DISTRIBUTION OF LINEAR CONTRASTS OF ORDER STATISTICS[1]

by

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INTRODUCTION

It can be said that, up to the last decades or so, two different types of approaches have been used in the tackling of practical and theoretical problems of statistical nature.

The first, mostly applied to data of the quantitative type, relied upon properties of the assumed distribution of the parent population, and on the properties of the sample-distributions of the statistics derived from the observations. That type may be called the parametric approach. The second, mostly applied to data of qualitative flavor, relied more heavily upon the observations and their properties, especially the rankings. That is the nonparametric approach.

It was felt that a mixture of the two methods, using, for instance, the information contained in the ordered (or ranked) sample values, and some properties of the assumed distribution of the parent population, would provide answers that could not be obtained some other way.

Many authors, working abng that line, have brought forth a great wealth of precious information. Wilks [10] [1]

1. Numbers in square brackets refer to bibliography.

presented a comprehensive study of order statistics including a rather complete bibliography up to the time of its publication. Mosteller [12], Godwin [5], and Nair [15], studied the properties of particular combinations of "ordered" sample values, called, by Mosteller, "systematic statistics".

However, no unified treatment of a class of ordered statistics has ever been published, especially in non null situations, and it was thought worthy of investigation to study, in more detail, the properties of the distribution of order statistics.

In the following work, some properties of the distribution of linear contrasts of order statistics are investigated.

The null distribution of linear contrasts of order statistics is obtained, first, (Chapter I) in the cases of three and four variables; then (Chapter II), in the general case of n variables.

Chapter III, consists of a detailed study of the properties of the moments, and related quantities, of the linear contrasts of ordered statistics in the case of three dimensions.

In Chapter IV, the non null distribution of linear contrasts of order statistics, in the case of three variables, is derived, and two particular hypotheses are considered.

Finally, on the basis of the information gathered in the first four chapters, a decision rule to pick out the population with the largest mean is suggested in the fifth chapter, and some properties of the decision rule are discussed.

CHAPTER I

NULL DISTRIBUTION OF LINEAR CONTRASTS

OF ORDER STATISTICS. CASE OF THREE AND FOUR VARIABLES.

1. Case of three variables. Suppose we are given x_0, x_1 and x_2, three independent random variables, normally distributed with unknown means, m_0, m_1 and m_2, respectively, and with a known common variance σ^2. We shall assume, for the present, that $\sigma^2 = 1$.

We may order the set of values, in a random sample of the above variates, from greatest to least, and denote the ranked values by $x_{(0)} > x_{(1)} > x_{(2)}$. It is to be noted that the probability of ties is zero in the continuous case; however, because of the limitations of the measuring instruments, ties will occur in experimental cases. In those situations, the tied sample values will be "ranked" using a random procedure assigning equal probability to each ranking.

Any combination of the ordered sample values of random variables is called "order statistic." In this work, we shall be concerned with linear contrasts of order statistics. A linear contrast is defined as follows: consider a set of random variables $y_1, y_2, \ldots, y_n$. Any linear combination $a_1y_1 + a_2y_2 + \ldots + a_ny_n$ is a linear contrast, provided $\sum_{i=1}^{n} a_i = 0$.

In the case of three random variables, we may consider the following contrast $c_0x_{(0)} + c_1x_{(1)} + c_2x_{(2)}$, where $\sum_{i=0}^{2} c_i = 1$. It will be convenient to set $c_0 = 1$; so we shall, henceforth, consider the expression

$$x_{(0)} - c_1x_{(1)} - c_2x_{(2)} \quad , \tag{1.1.1}$$

where $c_1 + c_2 = 1$, $c_i \geq 0$, $i = 1, 2$.

The above expression can be written, in terms of a single unspecified parameter, as follows:

$$x_{(0)} - cx_{(1)} - (1-c)x_{(2)} \;, \quad 0 \leq c \leq 1 \;. \tag{1.1.2}$$

We shall, presently, derive the probability density function of (1.1.2) under the null hypothesis:

$$H_0\colon \; m_0 = m_1 = m_2 = 0 \quad (\text{say}) \;. \tag{1.1.3}$$

The joint probability density of $x_{(0)}$, $x_{(1)}$ and $x_{(2)}$ may be written, (see Wilks [19]), :

$$f(x_{(0)}, x_{(1)}, x_{(2)}) = 3! \, [2\pi]^{-3/2} \exp [-\tfrac{1}{2} \sum_{i=0}^{2} x_{(i)}^2] . \tag{1.1.4}$$

First, let us introduce the following transformation

$$\begin{aligned} u_0 &= x_{(0)} + x_{(1)} + x_{(2)} \\ u_1 &= x_{(0)} - x_{(1)} \\ u_2 &= x_{(1)} - x_{(2)} \quad . \end{aligned} \tag{1.1.5}$$

In matrix notation, the above can be written,

$$U^* = AX \quad , \tag{1.1.6}$$

where U* and X are column vectors; i.e.,

$$U^* = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix} , \qquad X = \begin{bmatrix} x_{(0)} \\ x_{(1)} \\ x_{(2)} \end{bmatrix} ,$$

and where A is the square matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} .$$

The Jacobian of the transformation is easily checked to be equal to 1/3. We thus have

$$\sum_{i=0}^{2} x_{(i)}^2 = X'X = (A^{-1}U^*)'\,A^{-1}U^* = U^{*\prime}\,B^*\,U^* \text{ , where } B^* = (A^{-1})'\,A^{-1}.$$

It is readily found that

$$B^* = 1/3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = 1/3\ B(\text{say}), \text{ the definition}$$

of B being obvious.

We thus have, for the joint density of the u-variables, the following expression,

$$(1.1.7)\quad f(u_0, u_1, u_2) = 2\,[2\pi]^{-3/2} \exp\left(-\tfrac{1}{6} U^{*\prime} B U^{*}\right),$$

where

$$-\infty < u_0 < +\infty \quad ; \quad u_i > 0, \quad i = 1, 2.$$

The variable u_0 is readily integrated out; and, since,

$$\int_{-\infty}^{+\infty} (2\pi)^{-\frac{1}{2}} \exp(-u_0^2/6)\, du_0 = \sqrt{3},$$

we get

$$(1.1.8)\quad f(u_1, u_2) = 2(2\pi)^{-1}\sqrt{3}\exp\left(-\tfrac{1}{6} U' C U\right),$$

where

$$U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix};$$

U being a column vector, and C a symmetric matrix obtained from the matrix B by deleting the first row and first column.

We are now in a position to introduce our linear contrast given by (1.1.2).

Let

$$(1.1.10)\quad \begin{aligned} z_1 &= u_1 + (1-c)u_2 \\ z_2 &= u_2, \end{aligned} \qquad \text{where } 0 \leq c \leq 1.$$

The variate z_1 is precisely the linear contrast (1.1.2). The Jacobian of the transformation is obviously equal to one. In matrix notation, (1.1.10) becomes

$$Z = D\,U \quad , \tag{1.1.11}$$

where

$$Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} , \qquad D = \begin{bmatrix} 1 & 1-c \\ 0 & 1 \end{bmatrix} .$$

It follows that

(1.1.12) $\quad U'\,C\,U = Z'(D^{-1})'\,C\,D^{-1}\,Z = Z'\,M\,Z,\quad$ where
$M = (D^{-1})'\,C\,D^{-1}$.

It is readily verified that the symmetric matrix M has the following form

$$M = \begin{bmatrix} 2 & 2c-1 \\ 2c-1 & 2(c^2-c+1) \end{bmatrix} . \tag{1.1.13}$$

The mapping of the u-space onto the z-space, involved in the transformation (1.1.11), is rather simple; the region of variation in the latter space being a wedge-shaped region, in the upper right hand quadrant, limited by the lines $z_2 = 0$, and $z_2 = z_1/(1-c)$, $c \neq 1$. Thus, the joint density of z_1 and z_2 is

(1.1.14) $f(z_1, z_2) = 2\sqrt{3}\ (2\pi)^{-1} \exp\left(-\frac{1}{6}\ Z'\,M\,Z\right),$

where M is defined by (1.1.13); the limits of variation being given by,

(1.1.15) $0 < z_2 < z_1/(1-c)\quad, \; 0 < z_1 \quad ; \qquad c \neq 1.$

The density function of $z_1 = x_{(0)} - cx_{(1)} - (1-c)x_{(2)}$ will be obtained from (1.1.14) by integrating out the variable z_2 ; formally, we have

(1.1.16) $$g(z_1) = \int_0^{z_1/(1-c)} f(z_1, z_2)\, dz_2 \qquad .$$

In the present case, the integration is easily carried out. Expanding the matrix M and collecting terms, we can write $f(z_1, z_2)$ as follows:

(1.1.17) $$f(z_1, z_2) = 2\sqrt{3}\ (2\pi)^{-1} \exp\left(-z_1^2/3\right)$$

$$\exp\left[-\frac{1}{3}\left\{(c^2 - c + 1)\, z_2^2 + (2c - 1)\, z_1\, z_z\right\}\right] .$$

Since

$$(c^2-c+1)z_2^2 + (2c-1)z_1z_2 = \left[(c^2-c+1)^{\frac{1}{2}}z_2 + (2c-1)z_1/2(c^2-c+1)^{\frac{1}{2}}\right]^2$$

$$- (2c-1)^2\, z_1^2/4(c^2-c+1),$$

we get, after rearranging the terms,

$$(1.1.18)\quad f(z_1, z_2) = 2\sqrt{3}\,(2\pi)^{-1} \exp\left\{-z_1^2/4(c^2-c+1)\right\}$$

$$\cdot \exp\left\{-\tfrac{1}{3}\left[(c^2-c+1)^{\frac{1}{2}} z_2 + (2c-1)z_1/2(c^2-c+1)^{\frac{1}{2}}\right]^2\right\}.$$

Hence, we can write formally

$$g(z_1) = \frac{2\sqrt{3}}{\sqrt{2\pi}} \exp\{-z_1^2/4(c^2-c+1)\}$$

$$\cdot \int_0^{z_1/(1-c)} (2\pi)^{-\frac{1}{2}} \exp\left\{-\tfrac{1}{3}\left[(c^2-c+1)^{\frac{1}{2}} z_2 + (2c-1)z_1/2(c^2-c+1)^{\frac{1}{2}}\right]^2\right\} dz_2.$$

Making use of the transformation

$(2/3)^{\frac{1}{2}} \left[(c^2-c+1)^{\frac{1}{2}} z_2 + (2c-1)z_1/2(c^2-c+1)^{\frac{1}{2}}\right] = t$, we can,

finally, write the density of $z_1 = x_{(0)} - cx_{(1)} - (1-c)x_{(2)}$ as follows:

$$(1.1.19)\quad g(z_1) = 3\left[\pi(c^2-c+1)\right]^{-\frac{1}{2}} \exp\{-z_1^2/4(c^2-c+1)\}$$

$$\cdot \int_{(2c-1)z_1/[6(c^2-c+1)]^{\frac{1}{2}}}^{(c+1)z_1/(1-c)[6(c^2-c+1)]^{\frac{1}{2}}} (2\pi)^{-\frac{1}{2}} \exp\left[-t^2/2\right] dt.$$

From expression(1.1.19)several important particular cases can be derived.

(i) If we set $c = 0$, we get the case of the range

(1.1.20) $\qquad w = x_{(0)} - x_{(2)}$, and the density comes out as

$$g(w) = \frac{3}{\sqrt{\pi}} \exp\left[-w^2/4\right] \int_{-w/\sqrt{6}}^{w/\sqrt{6}} (2\pi)^{\frac{1}{2}} \exp\left[-t^2/2\right] dt \quad ,$$

or,

(1.1.21)

$$g(w) = \frac{6}{\sqrt{\pi}} \exp\left[-w^2/4\right] \int_{0}^{w/\sqrt{6}} (2\pi)^{-\frac{1}{2}} \exp\left[-t^2/2\right] dt \quad ,$$

which is the form obtained by McKay and Pearson [11].

(ii) Setting $c = \frac{1}{2}$, we get

(1.1.22) $$v = x_{(0)} - (x_{(1)} + x_{(2)})/2 = 3u/2 \ ,$$

where

(1.1.23) $$u = x_{(0)} - (x_{(0)} + x_{(1)} + x_{(2)})/3 \quad .$$

The statistic **u** has been studied by McKay [10], and the studentized form of **u** by Nair [14] who called it the "extreme deviate from the sample mean". It is readily seen that

$$g(v) = \frac{\sqrt{6}}{\pi} \exp(-v^2/3) \int_0^{v\sqrt{2}} \exp(-t^2/2) \ dt \quad ;$$

putting $t = \sqrt{2}\, t'$, and substituting in the above expression, we get,(dropping the prime),

$$g(v) = \sqrt{6}\, \pi^{-1} \exp(-v^2/3) \int_0^{v} \exp(-t^2) \ dt \quad .$$

It follows that the density for the statistic u is given by

$$(1.1.24)\quad g(u) = (3)^{\frac{3}{2}} (2)^{-\frac{1}{2}}\pi^{-1} \exp(-3u^2/4) \int_0^{3u/2} \exp(-t^2)\, dt ,$$

which is the form gotten by McKay [10].

(iii) The case of the difference between the two largest sample values can be obtained, as a limiting case from (1.1.19), allowing $c \longrightarrow 1$.

Setting $y = x_{(0)} - x_{(1)}$, we get

$$(1.1.25)\quad g(y) = 3\pi^{-1}(2)^{-\frac{1}{2}} \exp(-y^2/4) \int_{y/\sqrt{6}}^{+\infty} \exp(-t^2/2)\, dt .$$

Ordinates of $g(z_1)$ have been obtained for values of z_1, 0(0.2)4., with the help of tables of the normal probability function, [17], [18]; and, tables of the exponential function [16]. Table I summarizes the results for several values of the parameter c.

TABLE I

Table of ordinates of $g(z_1)$, for various values of the constant

c, where $z_1 = x_{(0)} - cx_{(1)} - (1-c)x_{(2)}$

	g(z)							
z \ c	0	0.1	0.2	0.4	0.6	0.8	0.9	1.0
0.0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.84628
0.2	.10917	.12101	.13709	.17898	.26659	.49282	.73839	.78334
0.4	.21095	.23318	.26050	.33877	.47562	.70969	.75554	.70763
0.6	.29932	.32877	.36410	.45941	.59859	.71048	.67281	.62378
0.8	.36927	.40194	.43988	.53834	.64049	.63299	.58344	.53652
1.0	.41774	.44958	.48459	.55897	.60386	.54116	.49344	.45022
1.2	.44376	.47102	.49861	.54388	.53555	.45020	.40687	.36855
1.4	.44833	.46822	.48548	.49838	.45187	.36497	.32709	.29429
1.6	.43408	.44502	.45086	.43473	.36725	.28832	.25636	.22920
1.8	.40476	.40647	.40149	.36270	.28937	.22196	.19588	.17410
2.0	.36474	.35800	.34410	.29160	.22168	.16649	.14590	.12896
2.2	.31842	.30485	.28468	.22659	.16529	.12170	.10593	.09315

Table I (continued)

	g(z)							
z \ c	0	0.1	0.2	0.4	0.6	0.8	0.9	1.0
2.4	.26981	.25145	.23115	.17064	.12000	.08668	.07497	.06560
2.6	.22221	.20121	.17665	.12479	.08484	.06016	.05171	.04504
2.8	.17809	.15639	.13290	.08871	.05840	.04068	.03477	.03016
3.0	.13903	.11819	.09715	.06135	.03918	.02679	.02278	.01968
3.2	.10580	.08692	.06904	.04130	.02556	.01720	.01455	.01252
3.4	.07853	.06225	.04775	.02707	.01625	.01076	.00905	.00777
3.6	.05690	.04345	.03215	.01727	.01006	.00656	.00548	.00469
3.8	.04026	.02975	.02109	.01073	.00606	.00389	.00327	.00277
4.0	.02782	.01971	.01348	.00650	.00356	.00225	.00188	.00159

2. Case of four dimensions. Suppose, now, that we have four independent random variables x_0, x_1, x_2, and x_3, normally distributed with unknown means, m_0, m_1, m_2 and m_3 respectively, and with a known common variance $\sigma^2 = 1$ (say). Let us denote by $x_{(0)} > x_{(1)} > x_{(2)} > x_{(3)}$, the ordered sample values of the above variates; we shall derive the probability density function of the linear contrast $x_{(0)} - c_1 x_{(1)} - c_2 x_{(2)} - c_3 x_{(3)}$;

$\sum_{i=1}^{3} c_i = 1$; $0 \leq c_i \leq 1$, $(i = 1, 2, 3)$; or, more precisely, of

its equivalent form: $x_{(0)} - c_1 x_{(1)} - c_2 x_{(2)} - (1 - c_1 - c_2) x_{(3)}$, $c_1 + c_2 \leq 1$, $0 \leq c_i \leq 1$, $(i = 1, 2)$, under the null hypothesis H_0: $m_0 = m_1 = m_2 = m_3 = 0$ (say).

