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PREEMPTIVE DISCIPLINES FOR QUEUES AND STORES

by

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CHAPTER I

INTRODUCTION TO QUEUES WITH PREEMPTIVE PRIORITY DISCIPLINES¹

1. Introduction. A queue is the term used to describe the line of customers that forms at a service mechanism according to specific rules of:

- (i) arrival: which is the input of customers to a line of customers that demand service at a service mechanism;
- (ii) the service mechanism: which supplies a service to each customer during a period of time called the service time and limits the number of customers that can receive service simultaneously;
- (iii) the queue discipline: which is the set of rules governing the manner each customer joins other customers in the queue, waits for service and receives service.

The customers who have arrived at the queue prior to time t but have not completed service by time t are said to be in the queueing system at this time. Thus the queue discipline governs the behavior and the order of service of each customer with respect to other customers at all times while in the queueing system.

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The study of the analytic properties of queues, called queueing theory, has interested many authors because of the variety of industrial and military applications. Certainly a preponderance of the literature has dealt with variations in the rules governing arrivals and the service mechanism under a strict head-of-the-line queue discipline. This type of discipline requires that customers enter the service mechanism in exactly the same order in time as their respective arrivals at the queueing system. In this dissertation we will consider other possible queue disciplines where k different classes or types of customers, say $c_1, c_2, c_3, \dots, c_k$, are serviced by the same service mechanism.

2. Arrivals. The arrivals of customers may be formulated mathematically in terms of the r -th customer to arrive after time zero. Let t_r be the time of arrival of the r -th customer, irrespective of class. Define $t_0 = 0$. Hence customers arrive at times t_1, t_2, t_3, \dots . The time interval, $t_r - t_{r-1}$, between the $(r-1)$ -th and r -th arrivals is called the interarrival time of the r -th customer. Considering the class c_i customers only, let t_1^i, t_2^i, \dots be their arrival times. If $t_0^i = 0$, then the interarrival time of the r -th class c_i customer is $t_r^i - t_{r-1}^i$, $r = 1, 2, \dots$. It is clear that the k sequences $\{t_r^i\}$, $i = 1, 2, \dots, k$ of arrival times for the k classes of customers are subsequences of the sequence of arrival times $\{t_r\}$ for the queueing system. We will assume that the interarrival

times of class c_i customers is a sequence of independent and identically distributed random variables. We will also assume that the sequence of interarrival times for any one class of customers is independent of the sequence of interarrival times for each other class of customers. Thus the terms of the sequence $\{t_r^i - t_{r-1}^i\}$ are independently distributed with distribution function $A_i(t)$ and mean interarrival time $1/\lambda_i$, $i = 1, 2, \dots, k$. In general, the distribution functions and mean interarrival times will be different for different classes of customers. However, we will make certain restricting assumptions which will be described in Section 6.

3. The Service Mechanism. Throughout this dissertation it will be assumed that the service mechanism provides service for only one customer at any one time. Some of the queue disciplines to be discussed involve interruptions in service of certain customers. That is, under circumstances specified by the queue discipline, a customer will enter the service mechanism, remain a period of time and return to the waiting line without completion of service. For a customer who is not interrupted while in the service mechanism, the service time can be formulated mathematically in terms of the r -th customer to arrive after time zero. Let S_r^i be the time in the service mechanism of the r -th class c_i customer to arrive after time

zero, provided service of this customer is not interrupted. We will call this the uninterrupted service time of the r -th arrival in class c_i . Suppose the uninterrupted service times $\{S_r^i\}$ in class c_i is a sequence of independent and identically distributed random variables with distribution function $B_i(s)$ and mean $1/\mu_i$, $i = 1, 2, \dots, k$. In general, the uninterrupted service times of different classes of customers will be independent with different distributions. Discussion of the total service times of customers with interrupted service is deferred to the next section.

4. The Queue Discipline. We will be considering queue disciplines for the k classes, c_1, c_2, \dots, c_k , of customers where some classes have a priority to service with respect to other classes. For definiteness, suppose that the subscript i of the class c_i indicates the order of priority. Thus c_i customers have priority over customers of classes $c_{i+1}, c_{i+2}, \dots, c_k$. Cobham [1]¹ introduced the head-of-the-line priority queue discipline. Under this rule, customers of the same class are served according to a head-of-the-line discipline. Customers of class c_i enter the service mechanism at

1. Numbers in square brackets refer to bibliography.

those moments only when service of another customer is completed and customer classes c_1, c_2, \dots, c_{i-1} are empty. Thus if a customer of class c_i arrives during the service of a customer of class c_j and $i < j$, we say that a higher priority customer has arrived. The higher priority customer must wait at least until the lower priority customer completes service. In fact, the c_i customer must wait until the service mechanism is empty and there are no customers of classes c_1, c_2, \dots, c_{i-1} and no prior arrivals of class c_i in the waiting line.

White and Christie [14] introduced the preemptive priority queue discipline. Under this rule, customers of the same class are serviced according to a head-of-the-line discipline. Customers of different priority classes enter the service mechanism, when it is free, in accordance with their respective priority classes. However, if a customer of class c_i arrives during service of a customer of lower priority class c_j (i.e. $i < j$) then the higher priority customer immediately enters the service mechanism. The lower priority customer returns to the head of the waiting line of c_j customers. Thus customers of class c_j are eligible for service only during those time periods that the queueing system contains no higher priority customers, i.e. no customers of classes c_1, c_2, \dots, c_{j-1} . For simplicity, we refer to this as the preemptive queue

discipline. Other authors have employed the more descriptive and unwieldy phrase "preemptive priority queue discipline".

We have assumed that the uninterrupted service time of each customer in class c_i is a random variable S with distribution function $B_i(s)$. If the service time of a class c_i customer is interrupted then the total time U that the customer is in the service mechanism is governed by one of the following disciplines:

- (i) Preemptive Resume where the total service time U is independent of the number of interruptions. Hence a preempted customer returns to the service mechanism to complete that portion of the service time remaining at the time of preemption and $U = S$.
- (ii) Preemptive Repeat (identical) means that a class c_i customer, upon initial entry to the queueing system, is assigned an uninterrupted service time S with distribution function $B_i(s)$. The customer must remain in the service mechanism without interruption a period S in order to complete service. Suppose a customer is interrupted N times and is in the service mechanism a period V_j before the j -th interruption, $j = 1, 2, \dots, N$. Thus the total service time $U = \sum_{j=1}^N V_j + S$ where $V_j < S$, $j = 1, 2, \dots, N$.
- (iii) Preemptive Repeat (independent) specifies that each class c_i customer is assigned an uninterrupted service time $S(j)$ with distribution function $B_i(s)$

upon the j -th entry to the service mechanism. Thus the customer must remain in the service mechanism without interruption a time period $S(j)$ after the j -th entry in order to complete service. Suppose the c_i customer is interrupted N times while in service and is in the service mechanism a period V_j before the j -th interruption, $j = 1, 2, \dots, N$. Thus the total service time $U = \sum_{j=1}^N V_j + S(N+1)$ where $V_j < S(j)$ and $S(j)$, $j = 1, 2, \dots, N+1$, are independent and identically distributed random variables.

It is clear that the preemptive repeat (identical) and (independent) disciplines are equivalent if the uninterrupted service times are constants for each class of customers. White and Christie [14] compared, in a general discussion, the preemptive resume and repeat disciplines. Gaver [2] introduced the terminology employed here to differentiate between the two preemptive repeat disciplines.

5. Theory of Queues. The theory of queues deals with the study of the properties of the probability distribution of a stochastic process $\{Z_t, t \in T\}$ where T is a subset of or the entire positive time axis. The process variable Z_t is either a real valued or a vector of real valued random variables defined over the set of time points T in such a way as to describe some aspect of interest of the queue.

In the application of queueing theory for different priority classes of customers many aspects of the queue may be of interest.

Thus the study of such queues could deal with the probability distribution of any of the following processes:

- (a) The queueing time of a customer of class c_i arriving at time t where queueing time is defined as the elapsed time between arrival and initial entry to the service mechanism.
- (b) The waiting time of a customer of class c_i arriving at time t where waiting time is the total time elapsed while in the queueing system. (Some confusion exists in the literature on Queueing Theory due to the interchange, by some authors, of the definitions of queueing time and waiting time. We will use the nomenclature defined here when referring to other authors' results regardless of whether or not this agrees with the terminology of the cited work).
- (c) The time-in-service of a customer of class c_i arriving at time t where the time-in-service, as defined by Miller [8], is the total elapsed time while the customer is actually in the service mechanism.
- (d) The completion time of a customer of class c_i , arriving at time t , where the completion time, as defined by Gaver [2], is the elapsed time from initial entry into the service mechanism until completion of service.
- (e) The queue size at time t is a k term vector of the number of customers in the queueing system of each

class of customer (excluding the customer in the service mechanism).

- (f) The busy period commencing at time t with arrival of a c_i customer at an empty queue where the busy period is the shortest elapsed time between two consecutive instants when the queueing system contains no customers.
- (g) The clearing time of the c_i customers of a queue at time t is the total time required to service all c_i customers already in the queueing system provided no additional customers arrive and the queue contains no c_1, c_2, \dots, c_{i-1} customers.
- (h) The number of preemptions of a class c_i customer while in some state of the queueing system. The number of preemptions is defined as the number of higher priority customers who arrive after the c_i customer and complete service prior to his departure from this state.

For a given queue discipline pertaining to priority classes of customers some of these aspects of the queueing system are closely related to other aspects and the properties of the probability distributions involved are derivable one from the other. For example, the completion time plus the queueing time of a customer is the waiting time of that customer. In the most general situation, it is not possible to say that a study of one aspect of the queueing

system is more important than the study of some other aspect. This depends entirely on the field of application of the queueing theory.

The study of the properties of the probability distribution of a stochastic process $\{Z_t, t \in T\}$ is divided into two areas. If the set T is restricted to a finite interval, or subset thereof, of the positive time axis, the transient behavior of the process is studied. The limiting behavior is the study of the limit, as $t \rightarrow \infty$, of the probability distribution for Z_t . The study of the limit of the probability distribution involves two questions: (i) Is the limit a proper probability distribution? (ii) If so, what are its properties? In this dissertation we will concern ourselves primarily with the limiting behavior of certain stochastic processes arising in the theory of queues with k priority classes of customers under preemptive resume or repeat disciplines. We shall consider also some closely related problems. In Chapters II and III we will assume the existence of the limiting distributions under investigation and obtain certain properties of these distributions. In Chapter IV we will derive the necessary and sufficient conditions for the existence of these limiting distributions.

6. Theory of Priority Queues. In all studies of queues with priority classes of customers, arrivals are assumed to follow an independent Poisson process for each class of customers. This means that the probability of n arrivals of class c_i customers in a time period t is $e^{-\lambda_i t} (\lambda_i t)^n / n!$ where λ_i is called the arrival rate.

This is equivalent to saying that the interarrival times of class c_i are independent and identically distributed random variables with the negative exponential distribution, i.e. $A_i(t) = 1 - e^{-\lambda_i t}$ with mean $1/\lambda_i$. This type of arrivals is often referred to as random arrivals with rate λ_i . We shall adopt this terminology.

The moment generating function (m.g.f.) for a Poisson process is $e^{-\lambda_i(1-\theta)t}$, from which it is apparent that the m.g.f. of the sum of k independent Poisson processes with rates $\lambda_1, \lambda_2, \dots, \lambda_k$ is $e^{-\lambda(1-\theta)t}$, the m.g.f. of a Poisson process with rate $\lambda = \sum_{i=1}^k \lambda_i$.

It would be difficult to summarize completely the known results and their derivations for queues with priority classes of customers. The derivations, in many cases, are quite long and involved. However, the major results are listed in Table 1.1 for the limiting behavior of descriptive aspects of the queue.

In this dissertation we develop a powerful method for investigating certain multivariate limiting distributions of aspects describing single server queues with k priority classes of customers. The utility and generality of the method become apparent as we employ it to consider some hitherto unsolved queueing and storage problems.

TABLE 1.1

MAJOR RESULTS FOR PREEMPTIVE AND PRIORITY QUEUES

Author and Reference	Service Time Distribution	Queue Discipline	Results
Cobham [1]	negative exponential	head-of-the-line priority	Mean queueing times for each of k classes of customers.
Morse [9]	negative exponential	head-of-the-line priority	m.g.f. of queue size and associated moments.
Kesten and Runnenberg [7]	general independent	head-of-the-line priority	Functional form of Laplace-Stieltjes transform of queueing time and the associated moments for low priority customers.
White and Christie [14]	negative exponential	Preemptive Resume	m.g.f. of queue size, moments of queueing time and waiting time.
Miller [8]	general independent	Preemptive Resume	results with respect to the queueing time, time-in-service, busy periods, and number of customers served during a busy period.
Heathcote [5]	convolutions of finite number of negative exponential distributions	Preemptive Resume	m.g.f. of queue size and associated moments.

7. Queues and Stores. Suppose loads of a material arrive at a store or warehouse. The supply in the store of this material is subject to a constant rate of depletion and the store has infinite capacity. The analytic study of the amount of material in the store, (called the store content) with respect to time is the storage problem. Smith [10] noted that the storage problem is mathematically equivalent to the study of the queueing time of a single server queue with one type of customer. The arrivals of customers coincide with the arrivals of loads of material. The service times of customers coincide with the load sizes. The uniform rate of reduction of the store content is equivalent to the operation of the service mechanism at a uniform rate. Thus the store content, at any time, is equivalent to the queueing time of a customer arriving at that time.

Suppose the store receives loads of k different types of material, say c_1, c_2, \dots, c_k , where each load is homogeneous of one type only. Further suppose the store content is diminished at a uniform rate. The store content of material c_i is reduced only during those time periods when the store contains no c_1, c_2, \dots, c_{i-1} material. This rule of operation we shall call a pre-emptive rule. It is evident that the store content of material c_i is equivalent to the clearing time of class c_i customers for a pre-emptive resume queue. In Chapter II we investigate the transient and limiting behavior of the joint distribution of the store content for k different types of material. We will then employ these results to investigate the distribution of the queueing times for a

queue with k classes of customers and three preemptive queue disciplines: resume, repeat (identical), and repeat (independent). Miller [8] obtained the Laplace-Stieltjes transform and the first two moments of the queueing time for the preemptive resume discipline only by a more direct approach. Therefore we will only indicate briefly the applicability to this case of our approach in order to demonstrate its generality. The same method will be applied to consider the hitherto unsolved problems of the queueing time for the two preemptive repeat disciplines. In the course of this we will also consider certain problems involved in a head-of-the-line queue with a single service mechanism which is subject to breakdowns.

8. Service Mechanisms Subject to Breakdowns. White and Christie [14] noted the analogy between a preemptive queue discipline for two classes of customers and a head-of-the-line queue with a single service mechanism which is subject to breakdowns. The customers with preemptive rights are considered as breakdowns of the service mechanism. In Chapter III we consider four possible types of occurrence of breakdowns. A variation of the method developed in Chapter II is employed to investigate queues under these types of breakdown. In Chapter IV the necessary and sufficient conditions are determined for the existence of a limiting distribution for a Poisson Reduction Process. The Poisson Reduction Process is defined in this chapter and shown to include, as special cases, all processes considered in Chapters II and III. The numbers of breakdowns during queueing time and during service time of a customer are considered in Chapter V.

CHAPTER II

THE PREEMPTIVE STORAGE PROBLEM

1. Introduction. A store receives loads of k different types of material, say c_1, c_2, \dots, c_k . Material $c_i (1 \leq i \leq k)$ arrives at the store in homogeneous loads at random rate λ_i . The sizes of consecutive loads of material c_i are a sequence of independent and identically distributed random variables with distribution function $B_i(s)$ and mean $1/\mu_i$, $i = 1, 2, \dots, k$. The load sizes for two different types of material are independent but not necessarily identically distributed sequences of random variables. Thus, in general, $B_i(s) \neq B_j(s)$ for $i \neq j$. The contents of the store are diminished at unit rate in accordance with a preemptive rule. That is, material c_i is eligible for depletion only when the store is empty of materials c_1, c_2, \dots, c_{i-1} . We call this the preemptive storage system with k materials.

Suppose the store has infinite capacity. We noted in Section 7, Chapter I that the preemptive storage problem is mathematically equivalent to a consideration of the clearing times for a single server queue with random arrivals, independent service times, preemptive resume queue discipline and k classes of customers.

For the purposes of definiteness and simplicity the preemptive storage problem is considered for $k = 2$. Then the results are extended in Section 5 for larger values of k .

2. The Preemptive Storage Problem for $k = 2$. Let X_t be the store content of material c_1 at time t which has preemptive rights over Y_t , the store content of material c_2 at time t . Define the following components of the bivariate distribution of the store content vector $Z_t = (X_t, Y_t)$ as:

$$\begin{aligned} F(x, y; t) &= P(0 < X_t \leq x, 0 < Y_t \leq y) && ; \\ G(y; t) &= P(X_t = 0, 0 < Y_t \leq y) && ; \\ H(x; t) &= P(0 < X_t \leq x, Y_t = 0) && ; \\ J(z, t) &= P(X_t + Y_t \leq z, X_t \neq 0, Y_t \neq 0) && ; \\ \pi(t) &= P(X_t = 0, Y_t = 0). \end{aligned}$$

Thus

$$F(x, y; t) + H(x; t) + G(y; t) + \pi(t) = P(X_t \leq x, Y_t \leq y) = P_t(x, y), \text{ say.}$$

Also

$$J(z; t) + H(z; t) + G(z; t) + \pi(t) = P(X_t + Y_t \leq z) .$$

Loads of material c_i arrive at random rate λ_i at times $t_1^i, t_2^i, t_3^i, \dots$, in loads of sizes $S_1^i, S_2^i, S_3^i, \dots$, $i = 1, 2$. The sequence $\{S_r^i\}$ is a sequence of independent and identically distributed random variables with mean $1/\mu_i$ and distribution function $B_i(s)$, $i = 1, 2$. The sequence $\{S_r^1\}$ is independent of the sequence $\{S_r^2\}$. It is assumed that each load of material is greater than zero with probability one, i.e., $B_i(0+) = 0$, $i = 1, 2$.

The store content $Z_t = (X_t, Y_t)$ is summarized in Figure 2.1 for a specific realization of the process.

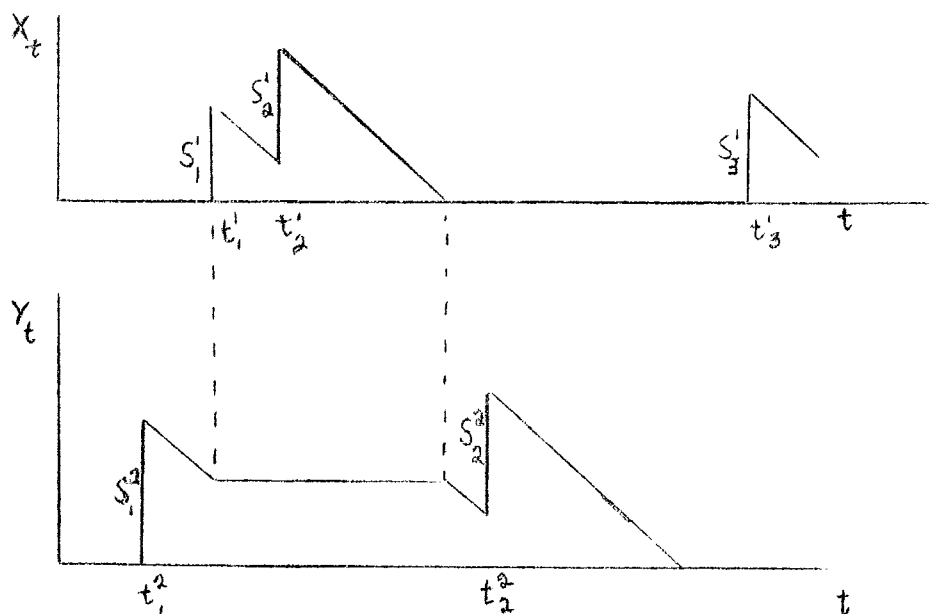


Figure 2.1

The process $Z_t = (X_t, Y_t)$ is a continuous parameter Markov process. This fact suggests an investigation of the time derivatives of the components of the instantaneous distribution function. We will prove first the following useful lemma.

