satisfied, we have*

$$
\frac{E_6^{\varepsilon_x,\varepsilon_t}(u_h,u;\tau)}{W(\tau)} \le \frac{c(d,k)\,\sigma}{\varepsilon_x+\varepsilon_t}\delta_2\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e |\,f_{e,T}^{LF}(u_h^T,u_h^{T_e})-\tilde{f}(u_h^T)\cdot n_{e,T}\,|\,d\lambda.
$$

The proof is similar to the proof of Lemma 4.8.

LEMMA 4.10. *We have*

$$\frac{E_7^{\varepsilon_x,\varepsilon_t}(u_h,u;\tau)}{W(\tau)} \leq \frac{c(d,k)}{\varepsilon}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T |A(u_h)|\, h_T \,\|\nabla'\hat{u}_h\|_{\mathbb{P}_h} dx'dt' + \frac{c(d,k)}{\varepsilon_x+\varepsilon_t}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T |A(u_h)|\, h_T\, dx'dt'.$$

This result follows from a direct application of Lemma 4.2.

LEMMA 4.11. *We have*

$$\begin{aligned}\frac{E_8^{\varepsilon_x,\varepsilon_t}(u_h,u;\tau)}{W(\tau)} &\leq \frac{c(d,k)}{\varepsilon}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e |f_{e,T}^{LF}(u_h^T,u_h^{T_e}) - \tilde{f}(u_h^T)\cdot n_{e,T}|\, h_T\,\|\nabla_e\hat{u}_h^T\|_{\mathbb{P}_h} d\lambda' \\ &+ \frac{c(d,k)}{\varepsilon_x+\varepsilon_t}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e |f_{e,T}^{LF}(u_h^T,u_h^{T_e}) - \tilde{f}(u_h^T)\cdot n_{e,T}|\, h_T\, d\lambda'.\end{aligned}$$

The proof is similar to the proof of the previous result.

LEMMA 4.12. *If Π_h' is the L^2-projection onto the space of piecewise-constant functions, then*

$$E_9^{\varepsilon_x,\varepsilon_t}(u_h,u;\tau) = 0.$$

If Π_h' is the classical nodal interpolation operator, then, if the condition (2.1b) *on the triangulations is satisfied, we have*

$$\begin{aligned}\frac{E_9^{\varepsilon_x,\varepsilon_t}(u_h,u;\tau)}{W(\tau)} &\leq c(d,k)(1+\|f'\|)\,\sigma\frac{\delta}{\varepsilon}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T |A(u_h)|\,\|\nabla'\hat{u}_h\|_{\mathbb{P}_h} dx'dt' \\ &+ c(d,k)(1+\|f'\|)\,\sigma\frac{\delta}{\varepsilon_x+\varepsilon_t}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T |A(u_h)|\, dx'dt'.\end{aligned}$$

Proof. The first equality follows trivially. Let us consider the case in which Π_h is the classical nodal interpolation operator. By definition, we have

$$\begin{aligned}E_9^{\varepsilon_x,\varepsilon_t}(u_h,u;\tau) &= -\delta\int_0^{t_{N_\tau}}\int_{\mathbb{R}^d}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T A(u_h)\,\tilde{A}\Big(u_h,\Pi_h'[U'(u_h-u)\,\varphi]\Big)\,dx'dt'dtdx \\ &\leq \delta\int_0^{t_{N_\tau}}\int_{\mathbb{R}^d}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T |A(u_h)|\,\Big|\tilde{A}\Big(u_h,\Pi_h'[U'(u_h-u)\,\varphi]\Big)\Big|\,dx'dt'dtdx \\ &= \delta\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T |A(u_h)|\left\{\int_0^\tau\int_{\mathbb{R}^d}|\Gamma|\,dtdx\right\}dx'dt',\end{aligned}$$

where $\Gamma = \tilde{A}(u_h,\Pi_h'[U'(u_h-u)\,\varphi])$. Taking into account the linearity of $\tilde{A}(u_h,\cdot)$ and the definition of Π_h', we get

$$\begin{aligned}\Gamma = \tilde{A}(u_h,\Pi_h'[U'(u_h-u)\,\varphi]) &= \sum_{i=1}^{N_{d+1,k}} U'(u_{h,i}-u)\,\varphi_i\tilde{A}(u_h,\Psi_i^T(t',x')) \\ &= \sum_{i=1}^{N_{d+1,k}}(U'(u_{h,i}-u)\,\varphi_i - U'(u_h-u)\,\varphi)\tilde{A}(u_h,\Psi_i^T(t',x'))\end{aligned}$$

since $\sum_{i=1}^{N_{d+1,k}} \Psi_i^T(t',x')) \equiv 1$ on T. Moreover, since $|\tilde{A}(u_h, \Psi_{i,n}^T)| \le c(d,k)(1+\|f'\|)/\rho_T$ on T, we obtain

$$|\Gamma| \le \frac{c(d,k)(1+\|f'\|)}{\rho_T} \sum_{i=1}^{N_{d+1,k}} |U'(u_{h,i}-u)\varphi_i - U'(u_h-u)\varphi|,$$

and since, by (3.3), $|U'| \le 1$ and $|U''| \le c_1/\varepsilon$, we get

$$\begin{aligned}|\Gamma| \le{}& \frac{c(d,k)(1+\|f'\|)}{\rho_T\,\varepsilon} \sum_{i=1}^{N_{d+1,k}} |u_{h,i}-u_h|\,\varphi \\ &+ \frac{c(d,k)(1+\|f'\|)}{\rho_T} \sum_{i=1}^{N_{d+1,k}} |\varphi_i - \varphi|.\end{aligned}$$

Finally, Lemma 4.1, conditions (2.1b), and (3.1) lead to

$$\begin{aligned}\int_0^{t_{N_\tau}}\int_{\mathbb{R}^d} |\Gamma| &\le c(d,k)(1+\|f'\|)\frac{h_T}{\rho_T}\left(\frac{1}{\varepsilon}\|\nabla'\hat{u}_h\|_{\mathbb{P}_h} + \frac{1}{\varepsilon_x+\varepsilon_t}\right) W(t_{N_\tau}) \\ &\le c(d,k)(1+\|f'\|)\,\sigma\left(\frac{1}{\varepsilon}\|\nabla'\hat{u}_h\|_{\mathbb{P}_h} + \frac{1}{\varepsilon_x+\varepsilon_t}\right) W(\tau),\end{aligned}$$

and the result follows. □

Finally, in order to use Lemma 3.1, we need an estimate on the term involving $D_t(u_h,u)$.

LEMMA 4.13. *We have, for* $\tau \in (0, T_\infty)$,

$$\begin{aligned}\chi_{(\varepsilon_t,\infty)}(\tau)\,\frac{1}{\varepsilon_t}\int_0^{\varepsilon_t} \frac{D_t^\infty(u_h,u)}{W^\infty(t)}dt \le{}& \frac{2}{\varepsilon_t}\sum_{n=0}^{N_\tau}\Delta t_n \sum_{T\in\mathcal{T}_{h,n}} \int_T |A(u_h)|dt'dx' \\ &+ \frac{2\,(c_\infty - c_\star)}{\varepsilon_t}\sum_{n=0}^{N_\tau}\Delta t_n \sum_{e\in\mathcal{E}_n}\int_e |u_h^T - u_h^{T_e}|d\lambda' \\ &+ \frac{2\,L}{\varepsilon_t}\sum_{n=0}^{N_\tau}\Delta t_n \int_{\mathbb{R}_n^d} |u_{h,+} - u_{h,-}|dx',\end{aligned}$$

where $\Delta t_n = t_{n+1} - t_n$ *is the width of the slab* S_n.