First, the joint density of $x_{(0)}$, $x_{(1)}$, $x_{(2)}$ and $x_{(3)}$ is given by

$$(1.2.1)\quad f(x_{(0)}, x_{(1)}, x_{(2)}, x_{(3)}) = 4!\ (2\pi)^{-2} \exp(-\tfrac{1}{2} \sum_{i=0}^{3} x_{(i)}^2).$$

Proceeding as in the previous case, we now make use of the transformation

$$(1.2.2)\quad u_0 = x_{(0)} + x_{(1)} + x_{(2)} + x_{(3)},$$

$$u_i = x_{(i-1)} - x_{(i)}, \qquad (i = 1,2,3).$$

In matrix notation, (1.2.2) becomes

(1.2.3) $$U^* = AX,$$

where U^* and X are column vectors, i.e.,

$$U = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}, \qquad X = \begin{bmatrix} x_{(0)} \\ x_{(1)} \\ x_{(2)} \\ x_{(3)} \end{bmatrix},$$

and A is the following matrix,

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

The Jacobian of the transformation is equal to 1/4. It follows that $\sum_{i=0}^{3} x_{(i)}^2 = X'X = U^{*\prime}(A^{-1})' A^{-1}U^* = U^{*\prime}B^*U^*$, the definition of B^* being obvious. A little algebra shows that

$$(1.2.4)\quad B^* = \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix} \equiv \frac{1}{4} B \text{ (say)} .$$

Consequently, we can write

$$(1.2.5)\qquad f(u_0,u_1,u_2,u_3) = 3!(2\pi)^{-2} \exp \left[- \frac{1}{8} U^{*'} B\, U^* \right] ,$$

where

$-\infty < u_0 < +\infty$; $u_i > 0$, $(i=1,2,3)$. Since u_0 is orthogonal to u_i, $(i=1,2,3)$, it can easily be integrated out.

But

$$\int_{-\infty}^{\infty} (2\pi)^{-\frac{1}{2}} \exp \left[-u_0^2/8 \right] du_0 = 2,$$

so we have

$$(1.2.6) \quad f(u_1,u_2,u_3) = 3!\ 2(2\pi)^{-\frac{3}{2}} \exp\left[-\tfrac{1}{8} U'C\, U\right] ,$$

where

$$U = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

; the matrix C being obtained from B by deleting the first row and first column.

We now introduce our statistic $x_{(0)}-c_1x_{(1)}-c_2x_{(2)}-(1-c_1-c_2)x_{(3)}$. Let us put

$$(1.2.7) \quad \begin{aligned} z_1 &= u_1 + (1-c_1)u_2 + (1-c_1-c_2)u_3 \\ z_i &= u_i, \qquad (i=2,3); \end{aligned}$$

the variate z_1 being precisely the linear contrast we are interested in. In matrix notation, the above transformation can be written $Z = DU$, where Z and U are column vectors, and D is given by

$$D = \begin{bmatrix} 1 & 1-c_1 & 1-c_1-c_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

The Jacobian of the transformation is obviously equal to unity. We thus have

$$U'C\,U = Z'(D^{-1})'\,C\,D^{-1}\,Z = Z'M\,Z \quad ,$$

where

$$M = (D^{-1})'\,C\,D^{-1} \quad .$$

Algebraic manipulation leads to the following form for the symmetric matrix M:

$$(1.2.8)\quad M = \begin{bmatrix} 3 & 3c_1-1 & 3c_1+3c_2-2 \\ & (3c_1-1)(c_1-1)+2(c_1+1) & (3c_1-1)(c_1+c_2-1)+c_1+1 \\ & & (3c_1+3c_2-2)(c_1+c_2-1)+c_1+c_2+2 \end{bmatrix} .$$

N.B. Some of the elements of the matrix M can be expanded and written down as polynomials; however, the above form sheds a little more light on the way the elements are built up. We now can write

$$(1.2.9) \qquad f(z_1,z_2,z_3) = 3!\, 2\, (2\pi)^{-\frac{3}{2}} \exp\left[-\frac{1}{8} Z' M Z\right] .$$

In order to know the domain of variation of the variables z, we have to investigate the mapping involved in the transformation (1.2.7). The relevant region of variation in the u-space consists of the octant limited by the positive portion of the axes Ou_i, $(i=1,2,3)$. The mapping of these three axes, involved in the transformation (1.2.7), proceeds as follows:

(i) The semi-axis Ou_1, determined by $u_2 = u_3 = 0$, is mapped into the semi-axis $O'z_1$: $z_2 = z_3 = 0$.

(ii) The semi-axis Ou_2 is determined by $u_1 = u_3 = 0$; the transformation (1.2.7), in this case, implies

$$(1.2.10) \qquad z_2 = \frac{z_1}{(1-c_1)} \quad , \quad z_3 = 0 \; ; \quad c_1 \neq 1 \quad .$$

Consequently Ou_2 is mapped into $O'L_2$, defined by the equations (1.2.10).

(iii) Finally, for the axis Ou_3, defined by $u_1 = u_2 = 0$, the transformation (1.2.7) implies

$$(1.2.11) \qquad z_2 = 0 \ , \ z_3 = \frac{z_1}{(1-c_1-c_2)} \ , \quad c_1 + c_2 \neq 1;$$

hence, Ou_3 is mapped into the line $O'L_3$ defined by (1.2.11).

Thus, in the z-space, the region of variation is a wedge-shaped region generated by the lines

$$O'z_1, \ O'L_2, \ O'L_3 \quad .$$

Since the equation of the plane determined by

$O'L_2$ and $O'L_3$ is $z_1 - (1-c_1)z_2 - (1-c_1-c_2)\, z_3 = 0$,

the variables z have the following limits of variation:

$$0 < z_3 < [z_1 - (1 - c_1)z_2]/(1 - c_1 - c_2),$$

(1.2.12) $$0 < z_2 < z_1/(1 - c_1) \; ,$$

$$0 < z_1 \quad .$$

In order to get an expression for the density of $z_1 = x_{(0)} - c_1 x_{(1)} - c_2 x_{(2)} - (1 - c_1 - c_2)x_{(3)}$, we have to integrate out the variables z_2 and z_3. This is best accomplished if, first, we reduce the matrix M to a diagonal form (except possibly, the first row and first column). That reduction is obtained by making use of the following transformation

(1.2.13)
$$\begin{aligned} v_1 &= z_1 , \\ v_2 &= (3c_1^2 - 2c_1 + 3)z_2 + (3c_1^2 + 3c_1c_2 - 3c_1 - c_2 + 2)z_3 , \\ v_3 &= z_3 ; \end{aligned}$$

the polynomial coefficients of z_2 and z_3 being merely the expanded form of the elements m_{22} and m_{23} of the matrix M, (expression 1.2.8). In matrix notation, the above transformation can be written V = NZ, where

(1.2.14)
$$N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (3c_1^2-2c_1+3) & (3c_1^2+3c_1c_2-3c_1-c_2+2) \\ 0 & 0 & 1 \end{bmatrix} .$$

The Jacobian of the transformation is obviously equal to $(3c_1^2 - 2c_1 + 3)^{-1}$, where $3c_1^2 - 2c_1 + 3 > 0$, for all values of c_1. The expression Z'MZ in (1.2.9) becomes $Z'MZ = V'(N^{-1})'MN^{-1}V = V'PV$, where $P = (N^{-1})'MN^{-1}$. Actually, the symmetric matrix P takes the form

(1.2.15)

$$P = \frac{1}{3c_1^2-2c_1+3}\begin{bmatrix} 3(3c_1^2-2c_1+3) & 3c-1 & 4(c_1+2c_2-1) \\ & 1 & 0 \\ & & 8(c_1^2+c_2^2+c_1c_2-c_1-c_2+1) \end{bmatrix};$$

the relevant elements, only, having been written. Consequently, the transformation (1.2.13), together with the density (1.2.9), lead to

$$(1.2.16)\quad f(v_1,v_2,v_3) = 3!2(2\pi)^{-3/2}(3c_1^2-2c_1+3)^{-1}\exp\left[-\tfrac{1}{8}V'PV\right];$$

or, to its expanded form

(1.2.17)

$$f(v_1,v_2,v_3) = \frac{3!\,2}{(2\pi)^{3/2}(3c_1^2-2c_1+3)}\exp(-3v_1^2/8)$$

$$\cdot\exp\{-[v_2^2+2(3c_1-1)v_1v_2]/8(3c_1^2-2c_1+3)\}$$

$$\cdot\exp\{-[(c_1^2+c_2^2+c_1c_2-c_1-c_2+1)v_3^2+(c_1+2c_2-1)v_1v_3]/(3c_1^2-2c_1+3)\}.$$

Formally, we can get, from (1.2.17), the density of

$v_1 = x_{(0)} - c_1x_{(1)} - c_2x_{(2)} - (1 - c_1 - c_2)x_{(3)}$, by integrating out the variables v_2 and v_3 over the proper domain. This domain will be given by the mapping, of the z-space onto the v-space, brought forth by the transformation (1.2.13). We shall, presently, study the mapping involved in this case.

Transformation (1.2.13) can be written

$$\text{(1.2.18)} \qquad \begin{aligned} v_1 &= z_1 \\ v_2 &= az_2 + bz_3 \\ v_3 &= z_3 \ ; \quad \text{where} \end{aligned}$$

$$\text{(1.2.19)} \qquad \begin{aligned} a &= 3c_1^2 - 2c_1 + 3 \\ b &= 3c_1^2 + 3c_1c_2 - 3c_1 - c_2 + 2. \end{aligned}$$

We shall now see how the three lines $O'z_1$, $O'L_2$ and $O'L_3$, which determine the wedge-shaped region in the z-space, are mapped into the v-space.

(i) Case of line $O'z_1$ defined by $z_2 = z_3 = 0$.

The transformation (1.2.18) implies that $v_2 = v_3 = 0$, hence, the line $O'z_1$ is mapped into the line $O''v_1$.

(ii) Case of line $O'L_2$ defined by $z_3 = 0$, $z_2 = z_1/(1 - c_1)$.

The transformation implies now

$$\text{(1.2.20)} \qquad v_3 = 0 \ , \qquad v_2 = av_1/(1 - c_1) \ ;$$

relations determining the image $O''L_2$ (in the v-space) of $O'L_2$ (in the z-space).

(iii) Case of the line $O'L_3$ determined by $z_2 = 0$, $z_3 = z_1/(1 - c_1 - c_2)$.

This time, the transformation implies

$$(1.2.21) \qquad v_3 = v_1/(1-c_1-c_2), \; v_2 = bv_1/(1-c_1-c_2), \quad c_1+c_2 \neq 1;$$

relations defining $O''L_3$, the image of $O'L_3$. The orthogonal projection, $O''L_4$, of the line $O''L_3$ on the plane $v_3 = 0$, is given by the equations

$$(1.2.22) \qquad v_2 = bv_1/(1-c_1-c_2), \quad v_3 = 0 \quad ;$$

consequently, according as $b/(1-c_1-c_2)$ is greater than or less than $a/(1-c_1)$, we have two analogous wedge-shaped regions, one of them being convex. It is quite evident that, for the purpose of integrating out the variables v_2 and v_3, the two regions coalesce. For the sake of discussion, we shall use the region obtained when

$$b/(1-c_1-c_2) < a/(1-c_1) \quad .$$

This wedge-shaped convex region, determined by the axis $O''v_1$, and by the straight lines $O''L_2$ and $O''L_3$, can be split up into two regions, R_1 and R_2. R_1 is generated by the axis $O''v_1$, and the straight lines $O''L_3$ and $O''L_4$; while R_2 is generated by the straight lines $O''L_2$, $O''L_3$ and $O''L_4$.

The equations of the flats bounding R_1 are

(i) $v_3 = 0$, determined by $O''v_1$ and $O''L_4$,

(ii) $v_2 = bv_1/(1-c_1-c_2)$, determined by $O''L_3$ and $O''L_4$,

(iii) $v_3 = v_2/b$, $b \neq 0$, determined by $O''v_1$ and $O''L_3$.

In the case of R_2, the equations of the flats are

(i) $v_3 = 0$, determined by $O''L_4$ and $O''L_2$,

(ii) $v_2 = bv_1/(1-c_1-c_2)$, determined by $O''L_4$ and $O''L_3$,

(iii) $av_1 - (1-c_1)v_2 + \{(a-b)c_1 + ac_2 + b - a\} v_3 = 0$,

determined by $O''L_3$ and $O''L_2$.

Consequently we have the following limits for the variables;

case of R_1:

$$0 < v_3 < v_2/b$$

(1.2.23)

$$0 < v_2 < bv_1/(1-c_1-c_2) \;;$$

inequalities defining the region T_1 (say).

<u>Note</u>: $b = 3c_1^2 + 3c_1c_2 - 3c_1 - c_2 + 2 > 0$, for $0 \leq c_i \leq 1$, $i = 1,2$.

Case of R_2:

$$0 < v_3 < [av_1 + (c_1-1)v_2]/[(b-a)c_1-ac_2+a-b]$$

(1.2.24)

$$bv_1/(1-c_1-c_2) < v_2 < av_1/(1-c_1),$$

inequalities defining the region T_2 (say);

$(b-a)c_1 - ac_2 + a - b \neq 0,\ c_1 + c_2 < 1,\ c_1,\ c_2 > 0.$

In both cases, $v_1 = x_{(0)} - c_1x_{(1)} - c_2x_{(2)} - (1-c_1-c_2)x_{(3)} > 0.$

Thus,formally, we have

(1.2.25)

$$f(v_1) = 3!2(2\pi)^{-3/2}(3c_1^2-2c_1+3)^{-1} \int_{T_1+T_2} \exp\left(-\frac{1}{8}V'PV\right) dv_2dv_3 .$$

Making use of the expanded form (1.2.17) of $f(v_1, v_2, v_3)$, the above expression becomes

(1.2.26)

$$f(v_1) = \frac{3!2 \cdot \exp(-3v_1^2/8)}{(2\pi)^{3/2}(3c_1^2-2c_1+3)}$$

$$\cdot \left[\int_0^{\frac{bv_1}{1-c_1-c_2}} \gamma_1(v_1,v_2)dv_2 \int_0^{\frac{v_2}{b}} \gamma_2(v_1,v_3)dv_3 + \int_{\frac{bv_1}{(1-c_1-c_2)}}^{\frac{av_1}{1-c_1}} \gamma_1(v_1,v_2)dv_2 \int_0^{\frac{av_1+(c_1-1)v_2}{(b-a)c_1-ac_2+a-b}} \gamma_2(v_1,v_3)dv_3 \right],$$

where

(1.2.27) $\gamma_1(v_1, v_2) = \exp\left\{ - [v_2^2 + (6c_1-2)v_1v_2] / [8(3c_1^2-2c_1+3)] \right\}$,

(1.2.28)

$$\gamma_2(v_1, v_3) = \exp\{-[(c_1^2+c_2^2+c_1c_2-c_1-c_2+1)v_3^2+(c_1+2c_2-1)v_1v_3]/[3c_1^2-2c_1+3]\}.$$