Lemma 2.1. Consider a time interval $[0, \tau]$ and two non-negative numbers S and θ , $0 \leq \theta \leq \tau$. Define a process

$$\begin{aligned} \xi_\delta(S, \theta) &= \text{Max}(0, \xi_0 - \delta), & 0 \leq \delta < \theta \\ &= \text{Max}(0, \xi_{\theta-\delta} + S - \delta + \theta), & \theta \leq \delta \leq \tau, \end{aligned}$$

where ξ_0 is any prescribed non-negative number. Then

$$(2.1) \quad \xi_0 + S - \tau \leq \xi_\tau(S, 0) \leq \xi_\tau(S, \theta) \leq \xi_\tau(S, \tau) \leq \xi_0 + S.$$

Proof. Now

$$\xi_{0^*-0}(S, \theta^*) = \text{Max} (0, \xi_0 - 0^*)$$

for $\theta^* = 0, \theta, \tau$. Thus, from the definition,

$$\xi_\tau(S, 0) = \text{Max} (0, \xi_0 + S - \tau) \quad ;$$

$$\xi_\tau(S, \theta) = \text{Max} (0, \xi_0 + S - \tau, S - \tau + \theta) \quad ;$$

and

$$\xi_\tau(S, \tau) = \text{Max} (0, \xi_0 + S - \tau, S) \quad .$$

The lemma is a direct result of these three equations and the fact that ξ_0 , S , and τ are all non-negative numbers.

Consider the left open, right closed time interval $(t, t + \tau]$ where $\tau > 0$. The store content process $Z_{t+\tau} = (X_{t+\tau}, Y_{t+\tau})$ is at the point $(0, 0)$ if one of the following mutually exclusive and exhaustive events occurs:

(i) Zero arrivals occur in the interval $(t, t + \tau]$ and

$$0 \leq X_t + Y_t \leq \tau.$$

(ii) One arrival of load size S occurs at time $t + \theta$,

$$0 < \theta \leq \tau, \text{ and } \xi_\tau(S, \theta) = 0 \text{ where } X_t + Y_t = \xi_0 \text{ and}$$

the function $\xi_\tau(S, \theta)$ satisfies the definition of lemma 2.1.

(iii) Two or more arrivals of material occur with certain restrictions on $X_t + Y_t$ and the sum of the load sizes.

In an interval of length τ the probability of zero arrivals is $\exp(-\lambda_1 \tau - \lambda_2 \tau)$ and the probability of one arrival is $\exp(-\lambda_1 \tau - \lambda_2 \tau) \cdot (\lambda_1 + \lambda_2) \tau$. The probability of two or more arrivals is $O(\tau^2)$.

Thus

$$(2.2) \quad \pi(t+\tau) \geq e^{-\lambda\tau} [\pi(t) + H(\tau;t) + G(\tau;t) + J(\tau;t)]$$

and, by lemma 2.1

$$(2.3) \quad \pi(t+\tau) \leq e^{-\lambda\tau} [\pi(t) + H(\tau;t) + G(\tau;t) + J(\tau;t) + \lambda_1 \tau B_1(\tau) + \lambda_2 \tau B_2(\tau)] \\ + o(\tau^2),$$

where $\lambda = \lambda_1 + \lambda_2$.

Hence we may write, as $\tau \rightarrow 0$,

$$(2.4) \quad [\pi(t+\tau) - \pi(t)]/\tau = [e^{-\lambda\tau} - 1] \pi(t)/\tau \\ + e^{-\lambda\tau} [G(\tau;t) + H(\tau;t) + J(\tau;t)]/\tau \\ + o(1).$$

Define the Laplace-Stieltjes transforms of the components of the distribution of store content as follows:

$$H^x(u;t) = \int_0^{\infty} e^{-ux} d_x H(x;t) \quad ;$$

$$G^y(v;t) = \int_0^{\infty} e^{-vy} d_y G(y;t) \quad ;$$

$$F^x(u,y;t) = \int_0^{\infty} e^{-ux} d_x F(x,y;t) \quad ;$$

$$F^{xy}(u,v;t) = \int_0^{\infty} e^{-vy} d_y F^x(u,y;t) \quad ;$$

where u and v are restricted to the class of real non-negative numbers.

The store content $Z_{t+\tau}$ is in the region $(0 < X_{t+\tau} < x, Y_{t+\tau} = 0)$ if one of the following mutually exclusive and exhaustive events occurs:

- (i) There are no arrivals of material in the time interval $(t, t+\tau]$ and $\tau < X_t \leq x + \tau, Y_t = 0$.
- (ii) A load of c_1 material of size S arrives at time $t + \theta$, $0 < \theta \leq \tau$, and;
 - (a) $X_t = Y_t = 0, \tau - \theta < S \leq x + \tau - \theta$, or
 - (b) $0 < X_t \leq x + \tau, Y_t = 0, 0 < \xi_\tau(S, \theta) \leq x$, or
 - (c) $0 < X_t + Y_t \leq \theta, Y_t > 0, 0 < \xi_\tau(S, \theta) \leq x$,
 where we define $\xi_0(S, \theta) = X_t + Y_t$.
- (iii) Two or more arrivals occur in the time interval $(t, t+\tau]$ with certain restrictions on Z_t and the load sizes.

Thus, employing Lemma 2.1 ,

$$(2.5) \quad H(x; t+\tau) \geq e^{-\lambda\tau} [H(x+\tau; t) - H(\tau; t) + \lambda_1\tau \pi(t) \{B_1(x) - B_1(\tau)\}] \\ + \lambda_1\tau \int_{\tau}^x H(x-z; t) d B_1(z)$$

and

$$(2.6) \quad H(x; t+\tau) \leq e^{-\lambda\tau} [H(x+\tau; t) - H(\tau; t) + \lambda_1\tau \pi(t) B_1(x+\tau)] \\ + \lambda_1\tau \int_0^{x+\tau} H(x+\tau-z; t) d B_1(z) \\ + \lambda_1\tau \{J(\tau; t) + G(\tau; t)\} B_1(x+\tau) \\ + O(\tau^2) .$$

We note that

$$\begin{aligned} 0 &\leq \lambda_1 \tau e^{-\lambda\tau} \int_0^\tau \pi(t) B_1(\tau) + \int_0^\tau H(x-z;t) d B_1(z) \\ &\leq \lambda_1 \tau e^{-\lambda\tau} B_1(\tau) = o(\tau) \end{aligned}$$

uniformly in t . Similarly the terms in (2.3) and (2.6) designated $O(\tau^2)$ are probabilities that are dominated by the probability that two or more arrivals of material occur in an interval of length τ . Thus both these terms are dominated by a term which is $o(\tau)$ uniformly in t and in the case of (2.6), uniformly in x .

In order to take Laplace-Stieltjes transforms of (2.5) and (2.6), we require the following lemma.

Lemma 2.2. If $H_1(x) \leq H_2(x)$, for all x and $H_1(0+) = H_2(0+) = 0$
then the Laplace Stieltjes transforms $H_1^x(u)$ and $H_2^x(u)$, respectively,
satisfy

$$H_1^x(u) \leq H_2^x(u)$$

where the transform variable u is real and positive.

Proof. Since $H_1(x) \leq H_2(x)$, then

$$\int_0^\infty e^{-ux} H_1(x) dx \leq \int_0^\infty e^{-ux} H_2(x) dx .$$

This implies, upon integrating by parts,

$$\frac{1}{u} [H_1(0+) + H_1^x(u)] \leq \frac{1}{u} [H_2(0+) + H_2^x(u)]$$

and the lemma is proved.

Inequalities (2.5) and (2.6) become, after taking Laplace-Stieltjes transforms with respect to x ,

$$(2.7) \quad H^x(u;t+\tau) \geq e^{(u-\lambda)\tau} \left[H^x(u;t) - \int_0^\tau e^{-ux} d_x H(x;t) \right] \\ + e^{-\lambda\tau} \lambda_1 \tau \left[\pi(t) + H^x(u;t) \right] B_1^x(u) \\ - o(\tau) \quad ,$$

and

$$(2.8) \quad H^x(u;t+\tau) \leq e^{(u-\lambda)\tau} \left[H^x(u;t) - \int_0^\tau e^{-ux} d_x H(x;t) \right] \\ + \lambda_1 \tau e^{(u-\lambda)\tau} \left[\pi(t) + J(\tau;t) + G(\tau;t) + H^x(u;t) \right] \\ \cdot B_1^x(u) \\ + o(\tau) \quad ,$$

where the terms $o(\tau)$ are uniformly $o(\tau)$ in t .

The store content $Z_{t+\tau}$ is in the region ($X_{t+\tau} = 0, 0 < Y_{t+\tau} \leq y$) if one of the following mutually exclusive and exhaustive events occurs:

- (i) There are zero arrivals of material in the interval $(t, t+\tau]$ and $0 \leq X_t \leq \tau, \tau < X_t + Y_t \leq y + \tau$.
- (ii) One load of material c_1 arrives in the interval $(t, t+\tau]$ at time $t + \theta, 0 < \theta \leq \tau$, of size S and $S \leq \tau - \theta, X_t + S \leq \tau, \tau < X_t + Y_t + S \leq y + \tau$.
- (iii) One load of material c_τ of size S arrives in the interval $(t, t+\tau]$ at time $t + \theta, 0 < \theta \leq \tau$, and $0 < \xi_\tau(S, \theta) \leq y$ where $\xi_0 = X_t + Y_t$. This requires either
 - (a) $X_t = Y_t = 0$, or

$$(b) \quad 0 < X_t \leq \tau, Y_t = 0, \text{ or}$$

$$(c) \quad 0 < X_t + Y_t \leq y + \tau, X_t \leq \tau, Y_t > 0.$$

(iv) Two or more loads of material arrive in the interval $(t, t+\tau]$ with certain restrictions on sum of load sizes and Z_t .

The probabilities arising under (ii) and (iv) are dominated, uniformly in both t and y , by probabilities that are $o(\tau)$. Thus, employing lemma 2.1 and other elementary inequalities we have;

$$(2.9) \quad G(y; t+\tau) \geq e^{-\lambda\tau} [G(y+\tau; t) + F(\tau, y; t) - G(\tau; t) - J(\tau; t)] \\ + e^{-\lambda\tau} \lambda_2 \tau [\pi(t) B_2(y) + \int_0^y G(y-z; t) d B_2(z)] \\ - o(\tau),$$

and

$$(2.10) \quad G(y; t+\tau) \leq e^{-\lambda\tau} [G(y+\tau; t) + F(\tau; y+\tau; t) - G(\tau; t) - J(\tau; t)] \\ + e^{-\lambda\tau} \lambda_2 \tau [\pi(t) B_2(y+\tau) + \int_0^{y+\tau} G(y+\tau-z; t) d B_2(z)] \\ + e^{-\lambda\tau} \lambda_2 \tau \int_0^\infty F(\tau, y + \tau - z; t) d B_2(z) \\ + o(\tau),$$

where the terms $o(\tau)$ are uniform in t . Upon taking Laplace-Stieltjes transforms with respect to y and upon applying lemma 2.2, inequalities (2.9) and (2.10) become;

$$\begin{aligned}
 (2.11) \quad G^y(v; t+\tau) &\geq e^{(v-\lambda)\tau} \left[G^y(v; t) - \int_0^\tau e^{-vy} d_y G(y; t) \right] \\
 &+ e^{-\lambda\tau} F^y(\tau, v; t) \\
 &+ e^{-\lambda\tau} \lambda_2 \tau \left[\pi(t) + G^y(v; t) \right] B_2^y(v) \\
 &- o(\tau) \quad ,
 \end{aligned}$$

and

$$\begin{aligned}
 (2.12) \quad G^y(v; t+\tau) &\leq e^{(v-\lambda)\tau} \left[G^y(v; t) - \int_0^\tau e^{-vy} d_y G(y; t) \right] \\
 &+ e^{(v-\lambda)\tau} \left[F^y(\tau, v; t) - \int_0^\tau e^{-vy} d_y F(\tau, y; t) \right] \\
 &+ e^{(v-\lambda)\tau} \lambda_2 \tau \left[\pi(t) + G^y(v; t) \right] B_2^y(v) \\
 &+ o(\tau) \quad .
 \end{aligned}$$

Again the terms $o(\tau)$ are uniform in t .

The store content $Z_{t+\tau}$ is in the region $(0 < X_{t+\tau} \leq x, 0 < Y_{t+\tau} \leq y)$ if one of the following mutually exclusive and exhaustive events occurs:

- (i) No loads of material arrive in the interval $(t, t+\tau]$ and $\tau < X_t \leq x + \tau, 0 < Y_{t+\tau} \leq y$.
- (ii) One load of c_1 material of size S arrives at time $t + \theta$, $0 < \theta \leq \tau$ and either
 - (a) $X_t = 0, \theta < Y_t \leq y + \theta, \tau - \theta < S \leq x + \tau - \theta$, or
 - (b) $0 < X_t \leq \theta_1, \theta - \theta_1 < Y_t \leq y + \theta - \theta_1, \tau - \theta < S \leq x + \tau - \theta, 0 < \theta_1 \leq \theta$, or
 - (c) $\theta < X_t \leq x + \tau - S, 0 < Y_t \leq y$.

- (iii) One load of c_2 material of size S arrives at time $t + \theta$, $0 < \theta \leq \tau$ and $\tau < X_t \leq x + \tau$, $0 < Y_t + S \leq y$.
- (iv) Two or more loads of material arrive in the time interval $(t, t + \tau]$ with certain restrictions on Z_t and the load sizes.

As before, the application to these four events of lemma 2.1 leads to the results

$$\begin{aligned}
 (2.13) \quad F(x, y; t + \tau) &\geq e^{-\lambda\tau} [F(x + \tau, y; t) - F(\tau, y; t)] \\
 &+ e^{-\lambda\tau} \lambda_1 \tau [G(y; t) B_1(x) + \int_0^x F(x - z, y; t) d B_1(z)] \\
 &+ e^{-\lambda\tau} \lambda_2 \tau [H(x; t) B_2(y) + \int_0^y F(x, y - z; t) d B_2(z)] \\
 &- o(\tau)
 \end{aligned}$$

and

$$\begin{aligned}
 (2.14) \quad F(x, y; t + \tau) &\leq e^{-\lambda\tau} [F(x + \tau, y; t) - F(\tau, y; t)] \\
 &+ e^{-\lambda\tau} \lambda_1 \tau [G(y + \tau; t) B_1(x + \tau) + \int_0^{x + \tau} F(x + \tau - z, y + \tau; t) d B_1(z)] \\
 &+ e^{-\lambda\tau} \lambda_2 \tau [H(x + \tau; t) B_2(y) + \int_0^y F(x + \tau, y - z; t) d B_2(z)] \\
 &+ o(\tau) .
 \end{aligned}$$

The Laplace-Stieltjes transforms of (2.13) and (2.14) reduce to, after applying lemma 2.2 and some elementary inequalities ,

$$\begin{aligned}
(2.15) \quad F^{xy}(u,v;t+\tau) &\geq e^{(u-\lambda)\tau} \left[F^{xy}(u,v;t) - \int_0^\tau e^{-ux} d_x F^y(x,v;t) \right] \\
&\quad + \lambda_1 \tau e^{-\lambda\tau} \left[G^y(v;t) + F^{xy}(u,v;t) \right] B_1^x(u) \\
&\quad + \lambda_2 \tau e^{-\lambda\tau} \left[H^x(u;t) + F^{xy}(u,v;t) \right] B_2^y(v) \\
&\quad - o(\tau) \quad ,
\end{aligned}$$

and

$$\begin{aligned}
(2.16) \quad F^{xy}(u,v;t+\tau) &\leq e^{(u-\lambda)\tau} \left[F^{xy}(u,v;t) - \int_0^\tau e^{-ux} d_x F^y(x,v;t) \right] \\
&\quad + \lambda_1 \tau e^{(u+v-\lambda)\tau} \left[G^y(v;t) + F^{xy}(u,v;t) \right] B_1^x(u) \\
&\quad + \lambda_2 \tau e^{(u+v-\lambda)\tau} \left[H^x(u;t) + F^{xy}(u,v;t) \right] B_2^y(v) \\
&\quad + o(\tau) \quad .
\end{aligned}$$

As before, the terms $o(\tau)$ are uniform in t . Define

$$P_t^{xy}(u,v) = \int_0^\infty \int_0^\infty e^{-ux-vy} d_x d_y P_t(x,y) \quad .$$

Thus

$$P_t^{xy}(u,v) = \pi(t) + H^x(u;t) + G^y(v;t) + F^{xy}(u,v;t)$$

and

$$\begin{aligned}
(2.17) \quad &\left[P_{t+\tau}^{xy}(u,v) - P_t^{xy}(u,v) \right] / \tau \\
&= \left[\pi(t+\tau) - \pi(t) \right] / \tau + \left[H^x(u;t+\tau) - H^x(u;t) \right] / \tau \\
&\quad + \left[G^y(v;t+\tau) - G^y(v;t) \right] / \tau + \left[F^{xy}(u,v;t+\tau) - F^{xy}(u,v;t) \right] / \tau .
\end{aligned}$$

Upon combining inequalities (2.2), (2.7), (2.11) and (2.15) we obtain

$$\begin{aligned}
 (2.18) \quad & \left[P_{t+\tau}^{xy}(u,v) - P_t^{xy}(u,v) \right] / \tau \\
 & \geq \left[e^{-\lambda\tau} - 1 \right] \pi(t) / \tau + e^{-\lambda\tau} \left[G(\tau;t) + H(\tau;t) + J(\tau;t) \right] / \tau \\
 & \quad + \left[e^{(u-\lambda)\tau} - 1 \right] \left[H^x(u;t) + F^{xy}(u,v;t) \right] / \tau \\
 & \quad + e^{-\lambda\tau} \lambda_1 B_1^x(u) \left[\pi(t) + H^x(u;t) + G^y(v;t) + F^{xy}(u,v;t) \right] \\
 & \quad + \left[e^{(v-\lambda)\tau} - 1 \right] G^y(v;t) / \tau \\
 & \quad + e^{-\lambda\tau} \lambda_2 B_2^y(v) \left[\pi(t) + H^x(u;t) + G^y(v;t) + F^{xy}(u,v;t) \right] \\
 & \quad - e^{(u-\lambda)\tau} \left[\int_0^\tau e^{-ux} d_x H(x;t) + \int_0^\tau e^{-ux} d_x F^y(x,v;t) \right] / \tau \\
 & \quad - e^{(v-\lambda)\tau} \int_0^\tau e^{-vy} d_y G(y;t) / \tau \\
 & \quad - o(1) .
 \end{aligned}$$

Moreover we observe that

$$\begin{aligned}
 & e^{-\lambda\tau} \left[G(\tau;t) - e^{v\tau} \int_0^\tau e^{-vy} d_y G(y;t) \right] / \tau \\
 & \geq e^{-\lambda\tau} \left[G(\tau;t) - e^{v\tau} G(\tau;t) \right] / \tau \\
 & = o(1) .
 \end{aligned}$$

Similarly

$$e^{-\lambda\tau} \left[G(\tau;t) - e^{v\tau} \int_0^\tau e^{-vy} d_y G(y;t) \right] / \tau \leq 0 .$$

Thus

$$e^{-\lambda\tau} \int G(\tau;t) - e^{v\tau} \int_0^\tau e^{-vy} d_y G(y;t) / \tau = o(1),$$

as $\tau \rightarrow 0+$.