Proof. We only need to prove the result for $\tau \ge \varepsilon_t$. By the definition of $D_\tau^\infty(u_h,u)$, we have

$$\frac{1}{\varepsilon_t}\int_0^{\varepsilon_t}\frac{D_t^\infty(u_h,u)}{W^\infty(t)}dt = \int_0^{\varepsilon_t}\frac{2\,w_{\varepsilon_t}^\infty(t)}{W^\infty(t)} D_t^\infty(u_h,u)dt = I + II + III,$$

where

$$I = \int_0^{\varepsilon_t}\frac{2\,w_{\varepsilon_t}^\infty(t)}{W^\infty(t)}\int_0^t\int_{\mathbb{R}^d}\sum_{T\in\mathcal{T}_{h,N_t}}\int_T A(u_h)U'(u_h-u)\chi_{(t_{N_t},t]}(s')\,\varphi^\infty\,ds'dx'dsdxdt,$$

$$\begin{aligned}II = -\int_0^{\varepsilon_t}\frac{2\,w_{\varepsilon_t}^\infty(t)}{W^\infty(t)}\int_0^t\int_{\mathbb{R}^d}\sum_{e\in\mathcal{E}_{N_t}}\int_e&\Big(\tilde{F}(u_h^T,u) - \tilde{F}(u_h^{T_e},u)\Big) \\ &\cdot n_{e,T}\,\chi_{(t_{N_t},t]}(s')\,\varphi^\infty\,d\lambda'dsdxdt,\end{aligned}$$

$$III = \int_0^{\varepsilon_t}\frac{2\,w_{\varepsilon_t}^\infty(t)}{W^\infty(t)}\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}_{N_t}^d}\Big(U(u_{h,+}-u) - U(u_{h,-}-u)\Big)\chi_{(t_{N_t},t]}(t)\,\varphi^\infty\,dx'dsdxdt.$$

Using the properties of U and φ^∞, we obtain for I

$$\begin{aligned}
I &\le \int_0^{\varepsilon_t} \frac{2\,w_{\varepsilon_t}^\infty(t)}{W^\infty(t)} \sum_{T\in\mathcal{T}_{h,N_t}} \int_T |A(u_h)| \int_0^t w_{\varepsilon_t}^\infty(s'-s)\chi_{(t_{N_t},t]}(s')\,ds\,ds'\,dx'\,dt \\
&\le 4\int_0^{\varepsilon_t} w_{\varepsilon_t}^\infty(t) \sum_{T\in\mathcal{T}_{h,N_t}} \int_T |A(u_h)|\,ds'\,dx'\,dt \\
&\le \frac{2}{\varepsilon_t}\int_0^{\varepsilon_t} \sum_{T\in\mathcal{T}_{h,N_t}} \int_T |A(u_h)|\,ds'\,dx'\,dt \\
&\le \frac{2}{\varepsilon_t}\sum_{n=0}^{N_\tau} \Delta t_n \sum_{T\in\mathcal{T}_{h,n}} \int_T |A(u_h)|\,ds'\,dx'.
\end{aligned}$$

For the second term, we have

$$II \le \int_0^{\varepsilon_t} \frac{2\,w_{\varepsilon_t}^\infty(t)}{W^\infty(t)} \int_0^t \int_{\mathbb{R}^d} \sum_{e\in\mathcal{E}_{N_t}} \int_e \left| \Big(\tilde F(u_h^T,u) - \tilde F(u_h^{T_e},u)\Big) n_{e,T} \right| \chi_{(t_{N_t},t]}(s')\,\varphi^\infty\,d\lambda'\,ds\,ds\,dt.$$

We then observe

$$\tilde F(u_h^T,u) - \tilde F(u_h^{T_e},u) = \int_{u_h^{T_e}}^{u_h^T} \frac{d}{ds}\tilde F(s,u)\,ds = \int_{u_h^{T_e}}^{u_h^T} \tilde f'(s)U'(s-u)\,ds,$$

and thus, by (2.4b) and (2.4c), we get

$$\Big(\tilde F(u_h^T,u) - \tilde F(u_h^{T_e},u)\Big)\cdot n_{e,T} \le |\tilde f'|_{e,\infty}|u_h^T - u_h^{T_e}| \le (c_\infty - c_\star)|u_h^T - u_h^{T_e}|.$$

This leads to

$$\begin{aligned}
II &\le 2\,(c_\infty - c_\star)\int_0^{\varepsilon_t} \frac{w_{\varepsilon_t}^\infty(t)}{W^\infty(t)} \sum_{e\in\mathcal{E}_{N_t}} \int_e |u_h^T - u_h^{T_e}| \int_0^t w_{\varepsilon_t}^\infty(s'-s)\chi_{(t_{N_t},t]}(s')\,d\lambda'\,ds\,dt \\
&\le 4(c_\infty - c_\star)\int_0^{\varepsilon_t} w_{\varepsilon_t}^\infty(t) \sum_{e\in\mathcal{E}_{N_t}} \int_e |u_h^T - u_h^{T_e}|\,d\lambda'\,dt \\
&\le \frac{2\,(c_\infty - c_\star)}{\epsilon_t} \sum_{n=0}^{N_\tau} \Delta t_n \sum_{e\in\mathcal{E}_{N_t}} \int_e |u_h^T - u_h^{T_e}|\,d\lambda'.
\end{aligned}$$

If we now turn to the third term, we have

$$III \le L\int_0^\tau \frac{2\,w_{\varepsilon_t}^\infty(t)}{W^\infty(t)} \int_{\mathbb{R}^d_{N_t}} |u_{h,+} - u_{h,-}| \int_0^t w_{\varepsilon_t}^\infty(t_{N_t} - s)\,dx'\,ds\,dt.$$

Proceeding as above yields

$$III \le \frac{2L}{\varepsilon_t}\sum_{n=0}^{N_\tau} \Delta t_n \int_{\mathbb{R}^d} |u_{h,+} - u_{h,-}|\,dx'.$$

The lemma is proved. □

As can be seen from the proof, we can replace N_τ by N_{ε_t} in the statement of Lemma 4.13; however, this result is satisfactory for our purposes.

We are now ready to prove our a posteriori estimates.

4.3. Proofs of the a posteriori error estimates.

Proof of Theorem 2.1. To take advantage of the approximation inequality of Lemma 3.1, we have to estimate the entropy dissipation terms $\tilde{E}^{\varepsilon_t,\varepsilon_x}(u_h, u; \tau)$ and $D_\tau(u_h, u)$. Since in the present case the approximate solution is piecewise constant, we have by Lemmas 4.4-4.12

$$\tilde{E}^{\varepsilon_t,\varepsilon_x}(u_h, u; \tau)/W(\tau) \le E_8^{\varepsilon_x,\varepsilon_t}(u_h, u; \tau)/W(\tau)$$

$$\le \frac{c(d,k)}{\varepsilon_x+\varepsilon_t} \sum_{n=0}^{N_\tau-1} \sum_{T\in\mathcal{T}_{h,n}} \sum_{e\in\partial T} \int_e |f_{e,T}^{LF}(u_h^T, u_h^{T_e}) - \tilde{f}(u_h^T)\cdot n_{e,T}|\, h_T\, d\lambda'.$$

On the other hand, Lemma 4.13 yields

$$\chi_{(\varepsilon_t,\infty)}(\tau)\frac{1}{\varepsilon_t}\int_0^{\varepsilon_t} \frac{D_t^\infty(u_h,u)}{W^\infty(t)}dt$$

$$\le \frac{c(d)}{\varepsilon_t}\sum_{n=0}^{N_\tau} \Delta t_n \Bigl(\sum_{e\in\mathcal{E}_n}\int_e |u_h^T - u_h^{T_e}|d\lambda' + \int_{\mathbb{R}_n^d} |u_{h,+} - u_{h,-}|dx'\Bigr).$$