Noticing that $v_2^2+(6c_1-2)v_1v_2 = [v_2+(3c_1-1)v_1]^2 - (3c_1-1)^2v_1^2$, and

$$(c_1^2+c_2^2+c_1c_2-c_1-c_2+1)v_3^2 + (c_1+2c_2-1)v_1v_2$$

$$= [gv_3+(c_1+2c_2-1)v_1(2g)^{-1}]^2 - (c_1+2c_2-1)^2v_1^2/4g^2,$$ where

(1.2.29) $g = +\sqrt{c_1^2+c_2^2+c_1c_2-c_1-c_2+1}$, $c_1^2+c_2^2+c_1c_2-c_1c_2 > 0$ for

all c_1, c_2; the density (1.2.26) becomes, after some obvious simplifications,

$$f(v_1) = \frac{3!\, 2 \cdot \exp \left[- v_1^2/4g^2 \right]}{(2\pi)^{3/2}\, a}$$

$$\cdot \left\{ \int_0^{\frac{bv_1}{1-c_1-c_2}} \gamma_3(v_1,v_2)dv_2 \int_0^{\frac{v_2}{b}} \gamma_4(v_1,v_3)dv_3 + \int_{\frac{bv_1}{1-c_1-c_2}}^{\frac{av_1}{1-c_1-c_2}} \gamma_3(v_1,v_2)dv_2 \int_0^{\frac{av_1+(c_1-1)v_2}{(b-a)c_1-ac_2+a-b}} \gamma_4(v_1,v_3)dv_3 \right\} .$$

where

$$\gamma_3(v_1,v_2) = \exp \left[-\{v_2+(3c_1-1)v_1\}^2/8a \right] \tag{1.2.31}$$

$$\gamma_4(v_1,v_3) = \exp \left[-\{gv_3+(c_1+2c_2-1)v_1/2g\}^2/a \right] .$$

Setting

$$t_2 = \left[v_2 + (3c_1-1)v_1 \right]/2\sqrt{a} \tag{1.2.32}$$

$$t_3 = \left[2g^2v_3 + (c_1+2c_2-1 \right]/ g\sqrt{2a} ,$$

we can write

$$f(v_1) = \frac{3!2^{3/2}}{g\sqrt{2\pi}} \exp\left[-v_1^2/4g^2\right]$$

$$(1.2.33)\cdot\left\{\int_{h_4(v_1)}^{h_5(v_1)} \alpha(t_2)dt_2 \int_{h_1(v_1)}^{h_2(v_1,t_2)} \alpha(t_3)dt_3 + \int_{h_5(v_1)}^{h_6(v_1)} \alpha(t_2)dt_2 \int_{h_1(v_1)}^{h_3(v_1,t_2)} \alpha(t_3)dt_3\right\},$$

where $v_1 = x_{(0)} - c_1x_{(1)} - c_2x_{(2)} - (1-c_1-c_2)\,x_{(3)}$, and

$$\alpha(y) = (2\pi)^{-\frac{1}{2}} \exp \left[-y^2/2\right],$$

$$h_1(v_1) = (c_1 + 2c_2 - 1)\, v_1 / \left[g\sqrt{2a}\right],$$

$$h_2(v_1,t_2) = \frac{4g^2\sqrt{a}\; t_2 + \{2g^2(1-3c_1)+b(c_1+2c_2-1)\}v_1}{bg\sqrt{2a}} \quad ,$$

$$h_3(v_1,t_2) = \left[r_1v_1+r_2t_2\right] / \left[g\sqrt{2a}\;\{(b-a)c_1-ac_2+a-b\}\right] \quad ,$$

where

(1.2.34) $$r_1 = 2g^2\{a+(c_1-1)(1-3c_1)\}+(c_1+2c_2-1)\{(b-a)c_1-ac_2+a-b\} \;,$$

$$r_2 = 4g^2\sqrt{a}\;(c_1-1),$$

$$h_4(v_1) = (3c_1-1)v_1/2\sqrt{a} \quad ,$$

$$h_5(v_1) = \{b+(3c_1-1)(1-c_1-c_2)\}v_1/2\sqrt{a} \quad ,$$

$$h_6(v_1) = \{a+(3c_1-1)(1-c_1-c_2)\}v_1/2\sqrt{a} \quad .$$

Using numerical integration methods, graphs of $f(v_1)$ can be obtained for various values of c_1 and c_2, $0 < c_1, c_2 < 1$, $c_1 + c_2 < 1$.

We shall, presently, write down three particular cases of the density function defined by (1.2.33) and (1.2.34).

(i) Case of the range. Setting $c_1 = c_2 = 0$, we get $w = x_{(0)} - x_{(1)}$. Then, expressions (1.2.33) and (1.2.34), after a few obvious steps, simplify into

$$(1.2.35)\quad f(w) = \frac{6\sqrt{2}}{\pi\sqrt{\pi}} \exp\left[-w^2/4\right] \int_0^w \exp(-t^2/4) \left[\int_0^{(w-t)/2} \exp(-y^2/2)dy \right] dt,$$

which is the form given by Nair [14] .

(ii) Case of the "extreme deviate from the sample mean".

Letting, $c_1 = c_2 = 1/3$ in our linear contrast, we get

$$(1.2.36)\qquad v = x_{(0)} - (x_{(1)} + x_{(2)} + x_{(3)})/3 = 4u/3,$$

where

$$u = x_{(0)} - (x_{(0)} + x_{(1)} + x_{(2)} + x_{(3)})/4 \quad .$$

The statistic u was studied by McKay, [10], and its studentized form by Nair, [14]. The substitution of $c_1 = c_2 = 1/3$ in (1.2.33) and (1.2.34), yields

$$f(v) = \frac{3!2\cdot\sqrt{3}}{\sqrt{2\pi}} \exp(-3v^2/8)$$

$$(1.2.37)\cdot \left\{ \left[\int_0^{v/\sqrt{6}} \alpha(t_2)dt_2 \int_0^{32t_2/3\sqrt{6}} \alpha(t_3)dt_3 + \tfrac{1}{2}\int_{v/\sqrt{6}}^{2v/\sqrt{6}} \alpha(t_2)dt_2 \right\} \right].$$

Relation (1.2.36) readily leads to the density of the statistic u. In fact, we get

$$(1.2.37) \qquad f(u) = \frac{3!8}{\sqrt{6\pi}} \exp(-2u^2/3)\cdot I(u),$$

where $I(u)$ is given by the following expression:

$$I(u) = \left\{ \int_0^{4u/3\sqrt{6}} \alpha(t_2)dt_2 \int_0^{16\sqrt{2}\,t_2/3\sqrt{3}} \alpha(t_3)dt_3 + \frac{1}{2}\int_{4u/3\sqrt{6}}^{8u/3\sqrt{6}} \alpha(t_2)dt_2 \right\},$$

where $$\alpha(t) = (2\pi)^{-\frac{1}{2}} \exp(-t^2/2)$$

(iii) Case of the difference between the two largest sample values.

The linear contrast, in this case, becomes $y=x_{(0)}-y_{(1)}$. It is not permissable to put $c_1= 1$, $c_2= 0$ directly into the expressions (1.2.33) and (1.2.34). However, carrying the argument used previously, we obtain

$$(1.2.39)\quad f(y)= \frac{3!2}{\sqrt{\pi}}\exp(-y^2/4)\int_{y/2}^{\infty} \alpha(t_2)dt_2\int_{0}^{(2t_2-y)/\sqrt{2}} \alpha(t_3)dt_3 ,$$

where, as before

$$\alpha(t) = (2\pi)^{-\frac{1}{2}} \exp(-t^2/2) \quad .$$

CHAPTER II

NULL DISTRIBUTION OF LINEAR CONTRASTS OF ORDER STATISTICS. CASE OF n + 1 VARIABLES.

1. General linear contrast. Suppose we have $n + 1$ independent random variables, x_0, x_1, ..., x_n, normally distributed with unknown means, m_0, m_1, ..., m_n, respectively, and with a known common variance $\sigma^2 = 1$ (say). Denoting by $x_{(0)}>x_{(1)}>\ldots>x_{(n)}$, the ordered sample values of the above variates, we shall indicate how can be derived the probability density function of the linear contrast $x_{(0)} - c_1x_{(1)} - c_2x_{(2)} - \ldots - c_nx_{(n)}$; $\sum_{i=1}^{n} c_i = 1$, $0 \leq c_i \leq 1$, $(i = 1, 2, \ldots, n)$; or, more precisely, of its equivalent form

$$(2.1.1) \qquad x_{(0)} - c_1x_{(1)} - c_2x_{(2)} - \ldots -(1 - c_1 - c_2 - \ldots - c_{n-1})x_{(n)},$$

$$\sum_{i=1}^{n-1} c_i \leq 1, \quad 0 \leq c_i \leq 1, \quad (i = 1, 2, \ldots n-1),$$

under the null hypothesis $H_0 = m_0 = m_1 = \ldots = m_n = 0$ (say).

The joint density of $x_{(0)}$, $x_{(1)}$, ..., $x_{(n)}$ can be written as

$$(2.1.2)\quad f(x_{(0)}, \ldots, x_{(n)}) = (n+1)!\ (2\pi)^{-(n+1)/2} \exp(-\tfrac{1}{2} \sum_{i=0}^{n} x_{(i)}^2).$$

Let us consider the transformation

$$u_0 = x_{(0)} + x_{(1)} + \ldots + x_{(n)}$$

$$(2.1.3)\quad u_i = x_{(i-1)} - x_{(i)}, \quad (i = 1, 2, \ldots, n),$$

which can be written, in matrix notation, $U^* = AX$, where U^* and X are column vectors, i.e.:

$$U^* = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad X = \begin{bmatrix} x_{(0)} \\ x_{(1)} \\ \vdots \\ x_{(n)} \end{bmatrix};$$

and the matrix A is given by

$$A = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 & 1 & 1 \\ 1 & -1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & -1 & \ldots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 1 & -1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 & -1 \end{bmatrix}.$$

The Jacobian of the transformation equals $(n+1)^{-1}$.

It follows that

$$\sum_{i=0}^{n} x_{(i)}^2 = X'X = U^{*'}(A^{-1})'A^{-1}U^* = U^{*'}B^*U^*,$$

where $B^* = (A^{-1})'(A^{-1})$. Actually it turns out that $B^* = \frac{1}{n+1}B$, where the symmetric matrix B is given by

(2.1.4)

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ & n & n-1 & n-2 & n-3 & \cdots & 3 & 2 & 1 \\ & & 2(n-1) & 2(n-2) & 2(n-3) & \cdots & 6 & 4 & 2 \\ & & & 3(n-2) & 3(n-3) & \cdots & 9 & 6 & 3 \\ & & & & \cdot & \cdots & \cdot & \cdot & \cdot \\ & & & & & & (n-2)3 & (n-2)2 & n-2 \\ & & & & & & & (n-1)2 & n-1 \\ & & & & & & & & n \end{bmatrix},$$

only the relevant terms having been written. The density (2.1.2) thus becomes

(2.1.5) $$f(u_0, u_1, \ldots, u_n) = n!(2\pi)^{-(n+1)/2}\exp\left[-\frac{1}{2(n+1)} U^{*'}BU^*\right].$$

Integrating out the u_0 variable, we get

(2.1.6) $$f(u_1,u_2,\ldots,u_n) = n!\sqrt{n+1}\,(2\pi)^{-n/2}\exp\left[-\frac{1}{2(n+1)} U'CU\right],$$

where $U = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, and C is the symmetric matrix obtained from B by deleting the first row and first column.

We are now in a position to introduce our statistic (2.1.1). Let us put

$$\begin{aligned} z_1 &= u_1 + (1-c_1)u_2 + (1-c_1-c_2)u_3 + \dots + (1-c_1-c_2-\dots-c_{n-1})u_n \\ z_i &= u_i, \quad (i = 1, 2, \dots, n). \end{aligned} \tag{2.1.7}$$

We immediately notice that z_1 is precisely the linear contrast we are interested in, and that the Jacobian of the transformation is equal to unity. In matrix notation, expression (2.1.7) can be written $Z = DU$, where $Z = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$, and

$$D = \begin{bmatrix} 1 & 1-c_1 & 1-c_1-c_2 & \dots & 1-c_1-\dots-c_{n-1} \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots 1 & 0 \\ 0 & 0 & 0 & \dots 0 & 1 \end{bmatrix} . \tag{2.1.8}$$

It now follows that

$U'C\,U = Z'(D^{-1})'C\,D^{-1}\,Z = Z'M\,Z$, where $M = (D^{-1})'\,C\,D^{-1}$. The symmetric matrix M has the form

$$M = [m_{ij}] \quad , \quad i,j=1,2,\ldots,n,$$

where

$$(2.1.9)\quad m_{ij} = [nc_1+nc_2+\ldots+nc_{i-1}-(i-1)]\,(c_1+c_2+\ldots+c_{j-1}-1)$$

$$+(n-j+1)\,[c_1+c_2+\ldots+c_{i-1}+(i-1)] \quad ,$$

$$j \geq i, \quad i,j=1,2,\ldots n.$$

The mapping involved in the transformation (2.1.7) is a straightforward generalization of the one involved in the case of four dimensions, (see Chapter I, Section 2).
The axis Ou_1, determined by $u_i=0$, $(i=2,3,\ldots,n)$ is mapped into the axis $O'z_1$: $z_i=0$, $i=2,3,\ldots,n$. The axes Ou_i, $i=2,3,\ldots,n$, determined by $u_1=u_2=\ldots=u_{i-1}=u_{i+1}=\ldots=u_n=0$, $i=2,3,\ldots,n$, are mapped into the straight lines $O'L_i$ determined by

$$z_i = z_1/(1-c_1-c_2-\dots-c_{i-1}) \qquad i, j = 2,3,\dots,n.$$
$$z_j = 0, \quad j \neq i$$

Consequently, the region of variation of the variables z will be a wedge-shaped region, in the n-dimensional space, bounded by flats passing through the origin. Taking the variate z_n for instance, we have that

$$0 < z_n < [z_1-(1-c_1)z_2-\dots-(1-c_1-\dots-c_{n-2})z_{n-1}] \,/\, [1-c_1-\dots-c_{n-1}],$$

$1-c_1-\dots-c_{n-1} \neq 0$; for fixed z_i, $i = 2, 3, \dots, n-1$.

Similarly, we have

$$0 < z_{n-1} < [z_1-(1-c_1)z_2-\dots(1-c_1-\dots-c_{n-3})z_{n-2}] \,/\, [1-c_1-\dots-c_{n-2}],$$

$1-c_1-\dots-c_{n-2} \neq 0$, for fixed z_i, $i = 2, 3, \dots, n-2$.

This feature being general, we can write

$$0 < z_i < [z_1-(1-c_1)z_2-\dots-(1-c_1-\dots c_{i-2})z_{i-1}]/[1-c_1-\dots-c_{i-1}], \tag{2.1.10}$$

for fixed z_j, $j < i$, $i = 2, 3, \dots n$.

Of course, we have also $z_1 > 0$.

The above considerations allow us to write

$$(2.1.11)\quad f(z_1,z_2,\ldots,z_n) = n!\,\sqrt{n+1}\,(2\pi)^{-n/2}\exp\left[-\frac{1}{2(n+1)}Z'MZ\right];$$

M, given by (2.1.9), and the region of variation, given by (2.1.10).

Integrating out the variables $z_2,z_3,\ldots,z_n$ from (2.1.11) over the region (2.1.10) we get, formally, the density of $z_1 = x_{(0)} - c_1x_{(1)} - \ldots - c_nx_{(n)}$. It appears that a transformation which would reduce the matrix M to a diagonal matrix, (except possibly for the 1st row), would simplify matters considerably. Such a transformation exists and several practical methods yielding the desired transformation are known; for instance, Lagrange's and Kronecker's methods.

Suppose the transformation is of the form

$$(2.1.12)\quad \begin{aligned} v_1 &= z_1 \\ v_2 &= r_{22}z_2 + r_{23}z_3 + \ldots + r_{2n}z_n \\ v_3 &= r_{33}z_3 + \ldots + r_{3n}z_n \\ &\vdots \\ v_n &= r_{nn}z_n, \end{aligned}$$

where the coefficients r_{ij} are known functions of the elements m_{ij} of the matrix **M**.

