SOME FURTHER ASPECTS OF SEQUENTIAL ESTIMATION OF 1/p

by

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0. ABSTRACT

The investigation relating to a characterization of the class of sampling plans providing unbiased estimation of 1/p has some independent interest. It appears that this problem has not received much attention in the literature. In this article we review the available results and furnish some additional results. We also study the same problem under Markovian dependence set-up with reference to certain specific sampling plans.

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1. INTRODUCTION

We address ourselves to the problem of unbiased (sequential) estimation of the reciprocal of the Bernoulli parameter p. Under the set-up of independent Bernoulli trials, some aspects of inference on p and 1/p have been studied, among others, by De Groot (1959), Girshick, Mosteller and Savage (1946), Gupta (1967), Sinha and Sinha (1975), Wasan (1964) and Wolfowitz (1946, 1947). The investigation relating to a characterization of the *class* of sampling plans providing unbiased sequential estimation of 1/p has an independent interest. This specific problem was studied first by Gupta (1967) and, subsequently, by Sinha and Sinha (1975). Still open a problem is establishing the validity or otherwise of a conjecture on the nature of such sampling plans (Vide Sinha and Sinha (1975) - henceforth abbreviated as SS).

The purpose of this paper is two-fold. First, we provide some additional results as supplementary to those in SS. Secondly, we consider the same estimation problem under the Markovian set-up of dependence and study properties of some specific plans such as *inverse binomial* sampling plans. In Section 2, we present the nomenclature and also display all the *relevant* available results in this direction. Section 3 puts forward further results in the same direction. Section 4 is meant for a study of some specific sampling plans under Markovian dependence set-up.

2. NOMENCLATURE AND LITERATURE REVIEW

2.1. Nomenclature: The nomenclature is fairly standard by now. For the sake of completeness, we discuss them briefly.

The word *point* will refer to points in the XY-plane with positive integral co-ordinates. A *region* R is a set of points including the point (0,0). The point (x', y') is immediately beyond the point (x,y) if either $x'=x+1$, $y'=y$ or $x'=x$, $y'=y+1$. A *path* in R from the point a_0 to the point a_n is a sequence of points $a_0, a_1, \ldots, a_n$ such that a_i $(i>0)$ is immediately beyond a_{i-1}. We can order the points in the XY-plane using the following convention - (x',y') occurs after (x,y) if either $x<x'$ or $x=x'$, $y<y'$. A *boundary point* (an element on the boundary B of R) is a point not in R which is the last point of a path from the origin. *Accessible points* are the points in R which can be reached by paths from the origin. The *index* of a *point* is the sum of its co-ordinates. A *finite* (or

bounded) region is a region for which the indices of the accessible points are less than some number n. The probability of an accessible point or a boundary point $(x, y) = r$ included in a path from the origin is $P(r) = K(r)p^y q^x$, where $K(r)$ is the number of paths from the origin to the point r. A region for which $\sum_{r \in B} P(r) = 1$ will be called a *closed region*. The corresponding sampling plan is a closed plan. For any accessible point $t = (x,y)$ of a plan, $t(a)$ will denote the total number of ways of passing from t to the boundary point a of the plan only through *its accessible points*. Throughout the paper, unless otherwise stated, every plan with origin at (0,0) will be assumed to be closed. An *estimator* f is defined only at the boundary points $r \in B$; it is defined to be an unbiased estimator (ue) of $g(p)$ if and only if $\sum_{r \in B} f(r) \cdot k(r)p^y q^x = g(p)$ identically in p, $0 < p < 1$. Also we should restrict our attention to ue's f of $1/p$ which are *proper* in the sense that $f(r) \geq 1 \forall r \in B$. An estimator which is not proper is said to be *improper*.
Other definitions and notations will be incorporated in proper places.

2.2. <u>Literature review on unbiased sequential estimation of 1/p under the set-up of independent Bernoulli trials</u>: We state the available results one by one in a systematic way so that the reader may have a fair idea as to the development of the subject matter.