Using the two previous relations and setting $\varepsilon = 0$ and $\varepsilon_t = \varepsilon_x$, we obtain from Lemma 3.1

$$\| u_h(t_n) - u(t_n) \|_{L^1(\mathbb{R}^d)} \le 2\, \| u_{0h} - u_0 \|_{L^1(\mathbb{R}^d)} + 8\, L(1+\|f'\|)\, \varepsilon_t\, |u_0|_{TV(\mathbb{R}^d)}$$

$$+ \frac{c(d,k)}{\varepsilon_t}\,\Theta_0(u_h)\, h.$$

Theorem 2.1 follows by minimizing over ε_t. □

Proof of Theorem 2.4. Since in this case we take $\Pi_h'(u_h)$ piecewise constant, we have, by Lemmas 4.4-4.12,

$$\tilde{E}^{\varepsilon_x,\varepsilon_t}(u_h, u; \tau)/W(\tau) \le E_7^{\varepsilon_x,\varepsilon_t}(u_h, u; \tau)/W(\tau) + E_8^{\varepsilon_x,\varepsilon_t}(u_h, u; \tau)/W(\tau)$$

$$\le \frac{c(d,k)}{\varepsilon} \sum_{n=0}^{N_\tau-1} \sum_{T\in\mathcal{T}_{h,n}} \int_T |A(u_h)|\, h_T\, \| \nabla'\hat{u}_h \|_{\mathbb{P}_h} dx'dt'$$

$$+ \frac{c(d,k)}{\varepsilon_x+\varepsilon_t} \sum_{n=0}^{N_\tau-1} \sum_{T\in\mathcal{T}_{h,n}} \int_T |A(u_h)|\, h_T\, dx'dt'$$

$$+ \frac{c(d,k)}{\varepsilon} \sum_{n=0}^{N_\tau-1} \sum_{T\in\mathcal{T}_{h,n}} \sum_{e\in\partial T} \int_e |f_{e,T}^{LF}(u_h^T, u_h^{T_e}) - \tilde{f}(u_h^T)\cdot n_{e,T}|\, h_T\, \| \nabla_e\hat{u}_h^T \|_{\mathbb{P}_h} d\lambda'$$

$$+ \frac{c(d,k)}{\varepsilon_x+\varepsilon_t} \sum_{n=0}^{N_\tau-1} \sum_{T\in\mathcal{T}_{h,n}} \sum_{e\in\partial T} \int_e |f_{e,T}^{LF}(u_h^T, u_h^{T_e}) - \tilde{f}(u_h^T)\cdot n_{e,T}|\, h_T\, d\lambda'.$$

Moreover, Lemma 4.13 yields

$$\chi_{(\varepsilon_t,\infty)}(\tau)\frac{1}{\varepsilon_t}\int_0^{\varepsilon_t} \frac{D_t(u_h,u)}{W(t)}dt$$

$$\le \frac{c(d)}{\varepsilon_t}\sum_{n=0}^{N_\tau} \Delta t_n \Bigl(\sum_{e\in\mathcal{E}_n}\int_e |u_h^T - u_h^{T_e}|d\lambda' + \int_{\mathbb{R}_n^d} |u_{h,+} - u_{h,-}|dx'\Bigr)$$

$$+ \frac{c(d)}{\varepsilon_t}\sum_{n=0}^{N_\tau} \Delta t_n \sum_{T\in\mathcal{T}_{h,n}} \int_T |A(u_h)|dx'dt'.$$

Consequently, setting $\varepsilon_t = \varepsilon_x$, the approximation inequality of Lemma 3.1 leads to

$$\begin{aligned}\| u_h(t_{N_\tau}) - u(t_{N_\tau}) \|_{L^1(\mathbb{R}^d)} \leq\ & 2 \| u_{0h} - u_0 \|_{L^1(\mathbb{R}^d)} + 8\, L(1 + \| f' \|) \,| u_0 |_{TV(\mathbb{R}^d)}\, \varepsilon_t \\ & + \left(c_0 (2M)^d + 2\, c_1 \| f'' \|\, T \,| u_0 |_{TV(\mathbb{R}^d)}\right) \varepsilon \\ & + \frac{c(d,k)}{\varepsilon_t}\, \Theta_1(u_h) + \frac{c(d,k)}{\varepsilon}\, \Theta_2(u_h).\end{aligned}$$

Theorem 2.4 follows by minimizing over ε and ε_t. □

Proof of Theorem 2.7. The proof of this result is similar to that of Theorem 2.4. The only difference is that now the terms $E_5^{\varepsilon_x,\varepsilon_t}(u_h, u; \tau)$, $E_6^{\varepsilon_x,\varepsilon_t}(u_h, u; \tau)$, $E_9^{\varepsilon_x,\varepsilon_t}(u_h, u; \tau)$ are not equal to zero. The term $E_5^{\varepsilon_x,\varepsilon_t}(u_h, u; \tau)$ contributes with the term associated with the factor δ_1 in the definition of $\Theta_3(u_h)$, the term $E_6^{\varepsilon_x,\varepsilon_t}(u_h, u; \tau)$ contributes with the term associated with the factor δ_2 in the definition of $\Theta_3(u_h)$, and the term $E_9^{\varepsilon_x,\varepsilon_t}(u_h, u; \tau)$ contributes with the terms associated with the factor δ in the definition of both $\Theta_3(u_h)$ and $\Theta_4(u_h)$. □

5. Regularity of the approximate solution and proofs of the remaining results. In this section, we prove Propositions 2.2, 2.5, and 2.8. To do that, we must obtain upper bounds for the forms $\Theta_i(u_h)$, $i = 0, \dots, 4$ which follow from the regularity properties of the approximate solution. We obtain those regularity properties from an a priori estimate that follows easily from a standard L^2-stability argument.

LEMMA 5.1 (a priori estimate). *For any N_τ, $0 \leq N_\tau \leq N$, we have*

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^d_{N_\tau}} {u_h^T}^2 dx + \frac{1}{2}\sum_{n=0}^{N_\tau-1} \sum_{e\in\partial\mathcal{T}_{h,n}\cap\mathbb{R}^d_n} \int_e (u_h^T - u_h^{T_e})^2 dx \\ &+ \sum_{n=0}^{N_\tau-1}\sum_{e\in\mathcal{E}_n}\int_e\int_{u_h^{T_e}}^{u_h^T}\left(f_{e,T}^{LF}(u_h^T, u_h^{T_e}) - \tilde{f}(s)\cdot n_{e,T}\right) ds\, d\lambda \\ &+ \sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T \varepsilon_1(u_h) \| \nabla\hat{u}_h \|^2_{\mathbb{P}_h}\, dx dt + \sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e \varepsilon_2(u_h^T) \| \nabla_e \hat{u}_h^T \|^2_{\mathbb{P}_h} d\lambda \\ &+ \delta\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T A(u_h)^2\, dx dt \leq \frac{1}{2}\| u_0 \|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Proof. We consider only the nontrivially positive terms in (2.2) when v is chosen as u_h, namely,

$$\begin{aligned}\Psi^n = &\sum_{T\in\mathcal{T}_{h,n}}\int_T\left(\frac{1}{2}\partial_t u_h^2 + \sum_{i=1}^d f_i'(u_h)\, u_h\, \partial_i u_h\right) dx dt \\ &+ \sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e\left(f_{e,T}^{LF}(u_h^T, u_h^{T_e}) - \tilde{f}(u_h^T)\cdot n_{e,T}\right) u_h^T d\lambda.\end{aligned}$$

By introducing the auxiliary pair entropy-entropy flux $\tilde{F}_a(u) = (U_a(u), F_a(u))$, where

$$U_a(u) = \frac{1}{2}u^2 \quad \text{and} \quad F'_{a,i}(u) = f_i'(u) U_a'(u) = f_i'(u) u, \quad i = 1, \dots, d,$$

the term Ψ^n may be expressed as

$$\begin{aligned}\Psi^n &= \sum_{T\in\mathcal{T}_{h,n}}\int_T \nabla\tilde{F}_a(u_h) dx dt + \sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e\left(f_{e,T}^{LF}(u_h^T, u_h^{T_e}) - \tilde{f}(u_h^T)\cdot n_{e,T}\right) u_h^T d\lambda \\ &= \sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e\left(\tilde{F}_a(u_h^T)\cdot n_{e,T} + (f_{e,T}^{LF}(u_h^T, u_h^{T_e}) - \tilde{f}(u_h^T)\cdot n_{e,T}) u_h^T\right) d\lambda.\end{aligned}$$