Likewise

$$e^{-\lambda\tau} \int H(\tau;t) - e^{u\tau} \int_0^\tau e^{-ux} d_x H(x;t) / \tau = o(1)$$

and

$$e^{-\lambda\tau} \int F^y(\tau,v;t) - e^{u\tau} \int_0^\tau e^{-ux} d_x F^y(x,v;t) / \tau = o(1).$$

Let us now omit the non-negative term $e^{-\lambda\tau} J(\tau;t)/\tau$ from the right hand side of (2.18) and then allow τ to decrease to zero. We then discover that

$$(2.19) \quad \liminf_{\tau \downarrow 0+} \left[\frac{P_{t+\tau}^{xy}(u,v) - P_t^{xy}(u,v)}{\tau} \right] \\ \geq \pi(t) \int [-\lambda_1 - \lambda_2 + \lambda_1 B_1^x(u) + \lambda_2 B_2^y(v)] \\ + G^y(v;t) \int [v - \lambda_1 - \lambda_2 + \lambda_1 B_1^x(u) + \lambda_2 B_2^y(v)] \\ + \int [H^x(u;t) + F^{xy}(u,v)] \int [u - \lambda_1 - \lambda_2 + \lambda_1 B_1^x(u) + \lambda_2 B_2^y(v)].$$

Inequalities (2.3), (2.8), (2.12) and (2.16) are combined to form an upper bound for

$$\int [P_{t+\tau}^{xy}(u,v) - P_t^{xy}(u,v)] / \tau.$$

By an argument analogous to that employed for the lower bound it can be shown that this upper bound has a limit equal to the right hand side of inequality (2.19).

Theorem 2.1. The double Laplace-Stieltjes transform $P_t^{xy}(u,v)$ has a partial derivative with respect to time which satisfies the equation

$$(2.20) \quad \frac{\partial P_t^{xy}(u,v)}{\partial t} = -\phi P_t^{xy}(u,v) + v G^y(v;t) + u [H^x(u;t) + F^{xy}(u,v;t)],$$

where $\phi = \lambda_1 + \lambda_2 - \lambda_1 B_1^x(u) - \lambda_2 B_2^y(v)$.

Proof. That the right hand derivative exists and satisfies (2.20) has just been demonstrated. The existence of the left hand derivative, which also satisfies (2.20), follows by considering the upper and lower bounds for

$$[P_t^{xy}(u,v) - P_{t-\tau}^{xy}(u,v)] / \tau .$$

These bounds can be derived from (2.2), (2.3), (2.7), (2.8), (2.11), (2.12), (2.15) and (2.16) merely by replacing t with $t-\tau$. We note that these eight inequalities also imply

$$\begin{aligned} \pi(t) &= \pi(t-\tau) + o(1) , \\ H^x(u;t) &= H^x(u;t-\tau) + o(1) , \\ G^y(v;t) &= G^y(v;t-\tau) + o(1) , \\ F^{xy}(u,v;t) &= F^{xy}(u,v;t-\tau) + o(1) , \end{aligned}$$

which are employed in a straightforward manner to complete the proof of the theorem.

3. Limiting Behavior of the Preemptive Store for $k = 2$. The necessary and sufficient conditions for the existence of the limiting bivariate distribution of the store content are $\lambda_1/\mu_1 + \lambda_2/\mu_2 < 1$ and finite initial store content with probability one. It is recalled that $1/\mu_i$ is mean load size and λ_i is the arrival rate of material c_i , $i = 1, 2$. This is proved in Chapter IV. Here we assume that this condition is satisfied and obtain four functional equations involving the limits with respect to time of distribution components $\pi(t)$, $F(x,y;t)$, $G(y;t)$ and $H(x;t)$. Define

$$\begin{aligned} \pi &= \lim_{t \rightarrow \infty} \pi(t) & , \\ G(y) &= \lim_{t \rightarrow \infty} G(y;t) & , \\ H(x) &= \lim_{t \rightarrow \infty} H(x;t) & , \\ J(z) &= \lim_{t \rightarrow \infty} J(z;t) & , \\ F(x,y) &= \lim_{t \rightarrow \infty} F(x,y;t) & , \\ P(x,y) &= \lim_{t \rightarrow \infty} P_t(x,y) & . \end{aligned}$$

The limit as $t \rightarrow \infty$ of the individual terms of inequalities (2.2) and (2.3) allows us to write

$$(2.21) \quad \int [1 - e^{-\lambda\tau}] \pi/\tau = \int [G(\tau) + H(\tau) + J(\tau)] / \tau + o(1) ,$$

since the terms designated $o(\tau)$ in these inequalities are dominated by terms which are $o(\tau)$ uniformly in t . From (2.21) it is clear that

$$(2.22) \quad (\lambda_1 + \lambda_2) \pi = \lim_{\tau \downarrow 0+} \int [G(\tau) + H(\tau) + J(\tau)] / \tau \quad .$$

We denote the Laplace-Stieltjes transforms of the following components of the limiting distribution as

$$H^x(u) = \int_0^{\infty} e^{-ux} dH(x) \quad ,$$

$$G^y(v) = \int_0^{\infty} e^{-vy} dG(y) \quad ,$$

and

$$F^{xy}(u,v) = \int_0^{\infty} \int_0^{\infty} e^{-ux-vy} d_x d_y F(x,y) \quad .$$

The uniformity with respect to t , of the terms included in $o(\tau)$ in the inequalities (2.5) and (2.6), allows us to write the Laplace-Stieltjes transforms of the limits, as $t \rightarrow \infty$, of the inequalities as

$$(2.23) \quad H^x(u) \geq e^{(u-\lambda)\tau} \int_0^{\tau} H^x(u) - \int_0^{\tau} e^{-ux} dH(x) \quad] \\ + \lambda_1 \tau e^{-\lambda\tau} \int_0^{\tau} \pi + H^x(u) \quad] B_1^x(u) \quad -o(\tau)$$

and

$$(2.24) \quad H^x(u) \leq e^{(u-\lambda)\tau} \int_0^{\tau} H^x(u) - \int_0^{\tau} e^{-ux} dH(x) + \lambda_1 \tau \pi B_1^x(u) \\ + \lambda_1 \tau H^x(u) B_1^x(u) \quad] \quad + o(\tau) \quad .$$

Therefore

$$\begin{aligned}
 (2.25) \quad \lim_{\tau \downarrow 0+} \inf \left[\frac{e^{(u-\lambda)\tau}}{\tau} \int_0^{\tau} e^{-ux} dH(x) \right] \\
 \geq (u - \lambda) H^X(u) + \lambda_1 \int \pi + H^X(u) B_1^X(u) \\
 \geq \lim_{t \downarrow 0+} \sup \left[\frac{e^{(u-\lambda)\tau}}{\tau} \int_0^{\tau} e^{-ux} dH(x) \right] .
 \end{aligned}$$

The existence of the following limit has been demonstrated

$$\lim_{\tau \downarrow 0+} \left[\frac{e^{(u-\lambda)\tau}}{\tau} \int_0^{\tau} e^{-ux} dH(x) \right] = h, \text{ say .}$$

Furthermore, since

$$\frac{e^{-\lambda\tau}}{\tau} H(\tau) \leq \frac{e^{(u-\lambda)\tau}}{\tau} \int_0^{\tau} e^{-ux} dH(x) \leq \frac{e^{(u-\lambda)\tau}}{\tau} H(\tau)$$

it follows that

$$\lim_{\tau \downarrow 0+} \int H(\tau)/\tau = h .$$

Hence (2.25) reduces to

$$(2.26) \quad h = \int u - \lambda_1 - \lambda_2 + \lambda_1 B_1^X(u) H^X(u) + \lambda_1 \pi B_1^X(u) .$$

As before, we can write the Laplace-Stieltjes transforms of the limits, as $t \rightarrow \infty$, of inequalities (2.13) and (2.14) as

$$\begin{aligned}
(2.27) \quad F^{xy}(u,v) &\geq e^{(u-\lambda)\tau} \left[F^{xy}(u,v) - \int_0^\tau e^{-ux} d_x F^y(x,v) \right] \\
&\quad + e^{-\lambda\tau} \lambda_1 \tau \left[G^y(v) + F^{xy}(u,v) \right] B_1^x(u) \\
&\quad + e^{-\lambda\tau} \lambda_2 \tau \left[H^x(u) + F^{xy}(u,v) \right] B_2^y(v) - o(\tau),
\end{aligned}$$

and

$$\begin{aligned}
(2.28) \quad F^{xy}(u,v) &\leq e^{(u-\lambda)\tau} \left[F^{xy}(u,v) - \int_0^\tau e^{-ux} d_x F^y(x,v) \right] \\
&\quad + e^{(u+v-\lambda)\tau} \lambda_1 \tau \left[G^y(v) + F^{xy}(u,v) \right] B_1^x(u) \\
&\quad + e^{(u+v-\lambda)\tau} \lambda_2 \tau \left[H^x(u) + F^{xy}(u,v) \right] B_2^y(v) \\
&\quad + o(\tau).
\end{aligned}$$

Division of (2.27) and (2.28) by τ and the taking of limits as $\tau \downarrow 0+$ produces the bounds

$$\begin{aligned}
(2.29) \quad \liminf_{\tau \downarrow 0+} &\left[\frac{e^{(u-\lambda)\tau}}{\tau} \int_0^\tau e^{-ux} d_x F^y(x,v) \right] \\
&\geq \left[u - \lambda + \lambda_1 B_1^x(u) + \lambda_2 B_2^y(v) \right] F^{xy}(u,v) \\
&\quad + \lambda_1 B_1^x(u) G^y(v) + \lambda_2 B_2^y(v) H^x(u) \\
&\geq \limsup_{\tau \downarrow 0+} \left[\frac{e^{(u-\lambda)\tau}}{\tau} \int_0^\tau e^{-ux} d_x F^y(x,v) \right].
\end{aligned}$$

Thus we have verified the existence of

$$\lim_{\tau \downarrow 0+} \left[\frac{e^{(u-\lambda)\tau}}{\tau} \int_0^\tau e^{-ux} d_x F^y(x,v) \right] = f^y(v), \text{ say,}$$

and (2.29) may be rewritten

$$(2.30) \quad f^y(v) = \int [u - \lambda_1 - \lambda_2 + \lambda_1 B_1^x(u) + \lambda_2 B_2^y(v)] F^{xy}(u, v) \\ + \lambda_1 B_1^x(u) G^y(v) + \lambda_2 B_2^y(v) H^x(u) .$$

Consider the limit of the terms in (2.9) and (2.10) as $t \rightarrow \infty$. Due to the uniformity of convergence with respect to t of the terms $o(\tau)$ the Laplace-Stieltjes transforms of the limits are

$$(2.31) \quad G^y(v) \geq e^{(v-\lambda)\tau} \int_0^\tau G^y(v) - \int_0^\tau e^{-vy} dG(y) \\ + e^{-\lambda\tau} F^y(\tau, v) + \lambda_2 \tau e^{-\lambda\tau} \int [\pi + G^y(v)] B_2^y(v) \\ - o(\tau)$$

and

$$(2.32) \quad G^y(v) \leq e^{(v-\lambda)\tau} \left\{ G^y(v) - \int_0^\tau e^{-vy} dG(y) + F^y(\tau, v) - \int_0^\tau e^{-vy} d_y F(\tau, y) \right. \\ \left. + \lambda_2 \tau \int [\pi + G^y(v)] B_2^y(v) \right\} \\ + o(\tau) .$$

We note that

$$\frac{e^{-\lambda\tau} F^y(\tau, v)}{\tau} \leq \frac{e^{(u-\lambda)\tau}}{\tau} \int_0^\tau e^{-ux} d_x F^y(x, v) \leq \frac{e^{(u-\lambda)\tau} F^y(\tau, v)}{\tau} .$$

Hence

$$(2.33) \quad \lim_{\tau \downarrow 0+} \int F^y(\tau, v) / \tau = f^y(v) .$$

Thus, from (2.31) and (2.32)

$$\begin{aligned}
 (2.34) \quad \liminf_{\tau \downarrow 0+} & \left[\frac{e^{(v-\lambda)\tau}}{\tau} \int_0^\tau e^{-vy} dG(y) \right] \\
 & \geq \int v - \lambda + \lambda_2 B_2^y(v) G^y(v) + \lambda_2 \pi B_2^y(v) + F^y(v) \\
 & \geq \limsup_{\tau \downarrow 0+} \left[\frac{e^{(v-\lambda)\tau}}{\tau} \int_0^\tau e^{-vy} dG(y) \right] ,
 \end{aligned}$$

from which inequalities we can deduce the existence of

$$\lim_{\tau \downarrow 0+} \left[\frac{e^{(v-\lambda)\tau}}{\tau} \int_0^\tau e^{-vy} dG(y) \right] = \lim_{\tau \downarrow 0+} \int G(\tau)/\tau = g, \text{ say .}$$

Now $0 \leq J(\tau) \leq F(\tau, \tau)$ and so from (2.31) and (2.32) we obtain

$$\begin{aligned}
 0 & \leq e^{-\lambda\tau} J(\tau)/\tau \\
 & \leq e^{-\lambda\tau} F(\tau, \tau)/\tau \\
 & \leq \frac{e^{(v-\lambda)\tau}}{\tau} \int_0^\tau e^{-vy} d_y F(\tau, y) \\
 & \leq \int e^{(v-\lambda)\tau} G^y(v)/\tau + e^{(v-\lambda)\tau} \frac{1}{\tau} \int_0^\tau e^{-vy} dG(y) \\
 & \quad + e^{(v-\lambda)\tau} F(\tau, v)/\tau + \lambda_2 \int \pi + G^y(v) B_2^y(v) + o(1) ,
 \end{aligned}$$

where the limit, as $\tau \downarrow 0+$, of this upper bound is zero by virtue of

(2.34). It is now simple to deduce that

$$\lim_{\tau \downarrow 0+} \int J(\tau)/\tau = 0 .$$

Thus equation (2.22) reduces to

$$(2.35) \quad (\lambda_1 + \lambda_2) \pi = g + h .$$

We summarize the results contained in equations (2.26), (2.30), (2.34) and (2.35) in the following theorem.

Theorem 2.2. The components of the limiting distribution of the store content, under a preemptive rule of operation, satisfy the four functional equations ;

$$(2.36) \quad \pi = (g + h)/(\lambda_1 + \lambda_2) ,$$

$$(2.37) \quad H^x(u) = \int [h - \lambda_1 \pi B_1^x(u)] / \int [u - \lambda_1 - \lambda_2 + \lambda_1 B_1^x(u)] ,$$

$$(2.38) \quad G^y(v) = \int [g - f^y(v) - \lambda_2 \pi B_2^y(v)] / \int [v - \lambda_1 - \lambda_2 + \lambda_2 B_2^y(v)] ,$$

$$(2.39) \quad F^{xy}(u,v) = \int [f^y(v) - \lambda_1 B_1^x(u) G^y(v) - \lambda_2 B_2^y(v) H^x(u)] / \int [u - \lambda_1 - \lambda_2 + \lambda_1 B_1^x(u) + \lambda_2 B_2^y(v)] .$$

Corollary to Theorem 2.2 . The components of the limiting distribution of the store content are independent of the components of the initial distribution of the store content.

Proof of the Corollary. The components of the initial distribution of store content are $\pi(t)$, $H(x;t)$, $G(y;t)$ and $F(x,y;t)$ at $t = 0$. The proof of the corollary is completed by noting that these functions do not appear, either explicitly or implicitly, in the Laplace-Stieltjes transforms of the components of the limiting distribution, as given in equations (2.36) to (2.39).

The result of clearing of fractions in (2.36) to (2.39) and adding is

$$(2.40) \quad -\pi\phi + G^y(v) \int [v-\phi] + \int [H^x(u) + F^{xy}(u,v)] \int [u-\phi] = 0$$

where it is recalled that $\phi = \lambda_1 + \lambda_2 - \lambda_1 B_1^x(u) - \lambda_2 B_2^y(v)$.

4. Moments of the Limiting Distribution of Store Content.

Equation (2.40) will be employed in this section to obtain the first and second order moments of the limiting distribution of the vector of store content. It is possible to obtain the derivatives and hence the moments of the components of the distribution directly from the four equations (2.36) to (2.39). However, the unknown parameters g and h and the unknown function $f^y(v)$ make this approach lengthy and algebraically quite involved. Therefore we shall employ a less direct approach which employs equation (2.40) and simplifies the algebra immensely.

In order to shorten the notation let

$$G^y(v) = N(v) = N$$

and

$$H^x(u) + F^{xy}(u,v) = M(u,v) = M.$$

Thus we may write (2.40) as

$$(2.41) \quad M(u - \phi) + N(v - \phi) - \pi\phi = 0.$$

Also, in the remainder of this study, we will designate the order of partial derivatives with respect to u and v by post superscript

numerals. For example

$$M^{11} = \frac{\partial^2 M}{\partial u \partial v}$$

and

$$M^{12} = \frac{\partial^3 M}{\partial u \partial v^2}$$

The function, without a post superscript, will be the function without any operators. Also note that

$$M^{01} = \frac{\partial M}{\partial v}$$

as opposed to

$$M^{10} = \frac{\partial M}{\partial u}$$

Evaluation of any function of u and v at $u = v = 0$ will be designated by a post subscript zero, e.g.

$$M_o^{10} = \frac{\partial M}{\partial u}, \text{ evaluated at } u = v = 0.$$

We recall that the load size of material c_i is the random variable S_i with distribution function $B_i(s)$ and mean $1/\mu_i$, $i = 1, 2$. Thus

$$\phi_o^{10} = \lambda_1 E(S_1) = \lambda_1 / \mu_1 = m_1, \text{ say};$$

$$\phi_o^{20} = -\lambda_1 E(S_1^2) = -m_2, \text{ say};$$

$$\phi_o^{30} = \lambda_1 E(S_1^3) = m_3, \text{ say};$$

$$\phi_o^{01} = \lambda_2 E(S_2) = \lambda_2 / \mu_2 = n_1, \text{ say};$$

$$\phi_o^{02} = -\lambda_2 E(S_2^2) = -n_2, \text{ say};$$

$$\phi_o^{03} = \lambda_2 E(S_2^3) = n_3, \text{ say}.$$

Note that $\phi^{ij} = 0$ for both i and $j \geq 1$ and that $N^{ij} = 0$ for $i \geq 1$.

Theorem 2.3. The limiting distribution of store content satisfies the equations:

$$(2.42) \quad P(X = Y = 0) = \pi = 1 - m_1 - n_1 ,$$

$$(2.43) \quad P(X = 0) = 1 - m_1 ,$$

$$(2.44) \quad P(Y = 0) = \pi u^*/\lambda_2$$

where u^* is the unique positive real root of the equation

$$u - (\lambda_1 + \lambda_2) + \lambda_1 B_1^X(u) = 0 .$$

Proof. The first order partial derivatives of the terms of equation (2.41), with respect to u and v successively, are

$$(u - \phi)M^{10} + (1 - \phi^{10})M - \phi^{10}(N + \pi) = 0$$

and

$$(u - \phi)M^{01} - \phi^{01}M + (1 - \phi^{01})N + (v - \phi)N^{01} - \phi^{01}\pi = 0 .$$

These equations, evaluated at $u = v = 0$, are

$$(1 - m_1)M_0 - m_1N_0 - m_1\pi = 0$$

and

$$-n_1M_0 + (1 - n_1)N_0 - n_1\pi = 0 .$$

Also, we have

$$M_0 + N_0 + \pi = 1 ,$$

because we have assumed that the necessary and sufficient conditions

for the existence of the limiting distribution are satisfied. Addition of the respective sides of these three equations shows that

$$P(X = Y = 0) = \pi = 1 - m_1 - n_1 .$$

Next we remark that

$$\begin{aligned} P(X = 0) &= \pi + \int_0^{\infty} d G(y) \\ &= \pi + N_0 \\ &= 1 - m_1 - n_1 + N_0 \\ &= 1 - m_1 , \end{aligned}$$

since putting $\lambda_2 = 0$ has no effect on the store content of material c_1 . Thus

$$P(X = 0, Y > 0) = N_0 = n_1 .$$

Consider equation (2. 7). The denominator of the right hand side

$$u - \lambda_1 - \lambda_2 + \lambda_1 B_1^x(u) = D(u), \text{ say ,}$$

has derivatives

$$D^{10} = 1 - \lambda_1 \int_0^{\infty} e^{-ux} x d B_1(x)$$

and

$$D^{20} = \lambda_1 \int_0^{\infty} e^{-ux} x^2 d B_1(x) > 0 \text{ for } u \geq 0 .$$

Moreover $D_0 = -\lambda_1$ and $D(\infty) = \infty$. Thus $D(u)$ is a monotone increasing function as u goes from 0 to ∞ . Hence $D(u) = 0$ has a unique positive real root, say u^* . Now $H^x(u)$ is the generating

function of a component of a bivariate distribution and therefore is bounded for $u \geq 0$. Thus substitution of $u = u^*$ in (2.37) must yield an indeterminate form, in other words

$$h - \lambda_1 \pi B_1^x(u^*) = 0 .$$

Thus

$$H^x(u) = \lambda_1 \pi \frac{B_1^x(u^*) - B_1^x(u)}{u - \lambda_1 - \lambda_2 + \lambda_1 B_1^x(u)} .$$

Therefore

$$\begin{aligned} P(Y = 0) &= \pi + H^x(0) \\ &= \pi + \lambda_1 \pi \frac{B_1^x(u^*) - 1}{-\lambda_2} \\ &= \pi u^* / \lambda_2 . \end{aligned}$$

This completes the proof of the theorem.

