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ON THE FALSITY OF EULER'S CONJECTURE ABOUT THE NON-EXISTENCE OF TWO ORTHOGONAL LATIN SQUARES OF ORDER 4t+2

(Preliminary Report)

by

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Institute of Statistics
Mimeograph Series No. 220
March, 1959

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1. Introduction. The purpose of this paper is to prove a general theorem on the existence of pairwise orthogonal Latin squares (p.o.l.s.) of a given order and to give a counter example to Euler's [3] conjecture that there do not exist two p.o.l.s. of order 4t+2.

2. Definitions. An arrangement of v objects (called treatments) in b sets (called blocks) will be called a pairwise balanced design of index unity and type $(v; k_1, k_2, \ldots, k_m)$ if each block contains either $k_1, k_2, \ldots,$ or k_m treatments which are all distinct $(k_i \leq v,\ k_i \neq k_j)$, and every pair of distinct treatments occurs exactly in one block of the design. If the number of blocks containing k_i treatments is b_i, then clearly

$$(1) \qquad b = \sum_{i=1}^{m} b_i, \qquad v(v-1) = \sum_{i=1}^{m} b_i k_i (k_i - 1)$$

3. Lemma 1. Suppose there exists a set Σ of q-1 p.o.l.s. of order k, then we can construct a $q \times k(k-1)$ matrix P, whose elements are the symbols 1, 2, ..., k and such that any ordered pair $\binom{i}{j}$ $i \neq j$, occurs as a column exactly once in any two rowed submatrix of P.

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We can take the set Σ in the standard form in which the first row of each Latin square contains the symbols 1, 2, ..., k in that order. We then prefix to the set Σ a kxk square containing the symbol i in each position in the i-th column. If we then write the elements of each square in a single row such that the symbol in the i-th row and j-th column occupies the n-th position in the row, where $n = k(i-1) + j$ then we can display these squares as in [2] in the form of an orthogonal array $A[k^2, q, k, 2]$ of q rows. By deleting the first k columns, we get the matrix P with the required properties.

Let γ be a column of k distinct treatments $t_1, t_2, \ldots, t_k$ in that order, then we shall denote by $P(\gamma)$, the $q \times k(k-1)$ matrix obtained by replacing the symbol i in P, by the treatment t_i occupying the i-th position in $\gamma (i = 1,2,\ldots,k)$. Clearly every treatment occurs exactly k-1 times in every row of $P(\gamma)$, and any order pair $\binom{t_i}{t_j}$ occurs as a column exactly once in any two rowed submatrix of $P(\gamma)$.

4. <u>Theorem 1.</u> Let there exist a pairwise balanced design of index unity and type $(v; k_1,k_2,\ldots,k_m)$ and suppose there exist $q_i - 1$ p.o.l.s. of order k_i. If

$$q = \min(q_1,q_2,\ldots,q_m)$$

then there exist q-2 p.o.l.s. of order v.

Let the treatments of the design be $t_1, t_2, \ldots, t_v$, and let the blocks of the design (written out as columns) which contain k_i treatments be denoted by $\gamma_{i1}, \gamma_{i2}, \ldots, \gamma_{ib_i}$. Let P_i be the matrix of order $q_i \times k_i(k_i-1)$ defined in Lemma 1, the elements of P_i being the symbols 1, 2, ..., k_i. Let $C_{ij} = P_i(\gamma_{ij})$ be the matrix obtained from P_i and γ_{ij}.

Retain only q rows of C_{ij} to get C^*_{ij}. From (1) the matrix

$$C^* = [C^*_{11}, C^*_{12}, \ldots, C^*_{1b_1}, \ldots, C^*_{i1}, C^*_{i2}, \ldots, C^*_{ib_i}, \ldots, C^*_{m1}, C^*_{m2}, \ldots, C^*_{mb_m}]$$

is of order q x v(v-1), and is such that any ordered pair of treatments $\binom{t_i}{t_j}$, $i \neq j$ occurs as a column exactly once in any two rowed submatrix of C^*. Let C^*_o be a qxv matrix whose i-th column contains t_i in every position $(i = 1,2,\ldots,v)$. Then from [2], the matrix $[C^*_o, C^*]$ is an orthogonal array $A[v^2, q, v, 2]$. Using two rows to coordinatize we get a set of q-2 p.o.l.s. of order v.

5. <u>Counter examples to Euler's conjecture</u>. Consider the balanced incomplete block (BIB) design with parameters $v^* = 15$, $b^* = 35$, $r^* = 7$, $k^* = 3$, $\lambda^* = 1$. A resolvable solution is given in Table III of [1]. To each block of the i-th complete replication add a new treatment θ_i $(i = 1,2,\ldots,7)$ and take a new block consisting of the treatments θ_1, θ_2, θ_3, θ_4, θ_5, θ_6, θ_7. We then get a pairwise balanced design of index unity and type (22; 4, 7). Since there exist 3 p.o.l.s. of order 4, and 6 p.o.l.s. of order 7, it follows from the theorem that there exist two orthogonal Latin squares of order 22. The actual squares are given in the Appendix.

A detailed paper generalizing and improving the results of Mann [4, p. 105] and Parker [5] is being prepared where among other things it will be shown that there are an infinity of values of t for which there exist two or more p.o.l.s. of order 4t+2.

References

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[2] K. A. Bush, "Orthogonal arrays of index unity," Ann. Math. Stat., 23 (1952), pp. 426-434.