After reordering, we get

$$\Psi^n = -\frac{1}{2}\sum_{e\in\partial\mathcal{T}_{h,n}\cap\mathbb{R}^d_n}\int_e u_h^{T_e 2}dx + \frac{1}{2}\sum_{e\in\partial\mathcal{T}_{h,n}\cap\mathbb{R}^d_n}\int_e (u_h^T - u_h^{T_e})^2 dx + \frac{1}{2}\sum_{e\in\partial\mathcal{T}_{h,n}\cap\mathbb{R}^d_{n+1}}\int_e u_h^{T 2}dx$$
$$+\sum_{e\in\mathcal{E}_n}\int_e\Big((\tilde{F}_a(u_h^T) - \tilde{F}_a(u_h^{T_e}))\cdot n_{e,T} + f^{LF}_{e,T}(u_h^T, u_h^{T_e})(u_h^T - u_h^{T_e})$$
$$-(\tilde{f}(u_h^T)u_h^T - \tilde{f}(u_h^{T_e})u_h^{T_e})\cdot n_{e,T}\Big)d\lambda.$$

Since

$$\tilde{F}_a(u_h^T) - \tilde{F}_a(u_h^{T_e}) = \int_{u_h^{T_e}}^{u_h^T}\tilde{F}'_a(s)ds = \int_{u_h^{T_e}}^{u_h^T}\tilde{f}'(s)s\,ds$$
$$= -\int_{u_h^{T_e}}^{u_h^T}\tilde{f}(s)ds + \tilde{f}(u_h^T)u_h^T - \tilde{f}(u_h^{T_e})u_h^{T_e},$$

we get

$$\Psi^n = -\frac{1}{2}\sum_{e\in\partial\mathcal{T}_{h,n}\cap\mathbb{R}^d_n}\int_e u_h^{T_e 2}dx + \frac{1}{2}\sum_{e\in\partial\mathcal{T}_{h,n}\cap\mathbb{R}^d_n}\int_e (u_h^T - u_h^{T_e})^2 dx$$
$$+\frac{1}{2}\sum_{e\in\partial\mathcal{T}_{h,n}\cap\mathbb{R}^d_{n+1}}\int_e u_h^{T 2}dx + \sum_{e\in\mathcal{E}_n}\int_e\int_{u_h^{T_e}}^{u_h^T}\Big(f^{LF}_{e,T}(u_h^T, u_h^{T_e}) - \tilde{f}(s)\cdot n_{e,T}\Big)ds\,d\lambda.$$

Summing over n from 0 to $N_\tau - 1$ completes the proof. □

From the above a priori estimate, we can now obtain several key estimates for obtaining our error estimates.

COROLLARY 5.2. *Let the constants C_e^{LF} satisfy the condition* (2.4b). *Then*

$$\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\sum_{\substack{e\in\partial T\\ \partial T\notin\mathbb{R}^d_{N_\tau}}}\int_e |u_h^T - u_h^{T_e}|^2\,d\lambda' \le 2\|\,u_0\,\|^2_{L^2(\mathbb{R}^d)}/\min\{1, c_\star\}.$$

Proof. By the definition of the Lax-Friedrichs flux, (2.4a), we have, for $e\in\mathcal{E}_n$,

$$\int_e\int_{u_h^{T_e}}^{u_h^T}\Big(f^{LF}_{e,T}(u_h^T, u_h^{T_e}) - \tilde{f}(s)\cdot n_{e,T}\Big)ds\,d\lambda = \int_e C_e^{LF}(u_h^T - u_h^{T_e})^2\,d\lambda$$
$$+\int_e\int_{u_h^{T_e}}^{u_h^T}\Big(\frac{1}{2}(\tilde{f}(u_h^T) + \tilde{f}(u_h^{T_e}))\cdot n_{e,T} - \tilde{f}(s)\cdot n_{e,T}ds\Big)d\lambda$$
$$\ge \int_e C_e^{LF}(u_h^T - u_h^{T_e})^2\,d\lambda - \int_e\frac{1}{2}|\tilde{f}'|_{e,\infty}(u_h^T - u_h^{T_e})^2 d\lambda$$
$$\ge c_\star\int_e (u_h^T - u_h^{T_e})^2\,d\lambda,$$

by the condition (2.4b). If $e\in\mathbb{R}^d_n$ then, by (2.4a),

$$\int_e\int_{u_h^{T_e}}^{u_h^T}\Big(f^{LF}_{e,T}(u_h^T, u_h^{T_e}) - \tilde{f}(s)\cdot n_{e,T}\Big)ds\,d\lambda = \frac{1}{2}\int_e (u_h^T - u_h^{T_e})^2\,d\lambda.$$

Thus,

$$\begin{aligned}\min\{1, c_\star\} \sum_{n=0}^{N_\tau-1} \sum_{T\in\mathcal{T}_{h,n}} \sum_{\substack{e\in\partial T\\ e\notin\mathbb{R}^d_{N_\tau}}} \int_e |u_h^T - u_h^{T_e}|^2\, d\lambda \\ \le \sum_{n=0}^{N_\tau-1} \sum_{T\in\mathcal{T}_{h,n}} \sum_{e\in\partial T} \int_e \int_{u_h^{T_e}}^{u_h^T} \left(f^{LF}_{e,T}(u_h^T, u_h^{T_e}) - \tilde{f}(s)\cdot n_{e,T}\right) ds\, d\lambda \\ \le 2\| u_0 \|^2_{L^2(\mathbb{R}^d)},\end{aligned}$$

by Lemma 5.1. This completes the proof. □

Let us notice that if the SCSD method is considered, only the second part of the above proof is relevant, and in Corollary 5.2 $\min\{1, c_\star\}$ can be replaced by 1.

COROLLARY 5.3. *Let the constants C_e^{LF} satisfy the conditions* (2.4b) *and* (2.4c). *Then*

$$\begin{aligned}\Phi_1(u_h) = \sum_{n=0}^{N_\tau-1} \sum_{T\in\mathcal{T}_{h,n}} \sum_{\substack{e\in\partial T\\ e\notin\mathbb{R}^d_{N_\tau}}} \int_e | f^{LF}_{e,T}(u_h^T, u_h^{T_e}) - \tilde{f}(u_h^T)\cdot n_{e,T} |\, d\lambda' \\ \le c(d)\, \{(2M)^d\, T_\infty / \min\{1, c_\star\}\}^{1/2} (\| f' \|/2 + c_\infty) \left\{\frac{\sigma}{\bar{c}}\right\}^{1/2} \| u_0 \|_{L^2(\mathbb{R}^d)}\, h^{-1/2}.\end{aligned}$$