Corollary to Theorem 2.3 . The Laplace-Stieltjes transform of $H(x)$ depends on the load size distribution of non-priority material c_2 only through the first moment. The function $H(x)$ may be obtained in explicit form by inversion of its transform when the form of $B_2(s)$ is unknown.

Proof. The proof of the corollary is an evident result of the equation

$$H^x(u) = \lambda_1 \pi \frac{B_1^x(u^*) - B_1^x(u)}{u - \lambda_1 - \lambda_2 + \lambda_1 B_1^x(u)} ,$$

where u^* is the unique positive root of the denominator.

We desire the moments of the store content (X, Y) under the

limiting distribution. It is clear that

$$(2.45) \quad E(X^i Y^j) = (-1)^{i+j} (M_0^{ij} + N_0^{ij}) \\ = (-1)^{i+j} \frac{\partial^{i+j}}{\partial u^i \partial v^j} \int \pi + G^Y(v) + H^X(u) + F^{XY}(u,v) \Big|,$$

evaluated at $u = v = 0$. Consider the second order partial derivative of (2.41) with respect to u . This is

$$M^{20}(u-\phi) + 2M^{10}(1-\phi^{10}) + (-\phi^{20}) (M + N + \pi) = 0 ,$$

and on evaluation at $u = v = 0$ it reduces to

$$-M_0^{10} = E(X) = m_2/2 (1 - m_1) .$$

The reader will recognize this formula as the expected waiting time, under the limiting distribution, of an arriving customer in a single server head-of-the-line queue with random arrivals, a result that is easily obtained from formula (19) Takacs [13] page 110 .

We also derive from (2.41) the equation

$$M^{02}(u-\phi) + 2M^{01}(-\phi^{01}) + N^{02}(v-\phi) \\ + 2N^{01}(1-\phi^{01}) + (M + N + \pi) (-\phi^{02}) = 0 ,$$

and

$$M^{11}(u-\phi) + M^{10}(-\phi^{01}) + M^{01}(1-\phi^{10}) + N^{01}(-\phi^{10}) = 0 .$$

These equations give

$$\begin{aligned}
E(Y) &= -(M_0^{01} + N_0^{01}) \\
&= \int 2n_1 E(X) + n_2 \gamma / 2\pi \\
&= \int n_1 m_2 / (1-m_1) + n_2 \gamma / 2 \int \sqrt{1-m_1-n_1} \gamma .
\end{aligned}$$

Theorem 2.4. The first and second order moments of the limiting distribution of store content are:

$$(2.46) \quad E(X) = m_2 / 2(1 - m_1) \quad ;$$

$$(2.47) \quad E(Y) = \int n_1 m_2 / (1-m_1) + n_2 \gamma / 2\pi \quad ;$$

$$(2.48) \quad \text{Var}(X) = m_3 / 3(1-m_1) + \int m_2 / 2(1-m_1) \gamma^2 \quad ;$$

$$(2.49) \quad \text{Cov}(X, Y) = \int m_3 / 3(1-m_1) + \frac{1}{2}(m_2 / (1-m_1))^2 \gamma \int n_1 / 2(1-m_1) \gamma \quad ;$$

$$\begin{aligned}
(2.50) \quad \text{Var}(Y) &= n_1^2 m_3 / 3\pi(1-m_1)^2 + n_1^2 m_2^2 / 2\pi(1-m_1)^3 \\
&\quad + \int m_2 n_1 / 2\pi(1-m_1) \gamma^2 + \int n_2 / 2\pi \gamma^2 \\
&\quad + m_2 n_2 / 2\pi^2 + n_3 / 3\pi .
\end{aligned}$$

Proof. Equations (2.46) and (2.47) have been verified already. The third order partial derivative of equation (2.41),

$$(u-\phi)M^{21} + M^{20}(-\phi^{01}) + 2(1-\phi^{10})M^{11} + (M^{01} + N^{01})(-\phi^{20}) = 0$$

evaluated at $u = v = 0$ reduces to

$$(2.51) \quad M_0^{11} = E(XY) = n_1 E(X^2)/2(1-m_1) + E(X) E(Y) .$$

The third order partial derivative of (2.41),

$$(u-\phi)M^{30} + 3(1-\phi^{10})M^{20} + 3(-\phi^{20})M^{10} + (M + N + \pi) (-\phi^{30}) = 0 ,$$

evaluated at $u = v = 0$, reduces to

$$(2.52) \quad E(X^2) = m_3/3(1-m_1) + \frac{1}{2} \int m_2/(1 - m_1) \bar{7}^2 .$$

Consider the two remaining third order partial derivatives of (2.41),

$$(u-\phi)M^{12} + 2(-\phi^{01})M^{11} + (-\phi^{02})M^{10} + (1-\phi^{10})M^{02} + (-\phi^{10})N^{02} = 0$$

and

$$(u-\phi)M^{03} + 3(-\phi^{01})M^{02} + 3(-\phi^{02})M^{01} + (-\phi^{03})(M+N+\pi) \\ + (v-\phi)N^{03} + 3(1-\phi^{01})N^{02} + 3(-\phi^{02})N^{01} = 0 .$$

Addition of the first equation to one-third of the second equation and evaluation at $u = v = 0$ results in the equation

$$(2.53) \quad \pi \int M_0^{02} + N_0^{02} \bar{7} = \pi E(Y^2) \\ = 2n_1 E(XY) + n_2 E(X+Y) + n_3/3 .$$

Straightforward manipulation of results (2.51), (2.52) and (2.53) establishes the three results (2.48), (2.49) and (2.50) and the theorem is proved.

Theorem 2.5. If the first r moments exist for the load size distribution functions $B_1(s)$ and $B_2(s)$, then the first $(r-1)$ -th

order moments $E(X^{i-1} Y^{r-i})$, $i = 1, 2, \dots, r$ of the store content exist for the limiting distribution. Also, if the first $(r-1)$ -th order moments of the store content exist, they are derivable from equation (2.41) in a manner analogous to the derivation of the first and second order moments.

Proof. We will prove this theorem by induction on r . Assume that the theorem is valid for $r - 2$ ($r > 0$). Write (2.41) as

$$\int M + N + \pi \phi = u M + v N .$$

The $(r+1)$ partial derivatives with respect to u and v of the r -th order are

$$(2.54) \quad \sum_{j=0}^i \sum_{k=0}^{r-i} \frac{i!}{j!(i-j)!} \cdot \frac{(r-i)!}{k!(r-i-k)!} \frac{\partial^{r-k-j} \int M + N + \pi \phi}{\partial u^{i-j} \partial v^{r-i-k}} \phi^{ij}$$

$$= u M^{i,r-i} + v N^{i,r-i} + i M^{i-1,r-i} + (r-i) N^{i,r-i-1} ,$$

for $i = 0, 1, 2, \dots, r$. Recall that $\phi^{jk} = 0$ when both $j \geq 1$, $k \geq 1$, that $N^{i,r-i} = 0$ for $i \geq 1$ and that $\phi_0 = 0$. Thus for $u = v = 0$, $i = 0$ we have

$$\sum_{k=2}^r \frac{r!}{k!(r-k)!} (-1)^{r-k} E(Y^{r-k}) \phi_0^{ok}$$

$$= r N_0^{0,r-1} - r \phi_0^{01} (-1)^{r-1} E(Y^{r-1}) ,$$

and for $i = 1$

$$\begin{aligned} \sum_{k=1}^{r-1} \frac{(r-1)!}{k! (r-k-1)!} (-1)^{r-k} E(X Y^{r-k-1}) \phi_0^{ok} \\ = M_0^{o,r-1} - (-1)^{r-1} \phi_0^{10} E(Y^{r-1}) . \end{aligned}$$

Division of the first equation by r and addition of the two results in the equations leads to

$$\begin{aligned} f_1 &= (-1)^{r-1} (1 - \phi_0^{10} - \phi_0^{01}) E(Y^{r-1}) \\ &= (-1)^{r-1} (1 - m_1 - n_1) E(Y^{r-1}) , \end{aligned}$$

where f_1 is a function which is linear in the moments of store content of order $r-2$ or less and linear in the first r moments of the load size distributions.

Evaluation at $u = v = 0$ of the last $r-1$ equations ($i = 2, 3, \dots, r$) of the set (2.54) leads to

$$\begin{aligned} (2.55) \quad \sum_{j=1}^i \frac{i!}{j! (i-j)!} (-1)^{r-j} E(X^{i-j} Y^{r-i}) \phi_0^{j0} \\ + \sum_{k=1}^{r-i} \frac{(r-i)!}{k! (r-i-k)!} (-1)^{r-k} E(X^i Y^{r-i-k}) \phi_0^{ok} \\ = (-1)^{r-1} \cdot i \cdot E(X^{i-1} Y^{r-i}) , \quad i = 2, 3, \dots, r . \end{aligned}$$

After transferring all store content moments of order $(r-1)$ to the right hand side, this set of equations together with the immediately preceding single equation may be written in matrix form as

$$(2.56) \quad \begin{bmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ f_r \end{bmatrix} = (-1)^{r-1} \begin{bmatrix} 1-m_1-n_1, (1-r)n_1, 0, \dots, 0 \\ 0, 2(1-m_1), (2-r)n_1, \dots, 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0, \dots, 0, r(1-m_1) \end{bmatrix} \begin{bmatrix} E(Y^{r-1}) \\ E(XY^{r-2}) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ E(X^{r-1}) \end{bmatrix} .$$

The functions f_1, f_2, \dots, f_r are linear in the moments of store content of order $r-2$ or less and linear in the first r moments of the load size distributions. Thus, by assumption f_1, f_2, \dots, f_r are completely determined. The coefficient matrix on the right hand side of (2.56) is non-singular provided the diagonal elements, which are the latent roots, are non-zero. That is, provided

$$1 - m_1 - n_1 \neq 0$$

and

$$1 - m_1 \neq 0 .$$

Both of these conditions are satisfied if $m_1 + n_1 < 1$. This is a necessary condition for the limiting distribution of store content to exist, thus, we are assuming that this condition is satisfied throughout this chapter. The necessity of this condition will be proved in Chapter IV. The mathematical induction is completed by

reference to the results (cf. Theorem 2.4) for the first and second order moments of the store content. This completes the proof of Theorem 2.5.

5. Moments of the Limiting Distribution of Store Content for $k > 2$. The extension of the methods of this chapter, Sections 1 to 4, to a storage system with k types of material, $k > 2$, is straightforward. However, the number of equations involving the Laplace-Stieltjes transforms of the limiting distribution components (cf. equations (2.36) to (2.39)) is 2^k for a k -level preemptive storage system. The volume of work involved in rigorously deriving these equations rapidly becomes impractical. Even the algebra involved in obtaining moments of the limiting distribution from an equation similar to (2.41) would prove lengthy and tedious. However, a method exists for obtaining the moments of the limiting distribution of store content for $k > 2$, by employing the results for $k = 2$. In this section we will describe briefly the method for general k , and illustrate it for the case $k = 4$.

Consider a store with random arrivals at rates $\lambda_1, \lambda_2, \dots, \lambda_k$ of k types of material c_1, c_2, \dots, c_k with a preemptive rule of operation. The load sizes have distribution functions $B_1(s), B_2(s), \dots, B_k(s)$ with expected load sizes $1/\mu_1, 1/\mu_2, \dots, 1/\mu_k$. Define

$$\bar{\lambda}_i = \lambda_1 + \lambda_2 + \dots + \lambda_{i-1} \quad ,$$

and

$$\tau_j = \lambda_j / \bar{\lambda}_i, \quad j \leq i-1, i = 2, 3, \dots, k.$$

Also, let

$$m_{ij} = \lambda_i E(s_i^j), \quad j = 1, 2, \dots; i = 1, 2, \dots, k.$$

The essential point, in considering the store content of material c_i , is that only two groups of material types need be considered. These are (a) the group of materials c_1, c_2, \dots, c_{i-1} with preemptive rights over c_i , and (b) material type c_i . Let \bar{S}_i be the random variable of a load size of material in group c_1, c_2, \dots, c_{i-1} . Thus \bar{S}_i is a random variable with distribution function $\tau_1 B_1(s) + \tau_2 B_2(s) + \dots + \tau_{i-1} B_{i-1}(s)$ and random occurrence rate $\bar{\lambda}_i$. Thus

$$\begin{aligned} \bar{\lambda}_i E(\bar{S}_i^j) &= \bar{\lambda}_i \left[\tau_1 E(S_1^j) + \tau_2 E(S_2^j) + \dots + \tau_{i-1} E(S_{i-1}^j) \right] \\ &= m_{1j} + m_{2j} + \dots + m_{i-1,j} \\ &= \bar{m}_{ij}, \text{ say.} \end{aligned}$$

Let $Z_k = (X_1, X_2, \dots, X_k)$ be the random vector of the store content for k types of material. Define \bar{X}_i as the random variable, under the limiting distribution, of the store content of the group c_1, c_2, \dots, c_{i-1} of materials. Therefore

$$\bar{X}_i = \sum_{j=1}^{i-1} X_j.$$

Theorem 2.6. The first and second moments of the store content,

\bar{X}_i and X_i , under the limiting distribution, are:

$$(2.57) \quad E(\bar{X}_i) = \bar{m}_{i2}/2(1 - \bar{m}_{i1}) \quad ,$$

$$(2.58) \quad E(\bar{X}_i^2) = \bar{m}_{i3}/3(1 - \bar{m}_{i1}) + \frac{1}{2} \int \bar{m}_{i2}/(1 - \bar{m}_{i1}) \tau^2 \quad ,$$

$$(2.59) \quad E(X_i) = \int m_{i1} \bar{m}_{i2}/(1 - \bar{m}_{i1}) + m_{i2} \tau/2(1 - \bar{m}_{i1} - m_{i1}) \quad ,$$

$$(2.60) \quad E(X_i^2) = m_{i1}^2 \bar{m}_{i3}/3(1 - \bar{m}_{i1} - m_{i1})(1 - \bar{m}_{i1})^2 \\ + m_{i1}^2 \bar{m}_{i2}^2/2(1 - \bar{m}_{i1} - m_{i1})(1 - \bar{m}_{i1})^3 \\ + \frac{1}{2} \int m_{i1} \bar{m}_{i2}/2(1 - \bar{m}_{i1} - m_{i1})(1 - \bar{m}_{i1}) \tau^2 \\ + m_{i1} \bar{m}_{i2} m_{i2}/2(1 - \bar{m}_{i1} - m_{i1})^2(1 - \bar{m}_{i1}) \\ + \bar{m}_{i2} m_{i2}/2(1 - \bar{m}_{i1} - m_{i1})^2 \\ + \frac{1}{2} \int m_{i2}/(1 - \bar{m}_{i1} - m_{i1}) \tau^2 \\ + m_{i3}/3(1 - \bar{m}_{i1} - m_{i1}) \quad ,$$

$$(2.61) \quad E(\bar{X}_i X_i) = m_{i1} E(\bar{X}_i^2)/2(1 - \bar{m}_{i1}) + E(\bar{X}_i) E(X_i) \quad .$$

Proof. Formulae (2.57) and (2.58) are obvious extensions of formulae (2.46) and (2.47), respectively, when arrivals in the group c_1, c_2, \dots, c_{i-1} are considered as having a load size distribution function $\tau_1 B_1(s) + \tau_2 B_2(s) + \dots + \tau_{i-1} B_{i-1}(s)$. Similarly, formulae

(2.58), (2.60) and (2.61) are obtained directly from (2.52), (2.53) and (2.51), respectively.

Suppose we now consider arrivals in the group $c_i, c_{i+1}, \dots, c_{j-1}$ with load size $\bar{S}_j(i)$ with distribution function $\tau_i B_i(s) + \tau_{i+1} B_{i+1}(s) + \dots + \tau_{j-1} B_{j-1}(s)$. The arrivals occur with random rate $\lambda_i + \lambda_{i+1} + \dots + \lambda_{j-1} = \bar{\lambda}_j(i)$, say. Thus

$$\begin{aligned} \bar{\lambda}_j(i) E \left[(\bar{S}_j(i))^r \right] &= m_{ir} + m_{i+1,r} + \dots + m_{j-1,r} \\ &= \bar{m}_{jr} - \bar{m}_{ir} \\ &= \bar{m}_{jr}(i), \text{ say.} \end{aligned}$$

Therefore the store content

$$X_i + X_{i+1} + \dots + X_{j-1} = \bar{X}_j(i), \text{ say,}$$

has moments;

$$(2.62) \quad E(\bar{X}_j(i)) = \left[\bar{m}_{j1}(i) \bar{m}_{i2} / (1 - \bar{m}_{i1}) + \bar{m}_{j2}(i) \right] / 2(1 - \bar{m}_{j1}),$$

$$(2.63) \quad E(\bar{X}_i \bar{X}_j(i)) = \bar{m}_{j1}(i) E(\bar{X}_i^2) / 2(1 - \bar{m}_{i1}) \\ + E(\bar{X}_i) E(\bar{X}_j(i)),$$

and

$$\begin{aligned}
(2.64) \quad E \left[(\bar{X}_j(i))^2 \right] &= (\bar{m}_{j1}(i))^2 \bar{m}_{j3} / 3(1 - \bar{m}_{j1})(1 - \bar{m}_{i1})^2 \\
&+ (\bar{m}_{j1}(i))^2 \bar{m}_{i2}^2 / 2(1 - \bar{m}_{j1})(1 - \bar{m}_{i1})^3 \\
&+ \frac{1}{2} \left[\bar{m}_{j1}(i) \bar{m}_{i2} / (1 - \bar{m}_{j1})(1 - \bar{m}_{i1}) \right]^2 \\
&+ \bar{m}_{j1}(i) \bar{m}_{i2} \bar{m}_{j2}(i) / 2(1 - \bar{m}_{j1})^2(1 - \bar{m}_{i1}) \\
&+ \bar{m}_{i2} \bar{m}_{j2}(i) / 2(1 - \bar{m}_{j1})^2 \\
&+ \frac{1}{2} \left[\bar{m}_{j2}(i) / (1 - \bar{m}_{j1}) \right]^2 \\
&+ \bar{m}_{j3}(i) / 3(1 - \bar{m}_{j1}) \quad .
\end{aligned}$$

These are a direct consequence of Theorem 2.6. Formula (2.64) may be written more simply, from equation (2.53), as

$$\begin{aligned}
E \left[(\bar{X}_j(i))^2 \right] &= \left[2\bar{m}_{j1}(i) E(\bar{X}_i \bar{X}_j(i)) + \bar{m}_{j2}(i) E(\bar{X}_{j+1}) \right] \\
&+ \bar{m}_{j3}(i) / (1 - \bar{m}_{j1}) \quad .
\end{aligned}$$

We note that all cross product second moments of the limiting distribution of store content, for a k-level preemptive storage system, may be obtained directly from formula (2.64). For example, consider

$$E(X_i X_j) \quad , \quad 1 \leq i < j \leq k \quad .$$

We have

$$\begin{aligned}
(2.65) \quad E(X_i X_j) &= E[(X_i + X_{i+1} + \dots + X_{j-1}) X_j - (X_{i+1} + \dots + X_{j-1}) X_j] \\
&= E[\bar{X}_j(i) X_j - \bar{X}_j(i+1) X_j] \\
&= \frac{1}{2} E[(\bar{X}_{j+1}(i))^2 - (\bar{X}_j(i))^2 - 2X_j^2 \\
&\quad + (\bar{X}_{j+1}(i+1))^2 - (\bar{X}_j(i+1))^2]
\end{aligned}$$

For the case $k = 4$, the 6 cross products second moments are, from (2.65);

$$E(X_1 X_2) = \frac{1}{2} E[(X_1 + X_2)^2 - X_1^2 - X_2^2],$$

$$E(X_1 X_3) = \frac{1}{2} E[(X_1 + X_2 + X_3)^2 - (X_1 + X_2)^2 - 2X_3^2 + (X_2 + X_3)^2 - X_2^2],$$

$$\begin{aligned}
E(X_1 X_4) &= \frac{1}{2} E[(X_1 + X_2 + X_3 + X_4)^2 - (X_1 + X_2 + X_3)^2 - 2X_4^2 + (X_2 + X_3 + X_4)^2 \\
&\quad - (X_2 + X_3)^2],
\end{aligned}$$

$$E(X_2 X_3) = \frac{1}{2} E[(X_2 + X_3)^2 - X_2^2 - X_3^2],$$

$$E(X_2 X_4) = \frac{1}{2} E[(X_2 + X_3 + X_4)^2 - (X_2 + X_3)^2 - 2X_4^2 + (X_3 + X_4)^2 - X_3^2],$$

$$E(X_3 X_4) = \frac{1}{2} E[(X_3 + X_4)^2 - X_3^2 - X_4^2].$$

The reader will recognize that this is not the only algorithm that is valid for obtaining all of the cross product second moments for $k = 4$. For example, $E(X_1 X_2)$ could be evaluated directly from equation (2.51).