(a) <u>Nature of the plan</u> (Necessity)

(i) [Gupta] The sampling plan must be unbounded.

(ii) [SS] The sampling plan must be unbounded along the X-direction.

(b) <u>Closure of an unbounded plan</u> (Sufficiency)

(i) [Wolfowitz] A sufficient condition that region R be closed is that $\liminf_{n \to \infty} \frac{A(n)}{\sqrt{n}} < \infty$, where A(n) is the number of accessible points of index n.

(c) <u>Nature of the plan</u> (Sufficienty)

(i) [Gupta] If the closed plan with boundary $B = \{r_i = (x_i, y_i)\}$ be such that by changing its boundary points from r_i to $r_i' = (x_i, y_i + 1)$ we get a closed plan $B' = \{r_i' = (x_i, y_i + 1)\}$, then 1/p is estimable for the plan B. And the unbiased estimate is given by $f(r) = \frac{K'(r')}{K(r)}$, $r \in B$ where $K'(r')$ is the number of paths from the origin to $r' \in B'$.

(ii) [SS] A sufficient condition for an unbiased estimator of 1/p to exist is that no point on the line "$Y=1$" is inaccessible.

(iii) [SS] A sufficient condition for an unbiased estimator of 1/p to exist is that the transformed plan (Vide SS) $P^T(x_0,1)$ is closed. Here x_0 is the first inaccessible point on the line "$Y=1$".

In SS, it has been demonstrated that the conditions (i) and (iii) are equivalent. Moreover, it has been conjectured that the condition (i) is necessary as well.

3. SOME FURTHER RESULTS ON CLOSURE OF UNBOUNDED SAMPLING PLANS WITH REFERENCE TO ESTIMATION OF 1/p

3.1. Theorem 1 of Wolfowitz (1946), cited above under b(i), on closure of a plan applies, as a matter of fact, only to plans having boundaries determined through two infinite sequences of points $(0, a_0), (1,a_1), (2,a_2),\cdots$ and $(b_0,0)$, $(b_1,1), (b_2,2),\cdots$. Here $1 \le a_0 \le a_1 \le a_2 \le \cdots$ and $1 \le b_0 \le b_1 \le b_2 \le \cdots$ are two infinite sequences of positive integers. Such plans have been termed *doubly simple* (Vide Wolfowitz (1946)). The number A(n) refers to the number of accessible points of index n for such a plan. However, an arbitrary unbounded sampling plan need not be doubly simple and, hence, the condition $\liminf_{n\to\infty} A(n)/\sqrt{n} < \infty$ can be substantially improved for other types of unbounded plans. As a matter of fact, plans with $A(n) = O(n)$ can also be closed. The point to be noted is that the actual value of A(n) is *not always* an important factor to decide on closure or otherwise of a plan. Once an accessible point is reached by a path, only the nature of the remaining part of the sampling plan ahead of this point (Vide notion of transformed plan in SS) is relevant for the path to hit a boundary point and, hence, to lead eventually to closure of the plan. We provide two examples of such plans viz., P_1 and P_2 at the end.

3.2. As is well-known, the simplest example of a class of unbounded sampling plans providing unbiased estimation of 1/p is that of the *inverse binomial* plans $[S(c): B(c) = \{\gamma: Y(\gamma) = c\}, c = 1, 2, \cdots]$ i.e., for every $c = 1, 2, \cdots$, the points on the line $Y=c$ serve as the boundary points of the plan S(c). One might develop the following generalization of such plans.

For a fixed *finite* k, let $1 \le c_1 < c_2 < \cdots < c_k < \infty$ be a set of k positive integers. Further, let $0 \le n_1 < n_2 < \cdots < n_{k-1} < \infty$ be another set of positive integers.