Proof. Setting $C = \| f' \|/2 + c_\infty$, we have, by (2.4c) and Corollary 5.2,

$$\begin{aligned}\Phi_1(u_h) &\le C \sum_{n=0}^{N_\tau-1} \sum_{T\in\mathcal{T}_{h,n}} \sum_{\substack{e\in\partial T\\ e\notin\mathbb{R}^d_{N_\tau}}} \int_e |u_h^T - u_h^{T_e}|\, d\lambda' \\ &\le C \left\{ \sum_{n=0}^{N_\tau-1} \sum_{T\in\mathcal{T}_{h,n}} \sum_{\substack{e\in\partial T\\ e\notin\mathbb{R}^d_{N_\tau}}} \int_e |u_h^T - u_h^{T_e}|^2\, d\lambda' \right\}^{1/2} \left\{ \sum_{n=0}^{N_\tau-1} \sum_{T\in\mathcal{T}^M_{h,n}} \sum_{e\in\partial T} |e| \right\}^{1/2} \\ &\le C \left\{2\| u_0 \|^2_{L^2(\mathbb{R}^d)} / \min\{1, c_\star\}\right\}^{1/2} \left\{ \sum_{n=0}^{N_\tau-1} \sum_{T\in\mathcal{T}^M_{h,n}} \sum_{e\in\partial T} |e| \right\}^{1/2},\end{aligned}$$

where $\mathcal{T}^M_{h,n}$ is the set of the elements $T \in \mathcal{T}_{h,n}$ such that $|x| \le M$ for any $(t, x) \in T$. Finally, we have, by conditions (2.1b) and (2.1c),

$$\begin{aligned}\sum_{n=0}^{N_\tau-1} \sum_{T\in\mathcal{T}^M_{h,n}} \sum_{e\in\partial T} |e| = \sum_{n=0}^{N_\tau-1} \sum_{T\in\mathcal{T}^M_{h,n}} \left(\frac{\rho_T}{|T|} \sum_{e\in\partial T} |e|\right) \left(\frac{h_T}{\rho_T}\right) \left(\frac{h}{h_T}\right) |T|\, h^{-1} \\ \le c(d)\, (2M)^d\, T_\infty \frac{\sigma}{\bar{c}}\, h^{-1},\end{aligned}$$

and the result follows. □

COROLLARY 5.4. *Let the constants C_e^{LF} satisfy the conditions* (2.4b). *Then*

$$\begin{aligned}\sum_{n=0}^{N_\tau-1} \frac{\Delta t_n}{h} \left(\sum_{e\in\mathcal{E}_n} \int_e |u_h^T - u_h^{T_e}| d\lambda' + \int_{\mathbb{R}^d_n} |u_h^T - u_h^{T_e}| d\lambda' \right) \\ \le c(d)\, \{(2M)^d\, T_\infty / \min\{1, c_\star\}\}^{1/2}\, \bar{\bar{c}} \left\{\frac{\sigma}{\bar{c}}\right\}^{1/2} \| u_0 \|_{L^2(\mathbb{R}^d)}\, h^{-1/2}.\end{aligned}$$

Proof. We have by (2.1d)

$$\sum_{n=0}^{N_\tau-1} \frac{\Delta t_n}{h}\Big(\sum_{e\in\mathcal{E}_n}\int_e |u_h^T - u_h^{T_e}|d\lambda' + \int_{\mathbb{R}_n^d} |u_h^T - u_h^{T_e}|d\lambda'\Big)$$
$$\leq \frac{\bar{\bar{c}}}{2}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\sum_{\substack{e\in\partial T\\ \partial T\notin\mathbb{R}^d_{N_\tau}}}\int_e |u_h^T - u_h^{T_e}|d\lambda'.$$

We conclude as in the proof of Corollary 5.3. □

COROLLARY 5.5.

$$\Phi_2(u_h) = \sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T |A(u_h)|\,dx\,dt \leq C_{3b}\,\|u_0\|_{L^2(\mathbb{R}^d)}\,\delta^{-1/2},$$

where $C_{3b} = \{(2M)^d\,T_\infty/2\}^{1/2}$.

Proof. The result follows after a simple application of the Cauchy-Schwarz inequality and Lemma 5.1. □

COROLLARY 5.6.

$$\Phi_3(u_h) = \sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e |f_{e,T}^{LF}(u_h^T, u_h^{T_e}) - \tilde{f}(u_h^T)\cdot n_{e,T}|\,\|\nabla_e\hat{u}_h^T\|_{\mathbb{P}_h}\,d\lambda'$$
$$\leq \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{R}^d)}\,\delta_2^{-1} + \delta_4\,\Phi_1(u_h).$$

Proof. We have, by definition of ε_2,

$$\Phi_3(u_h) \leq \delta_2^{-1}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\sum_{e\in\partial T}\int_e \varepsilon_2\,\|\nabla_e\hat{u}_h\|^2_{\mathbb{P}_h}\,d\lambda + \delta_4\,\Phi_1(u_h)$$
$$\leq \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{R}^d)}\,\delta_2^{-1} + \delta_4\,\Phi_1(u_h),$$

by Lemma 5.1. □

COROLLARY 5.7.

$$\Phi_4(u_h) = \sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T |A(u_h)|\,\|\nabla'\hat{u}_h\|_{\mathbb{P}_h}\,dx\,dt \leq \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{R}^d)}\,\delta_1^{-1} + \delta_3\,\Phi_2(u_h).$$

Proof. We have, by definition of ε_1,

$$\Phi_4(u_h) \leq \delta_1^{-1}\sum_{n=0}^{N_\tau-1}\sum_{T\in\mathcal{T}_{h,n}}\int_T \varepsilon_1\,\|\nabla\hat{u}_h\|^2_{\mathbb{P}_h}\,dx\,dt + \delta_3\,\Phi_2(u_h)$$
$$\leq \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{R}^d)}\,\delta_1^{-1} + \delta_3\,\Phi_2(u_h),$$

by Lemma 5.1. □

5.1. Proofs of the upper bounds on the forms $\Theta_i(u_h)$. Proposition 2.2 follows directly from Corollaries 5.3 and 5.4. Proposition 2.5 and Proposition 2.8 follow from Corollaries 5.3-5.7.

6. Proof of the approximation inequality of Lemma 3.1. To prove Lemma 3.1, we follow the ideas of the proof of a simpler inequality obtained in [1, Prop. 3.1], which we cannot use as stated therein. Moreover, although such an inequality is correct, the proof displayed in [1] contains a mistake that we fix in this paper; see Lemma 6.1 and Proposition 6.2.

Taking into account that φ is symmetric in t and t' and in x and x', see (3.1) and (3.2), and that U is an even function, see (3.3), it is very easy to obtain the following identity:

$$T_{err}(u, v; T) = T_{U''}(u, v; T) + E^{\varepsilon_t,\varepsilon_x}(v, u; T) + E^{\varepsilon_t,\varepsilon_x}(u, v; T), \tag{6.1a}$$

where

$$\begin{aligned} T_{err}(u, v; T) = &+\int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} U\left(v_-(T, x') - u(t, x)\right) \varphi(t, x; T, x')dx'dxdt \\ &+\int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} U\left(v(t', x') - u(T, x)\right) \varphi(T, x; t', x')dxdx'dt' \\ &-\int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} U\left(v_-(0, x') - u(t, x)\right) \varphi(t, x; 0, x')dx'dxdt \\ &-\int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} U\left(v(t', x') - u(0, x)\right) \varphi(0, x; t', x')dxdx'dt' \end{aligned} \tag{6.1b}$$

and

$$\begin{aligned} T_{U''}(u, v; T) = -\int_0^T \int_{\mathbb{R}^d}\int_0^T \int_{\mathbb{R}^d} &\{F(v(t', x'), u(t, x))-F\left(u(t, x), v(t', x')\right)\} \\ &\cdot\nabla_x\varphi(t, x; t', x')dx'dt'dxdt. \end{aligned} \tag{6.1c}$$

The term $T_{err}(u, v; T)$ is expected to converge to

$$\int_{\mathbb{R}^d} U\left(v(T, x) - u(T, x)\right) dx - \int_{\mathbb{R}^d} U\left(v_-(0, x) - u(0, x)\right) dx,$$

and thus it will be called the error term. The expression in the right-hand side of (6.1c) is identically equal to zero if $U''(u) = 0$ for $u \neq 0$, as is the case for $U(u) = |u|$; this is why we denote it by $T_{U''}(u, v; T)$. In Lemma 6.3, we show that this term is of order ε.