It will also be noticed that the terms involving only X_1 , X_2 and X_3 are valid for $k = 3$ and all six terms are valid for $k > 4$. These are direct results of the fact that the store content X_i of material c_i is not affected by the store content of materials c_{i+1} , c_{i+2} , ..., c_k .

CHAPTER III

QUEUEING WITH BREAKDOWN

1. Introduction. In Section 8 of Chapter I we noted the analogy between a two level queue with a preemptive discipline and a head-of-the-line queue with a single server which is subject to breakdowns. White and Christie [14] derived the stationary moment generating function of the queue size distribution for random breakdowns and arrivals, and negative exponential repair and service times. In this Chapter we will employ the methods developed in Chapter II to consider a single server queue with random breakdowns and arrivals, and general repair and service times. The types of breakdown of a service mechanism considered will be:

- (a) Active Unit Breakdowns. These are breakdowns that occur only when the service mechanism is actively servicing a customer.
- (b) Active Component Breakdowns. These are breakdowns of components of the service mechanism and occur either during operation of the service mechanism (servicing a customer) or during repair time of another component. The service mechanism is considered inoperative while any component is being repaired.

- (c) Independent Unit Breakdowns. These are breakdowns that occur at any time except when the service mechanism is being repaired.
- (d) Independent Component Breakdowns. These are breakdowns that can occur at any time. The service mechanism is said to be in a state of breakdown when one or more components is being repaired.

A customer whose service is interrupted by a breakdown can reenter the service mechanism according to one of the three preemptive queue disciplines (defined in Chapter I):

- (i) preemptive resume;
- (ii) preemptive repeat (identical);
- (iii) preemptive repeat (different) .

We will consider the three preemptive disciplines with respect to each of the four types of queue breakdowns. In the study of component breakdown it will be assumed that the service mechanism has an infinite number of components which are subject to breakdown. This assumption is not absurd when considering some of the modern electronic equipment such as the large electronic computers. A word about the preemptive repeat disciplines also seems appropriate here. A program that is being processed on a computer may need to be repeated entirely if a computer breakdown occurs. This example of a preemptive repeat (identical) discipline was noted by Gaver [2]. The dialing of a telephone number with random access to a circuit for each digit dialed, where the circuit equipment is subject to

breakdown, is an example of a preemptive repeat (different) discipline (provided the dialer does not hang up the telephone in disgust when a breakdown occurs).

Independent component breakdown with a preemptive resume discipline is mathematically equivalent to the storage problem of Chapter II. This equivalence is:

- (i) the arrivals of loads of material c_1 at random rate λ_1 with load size distribution function $B_1(s)$ are considered as occurrences of component breakdowns with repair time distribution function $B_1(s)$.
- (ii) the arrivals of loads of material c_2 at random rate λ_2 with load size distribution function $B_2(s)$ are considered as the arrivals of customers with service time distribution function $B_2(s)$.
- (iii) X_t^- , the store content of material c_1 , is equivalent to the uninterrupted repair time required to make the service mechanism operative provided no additional breakdowns occur. We call X_t^- the unexpired repair time.
- (iv) Y_t^- , the store content of material c_2 , is equivalent to the clearing time of all customers in the queue, i.e. the time required to serve all the customers in the system provided no additional breakdowns occur and the service mechanism is immediately repaired.

Thus the results of Chapter II apply, without change, to this type of breakdown.

Suppose we consider independent component breakdown with a preemptive resume discipline. Let $Z_t = (X_t, Y_t)$ be the unexpired repair and clearing times of the queue. Further, suppose that the random rate of breakdown and the repair time distribution both depend on the status of the stochastic process $Z_t = (X_t, Y_t)$. That is;

if $X_t = 0, Y_t = 0$ breakdowns occur at rate λ_{10} with repair time d.f. $B_{10}(s)$,

if $X_t > 0, Y_t = 0$ breakdowns occur at random rate λ_{11} with repair time d.f. $B_{11}(s)$,

if $X_t = 0, Y_t > 0$ breakdowns occur at random rate λ_{12} with repair time d.f. $B_{12}(s)$, and

if $X_t > 0, Y_t > 0$ breakdowns occur at random rate λ_{13} with repair time d.f. $B_{13}(s)$.

At all times, customers arrive at random rate λ_2 with service time d.f. $B_2(s)$. Assume that

$$B_{10}(0+) = B_{11}(0+) = B_{12}(0+) = B_{13}(0+) = B_2(0+) = 0.$$

Then a completely analogous derivation to that employed for equation (2.20) shows that the transient behavior of the bivariate Laplace-Stieltjes transform of the distribution function of $Z_t = (X_t, Y_t)$ satisfies

$$\begin{aligned}
(3.1) \quad \frac{\partial P_t^{xy}(u,v)}{\partial t} &= \pi(t) \int [-\lambda_{10} - \lambda_2 + \lambda_{10} B_{10}^x(u) + \lambda_2 B_2^y(v)] \\
&+ H^x(u;t) \int [u - \lambda_{11} - \lambda_2 + \lambda_{11} B_{11}^x(u) + \lambda_2 B_2^y(v)] \\
&+ G^y(v;t) \int [v - \lambda_{12} - \lambda_2 + \lambda_{12} B_{12}^x(u) + \lambda_2 B_2^y(v)] \\
&+ F^{xy}(u,v;t) \int [u - \lambda_{13} - \lambda_2 + \lambda_{13} B_{13}^x(u) + \lambda_2 B_2^y(v)] .
\end{aligned}$$

The notation is identical to that defined in Chapter II. This equation does not appear particularly enlightening or useful in this form, except to indicate the complexity of the transient behavior.

Suppose that the stochastic process $Z_t = (X_t, Y_t)$ has a limiting bivariate distribution and $Z = (X, Y)$ is a random vector with this distribution. We then speak of (X, Y) as the unexpired repair time and clearing time of the queuing system. A completely analogous argument to that employed in the derivation of equations (2.36) to (2.39) demonstrates that the components of the distribution function of $Z = (X, Y)$ satisfy the functional equations;

$$(3.2) \quad \pi = (g + h) / (\lambda_{10} + \lambda_2) ,$$

$$(3.3) \quad H^x(u) = \int [h - \lambda_{10} \pi B_{10}^x(u)] / \int [u - \lambda_{11} - \lambda_2 + \lambda_{11} B_{11}^x(u)] ,$$

$$(3.4) \quad G^y(v) = \int [g - \lambda_{12} \pi B_{12}^y(v)] / \int [v - \lambda_{12} - \lambda_2 + \lambda_{12} B_{12}^y(v)] ,$$

$$\begin{aligned}
(3.5) \quad F^{xy}(u,v) &= \int [F^y(v) - \lambda_{12} B_{12}^x(u) G^y(v) - \lambda_2 B_2^y(v) H^x(u)] / \\
&\int [u - \lambda_{13} - \lambda_2 + \lambda_{13} B_{13}^x(u) + \lambda_2 B_2^y(v)] .
\end{aligned}$$

The equation analogous to (2.40) is therefore

$$(3.6) \quad \left. \begin{aligned} & \pi \int \left[-\lambda_{10} - \lambda_2 + \lambda_{10} B_{10}^X(u) + \lambda_2 B_2^Y(v) \right] \\ & + H^X(u) \int \left[u - \lambda_{11} - \lambda_2 + \lambda_{11} B_{11}^X(u) + \lambda_2 B_2^Y(v) \right] \\ & + G^Y(v) \int \left[v - \lambda_{12} - \lambda_2 + \lambda_{12} B_{12}^X(u) + \lambda_2 B_2^Y(v) \right] \\ & + F^{XY}(u, v) \int \left[u - \lambda_{13} - \lambda_2 + \lambda_{13} B_{13}^X(u) + \lambda_2 B_2^Y(v) \right] \end{aligned} \right\} = 0$$

Equations (3.1) to (3.6) apply to the distribution of unexpired repair time and clearing time for the four different types of breakdown, under a preemptive resume discipline, if

$B_{10}(s) = B_{11}(s) = B_{12}(s) = B_{13}(s) = B_1(s)$ and the rates of occurrence of breakdown, λ_{10} , λ_{11} , λ_{12} and λ_{13} take on values in accordance with table 3.1.

Type of Breakdown	λ_{10}	λ_{11}	λ_{12}	λ_{13}
(a) Active Unit	0	0	λ_1	0
(b) Active Component	0	0	λ_1	λ_1
(c) Independent Unit	λ_1	0	λ_1	0
(d) Independent Component	λ_1	λ_1	λ_1	λ_1

TABLE 3.1

2. Unexpired Repair and Clearing Times for Unit Breakdown.

Consider equation (3.6) for the case of active unit breakdown. This is

$$(3.7) \quad \pi \int_{-\lambda_2 + \lambda_2 B_2^Y(v)} + \int_{H^X(u) + F^{XY}(u,v)} \int_{u - \lambda_2 + \lambda_2 B_2^Y(v)} \\ + G^Y(v) \int_{v - \lambda_1 - \lambda_2 + \lambda_1 B_1^X(u) + \lambda_2 B_2^Y(v)} = 0 .$$

For this type of breakdown it is clear that $P\{X > 0, Y = 0\} = 0$ since breakdowns only occur when $Y > 0$ and X is reduced to zero while holding Y constant. Thus $H^X(u) \equiv 0$ and may be omitted from equation (3.7).

Theorem 3.1. The random vector (X, Y) of the unexpired repair and clearing times for active unit breakdown with a preemptive resume discipline has the following properties;

$$(3.8) \quad P(X = Y = 0) = \pi = 1 - n_1(1 + m_1) ,$$

$$(3.9) \quad P(X = 0, Y > 0) = n_1 ,$$

$$(3.10) \quad P(X > 0, Y = 0) = 0 ,$$

$$(3.11) \quad P(X > 0, Y > 0) = m_1 n_1 ,$$

$$(3.12) \quad E(X) = n_1 m_2 / 2 ,$$

$$(3.13) \quad E(Y) = \int_{n_1^2 m_2 + (1 + m_1)m_2} / 2\pi ,$$

$$(3.14) \quad E(X^2) = n_1 m_3 / 3 ,$$

$$(3.15) \quad E(XY) = \int_{n_1^2 m_3 / 6} + \int_{m_2(n_2 + m_2 n_1^3) / 2\pi} ,$$

$$(3.16) \quad E(Y^2) = 2n_1 E(XY) + n_2 E(X) + \int_{2n_1^2 E(XY) + n_2 E(X+Y) + n_3 / 3} \\ \cdot \int_{1+\pi} / \pi .$$

The notation is the same as defined in the previous chapter. The repair time S_1 has distribution function $B_1(s)$ and the service time S_2 has distribution function $B_2(s)$. Thus

$$\lambda_1 E(S_1^j) = m_j, \quad j = 1, 2, 3$$

and

$$\lambda_2 E(S_2^j) = n_j, \quad j = 1, 2, 3.$$

Proof. Equations (3.8) to (3.16) are derived employing the method used to prove Theorems 2.3 and 2.4. That is, all partial derivatives, with respect to u and v , of equation (3.7) of orders 1, 2 and 3 are evaluated at $u = v = 0$. The resulting nine equations are solved for the nine unknowns in the theorem subject to the constraint that $F^{xy}(0,0) + G^y(0) + \pi = 1$. This constraint is due to the assumption of the existence of a proper limiting distribution. The complete details of these steps required to prove the theorem are omitted since the details add nothing new.

For the case of Independent Unit Breakdown equation (3.6) reduces to

$$(3.17) \quad \left. \begin{aligned} & \pi \left[-\lambda_1 - \lambda_2 + \lambda_1 B_1^x(u) + \lambda_2 B_2^y(v) \right. \\ & + \left[H^x(u) + F^{xy}(u,v) \right] \left[u - \lambda_2 + \lambda_2 B_2^y(v) \right] \\ & \left. + G^y(v) \left[v - \lambda_2 + \lambda_2 B_2^y(v) \right] \right\} = 0. \end{aligned}$$

That is, breakdowns may occur any time that the unexpired repair time is zero.

Theorem 3.2. The random vector (X,Y) of the unexpired repair and clearing times, for Independent Unit Breakdown with a preemptive resume discipline, has the following properties;

$$(3.18) \quad P(X = Y = 0) = \pi = 1 - (m_1/(1+m_1)) - n_1 ,$$

$$(3.19) \quad P(X = 0, Y > 0) = n_1 ,$$

$$(3.20) \quad P(X > 0, Y \geq 0) = m_1/(1 + m_1) ,$$

$$(3.21) \quad P(X > 0, Y = 0) = \lambda_1 \pi \int [1 - E(e^{-\lambda_2 S_1})] / \lambda_2 ,$$

$$(3.22) \quad E(X) = m_2/2(1 + m_1) ,$$

$$(3.23) \quad E(Y) = \int n_1 E(X) / \pi(1 + m_1) + \int (1-n_1)n_2/2\pi ,$$

$$(3.24) \quad E(X^2) = m_3/3(1 + m_1) ,$$

$$(3.25) \quad E(XY) = \int (1-n_1)n_2/4\pi(1+n_1) + \int n_1^2 E(X)/2\pi(1+m_1) + \int n_1 E(X^2)/2 ,$$

$$(3.26) \quad E(Y^2) = \int n_3/3\pi + \int n_2 E(X) + 2n_1 E(XY) / \pi(1+m_1) .$$

Proof. The proof is exactly the same as for Theorem 3.1 employing equation (3.17) instead of (3.7) .

3. Unexpired Repair and Clearing Times for Component Breakdown.

In the case of active component breakdown equation (3.6) reduces to

$$(3.27) \quad \pi \int [-\lambda_2 + \lambda_2 B_2^y(v)] + G^y(v) \int [v - \lambda_1 - \lambda_2 + \lambda_1 B_1^x(u) + \lambda_2 B_2^y(v)] + F^{xy}(u,v) \int [u - \lambda_1 - \lambda_2 + \lambda_1 B_1^x(u) + \lambda_2 B_2^y(v)] = 0 .$$

Breakdowns may occur, in this case, only when the clearing time is greater than zero.

Theorem 3.3. The random vector (X, Y) of the unexpired repair and clearing times, for active component breakdown with a preemptive resume discipline, has the following properties;

$$(3.28) \quad P(X = Y = 0) = \pi = (1 - m_1 - n_1)/(1 - m_1) ,$$

$$(3.29) \quad P(X = 0, Y > 0) = n_1 ,$$

$$(3.30) \quad P(X > 0, Y > 0) = m_1 n_1 / (1 - m_1) ,$$

$$(3.31) \quad P(X > 0, Y = 0) = 0 ,$$

$$(3.32) \quad E(X) = n_1 m_2 / 2(1 - m_1)^2 ,$$

$$(3.33) \quad E(Y) = \int n_1^2 m_2 / (1 - m_1)^2 + n_2 \int / 2(1 - m_1 - n_1) ,$$

$$(3.34) \quad E(X^2) = n_1 m_2^2 / 2(1 - m_1)^3 + n_1 m_3 / (1 - m_1)^2 ,$$

$$(3.35) \quad E(XY) = \int n_1 E(X^2) + m_2 E(Y) \int / 2(1 - m_1) ,$$

$$(3.36) \quad E(Y^2) = \int 2n_1 E(XY) + n_2 E(X + Y) + n_3 \int / (1 - m_1 - n_1) .$$

Proof. The proof of this theorem is a direct result of the same operations employed in the proof of Theorem 3.1 but applied to equation (3.27), in this case.

Independent component breakdown with a preemptive resume discipline has been discussed in Chapter II under the identity of a two level storage problem. The mathematical equivalence of the two problems has been discussed in Section 1 of this chapter. Thus the results for this case are not repeated here.

4. Completion Times and Busy Periods. Gaver [2] defined the period of time between initial entry into a service mechanism (subject to interruptions) and completion of service as the completion time C , of a customer. He obtained the functional form of the Laplace-Stieltjes transform of the distribution of C , when interruptions (only while the customer is receiving service, i.e., unit breakdown) occur at random and the lengths of the interruption periods are independent random variables. This functional form was obtained for the completion time C and is different for each of the three preemptive disciplines, resume, repeat (identical) and repeat (independent). The results obtained in the next section depend heavily on Gaver's work so, for completeness, we repeat here his argument regarding completion times for a preemptive resume discipline. The approach, for each of the two preemptive repeat disciplines, is quite similar and the results only are quoted here.

Consider a customer with service time S_2 with distribution function $B_2(s)$. Let $S_1(i)$ be the duration of the i -th interruption of the service time S_2 . Thus, the completion time is

$$C = S_2 + \sum_{i=1}^N S_1(i) \quad ,$$

where N is a random variable. Define $B_i^*(u)$, $i = 1, 2$, as the Laplace-Stieltjes transforms of the repair time and service time distribution functions $B_1(s)$ and $B_2(s)$, respectively. A consideration of conditional expectations provides;

$$\begin{aligned} E(e^{-uC}/S_2, N) &= e^{-uS_2} E \left[e^{\sum_{i=1}^N S_1(i)} \right] \\ &= e^{-uS_2} [B_1^*(u)]^N . \end{aligned}$$

Thus

$$\begin{aligned} E(e^{-uC}/S_2) &= e^{-uS_2} \sum_{n=0}^{\infty} e^{-\lambda_1 S_2} \frac{(\lambda_1 S_2 B_1^*(u))^n}{n!} \\ &= e^{-(u+\lambda_1 - \lambda_1 B_1^*(u))S_2} , \end{aligned}$$

and

$$(3.37) \quad E(e^{-uC}) = B_2^*(u + \lambda_1 - \lambda_1 B_1^*(u)) ,$$

which is equation (4.6) of Gaver [2], page 10. This is for a preemptive resume discipline and unit breakdown. The completion time C , for a preemptive repeat (identical) discipline and unit breakdown satisfies

$$(3.38) \quad E(e^{-uC}) = \int_0^{\infty} \frac{e^{-(u+\lambda_1)s}}{1 - B_1^*(u) \frac{\lambda_1}{u+\lambda_1} [1 - e^{-(u+\lambda_1)s}]} d B_2(s) .$$

This is equation (4.17) of Gaver, in our notation. For a preemptive repeat (independent) discipline Gaver's equation (4.26) is, in our notation,

$$(3.39) \quad E(e^{-uC}) = \frac{B_2^*(u + \lambda_1)}{1 - B_1^*(u) \frac{\lambda_1}{u+\lambda_1} [1 - B_2^*(u+\lambda_1)]}$$

Consider the busy period of a single server queue with random arrivals at rate λ_1 , independent service times with distribution function $B_1(s)$ and a head-of-the-line discipline. A busy period begins when a customer arrives to find the server free; that busy period ends when the server is next free. Let the random variable D represent the length of a busy period. Kendall [6] showed that the Laplace-Stieltjes transform, $E(e^{-uD})$, of the busy period distribution function satisfies the equation

$$(3.40) \quad E(e^{-uD}) = B_1^* (u - \lambda_1 + \lambda_1 E(e^{-uD})) .$$

Table 3.2 lists the functions $\lambda_1 E(D^j)$, $j = 1, 2, 3$, of the first three moments of the busy period distribution. These are a direct result of equation (3.40) due to Kendall [6]. Also in table 3.2 we list the functions $\lambda_2 E(C^j)$, $j = 1, 2, 3$, of the completion time for each of the three preemptive disciplines and unit breakdown. The completion time moments involved are contained in Gaver [2], except for the third moments under the two preemptive repeat disciplines. These were obtained directly from (3.38) and (3.39) by differentiation. All these functions appear repeatedly throughout Section 5.