Consider the plan P 2 shown at the end. We may call such plans *finite-step generalizations* of the inverse binomial plans. The questions of closure of the plan and estimation of 1/p based on it are both immediately settled by an application of b(i) and c(i) cited above. However, the explicit form of the estimator is quite complicated. For $k = 2$, one may derive the following results:

$$k(r) = \begin{cases} \binom{i + c_1 - 1}{c_1 - 1} & \text{for } r = (i, c_1),\ 0 \le i \le n_1 \\ \sum_{i=0}^{c_1 - 1} \binom{n_1 + i}{i}\binom{j + c_2 - i - 2}{c_2 - i - 1} & \text{for } r = (n_1 + j, c_2),\ j \ge 1 \end{cases}$$

$$f(r) = \begin{cases} \dfrac{i + c_1}{c_1} \text{ for } r = (i, c_1),\ 0 \le i \le n_1 \\ \left\{ \sum_{i=0}^{c_1} \binom{n_1 + i}{i}\binom{j + c_2 - i - 1}{c_2 - i} \Big/ \sum_{i=0}^{c_1 - 1} \binom{n_1 + i}{i}\binom{j + c_2 - i - 2}{c_2 - i - 1} \right\} \text{ for } r = (n_1 + j, c_2), j \ge 1 \end{cases}$$

We indicate the steps involved in the calculation of k(r)'s and f(r)'s.

(i) For any $r = (i, c_1)$ with $0 \le i \le n_1$, the result is based on inverse binomial set-up.

(ii) For $r = (n_1 + j,\ c_2)$, consider the following mutually exclusive paths:

path p_i: $\underbrace{(0,o) \to (n_1, i)}_{\text{under binomial set-up}} \underbrace{\to (n_1 + 1, i)}_{\text{unique path}} \underbrace{\to (n_1 + j, c_2)}_{\text{under inverse binomial set-up}}$

This is true for $i = 0, 1, 2, \ldots, c_1 - 1$. Let $K(r|p_i) = \#$ of paths of the type p_i from (0,0) to r. The above consideration yields

$$K(r|p_i) = \binom{n_1 + i}{i} .1. \binom{j - 1 + c_2 - i - 1}{c_2 - i - 1} = \binom{n_1 + i}{i}\binom{j + c_2 - i - 2}{c_2 - i - 1}.$$

$$\text{Hence, } K(r) = \sum_{i=0}^{c_1 - 1} K(r|p_i) = \sum_{0}^{c_1 - 1} \binom{n_1 + i}{i}\binom{j + c_2 - i - 2}{c_2 - i - 1}.$$

Since the *nature* of the shifted plan B′ remains *unchanged*, we may similarly deduce expression for $K'(r')$ and, hence, that of $f(r)$. This result could, of course, be derived from the general path counting formula to be discussed in Subsection 3.4. However, the above argument is much simpler.

Next suppose there are two sequences of positive integers: $\{C_k \uparrow\}$ and $\{n_k \uparrow$ with $n_1 \geq 0\}$. Consider the readily extended form of the plan P3. Clearly, this will now be unbounded in both directions. Such plans may be termed *infinite-step generalizations* of the inverse binomial plans. We want to examine the conditions under which such plans will be closed. These are very similar to doubly simple regions with the exception that there are no lower boundary points. We state and prove the following result.

Theorem 1. *Let* $(n-d_n, d_n)$ *be the coordinate position of the boundary point on the line* $X+Y=n$. *If* $\underline{\lim}\, \frac{d_n}{n} = 0$, *then the plan is closed uniformly in* $p \in (0,1]$.

<u>Proof</u>. Denote by S_n the number of heads (along Y-axis) in n trials so that we need to show $P(S_n \geq d_n$ for some $n) = 1$ for all $p \in (0,1]$. Since $\underline{\lim}\, \frac{d_n}{n} = 0$, there exists a subsequence $n^{(1)} < n^{(2)} < n^{(3)} < \cdots$ such that $\underline{\lim}\, \frac{d_{n^{(i)}}}{n^{(i)}} = 0$. Therefore, $P\{S_n \geq d_n$ for some $n\} \geq P\{S_{n^{(i)}} \geq d_{n^{(i)}}$ for some $i\} = P\{\frac{S_{n^{(i)}}}{n^{(i)}} \geq \frac{d_{n^{(i)}}}{n^{(i)}}$ for some $i\} \geq P\{\frac{S_{n^{(i)}}}{n^{(i)}} \geq \frac{d_{n^{(i)}}}{n^{(i)}}$ i.o.$\}$ $= 1$ since $\underline{\lim}\, \frac{d_{n^{(i)}}}{n^{(i)}} = 0$ and $\frac{S_{n^{(i)}}}{n^{(i)}} \xrightarrow{\text{a. s.}} p \in (0,1]$.