Our treatment of the term $T_{err}(u, v; T)$ differs from the one used by Kuznetsov [8]. Our goal is to obtain a lower bound for $T_{err}(u, v; T)$ independent of the modulus of continuity of the function $v : [0, T] \to L^1(\mathbb{R}^d)$.

LEMMA 6.1 (lower bound for the error term $T_{err}(u, v; T)$). *Let u be the entropy solution of* (1.1), (1.2). *Then we have*

$$\begin{aligned} T_{err}(u, v; T) \geq\; &W(T) \int_{\mathbb{R}^d} U\left(v_-(T, x') - u(T, x')\right) dx' \\ &+\int_0^T w_{\varepsilon_t}(T - t') \left\{ \int_{\mathbb{R}^d} U\left(v(t', x') - u(t', x')\right) dx' \right\} dt' \\ &-W(T) \int_{\mathbb{R}^d} U\left(v_-(0, x') - u(0, x')\right) dx' \\ &-\int_0^T w_{\varepsilon_t}(t') \int_{\mathbb{R}^d} U\left(v(t', x') - u(t', x')\right) dx' \\ &- 4L\, W(T) \left\{\| f' \|\, \varepsilon_t + \varepsilon_x\right\} | u_0 |_{TV(\mathbb{R}^d)}, \end{aligned}$$

where $L = \sup U' = 1$, by (3.3).

Proof. We see, from (6.1b), that we can write $T_{err} = T_1 + T_2 + T_3 + T_4$, with the obvious notation. Let us start by estimating T_1. Since

$$\begin{aligned} U\left(v_-(T,x') - u(t,x)\right) = {} & U\left(v_-(T,x') - u(T,x')\right) \\ & +\{U(v_-(T,x') - u(T,x)) - U(v_-(T,x') - u(T,x'))\} \\ & +\{U(v_-(T,x') - u(t,x)) - U(v_-(T,x') - u(T,x))\} \\ \geq {} & U\left(v_-(T,x') - u(T,x')\right) \\ & -L|u(T,x) - u(T,x')| - L|u(t,x) - u(T,x)|, \end{aligned}$$

we get

$$\begin{aligned} T_1 = {} & \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} U\left(v_-(T,x') - u(t,x)\right) \varphi(t,x;T,x')\, dx'\, dxdt \\ \geq {} & \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} U\left(v_-(T,x') - u(T,x')\right) \varphi(t,x;T,x')\, dx'\, dxdt \\ & -L \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\, u(T,x') - u(t,x')\,|\, \varphi(t,x;T,x')\, dx'\, dxdt \\ & -L \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\, u(T,x') - u(T,x)\,|\, \varphi(t,x;T,x')\, dx'\, dxdt \\ \geq {} & W(T) \int_{\mathbb{R}^d} U\left(v_-(T,x') - u(T,x')\right) dx' - L\, W(T) \left\{\|\, f' \,\|\, \varepsilon_t + \varepsilon_x\right\} |\, u_0\, |_{TV(\mathbb{R}^d)}. \end{aligned}$$

Now, let us consider the second term, T_2. Proceeding as for the first term, we get

$$\begin{aligned} T_2 = {} & \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} U\left(v(t',x') - u(T,x)\right) \varphi(T,x,t',x') dx\, dx' dt' \\ \geq {} & \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} U\left(v(t',x') - u(t',x')\right) \varphi(T,x,t',x') dx\, dx' dt' \\ & -L \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\, u(t',x') - u(T,x')\,| \varphi(T,x,t',x') dx\, dx' dt' \\ & -L \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\, u(T,x') - u(T,x)\,| \varphi(T,x,t',x') dx\, dx' dt' \\ \geq {} & \int_0^T w_{\varepsilon_t}(T-t') \left\{ \int_{\mathbb{R}^d} U\left(v(t',x') - u(t',x')\right) dx' \right\} dt' \\ & -L\, W(T) \left\{\|\, f' \,\|\, \varepsilon_t + \varepsilon_x\right\} |\, u_0\, |_{TV(\mathbb{R}^d)}. \end{aligned}$$

To estimate T_3 and T_4 we proceed in a similar way. We obtain

$$\begin{aligned} T_3 \geq {} & -W(T) \int_{\mathbb{R}^d} U\left(v_-(0,x') - u(0,x')\right) dx' \\ & -L\, W(T) \left\{\|\, f' \,\|\, \varepsilon_t + \varepsilon_x\right\} |\, u_0\, \|_{TV(\mathbb{R}^d)}, \end{aligned}$$

$$\begin{aligned} T_4 \geq {} & -\int_0^T w_{\varepsilon_t}(t') \int_{\mathbb{R}^d} U\left(v(t',x') - u(t',x')\right) dx' \\ & -L\, W(T) \left\{\|\, f' \,\|\, \varepsilon_t + \varepsilon_x\right\} |\, u_0\, |_{TV(\mathbb{R}^d)}. \end{aligned}$$

The result is obtained by adding these inequalities. □

We are now ready to prove our basic approximation result.

PROPOSITION 6.2 (basic approximation inequality). *Let u be the entropy solution of* (1.1), (1.2). *Then for any $t_n \leq T$, we have*

$$\begin{aligned}
&\int_{\mathbb{R}^d} U(v_-(t_n, x') - u(t_n, x'))dx' \\
&\qquad \leq 2 \int_{\mathbb{R}^d} U\left(v_-(0, x') - u(0, x')\right) dx' \\
&\qquad + 8\, L(\varepsilon_x + \varepsilon_t \| f' \|) \, | u_0 |_{TV(\mathbb{R}^d)} \\
&\qquad + 2 \lim_{\ell \to \infty} \sup_{0 \leq t \leq T} \left\{ \left(T^{\ell}_{U''}(u, v; t)/W^{\ell}(t) + \tilde{E}^{\varepsilon_t, \varepsilon_x; \ell}(v, u; t)/W^{\ell}(t) \right) \right\} \\
&\qquad + \chi_{(\varepsilon_t, \infty)}(t_n) \, \frac{1}{\varepsilon_t} \int_0^{\varepsilon_t} \frac{D_t^{\infty}(v, u)}{W^{\infty}(t)} dt.
\end{aligned}$$

Proof. By (6.1a) and the fact that u is the entropy solution of (1.1), (1.2), we have, for $\tau > 0$,

$$T_{err}(u, v; \tau) \leq W(\tau) \sup_{0 \leq t \leq \tau} \left\{ \left(T_{U''}(u, v; t)/W(t) + \tilde{E}^{\varepsilon_t, \varepsilon_x}(v, u; t)/W(t) \right) \right\} + D_\tau(v, u).$$

Writing $e(s) = \int_{\mathbb{R}^d} U(v_-(s, x) - u(s, x))\, dx$, we easily obtain, from the above inequality and from Lemma 6.1, that, for $\tau \in (0, T)$,

$$W(\tau) e(\tau) + \int_0^{\tau} w_{\varepsilon_t}(\tau - t) e(t) dt \leq W(\tau)\, C(w) + D_\tau(v, u) + \int_0^{\tau} w_{\varepsilon_t}(t) e(t) dt,$$

where

$$\begin{aligned}
C(w) = &\int_{\mathbb{R}^d} U\left(v_-(0, x') - u(0, x')\right) dx' + 4L(\varepsilon_x + \varepsilon_t \| f' \|_{L^\infty}) | u_0 |_{TV(\mathbb{R}^d)} \\
&+ \sup_{0 \leq t \leq T} \left\{ \left(T_{U''}(u, v; t)/W(t) + \tilde{E}^{\varepsilon_t, \varepsilon_x}(v, u; t)/W(t) \right) \right\}.
\end{aligned}$$

Setting $w = w^{\ell}$ and passing to the limit, we get

$$(6.2) \quad W^{\infty}(\tau) e(\tau) + \int_0^{\tau} w^{\infty}_{\varepsilon_t}(\tau - t) e(t) dt \leq W^{\infty}(\tau)\, C + D^{\infty}_{\tau}(v, u) + \int_0^{\tau} w^{\infty}_{\varepsilon_t}(t) e(t) dt,$$

where $C \leq \lim_{\ell \to \infty} C(w^{\ell})$ and $w^{\infty} = \chi_{(-1,1)}/2$.