Define:

$$p = \left[\lambda_1 E(e^{-\lambda_1 S_2}) \right]^{-1}$$

$$q = 1 - E(e^{-\lambda_1 S_2})$$

$$r = 1 - \lambda_1 E(S_2 e^{-\lambda_1 S_2})$$

$$s = 1 - \lambda_1 E(S_2^2 e^{-\lambda_1 S_2})$$

$$t = (e^{\lambda_1 S_2} - 1) / \lambda_1$$

$$u = S_2 e^{\lambda_1 S_2} / \lambda_1$$

$$v = S_2^2 e^{\lambda_1 S_2} / \lambda_1$$

TABLE 3.2

	Busy Period of queue with random arrivals	Random Unit Breakdown	
			Preemptive Resume
$\lambda_1 E(D)$	$m_1 / (1 - m_1)$	$\lambda_2 E(C)$	$n_1 (1 + m_1)$
$\lambda_1 E(D^2)$	$m_2 / (1 - m_1)^3$	$\lambda_2 E(C^2)$	$n_2 (1 + m_1)^2 + n_1 m_2$
$\lambda_1 E(D^3)$	$m_3 / (1 - m_1)^4$ $+ 3m_2^2 / (1 - m_1)^5$	$\lambda_2 E(C^3)$	$n_3 (1 + m_1)^3 + 3m_2 n_2 (1 + m_1)$ $+ m_3 n_1$

MOMENTS OF BUSY PERIOD AND COMPLETION TIME

TABLE 3.2 (continued).

RANDOM UNIT BREAKDOWN AT RATE λ_1 AND REPAIR TIME d.f. $B_1(s)$

Preemptive Repeat (identical)	Preemptive Repeat (independent)
$\lambda_2^E \int [1 + m_1] t \int$	$\lambda_2^P q(1 + m_1)$
$\lambda_2^E \int [2(1+m_1) \{ t\lambda_1^{-1} + (1 + m_1)t^2 - u \} + m_2 t \int$	$\lambda_2^P \int [m_2 q + 2p(1+m_1)(m_1 q^2 + q + r - 1) \int$
$\lambda_2^E \int [- m_3 t + 3m_2(u - 2t^2 - t\lambda_1^{-1}) + 3m_1 \{ -v + 2u(4t + \lambda_1^{-1}) \} - 6m_1 t(t + \lambda_1^{-1})(3t + \lambda_1^{-1}) + 3 \{ -v + 2u(2t + \lambda_1^{-1}) - 2t(t + \lambda_1^{-1}) \}^2 - 6m_1^3 t^3 - 6m_1 m_2 t^2 - 6m_1^2 t(2u + 3t^2 + 2t \lambda_1^{-1})]$	$\lambda_2^P \int [- m_3 q + 3m_2(pq^2 + pr - \lambda_1^{-1}) + 6m_1 pr(2pq - \lambda_1^{-1} + pr) - 3m_1 p(4q\lambda_1^{-1} + 3s) + 3p(2pr^2 - 2r\lambda_1^{-1} - 3s) + 6pqm_1^2(2pr + pq^2 - 2\lambda_1^{-1}) + 6m_1^3 p^2 q + 6m_1 m_2 pq^2]$

MOMENTS OF COMPLETION TIME

5. Queueing Times for Active Breakdown. The queueing time of a customer is defined as the time period from arrival at the queue up to the time of his initial entry to the service mechanism. The queueing time of a customer is zero if and only if the customer arrives to find the unexpired repair and clearing times both zero. Thus the probability that the queueing time is zero, under the limiting distribution, has already been derived in Sections 2 and 3 for the four types of breakdown. In this section and Section 6 we will obtain a functional equation involving the Laplace-Stieltjes transform for the limiting distribution function of the queueing time for each of the four types of breakdown. These lead directly to the moments.

Consider the stochastic process $Z_t = (X_t, Y_t)$ for which the limiting distribution satisfies equation (3.6). Suppose that $\lambda_{10} = \lambda_{11} = \lambda_{12} = \lambda_{13} = 0$, which means that there are no breakdowns and $X_t \equiv 0$. Further, suppose that the increments to Y_t , which occur at random with rate λ_2 , are completion times of customers whose service is subject to interruptions or breakdowns. The customers' completion times depend on the preemptive discipline adopted. However, at present we will merely assume that the completion times are independent and the Laplace-Stieltjes transform of the completion time distribution function is $C^Y(v)$. From (3.6) it is easy to see that, under these conditions, the limiting distribution of Y_t satisfies the equation

$$(3.41) \quad \pi \int [-\lambda_2 + \lambda_2 G^Y(v)] + G^Y(v) \int [v - \lambda_2 + \lambda_2 G^Y(v)] = 0 .$$

The accumulation of completion times at the service mechanism at time t is Y_t ; this is the total time until the queueing system is empty of customers, provided no additional customers arrive. Thus Y_t is actually the queueing time, say Q_t , of a customer arriving at time t . Therefore

$$(3.42) \quad G^Y(v) + \pi = E(e^{-vQ}) = v\pi / \int [v - \lambda_2 + \lambda_2 G^Y(v)]$$

where $\pi = P(Q = 0)$. We have noted in Chapter II that the store content of material c_2 has no effect on the store content of material c_1 . Hence the results of Chapter II can be employed to consider the limiting distribution of the queueing time Q_t , which is subject to random increases at rate λ_2 of an amount C (the completion time for the appropriate discipline) and a uniform rate of reduction. This is accomplished by an identification of the queueing time Q , under the limiting distribution, with the store content X of material c_1 , under the limiting distribution. Thus a renaming of the functions involved in equations (2.43), (2.46) and (2.52) proves the following theorem.

Theorem 3.4. The queueing time Q , under the limiting distribution, of a customer of a queue subject to active unit breakdown and a pre-emptive discipline has the properties;

$$(3.43) \quad P(Q = 0) = \pi = 1 - \lambda_2 E(C) \quad ,$$

$$(3.44) \quad E(Q) = \lambda_2 E(C^2) / 2(1 - \lambda_2 E(C)) \quad ,$$

$$(3.45) \quad E(Q^2) = \lambda_2 E(C^3) / 3(1 - \lambda_2 E(C)) \\ + \frac{1}{2} \left[\lambda_2 E(C^2) / (1 - \lambda_2 E(C)) \right]^2 \quad .$$

Moments of the completion time C are listed in table 3.2 for each of the three preemptive disciplines.

The difference between unit and component breakdown is that in the latter case a breakdown may also occur while the service mechanism is being repaired. The moments of the completion time have been derived for unit breakdown. If the unit repair time of the service mechanism is considered as a busy period of a queue with random arrivals at rate λ_1 and service time distribution function $B_1(s)$ then the completion time actually allows for random component breakdowns at rate λ_1 and repair time distribution function $B_1(s)$. The commencement of "busy periods" of repair occur only during the service time of customers at random with rate λ_1 . The completion time for component breakdown will be designated by \bar{C} . Thus the values for $\lambda_2 E(\bar{C}^j)$, $j = 1, 2, 3$, may be read directly from table 3.2 upon replacing m_j by $\lambda_1 E(D^j)$ in the formulas for $\lambda_2 E(C^j)$, $j = 1, 2, 3$.

Theorem 3.5. The queueing time Q , under the limiting distribution, of a customer of a queue subject to active component breakdown under a preemptive discipline has the properties;

$$(3.46) \quad P(Q = 0) = \pi = 1 - \lambda_2 E(\bar{C}) \quad ,$$

$$(3.47) \quad E(Q) = \lambda_2 E(\bar{C}^2) / 2(1 - \lambda_2 E(\bar{C})) \quad ,$$

and

$$(3.48) \quad E(Q^2) = \lambda_2 E(\bar{C}^3) / 3(1 - \lambda_2 E(\bar{C})) \\ + \frac{1}{2} \left[\lambda_2 E(\bar{C}^2) / (1 - \lambda_2 E(\bar{C})) \right]^2 \quad .$$

Moments of the completion time \bar{C} depend on the preemptive discipline involved.

Proof. This theorem is a straightforward extension of Theorem 3.4.

6. Queueing Times for Independent Breakdown. Suppose the stochastic process $Z_t = (X_t, Y_t)$ is subject to the following rules;

- (i) X_t is subject to random increases at rate λ_1 only during the time periods that $Z_t = (0, 0)$. The increase at each occurrence is a random variable S_1 with distribution function $B_1(s)$.
- (ii) Y_t is subject, at all times, to random increases at rate λ_2 . Each increase is the random variable C of a completion time with unit breakdown.
- (iii) If $X_t > 0$, then it is decreasing at unit rate; if $X_t = 0, Y_t > 0$, then Y_t is decreasing at unit rate; if $X_t = Y_t = 0$ then both are held constant until a random increase occurs.

The limiting distribution of $Z_t = (X_t, Y_t)$ satisfies the special case of equation (3.6) where

$$\lambda_{10} = \lambda_1, \quad \lambda_{11} = \lambda_{12} = \lambda_{13} = 0,$$

and

$$B_{10}^X(u) = B_1^X(u), \quad B_2^Y(v) = C^Y(v).$$

Thus

$$(3.49) \quad \left. \begin{aligned} & \pi \int [-\lambda_1 - \lambda_2 + \lambda_1 B_1^X(u) + \lambda_2 C^Y(v)] \\ & + G^Y(v) \int [v - \lambda_2 + \lambda_2 C^Y(v)] \\ & + \int [H^X(u) + F^{XY}(u,v)] \int [u - \lambda_2 + \lambda_2 C^Y(v)] \end{aligned} \right\} = 0$$

The separate components X and Y , for which the distribution function satisfies (3.49), have rather obscure interpretations. However, it should be clear from the previous section that $(X + Y)$ is the random variable which satisfies the limiting distribution function of the queueing time of a customer arriving at a queue with independent unit breakdown.

Theorem 3.6. The queueing time Q of a customer of a queue which is subject to independent unit breakdown has the following properties, under the limiting distribution;

$$(3.50) \quad P(Q = 0) = \pi = \int [1 - \lambda_2 E(C)] / \int [1 + m_1],$$

$$(3.51) \quad E(Q) = \int \pi m_2 + \lambda_2 E(C^2) / 2 \int [1 - \lambda_2 E(C)],$$

and

$$(3.52) \quad E(Q^2) = \int [3\lambda_2 E(C^2)E(Q) + \lambda_2 E(C^3) + \pi m_3] / 3 \int [1 - \lambda_2 E(C)].$$

Proof. These expressions are obtained by setting $u = v$ in (3.49), then evaluating its first, second and third derivatives with respect to u at $u = 0$.

Equations (3.49) to (3.52) are transformed to the case of independent component breakdown by replacing the repair time distribution by the busy period distribution of the repair time. Therefore

$$(3.53) \quad \left. \begin{aligned} & \pi \left[-\lambda_1 - \lambda_2 + \lambda_1 E(e^{-uD}) + \lambda_2 E(e^{-v\bar{C}}) \right] \\ & + \left[H^x(u) + F^{xy}(u,v) \right] \left[u - \lambda_2 + \lambda_2 E(e^{-v\bar{C}}) \right] \\ & + G^y(v) \left[v - \lambda_2 + \lambda_2 E(e^{-v\bar{C}}) \right] \end{aligned} \right\} = 0$$

It is possible to obtain information about X and Y separately from this equation. However, because of their obscure interpretation, we pass immediately to the following theorem.

Theorem 3.7. The queueing time Q , under the limiting distribution, of a customer of a queue which is subject to independent component breakdown has the following properties;

$$(3.54) \quad P(Q = 0) = \pi = \left[1 - \lambda_2 E(\bar{C}) \right] \left[1 - m_1 \right],$$

$$(3.55) \quad E(Q) = \left[\pi m_2 / (1 - m_1)^3 + \lambda_2 E(\bar{C}^2) \right] / 2 \left[1 - \lambda_2 E(\bar{C}) \right],$$

and

$$(3.56) \quad E(Q^2) = \left[3\lambda_2 E(\bar{C}^2) E(Q) + \lambda_2 E(\bar{C}^3) + \frac{\pi m_3}{(1-m_1)^4} + \frac{3\pi m_2^2}{(1-m_1)^5} \right] / \cdot 3 \left[1 - \lambda_2 E(\bar{C}) \right].$$

Moments of the completion time \bar{C} depend on the preemptive discipline involved.

Proof. These results are obtainable directly from equation (3.53). It is also possible to obtain them from Theorem 3.6 by replacing the moments of the repair time by the moments of the busy period, and the moments of the completion time C for unit breakdown by the moments of the completion time \bar{C} for component breakdown.

As a result of the equivalence of queueing with independent component breakdown and queueing of two classes of customers (those customers with and those without preemption rights), theorem 3.7 applies to the 2-level queue with any one of the three preemptive disciplines. Thus (3.55) and (3.56) are the first two moments of the queueing time of the customers without preemption rights for the three different preemptive disciplines. Miller [8] derived these moments for a preemptive resume discipline employing a more specialized argument. These results for the two preemptive repeat disciplines are new. We note that the moments of the queueing time for the preemptive repeat disciplines depend on the form of $B_2(s)$ but only on the moments of $B_1(s)$.

The transient behavior of the different stochastic processes considered in this chapter are characterized, in each case, by a special case of equation (3.1). Also, the extension of the results contained in this chapter regarding queueing times to three or more classes of customers (when this has meaning) is not difficult. It is omitted here because it is tedious.

CHAPTER IV

THE EXISTENCE OF LIMITING DISTRIBUTIONS

1. Introduction. In Chapter II we considered the store content of k materials under a preemptive rule of operation. The functional form of the limit, as $t \rightarrow \infty$, of the distribution function of the store content for $k = 2$ was obtained under the assumption that such a limit actually existed. Specifically, we assumed that the limit, as $t \rightarrow \infty$, of the bivariate distribution of store content $Z_t = (X_t, Y_t)$ was a proper bivariate distribution.

In Chapter III we have considered a queue when the service mechanism is subject to breakdowns, active or independent, unit or component. Under a preemptive resume queue discipline, the limit, as $t \rightarrow \infty$, of the distribution function of the unexpired repair and service time was derived in functional form. We also obtained the functional form of the limit of the queueing time distribution function for the four different types of breakdown, each subject to the different preemptive queue disciplines (resume, repeat (identical), and repeat (independent)). In each case it was assumed that the limit existed as a proper distribution. The necessary and sufficient conditions for the existence, as proper distributions, of all these limits will be derived in this chapter as special cases of a more general theory.

Let $Z_t = (X_t, Y_t)$ be a stochastic process, for $0 < t \leq \infty$, defined in the four regions:

$$R_0 : X_t = Y_t = 0 ;$$

$$R_1 : X_t > 0, Y_t = 0 ;$$

$$R_2 : X_t = 0, Y_t > 0 ;$$

$$R_3 : X_t > 0, Y_t > 0 .$$

This stochastic process is subject to the following rules:

- (i) In region R_i there are two Poisson processes P_{1i} and P_{2i} ; the rate of occurrence of events in them is λ_{1i} and λ_{2i} , $i = 0, 1, 2, 3$.
- (ii) When an event in P_{1i} occurs the X_t coordinate of Z_t increases instantly by an amount S_{1i} , a random sample from a population with distribution function $B_{1i}(s)$ where $B_{1i}(0+) = 0$, $i = 0, 1, 2, 3$.
- (iii) When an event in P_{2i} occurs the Y_t coordinate of Z_t increases instantly by an amount S_{2i} , a random sample from a population with distribution function $B_{2i}(s)$ where $B_{2i}(0+)$, $i = 0, 1, 2, 3$.
- (iv) In regions R_1 and R_3 , Y_t is held constant between its upward jumps and X_t decreases uniformly (a unit amount per unit of time) between its upward jumps.

- (v) In region R_2 , X_t is held constant until its next upward jump and Y_t decreases uniformly between its upward jumps until the next upward jump of X_t (i.e. until Z_t jumps to region R_3).
- (vi) In region R_0 , both X_t and Y_t are held constant until the next occurrence of an event in either P_{10} or P_{20} (i.e. until the process Z_t jumps out of region R_0).

We will call Z_t a Poisson Reduction Process. Consideration of some generalizations of this process would be interesting. However, this would be an excursus since all stochastic processes considered in this dissertation are special cases of a Poisson Reduction Process, as we define it.

2. Necessary and Sufficient Conditions for Existence of Limiting Distributions of Poisson Reduction Processes. Consider the Poisson Reduction Process $Z_t = (X_t, Y_t)$. We wish to obtain the necessary and sufficient conditions for the existence of

$$(4.1) \quad \lim_{t \rightarrow \infty} P(Z_t \in A), \text{ for all } A \in \mathcal{A},$$

where \mathcal{A} is the class of sets $\{0 \leq X_t \leq x, 0 \leq Y_t \leq y\}$.

Theorem 2 proved by Smith [11] page 14 is extremely useful for proving the existence of limiting distributions for Equilibrium Processes. In the sequel we will show that a Poisson Reduction Process is a special case of an Equilibrium Process, which we now define.

Let $\{t_i\}$, $i = 1, 2, \dots$ be an infinite sequence of independent, non-negative, identically distributed random variables, each of which is not zero with probability one. Let t_0 be a non-negative random variable, independent of and not necessarily distributed like t_i , $i \geq 1$. Smith [11] calls the sequence $\{t_i\}$, $i = 0, 1, 2, \dots$ a general renewal process. For any general renewal process $\{t_i\}$ define $T_{-1} = 0$, $T_k = t_0 + t_1 + \dots + t_k$, $k = 0, 1, 2, \dots$. For each $t \geq 0$ define the random variable n_t as the largest integer k such that $T_{k-1} \leq t$. Suppose Z_0 is a random variable, defined over the space z , specifying the initial condition of a stochastic process Z_t . This process Z_t is called an Equilibrium Process with respect to Z_0 over the sets \mathcal{A} if, for all $Z_0 \in z$, $A \in \mathcal{A}$, there exists a function $\phi_A(\cdot)$, depending only on A such that

$$(4.2) \quad P(Z_t \in A | Z_0 \in z; n_t > 0; T_{n_t}) = \phi_A(t - T_{n_t})$$

is a valid representation of the conditional probability. The Equilibrium Process Z_t is abbreviated $\mathcal{E}(z, \mathcal{A}, \{t_i\})$. The definitions of this paragraph are due to Smith [11] pages 9, 12 and 13.

Now consider the Poisson Reduction Process $Z_t = (X_t, Y_t)$ and let the time points T_0, T_1, T_2, \dots be the instants when the process enters the region R_0 . Thus T_0 is the least value of t such that $X_t = Y_t = 0$, but $X_{t-\epsilon} + Y_{t-\epsilon} > 0$ for all sufficiently small

$\epsilon > 0$; T_1 is the second smallest such value of t , and so on. For example, in the store content process of Chapter II the time points T_0, T_1, T_2, \dots are the instants when the store becomes empty.