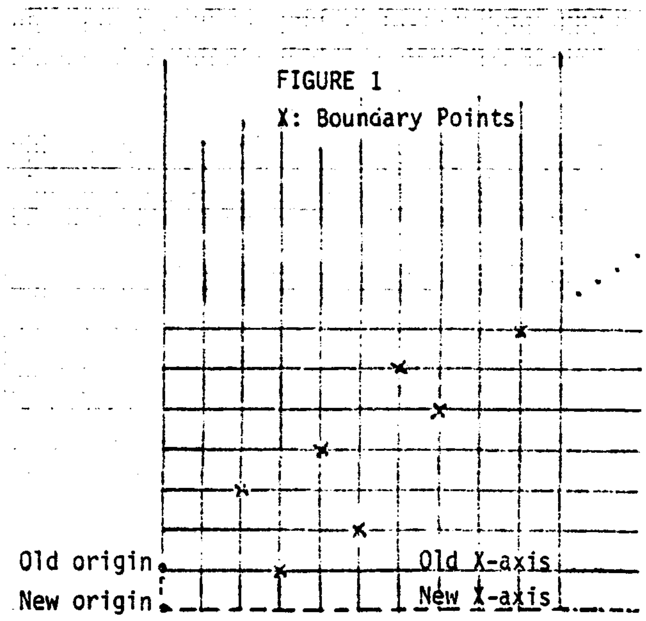
We also note the following results.

<u>Corollary 1</u>. *If* $\lim \frac{C_{k+1}}{n_k} = 0$, *then the plan is closed.*

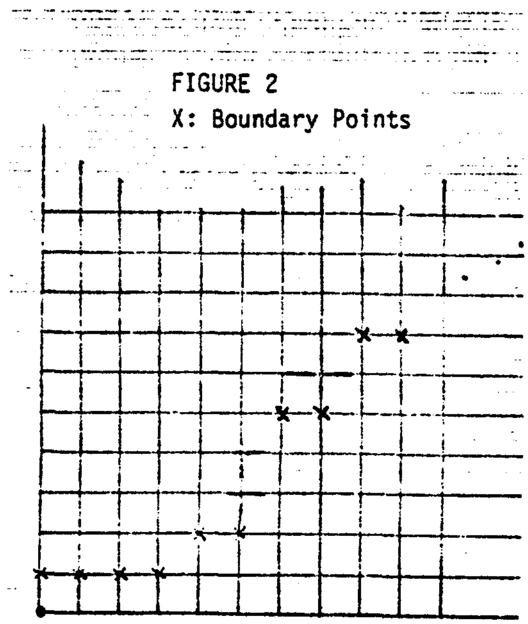
<u>Corollary 2</u>. *Suppose* $C_k = kc$, $k = 1, 2, \cdots$, *and* $n_k = 0(k^{1+\epsilon})$, $\epsilon > 0$. *Then the plan is closed.*

For example, the plan with $c=1$ and $n_k = k(k+1)$ is closed. Moreover, we can show by direct argument that a plan with $c_k = kc$ and $n_k = k\delta$ is *not* closed for any value of c and δ.

3.3. Lemma (3.2) in SS, cited under c(ii) above, has an immediate generalization in the sense that whenever no point on *some* line (not necessarily on the line $Y=1$) is inaccessible, unbiased estimation of $1/p$ is possible. With the exception of such plans, therefore, *every* closed X-unbounded (i.e., unbounded along X-direction) sampling plan contains *one* (and, hence, an infinite number of) *inaccessible* point(s) on *every* line '$Y=c$', $c=0, 1, 2, \cdots$. Let us examine the closure or otherwise the 'shifted' form B′ of such a plan B. Here by 'shifted' form we mean the one used in Gupta (1967) in arriving at an estimate of $1/p$. (Vide (c)(i) above.).