[3] L. Euler, "Recherches sur une nouvelle espece de quarres magiques," Verh. Genootsch. der Wet. Vlissingen, 9 (1782), pp. 85-232.

[4] H. B. Mann, Analysis and design of experiments, Dover, 1949, p. 105.

[5] E. T. Parker, "Construction of some sets of pairwise orthogonal Latin squares," Am. Math. Soc. Notices, 5 (1958) p. 815.

APPENDIX

(L_1)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
1	1	4	7	16	6	20	22	15	19	21	12	18	10	9	17	2	8	11	14	5	13	3
2	16	2	5	1	17	7	21	10	15	20	22	13	19	11	18	4	3	9	12	8	6	14
3	22	17	3	6	2	18	1	12	11	15	21	16	14	20	19	8	5	4	10	13	9	7
4	2	16	18	4	7	3	19	21	13	12	15	22	17	8	20	1	9	6	5	11	14	10
5	20	3	17	19	5	1	4	9	22	14	13	15	16	18	21	11	2	10	7	6	12	8
6	5	21	4	18	20	6	2	19	10	16	8	14	15	17	22	9	12	3	11	1	7	13
7	3	6	22	5	19	21	7	18	20	11	17	9	8	15	16	14	10	13	4	12	2	1
8	17	20	16	14	22	11	13	8	5	2	19	3	18	21	1	12	15	7	6	10	4	9
9	14	18	21	17	8	16	12	22	9	6	3	20	4	19	2	10	13	15	1	7	11	5
10	13	8	19	22	18	9	17	20	16	10	7	4	21	5	3	6	11	14	15	2	1	12
11	18	14	9	20	16	19	10	6	21	17	11	1	5	22	4	13	7	12	8	15	3	2
12	11	19	8	10	21	17	20	16	7	22	18	12	2	6	5	3	14	1	13	9	15	4
13	21	12	20	9	11	22	18	7	17	1	16	19	13	3	6	5	4	8	2	14	10	15
14	19	22	13	21	10	12	16	4	1	18	2	17	20	14	7	15	6	5	9	3	8	11
15	8	9	10	11	12	13	14	17	18	19	20	21	22	16	15	7	1	2	3	4	5	6
16	4	1	12	2	13	10	15	3	6	9	5	8	11	7	14	16	18	20	22	17	19	21
17	15	5	2	13	3	14	11	1	4	7	10	6	9	12	8	22	17	19	21	16	18	20
18	12	15	6	3	14	4	8	13	2	5	1	11	7	10	9	21	16	18	20	22	17	19
19	9	13	15	7	4	8	5	11	14	3	6	2	12	1	10	20	22	17	19	21	16	18
20	6	10	14	15	1	5	9	2	12	8	4	7	3	13	11	19	21	16	18	20	22	17
21	10	7	11	8	15	2	6	14	3	13	9	5	1	4	12	18	20	22	17	19	21	16
22	7	11	1	12	9	15	3	5	8	4	14	10	6	2	13	17	19	21	16	18	20	22

(L_2)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
1	1	16	22	2	20	5	3	17	14	13	18	11	21	19	8	4	15	12	9	6	10	7
2	4	2	17	16	3	21	6	20	18	8	14	19	12	22	9	1	5	15	13	10	7	11
3	7	5	3	18	17	4	22	16	21	19	9	8	20	13	10	12	2	6	15	14	11	1
4	16	1	6	4	19	18	5	14	17	22	20	10	9	21	11	2	13	3	7	15	8	12
5	6	17	2	7	5	20	19	22	8	18	16	21	11	10	12	13	3	14	4	1	15	9
6	20	7	18	3	1	6	21	11	16	9	19	17	22	12	13	10	14	4	8	5	2	15
7	22	21	1	19	4	2	7	13	12	17	10	20	18	16	14	15	11	8	5	9	6	3
8	15	10	12	21	9	19	18	8	22	20	6	16	7	4	17	3	1	13	11	2	14	5
9	19	15	11	13	22	10	20	5	9	16	21	7	17	1	18	6	4	2	14	12	3	8
10	21	20	15	12	14	16	11	2	6	10	17	22	1	18	19	9	7	5	3	8	13	4
11	12	22	21	15	13	8	17	19	3	7	11	18	16	2	20	5	10	1	6	4	9	14
12	18	13	16	22	15	14	9	3	20	4	1	12	19	17	21	8	6	11	2	7	5	10
13	10	19	14	17	16	15	8	18	4	21	5	2	13	20	22	11	9	7	12	3	1	6
14	9	11	20	8	18	17	15	21	19	5	22	6	3	14	16	7	12	10	1	13	4	2
15	17	18	19	20	21	22	16	1	2	3	4	5	6	7	15	14	8	9	10	11	12	13
16	2	4	8	1	11	9	14	12	10	6	13	3	5	15	7	16	20	17	21	18	22	19
17	8	3	5	9	2	12	10	15	13	11	7	14	4	6	1	20	17	21	18	22	19	16
18	11	9	4	6	10	3	13	7	15	14	12	1	8	5	2	17	21	18	22	19	16	20
19	14	12	10	5	7	11	4	6	1	15	8	13	2	9	3	21	18	22	19	16	20	17
20	5	8	13	11	6	1	12	10	7	2	15	9	14	3	4	18	22	19	16	20	17	21
21	13	6	9	14	12	7	2	4	11	1	3	15	10	8	5	22	19	16	20	17	21	18
22	3	14	7	10	8	13	1	9	5	12	2	4	15	11	6	19	16	20	17	21	18	22