If $\tau < \varepsilon_t$, then the two integrals in (6.2) involving the error are equal, and we get

$$(6.3) \qquad e(\tau) \leq C + \frac{D^{\infty}_{\tau}(v, u)}{W^{\infty}(\tau)} \qquad \forall \tau < \varepsilon_t.$$

The above relation applied to $\tau = t_n$ yields

$$e(t_n) \leq C \qquad \forall t_n < \varepsilon_t.$$

If $\tau \geq \varepsilon_t$, then $W^{\infty}(\tau) = \frac{1}{2}$ and relation (6.2) reads

$$e(\tau) \leq C + 2\, D^{\infty}_{\tau}(v, u) + \frac{1}{\varepsilon_t} \int_0^{\varepsilon_t} e(t) dt.$$

If now $\tau = t_n \geq \varepsilon_t$ for some n, we get

$$e(t_n) \leq C + \frac{1}{\varepsilon_t} \int_0^{\varepsilon_t} e(t) dt$$

and thus, by (6.3),

$$e(t_n) \le C + \frac{1}{\varepsilon_t}\int_0^{\varepsilon_t}\left(C + \frac{D_t^\infty(v,u)}{W^\infty(t)}\right)dt$$
$$\le 2C + \frac{1}{\varepsilon_t}\int_0^{\varepsilon_t}\frac{D_t^\infty(v,u)}{W^\infty(t)}dt,$$

and the result follows. □

We want to stress the fact that the advantage of this result over the original approximation inequality of Kuznetsov [8] is that the modulus of continuity in time of v does not appear in the estimate. Error estimates can thus be obtained solely in terms of upper bounds for the term $T_{U''}(u, v; t)/W(t)$, for the entropy production term $E^{\varepsilon_t,\varepsilon_x}(v, u; t)/W(t)$, and for the term involving $D_t(v, u)$.

Next, we show how to estimate the term $T_{U''}(u, v; t)/W(t)$. We need a simple, but useful, representation result.

LEMMA 6.3. *We have*

$$\partial_u\big(F(u,c) - F(c,u)\big) = \int_c^u U''(s-u)\,[f'(u) - f'(s)]\,ds.$$

Proof. By the definition of F, (3.3b), we have

$$\Theta(u,c) = F(u,c) - F(c,u)$$
$$= \int_c^u \big(U'(s-c) + U'(s-u)\big)\, f'(s)\, ds.$$

Then, by using the fact that U is smooth and even, we get

$$\begin{aligned}\partial_u\Theta(u,c) &= \big(U'(u-c) + U'(0)\big)\, f'(u) - \int_c^u U''(s-u)\, f'(s)\, ds\\ &= \big(U'(0) - U'(c-u)\big)\, f'(u) - \int_c^u U''(s-u)\, f'(s)\, ds\\ &= \int_c^u U''(s-u)\, f'(u)\, ds - \int_c^u U''(s-u)\, f'(s)\, ds,\end{aligned}$$

and the result follows. □

COROLLARY 6.4. *Let n be an arbitrary unit vector in $\mathbb{R}^d$. Then*

$$|\partial_u\,(F(u,c) - F(c,u))\cdot n)\,| \le \frac{c_1}{2}\,\varepsilon\,\|f''\|,$$

where $c_1 = \sup_{|w|\le 1} G''(w)$.

Proof. From Lemma 6.3, we have

$$|\partial_u(F(u,c) - F(c,u))\cdot n| \le \|f''\| \int_c^u U''(s-u)\,(u-s)\,ds \le \frac{c_1}{2}\,\varepsilon\,\|f''\|,$$

by the definition of U, (3.3), and the definition of $\|f''\|$, (2.6b). □

We can now use this corollary to estimate $T_{U''}(u, v; t)/W(t)$.

LEMMA 6.5 (estimate of $T_{U''}(u, v; t)/W(t)$). *Let u be the entropy solution of* (1.1), (1.2). *Then*

$$T_{U''}(u, v; t)/W(t) \le c_1\big(T\,\|f''\|\,|\,u_0\,|_{TV(\mathbb{R}^d)}\big)\,\varepsilon,$$

where $c_1 = \sup_{|w|\le 1} G''(w)$.

Proof. Let us assume for the moment that u is smooth. Then, by (6.1c) one can write

$$T_{U''}(u, v; T) = -\int_0^T \int_{\mathbb{R}^d} \int_0^T \int_{\mathbb{R}^d} \nabla_x \cdot \Big(F(u(t,x), v(t',x')) - F(v(t',x'), u(t,x)) \Big) \cdot \varphi(t, x; t', x') dx'dt'dxdt,$$

so that, by Corollary 6.4,

$$\begin{aligned} T_{U''}(u, v; T) &\le \frac{c_1}{2}\, \varepsilon\, \| f'' \| \int_0^T \int_{\mathbb{R}^d} \int_0^T \int_{\mathbb{R}^d} |\nabla u(t,x)|\, \varphi(t, x; t', x') dx'dt'dxdt \\ &\le \frac{c_1}{2}\, \varepsilon\, \| f'' \| \int_0^T |\, u(t)\,|_{TV(\mathbb{R}^d)} \left\{ \int_0^T w_{\varepsilon_t}\,(t - t')\, dt' \right\} dt \\ &\le c_1\, \varepsilon\, \| f'' \|\, T\, |\, u_0\,|_{TV(\mathbb{R}^d)}\, W(T). \end{aligned}$$

The result for u not smooth follows from a classical density argument. □

We are now ready to prove Lemma 3.1. From Proposition 6.2 and Lemma 6.5, we have

$$\begin{aligned} \int_{\mathbb{R}^d} U(v_-(t_{N_T}, x') - u(t_{N_T}, x'))dx' \le\ & 2 \int_{\mathbb{R}^d} U(v_-(0, x') - u(0, x'))dx' \\ & + 8\, L(\varepsilon_x + \varepsilon_t \| f' \|)\, |\, u_0\,|_{TV(\mathbb{R}^d)} + 2\, c_1\, \| f'' \|\, T\, |\, u_0\,|_{TV(\mathbb{R}^d)}\, \varepsilon \\ & + 2 \lim_{\ell \to \infty} \sup_{0 \le t \le T} \tilde{E}^{\varepsilon_t, \varepsilon_x; \ell}(v, u; t) / W^{\ell}(t) \\ & + \chi_{(\varepsilon_t, \infty)}(t_{N_T}) \frac{1}{\varepsilon_t} \int_0^{t_{N_T}} \frac{D_t^{\infty}(v, u)}{W^{\infty}(t)}\, dt. \end{aligned}$$

Next, consider the function $B(w) = |\, w\,| - U(w)$. By the definition (3.3a) of G, $B(w) \ge 0$ and

$$B(w) = \varepsilon\, (|\, w/\varepsilon| - G(w/\varepsilon)) \le \varepsilon \sup_{v \in \mathbb{R}} \Big|\, |\, v\,| - G(v)\, \Big| \le \varepsilon \sup_{|\, v\,| \le 1} \Big|\, |\, v\,| - G(v)\, \Big| = c_0\, \varepsilon.$$

This implies that $|w| - c_0\, \varepsilon \le U(w) \le |w|$, and hence

$$\begin{aligned} \| u_h(t_{N_T}) - u(t_{N_T}) \|_{L^1(\mathbb{R}^d)} \le\ & 2 \| u_{0h} - u_0 \|_{L^1(\mathbb{R}^d)} + 8\, L(\varepsilon_x + \varepsilon_t \| f' \|)\, |\, u_0\,|_{TV(\mathbb{R}^d)} \\ & + \left(c_0 (2M)^d + 2\, c_1\, \| f'' \|\, T\, |\, u_0\,|_{TV(\mathbb{R}^d)} \right) \varepsilon \\ & + 2 \lim_{\ell \to \infty} \sup_{0 \le t \le T} \tilde{E}^{\varepsilon_t, \varepsilon_x; \ell}(v, u; t) / W^{\ell}(t) \\ & + \chi_{(\varepsilon_t, \infty)}(t_{N_T}) \frac{1}{\varepsilon_t} \int_0^{t_{N_T}} \frac{D_t^{\infty}(v, u)}{W^{\infty}(t)}\, dt. \end{aligned}$$

This completes the proof of Lemma 3.1.