Plans B & B′



Plan B″

With reference to new origin, the first move towards Y-axis will eventually lead to closure. Let $(n_0, 0)$ be the coordinate position of the boundary point on the X-axis and (n_j, j) be that of the first boundary point on the j^{th} parallel line ($j=1, 2, \cdots$) in the old system. Then any path in B′ hitting the old X-axis at a point *not* beyond the point $(n_0, 0)$ will *eventually* end up in hitting a boundary point of B′. As a matter of fact, any path in B′, whenever identifiable as merging with a path in B, will ultimately get absorbed into a boundary point of

B and, hence, of B′. Therefore, the closure or otherwise of B′ solely depends on its behavior with reference to *totally* new paths. It is not difficult to verify that the problem is *essentially equivalent* to that of examining the status of a derived plan B″ whose boundary points are defined through the following rule:

i) All points on the X-axis are accessible;

ii) On the first parallel line, the initial points $\{(0,1),(1,1),\cdots,(n_0,1)\}$ are all boundary points, the rest being accessible;

iii) For any $j \geq 2$, on the j^{th} parallel line, the initial points $\{(0,j),(1,j),\cdots,(\lambda_{j-1},j)\}$ are all inaccessible, then the points $\{(\lambda_{j-1}+1,j),\cdots,(\lambda_j-1,j)\}$ are all boundary points and the rest are all accessible where $\lambda_j = \lambda_{j-1}+1$ if $n_{j-1} \leq n_{j-2}$ and $\lambda_j = n_{j-1}+1$ if $n_{j-1} > n_{j-2}$.

One can readily identify B″ as an infinite-step generalized inverse binomial plan (Vide Subsection 3.2). This should enable one to construct a large number of interesting sampling plans providing unbiased estimators of $1/p$.

3.4. It would be highly desirable at this stage to discuss some features of path counting for arbitrary sampling plans. This aspect of the estimation problem has, however, received little attention in the works of Gupta and SS. An excellent reference in this direction is Mohanty (1979). Theorem 1 on pp 32 and Theorem 2 on pp 36 in Mohanty (1979) can profitably be applied in determining $K(\alpha)$ and $K'(\alpha')$. The calculations are quite involved but the point to be noted is that these Theorems are of wide applicability in general terms. We discuss a simple example below in order to explain the steps involved. First we introduce some definitions and notations.

Any path terminating at the point (m,n) can be represented as a vector $\underline{x} = (x_1, x_2, \cdots, x_n)$ where $x_i (i = 1, 2, \cdots, n)$ is the minimal distance, measured parallel to the X-axis, of the points $(m, n-i)$ from the path (Mohanty (1979, pp 14)). A path $\underline{x} = (x_1, x_2, \cdots, x_n)$ dominates another path $\underline{y} = (y_1, y_2, \cdots, y_n)$ if and only if $y_i \leq x_i$ for all i (Mohanty (1979, pp 18)). Let $(\underline{b}, \underline{a})$ be the set of paths that dominate $\underline{b}$ and are dominated by $\underline{a}$ and let $|(\underline{b}, \underline{a})|$ denote the cardinality of this set.

Theorem 1. (Mohanty (1979, pp 32)) $|(\underline{b}, \underline{a})| = \det_{m\times n}(c_{ij}) = \|c_{ij}\|_{n\times n}$ where

$$c_{ij} = \binom{a_i - b_j + 1}{j - i + 1}_+ \quad \text{or} \quad c_{ij} = \binom{a_{n-j+1} - b_{n-i+1} + 1}{j - i + 1}_+$$

and $\binom{y}{z}_+ = \binom{y}{z}$ when $y \geq z$; $= 0$ when $y < 0$ or $y < z$ or $z < 0$; $= 1$ when $z = 0$.