Write $t_0 = T_0, t_0 + t_1 = T_1, \dots, t_0 + t_1 + \dots + t_n = T_n$. Plainly $\{t_i\}$ forms a general renewal process as described above. Let us fix $Z_0 \in z$, where z is the non-negative quadrant of two-dimensional Euclidean space, and define $K_{Z_0}(t)$ as the distribution function of t_0 . Let $F(t)$ be the distribution function of t_i , for $i \geq 1$. We now apply the following, a part of Theorem 2, Smith [11].

Theorem. If (i) Z_t is $\mathcal{E}(z, \mathcal{A}, \{t_i\})$,
(ii) $F(t)$ is not a step function such that all jumps are over any subset of $\{nw\}$,
 $n = 0, 1, 2, \dots$ for any $w > 0$,
(iii) $K_{Z_0}(+\infty) = 1$,
(iv) $E(t_i) = \mu < \infty, i \geq 1$,
(v) the k -th convolution of $F(t)$ with itself has an absolutely continuous component for some finite k ,
(vi) $\phi_A(t)$ is measurable,

then for all $Z_0 \in z, A \in \mathcal{A}$

$$(4.3) \quad \lim_{t \rightarrow \infty} P(Z_t \in A | Z_0) = \frac{1}{\mu} \int_0^{\infty} \phi_A(v) [1 - F(v)] dv.$$

We are considering the Poisson Reduction Process $Z_t = (X_t, Y_t)$ over the sets $A \in \mathcal{A}$, where \mathcal{A} is the class of sets $\{0 \leq X_t \leq x, 0 \leq Y_t \leq y\}$ where x and y are finite. It is clear that the rules of the process completely determine the future of the process at times $T_i, i = 0, 1, 2, \dots$. The future is also independent of the past history at these times except for the fact that the process entered state $(0,0)$ at such times. Thus $\{t_i\}, i \geq 1$ are independent, identically distributed and independent of t_0 . Hence equation (4.2) holds for the class of sets \mathcal{A} . Therefore Z_t is $\mathcal{E}(z, \mathcal{A}, \{t_i\})$ and condition (i) of the theorem holds.

Suppose the process Z_t enters state $(0,0)$ at time T_i and next departs from this state a time period P_{i+1} later. The length of the period, P_{i+1} , is a random variable with the negative exponential distribution. For example, this time period is an idle period of the service mechanism for the Poisson Reduction Process of Chapter III. Define $t_i = P_i + D_i$. Thus, during the interval t_i , D_i is that segment for which $X_t + Y_t > 0$. For those cases considered in Chapter III this would be called a busy period (repair and service) of the service mechanism. We will call the periods D_i for all Poisson Reduction Process the busy periods. The amount of the instantaneous increment to $X_t + Y_t$ at time $T_{i-1} + P_i$ is independent of P_i . The future of the process (after time $T_{i-1} + P_i$) depends only on the size of the increment and the rules of the process. Thus P_i is independent of D_i . Hence the distribution

function of t_i is the convolution of the distribution function of D_i and the absolutely continuous distribution function of P_i . Therefore t_i , $i = 0, 1, \dots$ has an absolutely continuous distribution function and conditions (ii) and (v) of the theorem hold.

Now consider the sequence $P_1, D_1, P_2, D_2, P_3, D_3, \dots$. Let $(m-1)$ be the largest integer for which $S_{m-1}^{(p)} \leq \Delta$, where $S_{m-1}^{(p)} = \sum_{i=1}^{m-1} P_i$ and Δ is any finite positive number. Now the sequence of random variables P_1, P_2, P_3, \dots is a renewal process with $0 < E(P_i) < \infty$ and, for finite Δ , $0 \leq E(m-1) < \infty$. (cf. Smith [12] p. 245). Upon setting $X_0 + Y_0 = \Delta$ we have

$$S_{m-1}^{(P)} + S_{m-1}^{(D)} \leq T_m \leq S_m^{(P)} + S_m^{(D)}.$$

A direct application of Wald's equation provides

$$(4.4) \quad E(D) + E(m-1) [E(P) + E(D)] \leq E(T_m) \leq E(m) [E(P) + E(D)].$$

Thus $E(T_m)$ is finite if and only if $E(D) < \infty$. Hence the function $K_{Z_0} (+\infty) = 1$ if $E(D) < \infty$. This modification of an argument found in Smith [12] p. 260 establishes condition (iii) of the theorem if $E(D) < \infty$.

Thus the conditions of the theorem are known to hold (when $E(D) < \infty$) provided $\phi_A(t)$ is a measurable function. We will verify this by showing that it is a continuous function of t (i.e. a Baire function of class zero) and appealing to the theorem that all

Baire functions are measurable (cf. Goffman [3] Theorem 1, p. 183) .

Let $\lambda_1 = \max \lfloor \lambda_{1i} + \lambda_{2i} \rfloor$ and $\lambda_2 = \min \lfloor \lambda_{1i} + \lambda_{2i} \rfloor$. For $\tau > 0$, elementary probability considerations allow us to write

$$\phi_A(t+\tau) \geq \phi_A(t) e^{-\lambda_1 \tau} ,$$

and

$$\phi_A(t+\tau) \leq \lfloor \phi_A(t) + \phi_{A',-A}(t) \rfloor e^{-\lambda_2 \tau} + \lfloor 1 - e^{-\lambda_1 \tau} \rfloor ,$$

where $A' = \{0 \leq X_t \leq x + \tau, 0 \leq Y_t \leq y + \tau\}$. By definition, $\phi_A(t)$ for fixed t is a measure over the sets A and must be a continuous set function from both above and below (cf. Halmos [4] p. 39). Thus

$$\lim_{\tau \rightarrow 0} \phi_{A',-A}(t) = 0 .$$

The above two inequalities as $\tau \rightarrow 0$ allow us to write

$$\phi_A(t+0) = \phi_A(t) .$$

Manipulation of these two inequalities also allows us to write, for $\tau > 0$,

$$\phi_A(t) e^{\lambda_1 \tau} \geq \phi_A(t-\tau) ,$$

and

$$\lfloor \phi_A(t) - \phi_{A''}(t) \rfloor e^{\lambda_2 \tau} - \lfloor 1 - e^{-\lambda_1 \tau} \rfloor e^{\lambda_2 \tau} \leq \phi_A(t-\tau) ,$$

where $A'' = \{x-\tau < X_t \leq x, y-\tau < Y_t \leq y\}$.

Thus

$$\phi_A(t-0) = \phi_A(t) .$$

Therefore $\phi_A(t)$ is continuous and, as a result, a measurable function of t . Condition (vi) of the theorem holds. Note: Smith [11] has shown that conditions (v) and (vi) of Theorem 2 may be replaced by the condition that $\phi_A(t) [1 - F(t)]$ is of bounded variation in any finite t interval. Thus an alternative approach here would be to employ Lemma 2 Smith [11] p. 17 to demonstrate that the bounded variation property holds for Poisson Reduction processes.

We have now shown that all the conditions of Theorem 2 are satisfied if $E(D) < \infty$ and $(X_0 + Y_0) < \infty$ with probability one. Thus we have completed the proof of the sufficiency part of the following theorem.

Theorem 4.1. Necessary and sufficient conditions for the existence of the limiting distribution of a Poisson Reduction Process are;

- (i) $E(D) < \infty$, where D is the duration of the busy period,
and
- (ii) the initial status $(X_0 + Y_0)$ of the process is finite with probability one.

Proof. (of the necessary part). Consider the regeneration points T_0, T_1, T_2, \dots of the process and define n_t as the largest integer k such that $T_{k-1} \leq t$. Smith [11] p. 19 defines any set A as an α -set if there are real numbers $\tau, \epsilon > 0$ such that

$$P(n_{t+\tau} > n_t \mid Z_0 \in z, Z_t \in A) \geq \epsilon$$

for sufficiently large t and Z_t any equilibrium process. As we

have seen, a Poisson Reduction Process is an equilibrium process.

For a Poisson Reduction Process, $e^{-\lambda_1(x+y)}$ is a lower bound for the probability that zero Poisson events occur in a time period $x + y$.

Thus, for any set $A = \{0 \leq X_t \leq x, 0 \leq Y_t \leq y\}$, we have

$$(4.5) \quad P(n_{t+\tau} > n_t \mid Z_0 \in z, Z_t \in A) \geq e^{-\lambda_1(x+y)} > 0$$

for finite x and y and $\tau > x + y$. The set A is said to be a null set if

$$\lim_{t \rightarrow \infty} P(Z_t \in A \mid Z_0) = 0$$

Theorem (Smith [11] p. 19). If there exists a non-null α -set then

$$K_{Z_0}(+\infty) > 0 \text{ and } \mu < \infty, E(t_i) = \mu$$

By virtue of (4.5) the set A is an α -set which is non-null only if $E(D) \leq \mu < \infty$. Thus the set A is null under our assumption that $E(D) = \infty$ and

$$\lim_{t \rightarrow \infty} P(Z_t \in A \mid Z_0) = \lim_{t \rightarrow \infty} [\pi(t) + G(y;t) + H(x;t) + F(x,y;t)] = 0$$

That is, the limiting distribution does not exist as a proper distribution if $E(D) = \infty$.

Assume $X_0 + Y_0 = \infty$ with probability $p < 1$. Thus $P(X_t + Y_t = \infty) \geq p$ for all finite t . Thus

$$\pi(t) + G(y;t) + H(x;t) + F(x,y;t) \leq 1 - p < 1$$

for all finite x, y and t . Therefore by taking limits as

$x \rightarrow \infty, y \rightarrow \infty$

$$(4.6) \quad \pi(t) + G(+\infty; t) + H(+\infty; t) + F(+\infty, +\infty; t) \leq 1 - P < 1 \quad \text{for}$$

all finite t . Hence the limit, as $t \rightarrow \infty$, of the left hand side of (4.6) is bounded away from one and the limiting distribution does not exist as a proper distribution. This completes the proof of theorem 4.1.

It is not difficult to see how one would extend the notion of a Poisson Reduction Process to n dimensions and prove results corresponding to those of this section.

3. Necessary and Sufficient Conditions for a Finite Mean Busy Period of Certain Poisson Reduction Processes. The different Poisson Reduction Processes discussed in Chapter II and III will be considered individually in this section. Takacs [13] et. al. developed a certain method for establishing necessary and sufficient conditions for the finiteness of the mean busy period for the head-of-the-line queue with one server; we employ a similar method in this section.

Suppose we consider the special case of a Poisson Reduction Process, $Z_t = (X_t, Y_t)$ satisfying;

- (a) $X_t \equiv 0$,
- (b) Y_t is subject to instantaneous jumps at Poisson rate λ_2 , in all regions, of an amount C ; where C is the completion time of a customer with either unit or component

breakdown and a preemptive queue discipline specified,

(c) if $Y_t > 0$, then it decreases at unit rate.

Thus $X_t + Y_t = Q_t$ is the queueing time of the customer arriving at time t , in a queue with active breakdowns. The type of breakdowns considered (unit or component) and the preemptive discipline are allowed for by the completion time distribution. The busy periods of the process Q_t , with a preemptive resume discipline, are identical to the busy periods for the Poisson Reduction Process $Z_t = (X_t, Y_t)$ of the unexpired repair and service times with active breakdowns and a preemptive resume discipline. In fact the queueing time Q_t for any breakdown system is zero if and only if the unexpired repair and service times are both zero. Thus the conditions for finiteness of the mean busy period may be investigated under different guises. We will investigate the finiteness conditions with respect to queueing times, for the different types of breakdown and preemptive queue disciplines. These finiteness conditions also apply, when meaningful, to the unexpired repair and service times.

Consider a random variable D with distribution function $\phi(d)$.

Suppose

$$D = D_1 + \sum_{i=0}^N D_{2i}$$

where D_1 has distribution function $\phi_1(d)$, D_{2i} , $i = 1, 2, \dots, N$ are independent with the same distribution function $\phi_2(d)$ and N is the number of Poisson events occurring, with rate λ , in a time

period D_1 . That is

$$P(N = n | D_1) = e^{-\lambda D_1} (\lambda D_1)^n / n! .$$

A consideration of conditional expectations leads to

$$E(e^{-uD} | D_1, N) = e^{-uD_1} \int \phi_2^*(u) \tau^N ,$$

where $\phi_2^*(u) = E(e^{-uD_2})$.

Thus

$$E(e^{-uD} | D_1) = e^{-uD_1} \sum_{n=0}^{\infty} e^{-\lambda D_1} \frac{(\lambda D_1 \phi_2^*(u))^n}{n!} ,$$

and

$$(4.7) \quad E(e^{-uD}) = \phi^*(u) = \phi_1^*(u + \lambda - \lambda \phi_2^*(u)) .$$

Upon letting $u \rightarrow 0$ in (4.7) it is clear that $\phi^*(u)$ is the Laplace-Stieltjes transform of a proper distribution function if and only if $\phi_1(d)$ and $\phi_2(d)$ are proper distribution functions. (We are assuming that $\phi_1(0+) < 1$). Equation (4.7) also implies that

$$E(D) = \int [1 + \lambda E(D_2)] E(D_1) .$$

Let C be the random variable of a completion time with distribution function $\Theta(c)$ where the service mechanism is subject to unit breakdown and an unspecified preemptive queue discipline. Let \bar{C} and $\bar{\Theta}(c)$ represent the same quantities for component breakdown. The first three moments of the completion times C and \bar{C} are listed in Table 3.2 for each of the three preemptive disciplines.

Consider D , the random variable of the duration of a busy period of the queueing time process for active unit breakdowns. Continuing the notation and definitions of Chapter III we have, as a result of (4.7),

$$(4.8) \quad E(e^{-uD}) = \Theta^*(u + \lambda_2 - \lambda_2 E(e^{-uD})) .$$

Thus

$$(4.9) \quad E(D) = E(C)/(1 - \lambda_2 E(C)) .$$

and $E(D) < \infty$ if and only if $\lambda_2 E(C) < 1$. Takacs [13] derived a similar result for a head-of-the-line single server queue. It is apparent that a reference here to Takacs would suffice. However the inclusion of this argument should make the following more readable.

Now consider \bar{D} , the random variable of the duration of a busy period for the queueing time process with active component breakdowns. The argument of the previous paragraph carries over with little change to this case provided the completion time distribution is made applicable to component breakdown. Thus

$$(4.10) \quad E(\bar{D}) = E(\bar{C})/(1 - \lambda_2 E(\bar{C})) ,$$

which is finite if and only if $\lambda_2 E(\bar{C}) < 1$.

Suppose D is the length of a busy period of the queueing time process with Independent Unit breakdown. Let D_1 be the random variable of the busy period length when it commences with a breakdown. This occurs with probability $\lambda_1/(\lambda_1 + \lambda_2)$. Let D_2 be the random variable of the busy period duration when it commences with an arrival of a customer. This occurs with probability

$\lambda_2/(\lambda_1 + \lambda_2)$. For independent unit breakdown, the length D of a busy period has mean

$$E(D) = [\lambda_1 E(D_1) + \lambda_2 E(D_2)] / (\lambda_1 + \lambda_2) .$$

From (4.7)

$$(4.11) \quad E(e^{-uD_1}) = B_1^*(u + \lambda_2 - \lambda_2 E(e^{-uD_2})) ,$$

and

$$(4.12) \quad E(e^{-uD_2}) = \theta^*(u + \lambda_2 - \lambda_2 E(e^{-uD_2})) .$$

Hence

$$(4.13) \quad E(D_1) = E(S_1) / (1 - \lambda_2 E(C)) ,$$

and

$$(4.14) \quad E(D_2) = E(C) / (1 - \lambda_2 E(C)) .$$

Thus the mean busy period, $E(D)$, is finite if and only if $\lambda_2 E(C) < 1$.

For Independent Component breakdown (4.11) to (4.14) are easily transformed to

$$(4.15) \quad E(e^{-uD_1}) = \phi_I^*(u + \lambda_2 - \lambda_2 \bar{c}^*(u)) ,$$

$$(4.16) \quad E(e^{-uD_2}) = \bar{\theta}^*(u + \lambda_2 - \lambda_2 E(e^{-uD_2})) ,$$

$$(4.17) \quad E(D_1) = E(S_1) [1 + \lambda_2 E(\bar{C})] / (1 - \lambda_1 E(S_1)) ,$$

and

$$(4.18) \quad E(D_2) = E(\bar{C}) / (1 - \lambda_2 E(\bar{C})) ,$$

where $\phi_I(d)$ is the distribution function of a busy period of repair

times in isolation. Both $E(D_1)$ and $E(D_2)$ are finite if and only if $\lambda_2 E(\bar{C}) < 1$ (which implies that $\lambda_1 E(S_1) < 1$, see Table 3.2). Thus the mean busy period of the queueing process with independent component breakdown is finite if and only if $\lambda_2 E(\bar{C}) < 1$.

These results are summarized in Table 4.1. In all cases finiteness of the busy period of completion times in isolation guarantees finiteness of the mean busy period of the associated queueing time process. The finiteness condition relating to Independent Component breakdown and a preemptive resume discipline was derived by Miller [8]. Gaver [2] obtained the results for unit breakdown for all three preemptive disciplines.

Breakdown	Unit	Component
Preemptive Discipline	$\lambda_2 E(C) < 1$ if and only if	$\lambda_2 E(\bar{C}) < 1$ if and only if
Resume	$n_1(1 + m_1) < 1$	$m_1 + n_1 < 1$
Repeat (identical)	$\frac{\lambda_2}{\lambda_1} \frac{E(e^{-\lambda_1 S_2}) - 1}{E(e^{-\lambda_1 S_2})} < 1$	$\frac{\lambda_2}{\lambda_1} \frac{E(e^{-\lambda_1 S_2}) - 1}{E(e^{-\lambda_1 S_2})} < 1$ and $m_1 < 1$
Repeat (independent)	$\frac{\lambda_2}{\lambda_1} \left[\frac{1 - E(e^{-\lambda_1 S_2})}{E(e^{-\lambda_1 S_2})} \right] < 1$	$\frac{\lambda_2}{\lambda_1} \left[\frac{1 - E(e^{-\lambda_1 S_2})}{E(e^{-\lambda_1 S_2})} \right] < 1$ and $m_1 < 1$

TABLE 4.1

Necessary and Sufficient Conditions for Finiteness of Mean Busy Periods.

CHAPTER V

THE NUMBER OF BREAKDOWNS

1. Number of Breakdowns During Service of One Customer. In Chapter III we have discussed four different types of breakdowns of the service mechanism for single server queues. We also noted the equivalence of the mathematical model for independent component breakdown and the 2-level single server queue with preemptive disciplines. In this section we investigate the number of times, N , that one customer is interrupted while receiving service. An interruption (occurrence of a breakdown or arrival of a higher priority customer) will be considered for the three different preemptive disciplines. The number of interruptions N will be the same for both unit and component breakdown, active or independent. This follows from the fact that for unit breakdown N is the number of breakdowns while the customer is in the service mechanism, while on the other hand, for component breakdown, N is the number of interruptions by busy periods of the component repair time.

The breakdowns or interruptions occur at random with rate λ_1 during the actual service time S_2 with distribution function $B_2(s)$. It is assumed that the customer returns to the service mechanism in accordance with a preemptive resume discipline.

For a given uninterrupted service time S_2 ,

$$P(N = n) = e^{-\lambda_1 S_2} (\lambda_1 S_2)^n / n! .$$

Thus the probability generating function of N is

$$\begin{aligned} (5.1) \quad E(Z^N) &= \sum_{n=0}^{\infty} Z^n \int_0^{\infty} e^{-\lambda_1 s} \frac{(\lambda_1 s)^n}{n!} d B_2(s) \\ &= B_2^*(\lambda_1 - \lambda_1 Z) , \end{aligned}$$

where $B_2^*(u)$ is the Laplace-Stieltjes transform of the uninterrupted service time distribution function.