In matrix notation, (2.1.12) can be written $V = RZ$, where

$$V = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix},$$ and the triangular matrix R is given by

$$(2.1.13) \qquad R = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & r_{22} & r_{23} & \cdots & r_{2n} \\ 0 & 0 & r_{33} & \cdots & r_{3n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & r_{nn} \end{bmatrix} .$$

The Jacobian of the transformation is equal to

$$(2.1.14) \qquad \left[\prod_{i=2}^{n} |r_{ii}| \right]^{-1} .$$

We have also

$$Z'MZ = V'(R^{-1})'R^{-1}V = V'TV,$$

where $T = (R^{-1})' R^{-1}$, the symmetric matrix T being of the form

$$(2.1.15) \qquad T = \begin{bmatrix} t_{11} & t_{12} & t_{13} & \cdots & t_{1n} \\ & t_{22} & 0 & \cdots & 0 \\ & & t_{33} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & t_{nn} \end{bmatrix} .$$

The problem of mapping is rather complicated, at least in the present formulation. A general discussion would be pointless, since too many possibilities would have to be considered. However, for any given number of variates, the matrix R would be known; and, the mapping, in that case, would be as simple as in the cases discussed in the first chapter.

The density (2.1.11) becomes

(2.1.16)

$$f(v_1,\ldots,v_n) = n!\ \sqrt{n+1}(2\pi)^{-n/2}\left[\prod_{i=2}^{n} |r_{ii}|\right]^{-1} \exp\left[-\frac{1}{2(n+1)}\ V'TV\right].$$

In expanded form, we have

$$f(v_1,\ldots,v_n) = n!\ \sqrt{n+1}(2\pi)^{-n/2}\left[\prod_{i=2}^{n} |r_{ii}|\right]^{-1} \exp\left[-t_{11}v_1^2/2(n+1)\right]$$

$$\cdot \exp\left[-(t_{22}v_2^2 + 2t_{12}v_1v_2)/2(n+1)\right] \ldots$$

$$\cdot \exp\left[-(t_{nn}v_n^2 + 2t_{1n}v_1v_n)/2(n+1)\right].$$

It is obvious that by simple transformations we could introduce the normal probability function. Then the density of v_1 would be obtained by integrating out the variables $v_2,\ldots,v_n$, over the proper region. The density of v_1 would be given by an expression involving iterated integrals of the normal probability function, over a wedge-shaped region, given essentially by the mapping involved in the transformation (2.1.12).

We shall presently see, in the next section, how things are shaping up, by considering a special case of our linear contrast.

2. Null distribution of the difference between the two largest sample values in the general case. We are now concerned with the following contrast: $x_{(0)} - x_{(1)}$, where $x_{(0)}$ and $x_{(1)}$, $(x_{(0)}>x_{(1)})$, are the two largest values from a sample of n+1 independent random variables $x_0,x_1,\ldots,x_n$. We assume, as before, that the variates are normally distributed with unknown means, $m_0,m_1,\ldots,m_n$, respectively, and with a known common variance $\sigma^2=1$ (say). In order to get the density of

$$y = x_{(0)} - x_{(1)} , \tag{2.2.1}$$

it is not permissable to simply substitute

$$c_1 = 1, \quad c_i = 0, \qquad i = 2, 3, \ldots, n-1,$$

in the general expression (2.1.16). However, since the first few steps indicated in section 1 of this chapter do not impose any restrictions on the parameters $c_1,\ldots,c_{n-1}$, we may use expression (2.1.6) as a starting point. Thus we have

$$(2.2.2)\quad f(u_1,\ldots u_n) = n!\sqrt{n+1}(2\pi)^{-n/2} \exp\left[-\frac{1}{2(n+1)} U'CU\right],$$

where

$$(2.2.3)\qquad u_i = x_{(i-1)} - x_{(i)}, \quad i = 1,2,\ldots,n;$$

$$U = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix},$$

and the symmetric matrix C is given by

$$(2.2.4)\quad C = \begin{bmatrix} n & n-1 & n-2 & \cdots & 3 & 2 & 1 \\ & 2(n-1) & 2(n-2) & \cdots & 6 & 4 & 2 \\ & & 3(n-2) & \cdots & 9 & 6 & 3 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ & & & & (n-2)3 & (n-2)2 & n-2 \\ & & & & & (n-1)2 & n-1 \\ & & & & & & n \end{bmatrix}.$$

We immediately notice that the variate u_1 is precisely the linear contrast (2.2.1); hence, to get the density of u_1, we only have to integrate out the variates u_i, $i = 2,3,\ldots,n$, over the proper region. The process of integrating out the unwanted variates is easier to carry through, if the C matrix, given by (2.2.4), is simplified. The possibility of a reduction of the symmetric matrix to a simpler form is known to exist. Intensive investigations have shown that the following transformation will do the trick. Let us put

$$
\begin{aligned}
z_1 &= u_1 \\
z_2 &= (n-1)u_2 + (n-2)u_3 + \ldots + 2u_{n-1} + u_n \\
z_3 &= (n-2)u_3 + \ldots + 2u_{n-1} + u_n \\
&\vdots \\
z_{n-1} &= 2u_{n-1} + u_n \\
z_n &= u_n ;
\end{aligned}
$$

that is, in matrix notation,

$$(2.2.5) \qquad Z = DU,$$

where

$$Z = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix},$$

and, the triangular matrix D has the following form

$$(2.2.6)\quad D = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ & n-1 & n-2 & n-3 & \dots & 2 & 1 \\ & & n-2 & n-3 & \dots & 2 & 1 \\ & & \cdot & \cdot & \dots & \cdot & \cdot \\ & & & & & 2 & 1 \\ & & & & & & 1 \end{bmatrix}.$$

It is readily seen that the Jacobian of the transformation is equal to $1/(n-1)!$

It follows that

$$U'CU = Z'(D^{-1})'CD^{-1}Z = Z'MZ,$$

where

$$(2.2.7)\qquad M = (D^{-1})'CD^{-1}.$$

Since D^{-1} turns out to be of the form

$$D^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{n-1} & -\frac{1}{n-1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{1}{n-2} & -\frac{1}{n-2} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & & \frac{1}{2} & -\frac{1}{2} \\ & & & & & & 1 \end{bmatrix},$$

obvious algebraic steps lead to the following expression for the symmetric matrix M:

(2.2.8)

$$M = \begin{bmatrix} n & 1 & 0 & \cdots & 0 & 0 & 0 \\ & \frac{2}{n-1} & 0 & \cdots & 0 & 0 & 0 \\ & & \frac{n+1}{(n-1)(n-2)} & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & \frac{n+1}{4\cdot 3} & 0 & 0 \\ & & & & & \frac{n+1}{3\cdot 2} & 0 \\ & & & & & & \frac{n+1}{2\cdot 1} \end{bmatrix}.$$

We may, at this stage, investigate the mapping involved in the transformation (2.2.5). The axis ou_1 is given by $u_2 = \ldots = u_n = 0$. The transformation (2.2.5) implies $z_1 = u_1$, $z_i = 0$, $i = 2,3,\ldots n$. Hence, ou_1 is mapped into $o'z_1 : z_i = 0$, $i = 2,3,\ldots,n$. The axis ou_2 is given by $u_1 = u_3 = \ldots = u_n = 0$. Now, (2.2.5) implies $z_1 = z_3 = \ldots = z_n = 0$; so, ou_2 is mapped into $o'z_2$.

In the other cases, it is readily seen that ou_i is mapped into the line $o'L_i$, $i = 3,4,\ldots,n$; where $o'L_i$ is defined by

$$z_1 = z_{i+1} = \ldots = z_n = 0,$$

$$z_2 = z_3 = \ldots = z_{i-1}, \quad i = 3,4,\ldots,n.$$

One notices immediately that $o'L_{i-1}$ is the orthogonal projection of $o'L_i$ on the euclidian space of i-1 dimensions, $i = 4,5,\ldots,n$. In fact, $o'z_2$ itself is the orthogonal projection of $o'L_3$ on the euclidian two-dimensional space.

Consequently, the variates z_i, $i = 1,2,\ldots,n$, have the following domain of variation

$$0 < z_i < z_{i-1}, \qquad i = 3,4,\ldots,n,$$

(2.2.9)

$$0 < z_2, \; 0 < z_1.$$

The density (2.2.2) thus becomes

$$(2.2.10)\quad f(z_1,\ldots,z_n) = n\sqrt{n+1}(2\pi)^{-n/2} \exp\left[-\frac{1}{2(n+1)} Z'MZ\right],$$

where M is given by (2.2.8), and the domain of variation of $z_1,\ldots,z_n$ given by (2.2.9). Since M is very close to a diagonal matrix, it will be possible to introduce the normal probability function without messing up the limits of variation. Let us set

$$(2.2.11)\quad \begin{aligned} v_1 &= z_1 \\ v_2 &= \left[z_1\sqrt{n-1}\right]/\sqrt{2(n+1)} + \sqrt{2}\, z_2 / \sqrt{(n+1)(n-1)} \\ v_i &= z_3 / \sqrt{(n-i-1)(n-i+2)}\ ,\ i = 3,4,\ldots,n. \end{aligned}$$

In matrix notation, we have

$$V = NZ, \text{ where } V = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \text{ and}$$

(2.2.12)

$$N = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ \frac{\sqrt{n-1}}{\sqrt{2(n+1)}} & \frac{\sqrt{2}}{\sqrt{(n+1)(n-1)}} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{(n-1)(n-2)}} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \frac{1}{\sqrt{2\cdot 3}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

It is readily verified that

$$\frac{1}{n+1} Z'MZ = \frac{1}{n+1} V' (N^{-1})'MN^{-1}V = V'RV,$$

where

(2.2.13) $$R = \frac{1}{n+1} (N^{-1})'MN^{-1} .$$

Simple algebraic manipulation leads to the following form for R,

$$(2.2.14) \quad R = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} .$$

The limits of variation are easily seen to be

$$0 < v_i < \frac{\sqrt{n-i+3}}{\sqrt{n-i+1}} \, v_{i-1} , \qquad i = 4,5,\ldots ,n,$$

$$(2.2.15) \qquad 0 < v_3 < \frac{\sqrt{2(n+1)} \, v_2 - \sqrt{n+1} \, v_1}{2 \sqrt{n-2}} ,$$

$$\frac{v_1 \sqrt{n-1}}{\sqrt{2(n+1)}} < v_2 ,$$

$$0 < v_1 .$$

Consequently, we can write

$$(2.2.16)\quad f(v_1) = \frac{(n+1)!}{\sqrt{2\pi}\,\sqrt{2}} \exp\left[\frac{-v_1^2}{4}\right] \int_{(v_1\sqrt{n-1})/\sqrt{2(n+1)}}^{\infty} \alpha(v_2)dv_2 \int \cdots$$

$$\cdot \int_0^{\frac{\sqrt{n-i+3}\, v_{i-1}}{\sqrt{n-i+1}}} \alpha(v_i)dv_i \int \cdots \int_0^{\sqrt{3}\, v_{n-1}} \alpha(v_n)dv_n,$$

where $v_1 = x_{(0)} - x_{(1)}$, and

$$\alpha(t) = (2\pi)^{-\frac{1}{2}} \exp\left(-t^2/2\right) \quad .$$

It is readily seen that expressions (1.1.25) and (1.2.39) are particular cases of the general result (2.2.16).

CHAPTER III

MOMENTS OF THE NULL DISTRIBUTION OF LINEAR CONTRASTS OF ORDER STATISTICS.

1. General expression for μ_k' in the case of three dimensions.

Let us consider the linear contrast

$$z = x_{(0)} - cx_{(1)} - (1-c)x_{(2)}, \quad 0 \leq c \leq 1. \tag{3.1.1}$$

It was shown in Chapter I, Section 1, that, under the null hypothesis H_0: $m_0 = m_1 = m_2 = 0$ (say), the probability density function of z is given by

$$f(z) = \frac{3}{\pi\sqrt{2(c^2-c+1)}} \exp\left[-z^2/4(c^2-c+1)\right] I_1(z), \tag{3.1.2}$$

where $I_1(z)$ has the form

$$\int_{(2c-1)z/\sqrt{6(c^2-c+1)}}^{(c+1)z/(1-c)\sqrt{6(c^2-c+1)}} \exp\left[-t^2/z\right]dt.$$

Denoting by μ_k', the k-th moment about the origin, we have

(3.1.3)

$$\mu_k' = \frac{3}{\pi\sqrt{2(c^2-c+1)}}\int_0^{\infty} z^k \exp\left[-z^2/4(c^2-c+1)\right] I_1(z)dz \ .$$

It is to be noted that the above integral exists for any $k > 0$, since $I_1(z)$ exists, ($\leq \sqrt{2\pi}$), and

$$\int_0^{\infty} x^k \exp(-ax^2)dx < +\infty \quad \text{for } a > 0 \text{ and all } k > 0.$$

Let us now introduce a new variable v, using the relation

(3.1.4) $$t = vz, \quad \text{for every fixed } z.$$

It follows that $dt = z\,dv$, and the limits of v are respectively $(2c-1)/\sqrt{6(c^2-c+1)}$ and $(c+1)/(1-c)\sqrt{6(c^2-c+1)}$. We thus can write

$$\mu_k' = \frac{3}{\pi\sqrt{2(c^2-c+1)}}\int_0^{\infty} z^{k+1} \exp\left[-z^2/4(c^2-c+1)\right] I_2(z)\,dz \ ,$$

where $I_2(z)$ has the form

$$\int_{(2c-1)/\sqrt{6(c^2-c+1)}}^{(c+1)/(1-c)\sqrt{6(c^2-c+1)}} \exp[- v^2z^2/2] \, dv \quad .$$

The above can also be written

$$\mu_k' = \frac{3}{\pi\sqrt{2(c^2-1+1)}} \int_0^{\infty} I_3(z)\, dz \quad ,$$

where $I_3(z)$ has the form

$$\int_{(2c-1)/\sqrt{6(c^2-c+1)}}^{(c+1)/(1-c)\sqrt{6(c^2-c+1)}} z^{k+1} \exp\left\{- \frac{z^2\{1+2(c^2-c+1)v^2\}}{4(c^2-c+1)}\right\} dv.$$

Since, in the present case, it is permissable to interchange the two integral signs, we can write

$$\text{(3.1.5)} \qquad \mu_k' = \frac{3}{\pi\sqrt{2(c^2-c+1)}} \int_{\alpha_1(c)}^{\alpha_2(c)} I_4(v)\, dv,$$

where $I_4(v)$ has the form

$$\int_0^{\infty} z^{k+1} \exp\left[-\frac{z^2\{1+2(c^2-c+1)v^2\}}{4(c^2-c+1)}\right] dz \quad ,$$

and where

$$\alpha_1(c) = (2c-1)/\sqrt{6(c^2-c+1)} \quad , \quad \alpha_2(c) = (c+1)/(1-c)\sqrt{6(c^2-c+1)} \quad .$$

Setting

$$(3.1.6) \qquad z^2 [1+2(c^2-c+1)v^2]/4(c^2-c+1) = u \ ;$$

it follows that

$$zdz = 2(c^2-c+1)du/[1+2(c^2-c+1)v^2] \ , \quad \text{and}$$

$$z^k = z^k(c^2-c+1)^{k/2} u^{k/2} / [1+2(c^2-c+1) v^2]^{k/2} \quad .$$

Since

$$\Gamma\left(\frac{k+2}{2}\right) = \int_0^{\infty} u^{k/2} e^{-u} du, \quad \text{we have}$$

$$(3.1.7)\quad \mu_k' = \frac{3\cdot 2^k (c^2-c+1)^{\frac{(k+1)}{2}}\,\Gamma(\frac{k+2}{2})}{\pi} \int_{\alpha_1(c)}^{\alpha_2(c)} [1+2(c^2-c+1)v^2]^{\frac{-(k+2)}{2}}\,dv.$$

Now if we put

$v = \dfrac{\tan\theta}{[2(c^2-c+1)]^{1/2}}$, we get, after a few obvious steps,

$$(3.1.8)\quad \mu_k' = \frac{3\cdot 2^k (c^2-c+1)^{k/2}}{\pi}\,\Gamma(\tfrac{k+2}{2}) \int_{\theta_1}^{\theta_2} \cos^k\theta\,d\theta \quad ,$$

where

$$(3.1.9)\quad \theta_1 = \arc\tan\{(2c-1)/\sqrt{3}\}, \theta_2 = \arc\tan\{(c+1)/(1-c)\sqrt{3}\}\ .$$

Note: We should write $\mu_k' = \mu_k'(c)$, since μ_k' is, in fact, a function of the parameter **c**.