For the one-boundary case with dominating path $\underline{a}$, one gets (Mohanty (1979, pp 35-36)).

i) $|(\underline{0}, \underline{a})| = |(\underline{a})| = \det_{n\times n}(d_{ij}) = \|d_{ij}\|_{n\times n}$ where

$$d_{11} = \binom{a_n + n}{n}, \quad \underset{i\neq 1}{d_{i1}} = \binom{a_{n-i+1} + n-i}{n-i}_+, \quad \underset{j\neq 1}{d_{1j}} = \binom{a_n - a_{n-j+1} + j-1}{j}_+$$

and $$\underset{\substack{i\neq 1\\ j\neq 1}}{d_{ij}} = \binom{a_{n-1} - a_{n-j+1} + j-i-1}{j-i}_+ \;;$$

ii) (recurrence relation) $$|(a_1, a_2, \cdots, a_n)| = \binom{a_n + n}{n} - \binom{a_n - a_1 + n - 1}{n}_+ - \sum_{i=2}^{n-1} \binom{a_n - a_i + n - i}{n-i+1}_+ |(a_1, a_2, \cdots, a_{i-1})| \quad \text{for } n > 2 \text{ with}$$

$$|(a_1)| = \binom{a_1 + 1}{2} \text{ and } |(a_1, a_2)| = \binom{a_2 + 2}{2} - \binom{a_2 - a_1 + 1}{2}_+ .$$

Consider now the following sampling plan (Figure 3).

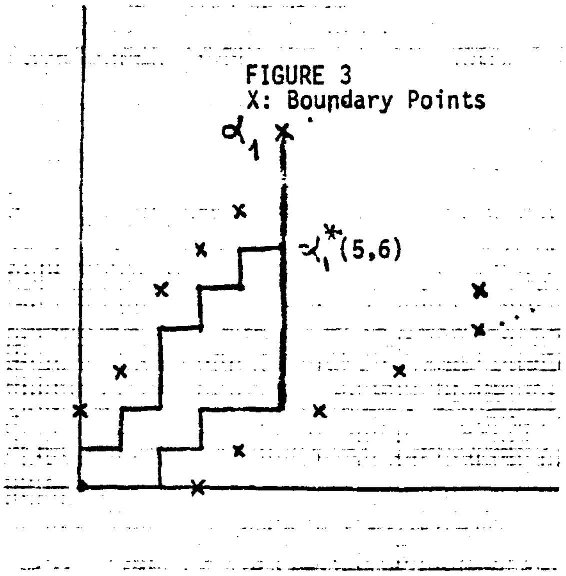


FIGURE 3
X: Boundary Points

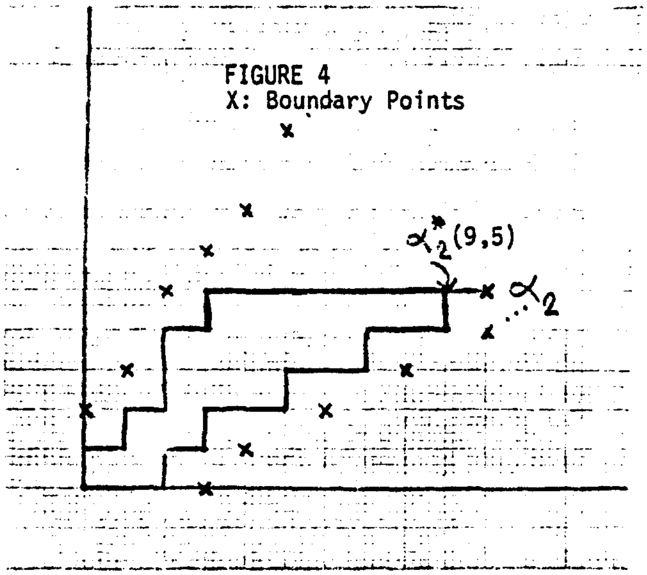


FIGURE 4
X: Boundary Points

For an upper boundary point, say, $\alpha_1 = (5,9)$, we determine $k(\alpha_1)$ as follows:

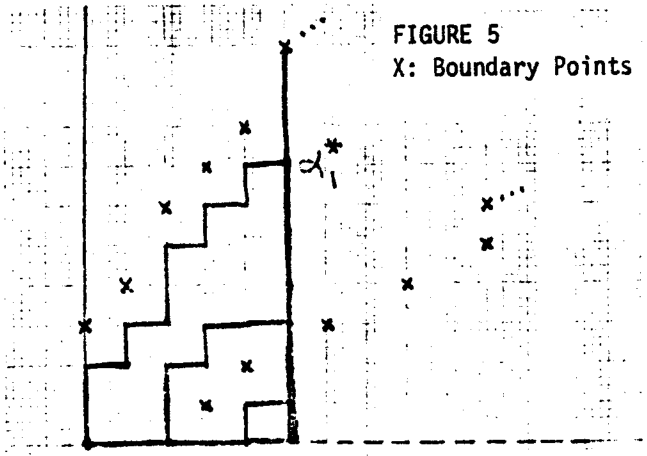
i) $k(\alpha_1) = k(\alpha_1^*)$ where $\alpha_1^* = (5,6)$ since the path from α_1^* to α_1 is unique.

ii) $k(\alpha_1^*) = |(\underline{a}, \underline{b})|$ where $\underline{a} = (1,2,3,3,4,5)$ and $\underline{b} = (0,0,0,0,2,3)$. The two paths have been indicated by heavy lines in Fig. 3. The rest is now clear.

For a lower boundary point, say, $\alpha_2 = (10,5)$, we have $k(\alpha_2) = k(\alpha_2^*)$ with $\alpha_2^* = (9,5)$. (See Fig. 4.) This gives $\underline{a} = (6,7,7,8,9)$ and $\underline{b} = (0,2,4,6,7)$.

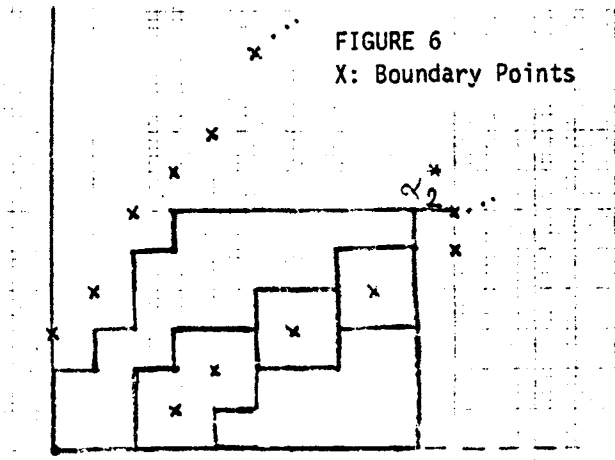
Next consider the 'shifted' form of the plan and suppose we want to calculate $k^*(\alpha_1^*)$ and $k^*(\alpha_2^*)$. We refer to Fig. 5 and Fig. 6 below.