7. Concluding remarks. Several versions of the SCSD and SCDG methods (i.e., several definitions of ε_1, ε_2, or ε_2^{SD}) can be found in the literature; see e.g. [3], [6], [10], [11], and [4]). It is interesting to note that the coefficients ε_1 and ε_2 considered in [4] are some kind of L^{∞} versions of our L^2 coefficients.

We want to emphasize that, to obtain our a posteriori results, the only property of the shock-capturing terms we require is that they satisfy a nonnegativity condition. This property is trivially satisfied for the SCDG method as a reflection of the fact that piecewise-constant functions belong to the finite element space. As a consequence, our results for the SCDG method hold for very general finite elements. For the SCSD method, this does not happen since the approximate solution is taken to be continuous in space. However, the nonnegativity of the shock-capturing terms can be proven, not only for acute $(d+1)$-simplices, as shown in this paper, but also (i) for prisms $T = K_d \times I$, where K_d is an acute d-dimensional simplex and I is an interval, together with local spaces of the form $P^{k_1}(T_d) \otimes P^{k_2}(I)$ (see [4]), and (ii) for $(d+1)$-dimensional cubes $T = I_1 \times \cdots \times I_{d+1}$ (or the corresponding parallelepipeds) with local spaces of the form $P^{k_1}(I_1) \otimes \cdots \otimes P^{k_{d+1}}(I_{d+1})$.

To obtain classical error estimates from the a posteriori error estimates, regularity properties of the approximate solution are needed. In this paper, we find a regularity property by a standard L^2-energy argument. (It is at this point that the definition of the shock-capturing terms is crucial.) This property can then be combined with the a posteriori estimates to produce rates of convergence. Since this regularity property is independent of the degree of the approximating polynomials (see the estimates of the terms $\Theta_i(u_h)$ in §2), so are the theoretical convergence rates. A better rate of convergence can be obtained with stronger regularity properties of the approximate solution.

Finally, we point out that our technique can be extended to other definitions of ε_1, ε_2, and ε_2^{SD}. If, for instance, we do not divide by the gradient of the approximate solution in the definition of ε_1, ε_2, or ε_2^{SD}, the theoretical orders of convergence (with δ, δ_1, and δ_2 of order h) are found to be the same as those in Corollaries 2.6 and 2.9. Moreover, if δ_1 and δ_2 are taken to be of order $h^{2-2\nu}$, for $\nu > 0$, the methods can be proven to converge with a rate of h^ν.

Appendix: Existence of a solution of the SCSD and SCDG methods. In this appendix, we prove that the SCSD method and the SCDG method, as defined in this paper, admit a solution. We only prove the result for the SCDG method.

We use the following result.

LEMMA. *Let $\mathbb{X}$ be a finite-dimensional vector space with inner product $(\cdot,\cdot)$. If $\mathbb{F} : \mathbb{X} \mapsto \mathbb{X}$ is continuous and satisfies $(\mathbb{F}x, x) \geq 0$ for all $x \notin \{y \in \mathbb{X} : (y, y) \leq r^2\} = B(0, r)$ for some $r > 0$, then $\mathbb{F}x = 0$ has a solution in $B(0, r)$.*

Proof. See [9, p. 164] for a proof of this lemma. □

We apply the lemma to $\mathbb{X} = V_h$, with $(\cdot,\cdot)$ the $L^2(\Omega)$-inner product for $\Omega = (0, T_\infty) \times \mathbb{R}^d$, and

$$(\mathbb{P}w, v) = \sum_{n=1}^{N_\tau - 1} \sum_{T \in \mathcal{T}_{h,n}} \int_T A(w)(v + \delta \tilde{A}(w, v))dxdt$$
$$+ \sum_{n=1}^{N_\tau - 1} \sum_{T \in \mathcal{T}_{h,n}} \sum_{e \in \partial T} \int_e \left(f_{e,T}^{LF}(w^T, w^{T_e}) - \tilde{f}(w^T) \cdot n_{e,T} \right) v^T d\lambda$$
$$+ \sum_{n=1}^{N_\tau - 1} \sum_{T \in \mathcal{T}_{h,n}} \int_T \varepsilon_1(w) \mathbb{P}_h(\nabla \hat{w} \cdot \nabla \hat{v})dxdt$$
$$+ \sum_{n=1}^{N_\tau - 1} \sum_{T \in \mathcal{T}_{h,n}} \sum_{e \in \partial T} \int_e \varepsilon_2(w^T) \mathbb{P}_h(\nabla_e \hat{w}^T \cdot \nabla_e \hat{v}^T)d\lambda.$$

By identifying V_h to its dual, we can easily see that $\mathbb{P}$ is a well-defined application from V_h to V_h. Moreover, if the viscosity terms are defined as in §2, that is, if

$$\varepsilon_1(v)|_T = \delta_1 \frac{|Av|}{\| \nabla \hat{v} \|_{\mathbb{P}_h} + \delta_3} \qquad \text{and} \qquad \varepsilon_2(v^T)|_e = \delta_2 \frac{|f_{e,T}^{LF}(v^T, v^{T_e}) - \tilde{f}(v^T) \cdot n_{e,T}|}{\| \nabla_e \hat{v}^T \|_{\mathbb{P}_h} + \delta_4},$$

where δ_3 and δ_4 are positive parameters, then $\mathbb{P}$ is continuous.

Finally, proceeding as in the proof of the a priori estimate of Lemma 5.1, we obtain

$$
\begin{aligned}
(\mathbb{P}v, v) = &\frac{1}{2}\int_{\mathbb{R}^d_{N_\tau}} {v^T}^2 dx + \frac{1}{2}\sum_{n=0}^{N_\tau-1} \sum_{e\in\partial\mathcal{T}_{h,n}\cap\mathbb{R}^d_n} \int_e (v^T - v^{T_e})^2 dx \\
&+ \sum_{n=0}^{N_\tau-1} \sum_{e\in\mathcal{E}_n} \int_e \int_{u_h^{T_e}}^{u_h^T} \left(f^{LF}_{e,T}(u_h^T, u_h^{T_e}) - \tilde{f}(s)\cdot n_{e,T} \right) ds\, d\lambda \\
&+ \sum_{n=0}^{N_\tau-1} \sum_{T\in\mathcal{T}_{h,n}} \int_T \varepsilon_1(v) \| \nabla\hat{v} \|^2_{\mathbb{P}_h} dx dt + \sum_{n=0}^{N_\tau-1} \sum_{T\in\mathcal{T}_{h,n}} \sum_{e\in\partial T} \int_e \varepsilon_2(v^T) \| \nabla_e \hat{v}^T \|^2_{\mathbb{P}_h} d\lambda \\
&+ \delta \sum_{n=0}^{N_\tau-1} \sum_{T\in\mathcal{T}_{h,n}} \int_T A(v)^2 - \frac{1}{2} \| u_{h0} \|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}
$$

Since, by (2.4), the second term of the right-hand side is positive, we have that if $\| v \|_{L^2(\Omega)}$ is large enough, then $(\mathbb{P}v, v) \geq 0$. We can thus apply the lemma and obtain a solution to our problem.

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