From relations (3.1.8) and (3 1.9) we can get expressions of μ_k' for the three particular cases that we have pointed out previously. For instance, we have

(i) case of the range, obtained from (3.1.1) by setting $c = 0$. $\mu_k'(c)$ becomes

$$\mu_k'(0) = \frac{3 \cdot 2^{k+1}\,\Gamma\left(\frac{k+2}{2}\right)}{\pi} \int_0^{\pi/6} \cos^k \theta \, d\theta \quad . \tag{3.1.10}$$

This is a result obtained by McKay and Pearson [11].

(ii) Case of the extreme deviate from the sample mean.

Setting $c = 1/2$ in (3.1.1) we get

$$v = x_{(0)} - \frac{1}{2}\, x_{(1)} - \frac{1}{2}\, x_{(2)} = 3u/2$$

where

$$u = x_{(0)} - \bar{x} \quad , \quad \bar{x} = \sum_{i=0}^{2} x_{(i)}/3 \quad .$$

The kth moment of v is readily obtained from (3.1.8) and (3.1.9); in fact, we have, after obvious simplifications,

$$\mu_k'(1/2) = \frac{3^{(k+2)/2}\,\Gamma(\frac{k+2}{2})}{\pi} \int_0^{\pi/3} \cos^k \theta \, d\theta \quad .$$

Now

$$\mu_k'(u) = E(u^k) = E(2v/3)^k = [2/3]^k \, E(v^k)$$

$$= (2/3)^k \, \mu_k'(1/2) \; ; \quad \text{hence}$$

$$(3.1.11) \qquad \mu_k'(u) = \frac{2^k\Gamma(\frac{k+2}{2})}{\pi\, 3^{\frac{(k-2)}{2}}} \int_0^{\pi/3} \cos^k \theta \, d\theta \quad .$$

(iii) Case of the difference between the two largest sample values. If we let c $\longrightarrow$ 1, we get

$$(3.1.12) \qquad \mu_k'(1) = \frac{3.2^k\,\Gamma(\frac{k+2}{2})}{\pi} \int_{\pi/6}^{\pi/2} \cos^k \theta \, d\theta \;.$$

2. Moments of low order. We shall now consider a few moments of low order, and study their properties, considering the moments as functions of the parameter c, $0 \leq c \leq 1$. It follows, from expressions (3.1.9), that

$$\sin \theta_1 = (2c-1)/2\sqrt{c^2-c+1}\;,$$

$$\cos \theta_1 = \sqrt{3}/2\sqrt{c^2-c+1}\;,$$

(3.2.1)

$$\sin \theta_2 = (c+1)/2\sqrt{c^2-c+1}\;,$$

$$\cos \theta_2 = (1-c)/2\sqrt{c^2-c+1}\;;$$

these four quantities being useful in the sequel.

<u>1st moment about the origin</u> $\mu_1'(c) = \mu_1(c)$.

From (3.1.8), we have

$$\mu_1(c) = \frac{6\sqrt{c^2-c+1}\,\Gamma(3/2)}{\pi} \int_{\theta_1}^{\theta_2} \cos\theta \, d\theta ;$$

the above, together with (3.2.1), lead to

$$\mu_1(c) = 3\Gamma(3/2)\pi^{-1}(2-c) \quad . \tag{3.2.2}$$

It is clear that $\mu_1(c)$ is a monotone decreasing function of c. Values of $\mu_1(c)$, for particular choices of c, are listed in table II.

<u>2nd moment about the origin:</u> $\mu_2'(c)$.

From (3.1.8) we get

$$\mu_2'(c) = \frac{3\cdot 2^2(c^2-c+1)\Gamma(2)}{\pi}\int_{\theta_1}^{\theta_2} \cos^2\theta \, d\theta \text{ ; after obvious}$$

steps, involving relations (3.2.1), we obtain

$$(3.2.3) \quad \mu_2'(c) = \frac{3!(c^2-c+1)\Gamma(2)}{\pi}\left[\frac{\sqrt{3}(2-2c-c^2)}{4(c^2-c+1)} + \theta_2 - \theta_1\right].$$

We shall now study the behavior of the function of c given by (3.2.3). First, we notice that $\frac{d[\theta_2 - \theta_1]}{dc} = 0$ for all values of c; i.e. $\theta_2 - \theta_1$ is a constant; its value is readily found to be $\pi/3$. Thus,

$$\frac{d\mu_2'(c)}{dc} = \Gamma(2)\left[(4\pi-3\sqrt{3})c-(2\pi+3\sqrt{3})\right] / \pi < 0, \text{ for all}$$

$0 \leq c \leq 1$; consequently, $\mu_2'(c)$ is a monotone decreasing function of c, $0 \leq c \leq 1$. Values of $\mu_2'(c)$, for a few particular c's, are listed in table II.

<u>3rd moment about the origin:</u> $\mu_3'(c)$.

From (3.1.8), we get

$$\mu_3'(c) = \frac{3\cdot 2^3(c^2-c+1)^{3/2}\Gamma(5/2)}{\pi}\int_{\theta_1}^{\theta_2} \cos^3\theta \, d\theta \quad ;$$

integrating the function of θ, and making use of (3.2.1), we have

$$\text{(3.2.4)} \qquad \mu_3'(c) = \left[(2-c)(5c^2-11c+11)\ \Gamma(5/2)\right] / \pi \quad .$$

It is readily found that

$$\frac{d\mu_3'(c)}{dc} = \Gamma(5/2)\{-15\,c^2+42c-33\}\;/\pi \quad < \; 0 \text{ for all values}$$

of c; hence, $\mu_3'(c)$ is a monotone decreasing function of c, $0 \leq c \leq 1$.

Table II contains a few values of $\mu_3'(c)$ for particular choices of the parameter c.

<u>4th moment about the origin:</u> $\mu_4'(c)$.

Again, from (3.1.8) we have

$$\mu_4'(c) = \frac{3\cdot 2^4(c^2-c+1)^2\Gamma(3)}{\pi}\int_{\theta_1}^{\theta_2} \cos^4\theta \, d\,\theta \quad ;$$

from this, we get, after a few obvious steps,

$$(3.2.5)\quad \mu_4'(c) = -\frac{3^{7/2}\Gamma(3)}{4\pi}\left[c^4-2c^2+4c-2\right]+\frac{3^2\cdot 2(c^2-c+1)^2(\theta_2-\theta_1)\Gamma(3)}{\pi}\,.$$

In order to study the behavior of $\mu_4'(c)$, for $0 \leq c \leq 1$, Let us consider $\frac{d\mu_4'(c)}{dc}$. We easily find

$$\frac{d\mu_4'(c)}{dc} = [3\sqrt{3}\, g(c)] / \pi \quad ,$$

where

$$g(c) = (8\pi - 9\sqrt{3})c^3 - 12\pi c^2 + (12\pi + 9\sqrt{3})c - (4\pi + 9\sqrt{3}) \; . \tag{3.2.6}$$

Thus, we have

$$\frac{dg(c)}{dc} = g'(c) = 3(8\pi - 9\sqrt{3})\, c^2 - 24\pi c + 12\pi + 9\sqrt{3} \quad ;$$

and,

moreover, $g'(c) > 0$ for all values of c. Consequently, $g(c)$ is a strictly increasing function of c; and, hence, possesses only one real root. Now, since $g(1) < 0$, it follows that $g(c) < 0$, $0 \leq c \leq 1$. The above argument thus shows that $\mu_4'(c)$ is a monotone decreasing function of c, $0 \leq c \leq 1$.

Using the above four moments, we shall now derive expressions for the moments about the mean and for the corresponding cumulants.

The case of the first moment about the mean is trivial. We always have

(3.2.7) $$\kappa_1(c) = 0 \quad .$$

For the second moment about the mean, identical in fact with the second cumulant, we have

$$\kappa_2(c) = \mu_2(c) = \mu_2'(c) - \mu_1^2(c) \quad ;$$

substituting the values of $\mu_1(c)$ and $\mu_2'(c)$ given by (3.2.2) and (3.2.3), we get, after simplifications,

(3.2.8) $$\kappa_2(c)=\mu_2(c)=[(8\pi-6\sqrt{3}-9)c^2+4(9-2\pi-3\sqrt{3})c+4(3\sqrt{3}+2\pi-9)]/4\pi.$$

Properties of $\mu_2(c) = \kappa_2(c)$.

It is readily verified that $8\pi-6\sqrt{3} - 9 > 0$, and

$$(9 - 2\pi - 3\sqrt{3})^2 - (8\pi - 6\sqrt{3} - 9)(3\sqrt{3} + 2\pi - 9) < 0;$$

consequently, $\mu_2(c) > 0$, for all values of c, which is what we would expect of an expression giving the variance of a variate. Moreover, from

$$\frac{d\mu_2(c)}{dc} = [(8\pi - 6\sqrt{3} - 9)\,c + 2\,(9 - 2\pi - 3\sqrt{3})]/2\pi\ ,$$

it follows that

$$\frac{d\mu_2(c)}{dc} < 0\ , \quad \text{if } 0 \leq c < \frac{2(3\sqrt{3} + 2\pi - 9)}{8\pi - 6\sqrt{3} - 9}\ ,$$

and

$$\frac{d\mu_2(c)}{dc} > 0\ , \quad \text{if } \frac{2(3\sqrt{3} + 2\pi - 9)}{8\pi - 6\sqrt{3} - 9} < c \leq 1\ ;$$

in fact

$$\frac{d\mu_2(c)}{dc} = 0, \quad \text{if and only if,} \quad c = \frac{2(3\sqrt{3} + 2\pi - 9)}{8\pi - 6\sqrt{3} - 9} \doteq 0.868.$$

Consequently, $\mu_2(c)$ is a monotone decreasing function of c, $0 \leq c < c_0$; a monotone increasing function of c, $c_0 < c \leq 1$; and a stationary function for $c = c_0$, where

$c_0 = \dfrac{2(3\sqrt{3} + 2\pi - 9)}{8\pi - 6\sqrt{3} - 9} \doteq 0.868$; in fact, $\mu_2(c)$ passes through a minimum at $c = c_0$. From the symmetry of $\mu_2(c)$ and the above considerations, it follows that $\mu_2(1) < \mu_2(0)$. Particular values of $\mu_2(c)$ are listed in Table II.

The third moment about the mean, identical with the third cumulant, is given by

$$(3.2.9) \quad \mu_3(c) = \kappa_3(c) = \mu_3'(c) - 3\mu_1(c)\mu_2'(c) + 2\mu_1^3(c) \ .$$

With the help of expressions (3.2.2), (3.2.3) and (3.2.4), we can write

(3.2.10)

$$\mu_3(c) = 3\,[(7\pi - 9\sqrt{3} - 9)c^3 + (54 - 15\pi)c^2 - 3(36 - 18\sqrt{3} - \pi)c + 2(36 - 18\sqrt{3} - \pi)]/\,4\pi^{3/2} \ .$$

In order to study the properties of $\mu_3(c) = \kappa_3(c)$, let us consider its first derivative with respect to the parameter c. We get

$$\frac{d\mu_3(c)}{dc} = g(c) = 9\,[(7\pi - 9\sqrt{3} - 9)c^2 + (36 - 10\pi)c - (36 - 18\sqrt{3} - \pi)]/4\pi^{3/2}.$$

It is readily found that

$$g(c) < 0\ , \quad \text{for } 0 \leq c < c_0;$$
$$g(c) > 0, \quad \text{for } c_0 < c \leq 1;$$

and

$$g(c) = 0 \ , \quad \text{for} \quad c = c_0,$$

where

$$c_0 = \frac{-(18 - 5\pi) - [(18 - 5\pi)^2 + (7\pi - 9\sqrt{3} - 9)(36 - 18\sqrt{3} - \pi)]^{\frac{1}{2}}}{(7\pi - 9\sqrt{3} - 9)}$$

$$\simeq 0.5201.$$

Moreover, $\mu_3(c_0) > 0$; and, since the coefficient of c^3 in (3.2.10) is negative, we have

$$\mu_3(c) > 0 \quad \text{for } 0 \leq c \leq 1.$$

So, $\mu_3(c)$ is a monotone decreasing function of c for $0 \leq c < c_0$, and a monotone increasing function of c, for $c_0 < c \leq 1$. Table II exhibits a few values of $\mu_3(c) = \kappa_3(c)$.

The fourth moment about the mean can be written as follows

$$(3.2.11) \quad \mu_4(c) = \mu_4'(c) - 4\mu_1(c)\mu_3'(c) + 6\mu_2'(c)\mu_1^2(c) - 3\mu_1^4(c).$$

With the help of expressions (3.2.2), (3.2.3), (3.2.4), and (3.2.5), we get

$$(3.2.12) \quad \mu_4(c) = 3(16\pi^2)^{-1}[b_0 + b_1c + b_2c^2 + b_3c^3 + b_4c^4],$$

where

$$b_0 = 16(4\pi^2 - 30\pi + 9\sqrt{3}\,\pi + 54\sqrt{3} - 81),$$

$$b_1 = -2b_0,$$

$$(3.2.13)\quad b_2 = 24(8\pi^2 - 21\pi + 6\sqrt{3}\,\pi + 27\sqrt{3} - 81),$$

$$b_3 = -8(16\pi^2 - 3\pi - 27\sqrt{3} - 81),$$

$$b_4 = 64\pi^2 + 24\pi - 72\sqrt{3}\,\pi - 108\sqrt{3} - 81.$$

In order to study the properties of $\mu_4(c)$, let us consider

$$(3.2.14)\quad \frac{d\mu_4(c)}{dc} = g(c) = 3(16\pi^2)^{-1}[4b_4c^3 + 3b_3c^2 + 2b_2c + b_1],$$

the b's being defined above. Since

$\frac{dg(c)}{dc} = g'(c) > 0$, for all values of c, it follows that g(c) is a strictly increasing function of c; hence, g(c) possesses only one real root. It is readily found that the value of the root, c_0, say, is in the interval $0.75 < c < 0.8$.

A more precise value of the root was not computed, since an exact knowledge of it is not too important in the case. Consequently, we have, from the above considerations, that $\mu_4(c)$ is a monotone decreasing function for $0 \leq c < c_0$, and a monotone increasing function for $c_0 < c \leq 1$; c_0, the real root of (3.2.14), being on the open interval (0.75, 0.8).

The value of $\kappa_4(c)$ can be otained from (3.2.8) and (3.2.12), using the relation

$$(3.2.15) \qquad \kappa_4(c) = \mu_4(c) - 3\,\mu_2^2(c).$$

A little algebra leads to the following result:

$$(3.2.16) \qquad \kappa_4(c) = 9[16\pi^2]^{-1}\{d_0 + d_1c + d_2c^2 + d_3c^3 + d_4c^4\},$$

where

$$d_0 = 16(2\pi - \pi\sqrt{3} + 36\sqrt{3} - 63,$$

$$d_1 = -2d_0,$$

$$(3.2.17) \qquad d_2 = 16(11\pi - 2\pi\sqrt{3} + 18\sqrt{3} - 54),$$

$$d_3 = -8(29\pi - 4\pi\sqrt{3} - 18\sqrt{3} - 36),$$

$$d_4 = 56\pi + 8\pi\sqrt{3} - 72\sqrt{3} - 90 \quad .$$

The properties of $\kappa_4(c)$ have not been investigated at this point, since $\kappa_4(c)$ will be considered, at a later stage, as a component of the quantity $\gamma_2 = \kappa_4/\kappa_2^2$.

3. <u>Brief study of the skewness and kurtosis of the distributions</u>.