Theorem 5.1. The number, N , of interruptions of a customer's service, under a preemptive resume discipline, satisfies;

$$(5.2) \quad P(N=0) = B_2^*(\lambda_1) ,$$

$$(5.3) \quad E(N) = \lambda_1 E(S_2) \quad \text{and}$$

$$(5.4) \quad E(N^2) = \lambda_1^2 E(S_2^2) + \lambda_1 E(S_2) ,$$

where S_2 is the uninterrupted service time with distribution function $B_2(s)$.

Proof. This theorem is an obvious consequence of (5.1) by differentiation with respect to Z and then evaluated at $Z = 1$.

Suppose interruptions occur at random with rate λ_1 for a customer with uninterrupted service time S_2 . The customer returns to service according to a preemptive repeat (identical) discipline. For

a given S_2 ;

$$P(N = n) = (1 - e^{-\lambda_1 S_2})^n e^{-\lambda_1 S_2} .$$

Thus

$$(5.5) \quad E(Z^N) = \int_0^{\infty} \frac{e^{-\lambda_1 s}}{1 - Z(1 - e^{-\lambda_1 s})} d B_2(s) .$$

Theorem 5.2. The number, N , of interruptions of one customer's service under a preemptive repeat (identical) discipline satisfies;

$$(5.6) \quad P(N = 0) = B_2^*(\lambda_1) ,$$

$$(5.7) \quad E(N) = E(e^{\lambda_1 S_2}) - 1 \\ = B_2^*(-\lambda_1) - 1 ,$$

if $B_2^*(-\lambda_1) < \infty$, and

$$(5.8) \quad E(N^2) = 2E(e^{2\lambda_1 S_2}) - 4E(e^{\lambda_1 S_2}) + 2 \\ = 2B_2^*(-2\lambda_1) - 4B_2^*(-\lambda_1) + 2 ,$$

if $B_2^*(-\lambda_1) < \infty$ and $B_2^*(-2\lambda_1) < \infty$.

Proof. The theorem is a direct consequence of equation (5.5) .

Suppose interruptions occur at random rate λ_1 for a customer with uninterrupted service time S_2 and the customer returns to service according to the preemptive repeat (independent) discipline. For a given sequence $S(1), S(2), \dots, S(n+1)$ of independent uninterrupted service times

$$P(N = n) = \prod_{i=1}^n (1 - e^{-\lambda_1 S(i)}) e^{-\lambda_1 S(n+1)} .$$

Thus

$$(5.9) \quad E(Z^N) = B_2^*(\lambda_1) / [1 - Z + Z B_2^*(\lambda_1)] .$$

The following theorem is a direct consequence of (5.9).

Theorem 5.3. The number, N, of interruptions of one customer's service with return to service according to a preemptive repeat (independent) discipline satisfies:

$$(5.10) \quad P(N = 0) = B_2^*(\lambda_1) ,$$

$$(5.11) \quad E(N) = [1 - B_2^*(\lambda_1)] / B_2^*(\lambda_1) \text{ and}$$

$$(5.12) \quad E(N^2) = 2 \left\{ [1 - B_2^*(\lambda_1)] / B_2^*(\lambda_1) \right\}^2 + [1 - B_2^*(\lambda_1)] / B_2^*(\lambda_1) .$$

Consider the queue with independent component breakdown or equivalently the queue with two classes of customers and a preemptive discipline. The interruption of service for a low priority customer is due to arrival of a high priority customer. The low priority customer re-enters the service mechanism at the end of a "busy period" of the high priority customers (busy period in the sense that only high priority customers are served). Let M be the number of high priority customers served during such a "busy period". Takacs [13] has shown that M has a probability generating function which satisfies

$$(5.13) \quad E(Z^M) = Z B_1^*(\lambda_1 - \lambda_1 E(Z^M)) .$$

Equation (5.13) may be obtained as follows: — Let S_1 be the service time of the first customer served during a busy period of high priority customers only. Suppose M_1 customers (high priority) arrive during the period S_1 . Define $p_1^{(m_1)}(m-m_1)$ as the probability that $m-m_1$ customers are served during m_1 independent busy periods. Now

$$P(M_1 = m_1/S_1) = e^{-\lambda_1 S_1} (\lambda_1 S_1)^{m_1} / m_1! .$$

Each of the M_1 customers that arrive during S_1 gives rise to a busy period. Thus

$$P(M = m + 1/S_1) = \sum_{m_1=0}^m e^{-\lambda_1 S_1} \frac{(\lambda_1 S_1)^{m_1} p_1^{(m_1)}(m-m_1)}{m_1!} .$$

Therefore

$$\begin{aligned} E(Z^M/S_1) &= \sum_{m=0}^{\infty} Z^{m+1} \sum_{m_1=0}^m e^{-\lambda_1 S_1} \frac{(\lambda_1 S_1)^{m_1} p_1^{(m_1)}(m-m_1)}{m_1!} , \\ &= Z \sum_{m_1=0}^{\infty} Z^{m_1} e^{-\lambda_1 S_1} (\lambda_1 S_1 E(Z^M))^{m_1} / m_1! \\ &= Z e^{-(\lambda_1 S_1 - \lambda_1 S_1 E(Z^M))} , \end{aligned}$$

and (5.13) follows upon taking expectations with respect to S_1 .

The following theorem is a direct consequence of (5.13).

Theorem 5.4. The number, M, of customers served during a busy period of a queue with arrivals at random rate λ_1 , service time distribution function $B_1(s)$ and no breakdowns satisfies: --

$$(5.14) \quad P(M = 0) = p_0 \quad \text{where } p_0 \text{ satisfies } p_0 = B_1^*(\lambda_1 - \lambda_1 p_0),$$

$$(5.15) \quad E(M) = (1 - m_1)^{-1}, \quad \text{and}$$

$$(5.16) \quad E(M^2) = \left[(1 + m_1)/(1 - m_1)^2 \right] + \left[m_2/(1 - m_1)^3 \right],$$

where $m_i = \lambda_1 E(S_1^i)$, $i = 1, 2$.

Thus the total number of top priority customers that arrive after a low priority customer enters the service mechanism and complete service earlier than the low priority customer, is $M \cdot N$. The number, M, of top priority customers serviced during a "busy period" of top priority customers only, is independent of the number, N, of such busy periods. Thus

$$(5.17) \quad E[(M \cdot N)^j] = E(M^j) E(N^j), \quad j = 1, 2,$$

where the factors on the right side are given by Theorems 5.1 to 5.4, depending on the preemptive discipline.

2. Number of Breakdowns During Queueing Time for the Preemptive Resume Discipline. In this section we will restrict our consideration to the preemptive resume discipline. The limiting distribution of the unexpired repair and service times (X, Y) was characterized in Chapters II and III for four types of breakdown. Suppose a cus-

customer arrives at the queue when the unexpired repair and service times satisfy the limiting distribution. This customer joins the waiting line for a period $X + Y$ plus a period of time equal to the repair times of all breakdowns that may arise in obtaining a period X , of repair and a period Y of useful service.

Let N_1 be the number of occurrences during a period X of a random event with rate λ_1 and let N_2 be the number of occurrences of this random event during a period Y . Thus

$$P(N_1 = n_1, N_2 = n_2 / X, Y) = e^{-\lambda_1(X+Y)} (\lambda_1 X)^{n_1} (\lambda_1 Y)^{n_2} / n_1! n_2! .$$

Therefore the joint probability generating function of N_1 and N_2 is

$$E(Z_1^{N_1} Z_2^{N_2}) = E \left[e^{-(\lambda_1 - \lambda_1 Z_1)X} \cdot e^{-(\lambda_1 - \lambda_1 Z_2)Y} \right] .$$

Whence

$$(5.18) \quad E(N_1) = \lambda_1 E(X) , \quad E(N_2) = \lambda_1 E(Y) ,$$

$$(5.19) \quad E(N_1^2) = \lambda_1^2 E(X^2) + \lambda_1 E(X) , \quad E(N_2^2) = \lambda_1^2 E(Y^2) + \lambda_1 E(Y) ,$$

and

$$(5.20) \quad E(N_1 N_2) = \lambda_1^2 E(XY) .$$

Suppose we consider a queue with unit breakdown. The breakdowns during the queueing time of a customer occur only during the period Y of unexpired service time. Thus, for unit breakdown, the number of breakdowns during queueing time is simply N_2 with moments given by (5.18) and (5.19). The moments of the unexpired service Y are

given by (3.13) and (3.16) for active unit and by (3.23) and (3.26) for independent unit breakdown.

Under a component breakdown system, the number of breakdown busy periods which occur during the queueing time of a customer is $N_1 + N_2$ with moments given by (5.18) to (5.20). The moments of X and Y are given by (3.32) to (3.36) for active component breakdown and by (2.46) to (2.50) for independent component breakdown. Each of these $N_1 + N_2$ busy periods of repair times in isolation is composed of M breakdowns where the moments of M are given by (5.15) and (5.16). Thus the total number of breakdowns that occur during the queueing time of a customer is the random variable $(N_1 + N_2)M$ for component breakdown. Each of these busy periods of repair times in isolation, is composed of M breakdowns where the moments of M may be evaluated from (5.15) and (5.16). Thus the total number of breakdowns that occur during the queueing time of one customer is the random variable $(N_1 + N_2)M$ for component breakdown. Clearly $(N_1 + N_2)$ and M are independent.

Let N be the number of breakdowns that occur during the queueing time of one customer, for a preemptive resume discipline. By virtue of the arguments advanced in the preceding two paragraphs, it is straightforward to obtain moments of the random variable N from known formulae. Hence we have the following theorem.

Theorem 5.5. The first two moments of N , the number of breakdowns during the queueing time of one customer, are listed in table 5.1.

TABLE 5.1
 MOMENTS OF N, THE NUMBER OF BREAKDOWNS DURING QUEUEING TIME

	UNIT		COMPONENT	
	Active	Independent	Active	Independent
$E(N)$	$\lambda_1 E(Y)$ (3.13)*	$\lambda_1 E(Y)$ (3.23)	$\lambda_1 E(X+Y) [1 - m_1]^{-1}$ (3.32) (3.33)	$\lambda_1 E(X+Y) [1 - m_1]^{-1}$ (2.46) (2.47)
$E(N^2)$	$\lambda_1^2 E(Y^2) + \lambda_1 E(Y)$ (3.13), (3.16)	$\lambda_1^2 E(Y^2) + \lambda_1 E(Y)$ (3.23), (3.26)	$\lambda_1^2 E(X+Y)^2 + \lambda_1 E(X+Y) \left[\frac{1 + m_1 - m_1^2}{(1 - m_1)^3} \right]$ (3.32) to (3.36)	$\lambda_1^2 E(X+Y)^2 + \lambda_1 E(X+Y) \left[\frac{1 + m_1 - m_1^2}{(1 - m_1)^3} \right]$ (2.46) to (2.50)

*The value of each entry is given by a general formula containing moments of unexpired repair and service times. For each entry, the values of the terms of the general formula are given by the equations for which the numbers are listed.

A similar argument for the preemptive repeat disciplines is not valid when using (X,Y) , the unexpired repair and completion times. This follows from the fact that the completion time for a customer contains the breakdown repair times for breakdowns that occur during service of that customer. Thus the length of the unexpired completion time is not independent of the number of breakdowns that occur during that period.

3. Miscellaneous Remarks Regarding Transient Behavior. Consider (X_t, Y_t) , the stochastic vector of unexpired repair and service times of a queue with independent component breakdown and a preemptive resume discipline. Recall that this stochastic vector was investigated in Chapter II as the store content of a mathematically equivalent storage process. The time dependent components of the distribution function of the store content were defined as:

$$\begin{aligned} \pi(t) &= P(X_t = Y_t = 0) \quad , \\ H(x;t) &= P(0 < X_t \leq x, Y_t = 0) \quad , \\ G(y;t) &= P(X_t = 0, 0 < Y_t \leq y) \quad , \\ F(x,y;t) &= P(0 < X_t \leq x, 0 < Y_t \leq y) \quad , \\ P_t(x,y) &= \pi(t) + H(x;t) + G(y;t) + F(x,y;t) \quad . \end{aligned}$$

The respective components of the limiting distribution function were defined as π , $H(x)$, $G(y)$, $F(x,y)$ and $P(x,y)$. The transient behavior of the process is characterized in terms of the time dependent components of the distribution function.

In Chapter II, probability considerations led to a set of four inequalities for each of the time dependent components (for example, see (2.5) and (2.6)). These inequalities were combined to provide a characterization of the transient behavior in equation (2.20).

This is

$$\frac{\partial P_t^{xy}(u,v)}{\partial t} = \int [u-\phi] [F^{xy}(u,v;t) + H^x(u;t)] + \int [v-\phi] G^y(v;t) - \phi \pi(t) ,$$

where u and v are Laplace-Stieltjes transform variables with respect to x and y , respectively and $\phi = \lambda_1 + \lambda_2 - \lambda_1 B_1^x(u) - \lambda_1 B_2^y(v)$.

Each of the four sets of inequalities were employed individually to derive a functional equation involving components of the limiting distribution (see (2.36) to (2.39)). It might be thought that each set of inequalities should lead to a characterization of the transient behavior for components of the distribution function. In general this is not true and in this section we illustrate reasons why this is not so.

Consider (2.7) and (2.8) where $\tau > 0$. Thus

$$\begin{aligned} \frac{H^x(u;t+\tau) - H^x(u;t)}{\tau} &\leq \left[\frac{e^{(u-\lambda)\tau} - 1}{\tau} \right] H^x(u;t) - \frac{e^{(u-\lambda)\tau}}{\tau} \int_0^\tau e^{-ux} d_x H(x;t) \\ &+ \frac{e^{-\lambda\tau}}{\tau} \int [\lambda_1 \tau \pi(t) B_1^x(u) + \lambda_1 \tau H^x(u;t) B_1^x(u)] \\ &+ o(1) , \end{aligned}$$

and

$$\begin{aligned} \frac{H^X(u;t+\tau) - H^X(u;t)}{\tau} &\geq \left[\frac{e^{(u-\lambda)\tau} - 1}{\tau} \right] H^X(u;t) - \frac{e^{(u-\lambda)\tau}}{\tau} \int_0^\tau e^{-ux} d_x H(x;t) \\ &\quad + \frac{e^{-\lambda\tau}}{\tau} \left[\lambda_1 \tau (\pi(t) + H^X(u;t)) B_1^X(u) \right] \\ &\quad - o(1) . \end{aligned}$$

Suppose we proceed formally to limits as $\tau \downarrow 0+$, under the assumption that all limits exist. Then the right hand time derivative satisfies:

$$(5.21) \quad \frac{\partial H^X(u;t)}{\partial t} = \left[u - \lambda_1 - \lambda_2 + \lambda_1 B_1^X(u) \right] H^X(u;t) \\ + \lambda_1 \pi(t) B_1^X(u) - h(0;t) ,$$

$$\text{where } h(0;t) = \lim_{\tau \downarrow 0+} \left[\frac{e^{(u-\lambda)\tau}}{\tau} \int_0^\tau e^{-ux} d_x H(x;t) \right]$$

by assumption. Since

$$\frac{e^{-\lambda\tau} H(\tau;t)}{\tau} \leq \frac{e^{(u-\lambda)\tau}}{\tau} \int_0^\tau e^{-ux} d_x H(x;t) \leq \frac{e^{(u-\lambda)\tau}}{\tau} H(\tau;t)$$

thus we have assumed that

$$\lim_{\tau \rightarrow 0+} \left[H(\tau;t) / \tau \right] = h(0;t) .$$

Thus equation (5.21) is valid for all t only if the limit, as $\tau \rightarrow 0+$, of $\left[H(\tau;t) / \tau \right]$ exists for all t . We will show that this limit does not exist for all t by two counterexamples.

Since $B_1(s)$ is a distribution function, the right hand derivative exists a.e. (s). Suppose that the right hand derivative is infinite at the countable sequence of points $s_1^*, s_2^*, s_3^*, \dots$. Further suppose that we consider the process (X_t, Y_t) started at time zero with the arrival at an empty store of c_1 material of load size S_1 with distribution function $B_1(s)$. Thus $0 < X_t \leq \tau, Y_t = 0$ if:

- (a) there are zero arrivals in the time interval $(0, t]$ and $t < S_1 \leq t + \tau$, or
- (b) there is at least one arrival in the time interval $(0, t)$ with appropriate limitations on the load sizes and order of arrivals.

Thus

$$H(\tau; t) \geq e^{-(\lambda_1 + \lambda_2)t} [B_1(t + \tau) - B_1(t)],$$

and

$$\frac{H(\tau; t)}{\tau} \geq e^{-(\lambda_1 + \lambda_2)t} \left[\frac{B_1(t + \tau) - B_1(t)}{\tau} \right].$$

From this it is clear that the limit of $[H(\tau; t)/\tau]$, as $\tau \rightarrow 0+$ does not exist for $t = s_1^*, s_2^*, s_3^*, \dots$.

For the second counterexample consider the initial distribution function of (X_t, Y_t) . Suppose its component $H(x; 0)$ has an infinite right hand derivative at the countable set of points $x_1^*, x_2^*, x_3^*, \dots$. Now $0 < X_t \leq \tau, 0 = Y_t$ if :

- (a) Zero arrivals occur in the time interval $(0, \underline{t}]$ and $t < X_0 \leq t + \tau$, $Y_0 = 0$.
- (b) One or more arrivals in time interval $(0, \underline{t}]$ with certain appropriate limitations on the process.

Thus

$$\frac{H(\tau; t)}{\tau} \geq e^{-(\lambda_1 + \lambda_2)t} \left[\frac{H(t+\tau; 0) - H(t; 0)}{\tau} \right].$$

Hence the limit of $\lfloor H(\tau; t)/\tau \rfloor$, as $\tau \downarrow 0+$ does not exist for $t = x_1^*, x_2^*, x_3^*, \dots$.

Therefore we have shown that necessary conditions for the validity of (5.21), for all t , are; right hand derivatives exist everywhere for the load size distribution function $B_1(s)$ and the component $H(x; 0)$ of the initial distribution function. It is not known whether these are sufficient conditions.

We have shown by counterexample that (5.21) may be invalid on certain sets of measure zero on the time axis. It is not known whether or not (5.21) actually is valid a.e. (t). The investigation of these questions is interesting; however, the resulting four equations characterizing the transient behavior are most intractable. We list below the other three equations analogous to (5.21) without any claim to their validity. Suppose the limits, as $\tau \rightarrow 0+$, are assumed to exist and defined as follows:

$$\lim_{\tau \rightarrow 0^+} \frac{\int G(\tau;t)/\tau}{\tau} = g(0;t) ,$$

$$\lim_{\tau \rightarrow 0^+} \frac{\int J(\tau;t)/\tau}{\tau} = j(0;t) ,$$

$$\lim_{\tau \rightarrow 0^+} \frac{\int F^y(\tau,v;t)/\tau}{\tau} = f^y(0,v;t) .$$

Then the right hand derivatives with respect to time satisfy;

$$(5.22) \quad \frac{\partial \pi(t)}{\partial t} = -(\lambda_1 + \lambda_2)\pi(t) + g(0;t) + h(0;t) + j(0;t) ,$$

$$(5.23) \quad \frac{\partial G^y(v;t)}{\partial t} = \int v - \lambda_1 - \lambda_2 + \lambda_2 B_2^y(v) \int G^y(v;t) + \lambda_2 \pi(t) B_2^y(v) \\ + f^y(0,v;t) - g(0;t) - j(0;t) ,$$

and

$$(5.24) \quad \frac{\partial F^{xy}(u,v;t)}{\partial t} = \int u - \lambda_1 - \lambda_2 + \lambda_1 B_1^x(u) + \lambda_2 B_2^y(v) \int F^{xy}(u,v;t) \\ + \lambda_1 B_1^x(u) G^t(v;t) + \lambda_2 B_2^y(v) H^x(u;t) \\ - f^y(0,v;t) .$$

The sum of the respective sides of the four equations (5.21) to (5.24) is equation (2.20), which is rigorously derived in Chapter II .

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