We may, for instance, use the coefficient

$$(3.3.1) \qquad \gamma_1(c) = \kappa_3/(\kappa_2)^{3/2} \quad ,$$

as a measure of the skewness of our distributions. From the expressions $\kappa_2(c)$ and $\kappa_3(c)$ previously derived, we get

$$(3.3.2) \qquad \gamma_1(c) = 6g_1(c) \,/\, [g_2(c)]^{3/2} \quad ,$$

where

(3.3.3)

$$g_1(c) = (7\pi-9\sqrt{3}-9)c^3+(54-15\pi)c^2-3(36-18\sqrt{3}-\pi)c+2(36-18\sqrt{3}-\pi),$$

$$g_2(c) = (8\pi-6\sqrt{3}-9)c^2+4(9-2\pi-3\sqrt{3})c + 4(3\sqrt{3}+2\pi-9).$$

Properties of $\gamma_1(c)$.

(i) $\gamma_1(c) > 0$, for $0 \leq c \leq 1$. This follows immediately from the previous discussion where it was shown that both $\kappa_3(c)$ and $\kappa_2(c)$ are positive, $0 \leq c \leq 1$. Our distributions are thus skewed to the right.

(ii) $\gamma_1(c)$ is a monotone increasing function of c, $0 < c \leq 1$; it is stationary at $c = 0$. This is easily shown, by considering the first derivative of $\gamma_1(c)$ with respect to c. After some obvious algebraic steps and simplifications, one gets

$$(3.3.4) \quad \frac{d\gamma_1(c)}{dc} = 54c \left\{ a_0 + a_1 c + a_2 c^2 \right\} / \left[g_2(c) \right]^{5/2} ,$$

where

$$a_0 = -20\pi^2 + 78\sqrt{3}\,\pi - 36\pi - 108,$$

$$a_1 = - a_0$$

$$a_2 = 4\pi^2 - 9\pi - 12\pi\sqrt{3} + 54.$$

Since, $a_0 + a_1 c + a_2 c^2 > 0$, for $0 \leq c \leq 1$, it follows that

$\dfrac{d\gamma_1(c)}{dc} > 0$, $\quad 0 < c \leq 1$; and $\dfrac{d\gamma_1(c)}{dc} = 0$, $c = 0$; hence, property (ii).

The coefficient $\gamma_2 = \kappa_4/\kappa_2^2$ is ordinarily used as a measure of kurtosis. Since $\gamma_2 = 0$ in the case of the normal distribution, then, if $\gamma_2 > 0$, we label the distribution as "leptokurtic," meaning that the "peakedness" of the distribution is greater than in the "normal" case. When $\gamma_2 < 0$, the distribution is labeled as "platykurtic," the overall "flatness" of the curve being somewhat more than that of the "normal" curve.

In the present case, we shall study the properties of $\gamma_2(c)$, considered as a function of the unspecified parameter c. To start with, we have

$$\gamma_2(c) = \frac{\kappa_4(c)}{[\kappa_2(c)]^2}$$, where $\kappa_2(c)$ and $\kappa_4(c)$ are given by the expressions (3.2.8) and (3.2.16), (3.2.17), respectively. We shall show that

(i) $\gamma_2(c)$ is a non decreasing function of c, $0 \leq c \leq 1$; in fact, $\gamma_2(c)$ is a monotone increasing function of c, $0 < c \leq 1$, and

(ii) $\gamma_2(c) > 0, \quad 0 \leq c \leq 1$.

In order to establish property (i), it suffices to consider $\frac{d\gamma_2(c)}{dc}$. Tedious algebraic steps lead to the following expression

$$(3.3.5) \quad \frac{d\gamma_2(c)}{dc} = 72c[a_0+a_1c+a_2c^2+a_3c^3]/[h_2(c)]^3 \quad ,$$

where

(3.3.6)

$$a_0 = 160\pi^2 + 32\sqrt{3}\,\pi^2 - 2280\pi\sqrt{3} + 2880\pi - 3240 + 2592\sqrt{3}\ ,$$

$$a_1 = -504\pi^2 + 2376\pi\sqrt{3} - 1620\pi - 2916\ ,$$

$$a_2 = 104\pi^2 + 64\pi^2\sqrt{3} - 132\pi\sqrt{3} - 720\pi + 324 + 468\sqrt{3}\ ,$$

$$a_3 = 120\pi^2 - 48\pi^2\sqrt{3} - 234\pi\sqrt{3} + 135\pi + 486\ ;$$

and

$$h_2(c) = (8\pi - 6\sqrt{3} - 9)c^2 + 4(9 - 2\pi - 3\sqrt{3})c + 4(3\sqrt{3} + 2\pi - 9),$$

where $h_2(c) > 0$ for $0 \leq c \leq 1$.

Some more algebra will show that

$$a_0 + a_1 c + a_2 c^2 + a_3 c^3 > 0\ ,\ \text{for } 0 \leq c \leq 1.$$

Hence we have

$\dfrac{d\gamma_2(c)}{dc} > 0,\quad 0 < c \leq 1,$ and $\dfrac{d\gamma_2(c)}{dc} = 0,\quad c = 0.$ These two expressions establish property (i).

Property (ii) follows immediately from property (i) and the fact that $\gamma_2(c) > 0$, for $c = 0$.

Table II exhibits a few values of $\gamma_2(c)$.

In the cases of more than three dimensions, general expressions for the κ-th moments about the origin can be derived, using methods similar to the one described previously. However, we cannot get exact results, since the expressions involve iterated integrals. The case of four dimensions would still be within the realm of possibility, assuming calculators of the desk type are available. However the process, involving quadrature, would be time consuming. Further research in this direction will be necessary before the results are worth the time employed to get them.

TABLE II

Table of Moments and Related Quantities

for the Linear Contrast

$$x_{(0)} - cx_{(1)} - (1-c)x_{(2)}.$$

	1	2	3	4	5	6
	c = 0	c = 0.5	c = 1	(c = 1)	u	(u)
μ_1'	1.69257	1.26943	0.84628	(0.8458)	0.84628	(.8463)
μ_2'	3.65399	2.12024	1.17301	(1.1730)	0.94233	--
μ_3'	9.30913	4.28431	2.11571	(2.1158)	1.26943	--
μ_4'	26.88588	10.00629	4.55706	(4.5575)	1.97655	--
$\kappa_2=\mu_2$	0.78920	0.50880	0.45681	(0.4577)	0.22613	(.2261)
$\kappa_3=\mu_3$	0.45296	0.30105	0.34983	(0.3495)	0.08920	--
μ_4	2.04688	0.96148	0.89690	(0.8988)	0.18992	--
κ_4	0.17838	0.18484	0.27087	--	0.03651	--
γ_1	0.63799	0.82949	1.13307	(1.28*)	0.82949	(.8296)
γ_2	0.28640	0.71400	1.29807	(1.29)	0.71400	(.7135)

* Obvious typographical error; the result should read 1.128

Remarks.

(i) columns 1, 2 and 3 contain results for the specified values of c;

(ii) column 4 contains the results obtained by Irwin [9], for the case of the difference between the two largest sample values, using approximations;

(iii) column 5 contains the values of the moments of the statistic $u = x_{(0)} - (x_0 + x_1 + x_2)/3$, studied by Nair [14], and later by Grubbs [6];

(iv) column 6 lists the results obtained by Grubbs [6], for the statistic u.

CHAPTER IV

NON NULL DISTRIBUTION OF LINEAR CONTRASTS OF ORDER STATISTICS.

1. Case of three dimensions; general considerations. We assume, as before, that we are dealing with three independent random variables, x_0, x_1 and x_2, normally distributed with unknown means, m_0, m_1 and m_2, respectively, and with a known common variance $\sigma^2 = 1$ (say). Let us denote by $x_{(0)} > x_{(1)} > x_{(2)}$ the sample values of the above variates. We shall be interested in finding the density function of the linear contrast

$$x_{(0)} - cx_{(1)} - (1-c)x_{(2)} , \tag{4.1.1}$$

where c is an arbitrary real number taking values on the closed interval $[0, 1]$. The joint density of $x_{(0)}$, $x_{(1)}$ and $x_{(2)}$ is given by

(4.1.2)

$$f(x_{(0)},x_{(1)},x_{(2)}) = \frac{1}{(2\pi)^{3/2}} \Sigma^* \exp\{- \tfrac{1}{2}[(x_{(0)}-m_{i_0})^2+\ldots+(x_{(2)}-m_{i_2})^2]\} ,$$

where Σ^* stands for the summation over all the permutations

i_0, i_1, i_2 of the numbers 0, 1 and 2.

Expanding the right hand side of (4.1.2), and collecting terms, we can write the density as follows

$$(4.1.3)\quad f(x_{(0)},x_{(1)},x_{(2)}) = \frac{1}{(2\pi)^{3/2}} \exp(-\tfrac{1}{2}\mu'\mu) \exp(-\tfrac{1}{2}X'X)$$

$$\cdot\ \Sigma^* \exp(\mu_i' X),$$

where μ, X and μ_i are the following column vectors

$$(4.1.4)\qquad \mu = \begin{bmatrix} m_0 \\ m_1 \\ m_2 \end{bmatrix},\quad X = \begin{bmatrix} x_{(0)} \\ x_{(1)} \\ x_{(2)} \end{bmatrix},\quad \mu_i = \begin{bmatrix} m_{i_0} \\ m_{i_1} \\ m_{i_2} \end{bmatrix}.$$

Proceeding as in the null case, let us put

$$(4.1.5)\qquad U^* = AX\ ,$$

where

$$U = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}.$$

We have, as before,

$$X'X = \frac{1}{3} U^{*'}BU^{*}, \quad \text{and} \quad \mu_i'X = \mu_i'A^{-1}U^{*},$$

where
$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Consequently, (4.1.3) becomes

$$(4.1.6) \quad f(u_0,u_1,u_2) = \frac{1}{3(2\pi)^{3/2}} \exp(-\tfrac{1}{2}\mu'\mu) \exp(-\tfrac{1}{6} U^{*'}BU^{*})$$

$$\cdot \Sigma^{*} \exp(\mu_i'A^{-1}U^{*}).$$

The variate u_0 can be separated from the others, and we can write

(4.1.7) $$f(u_0, u_1, u_2) = \frac{1}{3(2\pi)^{3/2}} \exp(-\tfrac{1}{2}\mu'\mu) \exp(-\tfrac{1}{6} U'CU)$$

$$\cdot \exp\left[-\tfrac{1}{6}(u_0^2 - 2mu_0)\right] \Sigma^* \exp(\tfrac{1}{3}\nu_i' U) \ ,$$

where

(4.1.8) $$U = \begin{bmatrix} u_1 \\ \\ u_2 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 \\ & \\ 1 & 2 \end{bmatrix}, \quad \nu_i = \begin{bmatrix} 2m_{i_0} - m_{i_1} - m_{i_2} \\ \\ m_{i_0} + m_{i_1} - 2m_{i_2} \end{bmatrix},$$

$$m = m_0 + m_1 + m_2 \ .$$

Since $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left[-\tfrac{1}{6}(u_0^2 - 2mu_0)\right] = \sqrt{3}\exp(m^2/6)$,

it follows that

(4.1.9) $$f(u_1, u_2) = \frac{1}{2\pi\sqrt{3}} \exp\{-\tfrac{1}{2}(\mu'\mu - m^2/3)\} \exp(-\tfrac{1}{6} U'CU)$$

$$\cdot \Sigma^* \exp(\tfrac{1}{3}\nu_i' U).$$

We may now set

(4.1.10) $$Z = DU \ ,$$

where

$$(4.1.11) \qquad Z = \begin{bmatrix} z_1 \\ \\ z_2 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 1-c \\ \\ 0 & 1 \end{bmatrix} .$$

It is readily seen that $z_1 = u_1 + (1-c)u_2$ is precisely the linear contrast (4.1.1).

The transformation (4.1.10) implies

$$U'CU = Z'(D^{-1})'CD^{-1}Z = Z'MZ,$$

where

$$(4.1.12) \qquad M = \begin{bmatrix} 2 & 2c-1 \\ \\ 2c-1 & 2(c^2-c+1) \end{bmatrix} ;$$

it also implies

$$v_i' \, U = v_i' \, D^{-1} Z = \lambda_i' \, Z \quad ,$$

where

$$(4.1.13) \qquad \lambda_i = \begin{bmatrix} 2m_{i_0} - m_{i_1} - m_{i_2} \\ \\ (2c-1)m_{i_0} + (2-c)m_{i_1} - (1+c)m_{i_2} \end{bmatrix} .$$

Consequently, (4.1.9) becomes

$$(4.1.14) \quad f(z_1,z_2) = \frac{1}{2\pi\sqrt{3}} \exp\left[-\tfrac{1}{2}(\mu'\mu - m^2/3)\right] \exp\left(-\tfrac{1}{6} Z'MZ\right)$$

$$\cdot\ \Sigma^* \exp\left(\tfrac{1}{3}\lambda_i' Z\right).$$

The limits of variation are, as in the null case, given by

$$(4.1.15) \qquad 0 < z_2 < z_1 / (1-c)\ , \quad z_1 > 0\ , \quad c \neq 1.$$

In order to get the density of the linear contrast $z_1 = x_{(0)} - cx_{(1)} - (1-c)x_{(2)}$, we have to integrate out the variate z_2 over the proper region. Expanding (4.1.14) and rearranging the terms, we have

$$(4.1.16) \quad f(z_1 z_2) = K\, \Sigma^* \Big\{ \exp\left[-\tfrac{1}{3}(z_1^2 - (2m_{i_0} - m_{i_1} - m_{i_2})z_1)\right]$$

$$\cdot \exp\left[-\tfrac{1}{3}\left(az_2^2 + bz_2\right)\right] \Big\},$$

where

$$K = \frac{1}{2\pi\sqrt{3}} \exp\left[-\tfrac{1}{2}(\mu'\mu - m^2/3)\right],$$

$$a = c^2 - c + 1\ ,$$

$$b = (2c-1)z_1 - \{(2c-1)m_{i_0} + (2-c)m_{i_1} - (1+c)m_{i_2}\}\ .$$

Now, using the identity

$$at^2 + bt = \left\{ \sqrt{a}\, t + \frac{b}{2\sqrt{a}} \right\}^2 - \frac{b^2}{4a} ,$$

and collecting terms, we have, after a few algebraic steps,

(4.1.18)

$$f(z_1, z_2) = K \Sigma^* \left\{ \left[\exp \left[-\{ z_1^2 - 2z_1 (m_{i_0} - cm_{i_1} - (1-c)m_{i_2}) \} / 4(c^2-c+1) \right] \right. \right.$$

$$\cdot \exp \left[\left\{ (2c-1)m_{i_0} + (2-c)m_{i_1} - (1+c)m_{i_2} \right\}^2 / 12(c^2-c+1) \right]$$

$$\left. \left. \cdot \exp \left[- \tfrac{1}{3} \{ \sqrt{a}\; z_2 + b/2\sqrt{a} \}^2 \right] \right] \right\} ,$$

where K, a and b are defined by (4.1.17).

Setting

(4.1.19) $$\sqrt{2}\, \{ \sqrt{a}\; z_2 + b/2\sqrt{a} \} / \sqrt{3} = t ,$$

and formally integrating out the variable t, we have

(4.1.20)

$$f(z_1) = K_1 \Sigma^* \left\{ \left[g_1(z_1, m_i, c) g_2(m_i, c) \int_{h_1(z_1)}^{h_2(z_1)} (2\pi)^{-\frac{1}{2}} \exp(-t^2/2) dt \right] \right\} ,$$

where

$$(4.1.21)\quad K_1 = \frac{1}{\sqrt{2\pi}\ \sqrt{2(c^2-c+1)}} \exp\left[-\tfrac{1}{2}(\mu'\mu - m^2/3)\right] ,$$

(4.1.22)

$$g_1(z_1,m_i,c) = \exp\left[-\left\{z_1^2-2z_1(m_{i_0}-cm_{i_1}-(1-c)m_{i_2})\right\} / 4(c^2-c+1)\right],$$

(4.1.23)

$$g_2(m_i,c) = \exp\left[\left\{(2c-1)m_{i_0} + (2-c)m_{i_1} - (1+c)m_{i_2}\right\}^2 / 12(c^2-c+1)\right],$$

(4.1.24)

$$h_1(z_1) = \left\{(2c-1)z_1-\left[(2c-1)m_{i_0}+(2-c)m_{i_1}-(1+c)m_{i_2}\right]\right\} / \sqrt{6(c^2-c+1)},$$

$$h_2(z_1) = \frac{(c+1)z_1-(1-c)\left[(2c-1)m_{i_0}+(2-c)m_{i_1}-(1+c)m_{i_2}\right]}{(1-c)\ \sqrt{6(c^2-c+1)}} .$$

It is to be noted that two expressions involving the population parameters are closely related. In fact, if we put

$$(4.1.25)\quad \begin{aligned} \lambda_1 &= m_{i_0} - cm_{i_1} - (1-c)m_{i_2} \\ \lambda_2 &= -(1-c)m_{i_0} + m_{i_1} - cm_{i_2} \end{aligned} ,$$

expressions (4.1.22), (4.1.23) and (4.1.24) reduce to

$$
(4.1.26)\quad
\begin{aligned}
g_3(z,m_i,c) &= \exp\left[-\{z^2-2\lambda_1 z\}/4(c^2-c+1)\right], \\
g_4(m_i,c) &= \exp\left[\{\lambda_1+2\lambda_2\}^2/12(c^2-c+1)\right], \\
h_3(z,m_i,c) &= \{(2c-1)z-(\lambda_1+2\lambda_2)\}/\sqrt{6(c^2-c+1)}, \\
h_4(z,m_i,c) &= \{(c+1)z-(1-c)(\lambda_1+2\lambda_2)\}/(1-c)\sqrt{6(c^2-c+1)} .
\end{aligned}
$$

Note: The subscript 1 has been dropped from the z for simplicity. The above quantities, λ_1 and λ_2, are related by a very simple cyclic permutation of i_0, i_1 and i_2.

The density (4.1.20), when expanded, takes the following form:

(4.1.27)

$$
f(z) = \frac{1}{\sqrt{2\pi}\sqrt{2(c^2-c+1)}} \exp\left[-\tfrac{1}{2}(\mu'\mu - m^2/3)\right]
$$

$$
\cdot\ \Sigma^* \left\{ \left[\exp\left[-\{z^2-2\lambda_1 z\}/4(c^2-c+1)\right] \exp\left[\{\lambda_1+2\lambda_2\}^2/12(c^2-c+1)\right] \int_{\frac{(2c-1)z-(\lambda_1+2\lambda_2)}{\sqrt{6(c^2-c+1)}}}^{\frac{(c+1)z-(1-c)(\lambda_1+2\lambda_2)}{(1-c)\sqrt{6(c^2-c+1)}}} \frac{\exp(-t^2/2)}{\sqrt{2\pi}}\, dt \right] \right\},
$$

the quantities μ, m, λ_1, and λ_2 having been defined previously.

From (4.1.27), we shall derive three cases of special interest.

(i) Case of the range.

Setting c = 0 in (4.1.1), we get

$$w = x_{(0)} - x_{(2)} \quad .$$

The density (4.1.27) simplifies out in the form

$$(4.1.28)\quad f(w) = \frac{1}{2\sqrt{\pi}} \exp\left[-\tfrac{1}{2}(\mu'\mu - m^2/3)\right] \exp(-w^2/4)$$

$$\cdot\Sigma^* \left\{ \left[\exp\{(m_{i_0}-m_{i_2})w/2\} \exp\{(m_{i_0}-2m_{i_1}+m_{i_2})^2/12\} \cdot \int_{\frac{-w+m_{i_0}-2m_{i_1}+m_{i_2}}{\sqrt{6}}}^{\frac{w+m_{i_0}-2m_{i_1}+m_{i_2}}{\sqrt{6}}} \frac{\exp(-t^2/2)}{\sqrt{2\pi}}\, dt \right] \right\} .$$

(ii) Case of the extreme deviate from the sample mean.

Setting c = 1/2 in (4.1.1), we get

$$v = x_{(0)}-(x_{(1)}+x_{(2)})/2 = 3u/2 \quad , \tag{4.1.29}$$

where

$$u = x_{(0)}-(x_0+x_1+x_2)/3 \quad .$$

It is readily found that (4.1.27) becomes, in this case,

$$f(v) = \frac{1}{\sqrt{3\pi}} \exp\left[-\tfrac{1}{2}\mu'\mu-m^2/3)\right] \exp(-v^2/3) \tag{4.1.30}$$

$$\cdot\Sigma^* \left\{ \begin{bmatrix} \exp\{(2m_{i_0}-m_{i_1}-m_{i_2})v/3\} \exp\{(m_{i_1}-m_{i_2})^2/4\} \\ \cdot \displaystyle\int_{\frac{-(m_{i_1}-m_{i_2})}{\sqrt{2}}}^{\frac{2v-m_{i_1}+m_{i_2}}{\sqrt{2}}} \frac{\exp(-t^2/2)}{\sqrt{2\pi}}\, dt \end{bmatrix} \right\} .$$

Finally, (4.1.30), together with (4.1.29), lead to

$$(4.1.31) \qquad f(u) = \frac{\sqrt{3}}{2\sqrt{\pi}} \exp\left[-\tfrac{1}{2}(\mu'\mu - m^2/3)\right] \exp(-3u^2/4)$$

$$\cdot \Sigma^* \left\{ \begin{array}{l} \exp\{(2m_{i_0} - m_{i_1} - m_{i_2})u/2\} \exp\{(m_{i_1} - m_{i_2})^2/4\} \\ \cdot \displaystyle\int_{-\frac{m_{i_1} - m_{i_2}}{\sqrt{2}}}^{\frac{3u - m_{i_1} + m_{i_2}}{\sqrt{2}}} \frac{\exp(-t^2/2)}{\sqrt{2\pi}}\, dt \end{array} \right\} .$$

(iii) Case of the difference between the two largest sample values.

Setting $c = 1$ in (4.1.1), we get

$$(4.1.32) \qquad y = x_{(0)} - x_{(1)} \quad .$$

It is not permissable to set c = 1 in the density (4.1.27). However, taking the limits, as $c \longrightarrow 1$, we get

$$(4.1.33) \qquad f(y) = \frac{1}{2\sqrt{\pi}} \exp\left[-\tfrac{1}{2}(\mu'\mu - m^2/3)\right] \exp(-y^2/4)$$

$$\cdot \Sigma^* \left\{ \begin{bmatrix} \exp\{(m_{i_0} - m_{i_1})y/2\} \exp\{(m_{i_0} + m_{i_1} - 2m_{i_2})^2/12\} \\ \\ \cdot \displaystyle\int_{\frac{y - m_{i_0} - m_{i_1} + 2m_{i_2}}{\sqrt{6}}}^{+\infty} \frac{\exp(-t^2/2)}{\sqrt{2\pi}}\, dt \end{bmatrix} \right\}.$$

2. <u>Case of three dimensions under two particular hypotheses.</u> In this section, the non null distribution of the linear contrast

$$z = x_{(0)} - cx_{(1)} - (1-c)x_{(2)} \tag{4.2.1}$$

will be obtained in two particular cases of special interest.

We shall, first, take the case of the distribution of (4.2.1) under the hypothesis

$$H_1\text{: } m_0 = \delta;\ m_1 = m_2 = 0;\ \delta > 0. \tag{4.2.2}$$

Under H_1, $\mu = \begin{bmatrix} m_0 \\ m_1 \\ m_2 \end{bmatrix}$ becomes $\mu = \begin{bmatrix} \delta \\ 0 \\ 0 \end{bmatrix}$, so $\mu'\mu = \delta^2$; also,

$m = m_0 + m_1 + m_2$ becomes $m = \delta$; so, we have

$$\mu'\mu - m^2/3 = 2\delta^2/3.$$

The quantities λ_1 and λ_2, defined by (4.1.25), cannot be written directly in terms of δ; however, taking the summation indicated by Σ^*, we get

$$f(z|H_1) = \frac{1}{\sqrt{\pi(c^2-c+1)}} \exp(-\delta^2/3)[g_1 + g_2 + g_3], \tag{4.2.3}$$

where $f(z \mid H_1)$ stands for the density of the statistic z, defined in (4.2.1), under the hypothesis H_1; and, g_1, g_2 and g_3 are functions of z and of the parameters δ and c defined by the expressions

$$g_1(z;\delta,c) = \exp\left[-\{3z^2-6\delta z-(2c-1)^2\delta^2\}/12(c^2-c+1)\right]\cdot I_1(z;\delta,c)$$

$$(4.2.4)\quad g_2(z;\delta,c) = \exp\left[-\{3z^2+6c\delta z-(2-c)^2\delta^2\}/12(c^2-c+1)\right]\cdot I_2(z;\delta,c)$$

$$g_3(z;\delta,c) = \exp\left[-\{3z^2+6(1-c)\delta z-(1+c)^2\delta^2\}/12(c^2-c+1)\right]I_3(z;\delta,c).$$

The functions I_1, I_2 and I_3 are given by

$$I_1 = \int_{\frac{(2c-1)z-(2c-1)\delta}{\sqrt{6(c^2-c+1)}}}^{\frac{(c+1)z-(1-c)(2c-1)\delta}{(1-c)\sqrt{6(c^2-c+1)}}} \frac{\exp(-t^2/2)}{\sqrt{2\pi}}\,dt \quad ,$$

$$(4.2.5)\quad I_2 = \int_{\frac{(2c-1)z-(2-c)\delta}{\sqrt{6(c^2-c+1)}}}^{\frac{(c+1)z-(1-c)(2-c)\delta}{(1-c)\sqrt{6(c^2-c+1)}}} \frac{\exp(-t^2/2)}{\sqrt{2\pi}}\,dt\ ,$$

$$I_3 = \int_{\frac{(2c-1)z+(1+c)\delta}{\sqrt{6(c^2-c+1)}}}^{\frac{(c+1)z+(1-c)(1+c)\delta}{(1-c)\sqrt{6(c^2-c+1)}}} \frac{\exp(-t^2/2)}{\sqrt{2\pi}}\,dt\ .$$

The density (4.2.3) has been evaluated, in the particular case $\delta = 1$, for certain values of z, and for a few values of the parameter c. The results are listed in Table III.

Let us, now, consider the distribution of (4.2.1) under the hypothesis

(4.2.6) $$H_2 \colon m_0 = 2\delta;\ m_1 = \delta,\ m_2 = 0;\ \delta > 0.$$

Then, the column vector μ becomes

$$\mu = \begin{bmatrix} 2\delta \\ \delta \\ 0 \end{bmatrix}, \text{ and hence } \mu'\mu = 5\delta^2;$$

also, the quantity m_1 becomes $m = 3\delta$. Consequently,

(4.2.7) $$\mu'\mu - m^2/3 = 2\delta^2.$$

The density (4.1.27) can now be written

(4.2.8) $$f(z \mid H_2) = \frac{1}{2\sqrt{\pi(c^2-c+1)}} \exp(-\delta^2)[r_1 + r_2 + \dots + r_6],$$

where the functions, $r_1, \dots, r_6$, are defined by

$$r_1(z;\delta,c) = \exp\left[-\{z^2-2(2-c)\delta z-3c^2\delta^2\}/4(c^2-c+1)\right]G_1(z;\delta,c),$$

$$r_2(z;\delta,c) = \exp\left[-\{z^2-2(1+c)\delta z-3\delta^2(1-c)^2\}/4(c^2-c+1)\right]G_2(z;\delta,c),$$

$$r_3(z;\delta,c) = \exp\left[-\{z^2+2(2c-1)\delta z-3\delta^2\}/4(c^2-c+1)\right]G_3(z;\delta,c),$$

(4.2.9)

$$r_4(z;\delta,c) = r_3(z;-\delta,c),$$

$$r_5(z;\delta,c) = r_2(z;-\delta,c),$$

$$r_6(z;\delta,c) = r_1(z;-\delta,c) ;$$

the functions G_1, G_2 and G_3 being given by the expressions

$$G_1(z;\delta,c) = \int_{\frac{(2c-1)z-3\delta c}{\sqrt{6(c^2-c+1)}}}^{\frac{(c+1)z-3\delta c(1-c)}{(1-c)\sqrt{6(c^2-c+1)}}} \frac{\exp(-t^2/2)}{\sqrt{2\pi}}\, dt ,$$

$$(4.2.10)\quad G_2(z;\delta,c)=\int_{\frac{(2c-1)z+3\delta(1-c)}{\sqrt{6(c^2-c+1)}}}^{\frac{(c+1)z+3\delta(1-c)^2}{(1-c)\sqrt{6(c^2-c+1)}}} \frac{\exp\,(-t^2/2)}{\sqrt{2\pi}}\,dt \quad ,$$

$$G_3(z;\delta,c)=\int_{\frac{(2c-1)z-3\delta}{\sqrt{6(c^2-c+1)}}}^{\frac{(c+1)z-3\delta(1-c)}{(1-c)\sqrt{6(c^2-c+1)}}} \frac{\exp(-t^2/2)}{\sqrt{2\pi}}\,dt \quad .$$

The density (4.2.8) has been evaluated, in the particular case $\delta = 1$, for certain values of z, and for a few values of the parameter c. The results are listed in table IV.

The non null distribution of linear contrasts, in the cases of four or more order statistics, can be obtained, using procedures similar to the one given in the case of three variates. The distributions were actually obtained; but the expressions are so bulky that they have not been included in the present work. Some more research on possible recurrence formulae needs to be done before the results can be put to use.

TABLE III

Table of ordinates of the density of the linear contrast

$z = x_{(0)} - cx_{(1)} - (1-c)x_{(2)}$, under the hypothesis

$H_1 : m_0 = \delta = 1;\ m_1 = m_2 = 0.$

z \ c	0.0	0.1	0.2	0.4	0.6	0.8	0.9	1.0
0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.69550
0.2	.07843	.08707	.09783	.12988	.19255	.36249	.58377	.65223
0.4	.15340	.16984	.19015	.24916	.35644	.56737	.63842	.60265
0.6	.22169	.24434	.27187	.34847	.47043	.60539	.58628	.54862
0.8	.28049	.30717	.33879	.42129	.52678	.56566	.52858	.49195
1.0	.32764	.35584	.38797	.46387	.53322	.50692	.46915	.43436
1.2	.36168	.38868	.41801	.47716	.51027	.44557	.40985	.37744
1.4	.38200	.40550	.42904	.46589	.45370	.38510	.35211	.32259
1.6	.38882	.40689	.42265	.43486	.39556	.32714	.29733	.27098
1.8	.38314	.39449	.40156	.39238	.33636	.27293	.24660	.22357
2.0	.36658	.37079	.36936	.34034	.27999	.22364	.20067	.18002
2.2	.34126	.33851	.32948	.38770	.22839	.17958	.16014	.14373
2.4	.30954	.30284	.28546	.23706	.18255	.14116	.12632	.11184
2.6	.27387	.26024	.24117	.19065	.14288	-	-	-
2.8	.23387	.21939	.19834	.14980	-	-	-	-
3.0	.19953	.18053	.15882	-	-	-	-	-
3.2	.16449	.14500	-	-	-	-	-	-
3.4	.13256	-	-	-	-	-	-	-

TABLE IV

Table of ordinates of the density of the linear contrast

$z = x_{(0)} - cx_{(1)} - (1-c)x_{(2)}$, under the hypothesis

$H_2 : m_0 = 2\delta;\ m_1 = \delta,\ m_2 = 0;\ \delta = 1.$

z \ c	0.0	0.1	0.2	0.4	0.6	0.8	0.9	1.0
0.0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	.54317
0.2	.04056	.03960	.05069	.06759	.10137	.20168	.38016	.52172
0.4	.08109	.09037	.10133	.13497	.20146	.37844	.51800	.49788
0.6	.12140	.13481	.15152	.20107	.29509	.47802	.50288	.47151
0.8	.16106	.17860	.20039	.26360	.37311	.49628	.47401	.44252
1.0	.19934	.22058	.24647	.31926	.42638	.47505	.44270	.41092
1.2	.23518	.25927	.28808	.36377	.45112	.44210	.40851	.37691
1.4	.26731	.29305	.32298	.39430	.44918	.40526	.37183	.34096
1.6	.29437	.32027	.34906	.40863	.42718	.36591	.33331	.30372
1.8	.31476	.33859	.36482	.40664	.39249	.32495	.29381	.26591
2.0	.32837	.34951	.36940	.38986	.35119	.28344	.25435	.22887
2.2	.33363	.35004	.36285	.36159	.30690	.23920	.21601	.19316
2.4	.33070	.34123	.34610	.32502	.26260	.19934	.17977	.15978
2.6	.31996	.32398	.32083	.28390	.21793	-	-	-
2.8	.30177	.29972	.28927	.24141	-	-	-	-
3.0	.27902	.27019	.25385	-	-	-	-	-
3.2	.25165	.23777	-	-	-	-	-	-
3.4	.22185	-	-	-	-	-	-	-

CHAPTER V

A DECISION RULE TO PICK OUT THE POPULATION WITH THE LARGEST MEAN.

1. General considerations. It has been realized, in the last decade or so, that the statistical techniques, based on the classical concepts of "testing of hypotheses" and "confidence intervals", do not always give the experimenter the answer that he was hoping to get. This applies, in particular, to the situations where the analysis of variance is routinely used. For instance, testing procedures tell the experimenter that all the "treatments", which are the subject of investigation, are not identical; or, at least, the hypothesis of equality, if it is true, has a probability α of being rejected, where α is pre-assigned by the statistician. Very often, an answer of this type is inadequate, as it may be conjectured, even before the experiment is carried out, that the treatments could be shown to be different from one another, by sufficiently increasing the size of the sample. In many instances the experimenter likes to take some decision; for example, to rank the treatments according to means, or to select the treatment with the largest mean. The problems are real and complex, and it may very well be that a unique tool does not exist which can provide an adequate answer to all "decision problems".

We shall, in this chapter, discuss the bearing of the work in the previous chapters, on the following problem. Let us assume that it is wanted to find the "best" of several populations, where, by the "best" population, we shall mean: the population having the greatest mean. Let us consider n+1 independent normal populations, each one of them being characterized by its mean. For the present, we assume that the common variance of the above population is known, ($\sigma^2 = 1$ say).

Let the unknown, ranked population means be $m_0 \geq m_1 \geq \cdots \geq m_n$. Suppose that a sample of size N is available from each population, and let the sample means be $x_0, x_1, \ldots, x_n$. Consider the sample point $X = (x_0, x_1, \ldots, x_n)$, defined by the set of sample means, and let W be a region in the euclidian (n+1)-dimensional space; then, the decision rule is of the following type:

i) if $X = (x_0, x_1, \ldots, x_n) \varepsilon W$, decide the population corresponding to the largest sample mean is the best;

ii) if $X = (x_0, x_1, \ldots, x_n) \not\varepsilon W$, do not decide (withhold judgment).

The region W is so determined that the probability of taking a wrong decision is always less than a preassigned value,

α_0 (say). The number $1-\alpha_0$ may be called the safety level of the decision rule. In the following, W will be defined by means of an auxiliary statistic.

Let $x_{(0)} > x_{(1)} > \dots > x_{(n)}$ be the ordered, (or ranked,) sample means. Of course, it is unknown whether or not $x_{(i)}$ comes from the population with mean m_i, $(i = 0, 1, \dots, n)$. Let us consider regions W based on the class of auxiliary statistics composed of linear contrasts of the ordered sample values. Let

$$(5.1.1) \quad g(x_{(0)}, \dots, x_{(n)}) = x_{(0)} - c_1 x_{(1)} - \dots - c_n x_{(n)} ,$$

where $\sum_{i=1}^{n} c_i = 1$, $c_i \geq 0$, $(i = 1, 2, \dots, n)$, and let the critical region W be of the form $g(x_{(0)}, \dots, x_{(n)}) > k$, where k is a positive number determined in such a way that the decision rule provides a given safety.

Now it is easy to show that all the linear contrasts of the form (5.1.1), except

$$(5.1.2) \qquad y = x_{(0)} - x_{(1)} ,$$

cannot, when used as an auxiliary statistic in our decision rule, provide a reasonable safety. In fact, consider the linear contrasts

$$(5.1.3) \qquad g_1(x_{(0)}, \ldots, x_{(n)}) = x_{(0)} - c_1 x_{(1)} - \cdots - c_n x_{(n)} ,$$

where

$$(5.1.4) \qquad c_1 < 1, \quad \sum_{i=1}^{n} c_i = 1, \quad c_i \geq 0, \; (i = 1, 2, \ldots, n) .$$

Let the population means be

$$(5.1.5) \qquad \begin{aligned} & m_0 = 0, \; m_1 = m_0 + \varepsilon , \\ & m_2 = m_3 = \ldots = m_n = m \quad \text{(say)}, \end{aligned}$$

where ε is a small positive number. Then, it follows that

$$P \left[g_1(x_{(0)}, \ldots, x_{(n)}) > k \right] \longrightarrow 1,$$

when $m \longrightarrow -\infty$, $\varepsilon \longrightarrow 0$, for any given k. This means that, when (5.1.5) is true and m is tending to $-\infty$, the probability of taking a decision tends to unity and hence, the probability of taking a "wrong decision" tends to 0.5. Thus, a reasonable safety is not provided by the class of auxiliary statistics given by (5.1.3) and (5.1.4). In particular, the range $w = x_{(0)} - x_{(n)}$, or Nair's "extreme deviate from the sample mean"

$$x_{(0)} - \bar{x} \; ; \quad \bar{x} = \sum_{i=0}^{n} x_{(i)}/(n+1) \; ,$$

are seen to belong to the discarded class of auxiliary statistics.

Consequently, only one member of (5.1.1) is left; that is,

$$y = x_{(0)} - x_{(1)} \; . \tag{5.1.6}$$

In order to use (5.1.6) as an auxiliary statistic in the proposed decision rule, properties of the distribution of (5.1.6) will have to be studied. The upper bound of the probability of wrong decision, corresponding to all possible spacings of the population means, has to be found out. This problem has not yet been solved. However, the case of three dimensions has been studied in more detail since the distribution of $x_{(0)} - x_{(1)}$, in the null and non-null cases, has previously been derived.

We shall indicate, in the next section, the results that are available at this moment.

2. Some properties of the suggested decision rule in the case of three populations. We have, now, three normal, independent populations with unknown means, $m_0 \geq m_1 \geq m_2$, and with a known common variance $\sigma^2 = 1$ (say). Let the observed means be x_0, x_1 and x_2, and the ranked means be $x_{(0)} > x_{(1)} > x_{(2)}$. As before, it is unknown whether or not $x_{(i)}$ comes from the population having the mean m_i , $(i = 0, 1, 2)$. Our auxiliary statistic will be

$$y = x_{(0)} - x_{(1)} . \tag{5.2.1}$$

The decision rule takes the form

(5.2.2)

i) if $y > k$, decide $x_{(0)}$ belongs to the "best" population,

ii) if $y \leq k$, do not decide,

where k will be determined in such a way that the probability of taking a wrong decision is less than a preassigned value,

$\alpha_0 > 0$ (say).

Of course, α_0 being given, to determine k we will have to find out the "worst" scattering of the true means; i.e., the one which will lead to the greatest probability of wrong decision. This investigation will be based on the non-null distribution of our auxiliary statistic.

Denoting by $f(y \mid \overline{H}_0)$ the probability density of (5.2.1) where the null hypothesis, $H_0 : m_0 = m_1 = m_2$, is not true, we have, from (4.1.33),

(5.2.3)

$$f(y \mid \overline{H}_0) = \frac{1}{2\sqrt{\pi}} \exp\left[-\frac{1}{2}(\mu'\mu - m^2/3)\right] \exp(-y^2/4)$$

$$\cdot \Sigma^* \left\{ \begin{array}{l} \exp\left[(m_{i_0} - m_{i_1})y/2\right] \exp\left[(m_{i_0} + m_{i_1} - 2m_{i_2})^2/12\right] \\ \cdot \displaystyle\int_{\{y-(m_{i_1}+m_{i_1}-2m_{i_2})\}/\sqrt{6}}^{\infty} \frac{\exp(-t^2/2)}{\sqrt{2\pi}}\, dt \end{array} \right\}$$

Setting up

(5.2.4) $\qquad m_0 - m_1 = \delta, \quad m_1 - m_2 = \gamma, \quad \delta, \gamma \geq 0$;

expression (5.2.3), after a few obvious steps, becomes

(5.2.5)

$$f(y \mid \overline{H}_0) = \frac{1}{\sqrt{\pi}} \exp\left[- \frac{1}{3}(\delta^2 + \gamma^2 + \gamma\delta)\right] \exp(-y^2/4)$$

$$\cdot \left\{ \begin{array}{l} \exp\left[(2\delta+\gamma)^2/12\right] \cosh \frac{\delta v}{2} \displaystyle\int_{\frac{v-2\gamma-\delta}{\sqrt{6}}}^{\infty} \frac{\exp(-t^2/2)}{\sqrt{2\pi}}\, dt \\ \\ + \exp\left[(\delta-\gamma)^2/12\right] \cosh \frac{v(\gamma+\delta)}{2} \displaystyle\int_{\frac{v-\delta+\gamma}{\sqrt{6}}}^{\infty} \frac{\exp(-t^2/2)}{\sqrt{2\pi}}\, dt \\ \\ + \exp\left[(\gamma+2\delta)^2/12\right] \cosh \frac{\gamma v}{2} \displaystyle\int_{\frac{v+\gamma+2\delta}{\sqrt{6}}}^{\infty} \frac{\exp(-t^2/2)}{\sqrt{2\pi}}\, dt \end{array} \right\} .$$

The null distribution is readily obtained from (5.2.5), by putting $\delta = \gamma = 0$.

Now suppose, for the moment, that the constant k in (5.2.2) is known. Then the probability of taking a decision, denoted by $P_d(\gamma, \delta)$, is given by

$$P_d(\gamma, \delta) = \int_k^{\infty} f(y \mid \bar{H}_0)\, dy. \tag{5.2.6}$$

This probability can be split up into two parts, (i) the probability of taking a wrong decision, denoted by $P_w(\gamma, \delta)$, (ii) the probability of taking a good decision, denoted by $P_g(\gamma, \delta)$. Explicitly, $P_g(\gamma, \delta)$ is the probability that $y > k$ and $x_{(0)}$ comes from the population with mean m_0. Formally, we have

$$P_g(\gamma,\delta) = \frac{1}{2\sqrt{\pi}} \exp\left[-\tfrac{1}{3}(\delta^2+\gamma^2+\delta\gamma)\right] \int_k^{\infty} H_1(y;\gamma,\delta)\, dy, \tag{5.2.7}$$

where

$$(5.2.8)\quad H_1(y;\gamma,\delta) = \exp\left[-\{3y^2-6\delta y-(2\gamma+\delta)^2\}/12\right]\int_{\frac{y-2\gamma-\delta}{\sqrt{6}}}^{\infty} \frac{1}{\sqrt{2\pi}}\exp(-t^2/2)\,dt$$

$$+\exp\left[-\{3y^2-6(\gamma+\delta)y-(\delta-\gamma)^2\}/12\right]\int_{\frac{y-\delta+\gamma}{\sqrt{6}}}^{\infty} \frac{1}{\sqrt{2\pi}}\exp(-t^2/2)\,dt.$$

Similarly, $P_W(\gamma,\delta)$ is the probability that $y > k$ and $x_{(0)}$ does not come from the population with mean m_0. Formally, we can write $P_W(\gamma,\delta)$ as follows:

$$(5.2.9)\quad P_W(\gamma,\delta) = \frac{1}{2\sqrt{\pi}}\exp\left[-\tfrac{1}{3}(\delta^2+\gamma^2+\delta\gamma)\right]\int_k^{\infty} H_2(y;\gamma,\delta)\,dy,$$

where

(5.2.10)

$$H_2(y;\gamma,\delta) = \exp\left[-(3y^2+6\delta y-(2\gamma+\delta)^2)/12\right]\int_{\frac{v-2\gamma-\delta}{\sqrt{6}}}^{\infty} \frac{1}{\sqrt{2\pi}}\exp(-t^2/2)\,dt$$

$$+\exp\left[-(3y^2+6(\gamma+\delta)y-(\delta-\gamma)^2)/12\right]\int_{\frac{v-\delta+\gamma}{\sqrt{6}}}^{\infty} \frac{1}{\sqrt{2\pi}}\exp(-t^2/2)\,dt$$

$$+\exp\left[-(3y^2-6\gamma y-(\gamma+2\delta)^2)/12\right]\int_{\frac{v+\gamma+2\delta}{\sqrt{6}}}^{\infty} \frac{1}{\sqrt{2\pi}}\exp(-t^2/2)\,dt$$

$$+\exp\left[-(3y^2+6\gamma y-(\gamma+2\delta)^2)/12\right]\int_{\frac{v+\gamma+2\delta}{\sqrt{6}}}^{\infty} \frac{1}{\sqrt{2\pi}}\exp(-t^2/2)\,dt\ .$$

It has not yet been possible to find analytically the values of γ and δ which will maximize the probability of taking a wrong decision. However, we have numerical evidence that, for a reasonable safety, (.90 or more), $\delta=0, \gamma=+\infty$ is the worst case.

Anyway it is easy to show that when $\delta=0$, $P_w(\gamma,0) \geq P_g(\gamma,0)$; in fact, from symmetry considerations, it follows that

$$(5.2.11) \qquad \frac{1}{2} P_d(\gamma, 0) \leq P_w(\gamma, 0) \leq \frac{2}{3} P_d(\gamma, 0) .$$

Numerical evidence was gathered in the following manner. The constant k was temporarily determined by

$$(5.2.12) \qquad P_d(0, 0) = \int_k^{\infty} f(y \mid H_0)\, dy = \alpha = 0.075 ,$$

using the null distribution of y, whose ordinates are listed in Table I, and numerical integration methods (Weddle's rule); k was found to be approximately equal to 1.95700. The value $\alpha = 0.075$ was chosen because, in this case, $P_w(0,0) = \frac{2}{3}\alpha = 0.05$.

Using the above value of k, the following six cases were considered:

$$H_1 : \delta = 1, \gamma = 0 ,$$

$$H_2 : \delta = 1, \gamma = 1 ,$$

$$H_3 : \delta = 0, \gamma = 0.6 ,$$

$$H_4 : \delta = 0, \gamma = 1.0 ,$$

$$H_5 : \delta = 0, \gamma = 1.4 ,$$

$$H_6 : \delta = 0, \gamma \longrightarrow +\infty .$$

First, the densities were obtained with the help of tables, [16], [17] and [18]; and then the quantities $P_d(\gamma, \delta)$ and $P_W(\gamma, \delta)$ were obtained by numerical integration methods. The results are summarized in Table V.

TABLE V

	$\delta = 1$ $\gamma = 0$	$\delta = 1$ $\gamma = 1$	$\delta = 0$ $\gamma=0.6$	$\delta = 0$ $\gamma=1.0$	$\delta = 0$ $\gamma=1.4$	$\delta = 0$ $\gamma \to +\infty$
$P_d(\gamma,\delta)$	0.13476	0.20622	0.08768	0.10196	0.12140	0.16642
$P_W(\gamma,\delta)$	0.015	0.014	0.0477	.0541	.0620	.0832

The above numerical evidence points out that H_6 is the "worst" case. The first two cases, H_1 and H_2, indicate that the power of the procedure is rather good and that $P_w(\gamma, 1)$ is in fact quite small.

3. Suggestions for further research, and concluding remarks.

It is conjectured that, if a reasonable safety is required, then, in the (n+1)-dimensional case, the maximum value of the probability of wrong decision, using a procedure based on a critical region of the type $x_{(0)} - x_{(1)} > k$, will still occur when $m_0 = m_1$, and $m_2 = m_3 = \ldots = m_n = m$ (say), where m is negatively infinite. Further investigation of the properties of the non-null distribution in the general case, will be required.

The next step will be to studentize the auxiliary statistic, using, in the analysis of variance situation, the available independent estimate of the common variance σ^2. This step should be easy to make, since methods of studentization are known; see, for instance, Hartley [7] and Nair [13], to mention only a few.

In conclusion, reference may be made to work along similar lines by other authors. A large number of testing procedures for "outliers", or "stragglers", have been proposed, see, for instance, Irwin [8] and [9], Grubbs [6], Dixon [2] and [3]. Recently, Duncan [4], has presented a procedure for ranking means, based on successive applications of the range. Finally, Bechhofer [1] has tackled the decision problem along lines somewhat different from ours.

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