FIGURE 5
X: Boundary Points



Shifted Plan:Path Description

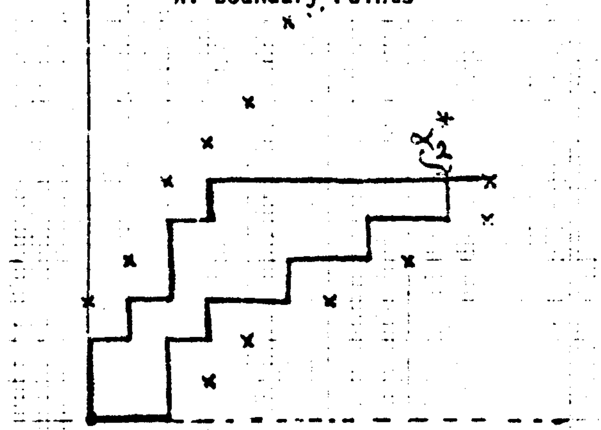
FIGURE 6
X: Boundary Points



Shifted Plan: Path Description

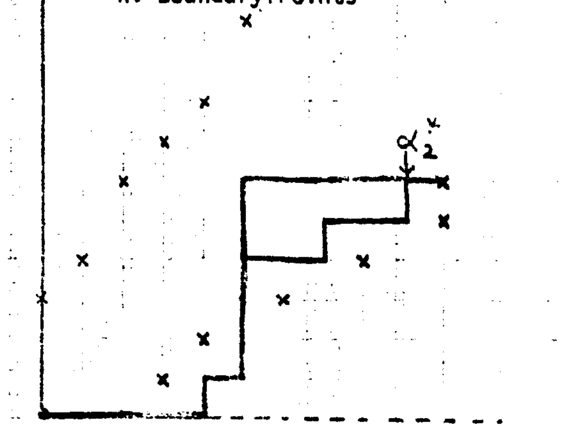
It is seen that there are two different types of paths terminating at α_1^*. One of them follows the earlier pattern while the other emerges due to the 'shift'. We have $k^*(\alpha_1^*) = k_1^*(\alpha_1^*) + k_2^*(\alpha_1^*) = |(a_1^*, b_1^*)| + |(a_2^*, b_2^*)|$ where $\underline{a}_1^* = (1,2,3,3,4,5,5)$, $\underline{b}_1^* = (0,0,0,0,2,3,5)$, $\underline{a}_2^* = (0,0,0,0,0,0,1)$ and $\underline{b}_2^* = (0,0,0,0,0,0,0)$. Next, note that there are altogether four different types of paths terminating at α_2^*. We show them separately in the following figures. Usual calculations will now yield $k^*(\alpha_2^*)$.

FIGURE 7:Path Type 1
X: Boundary Points



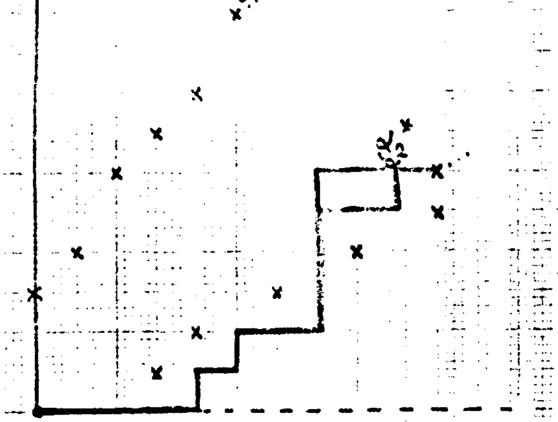
Shifted Plan:Path Description

FIGURE 8:Path Type 2
X: Boundary Points



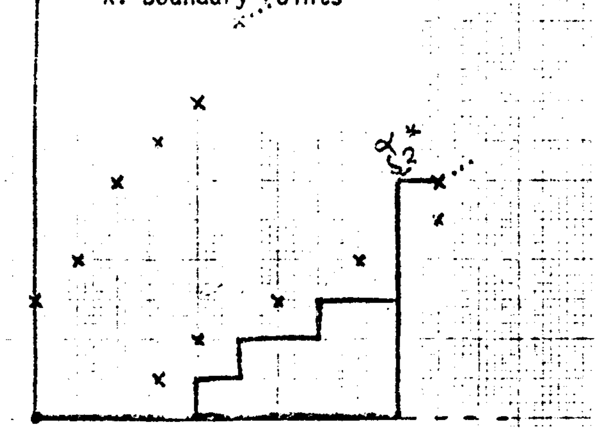
Shifted Plan:Path Description

FIGURE 9:Path Type 3
X: Boundary Points



Shifted Plan:Path Description

FIGURE 10:Path Type 4
X: Boundary Points



Shifted Plan:Path Description

4. ESTIMATION UNDER MARKOVIAN SET-UP

4.1. The model. Assume $P(X_1 = 1) = p = 1 - P(X_1 = 0)$ and $P(X_i = 1 | X_{i-1} = 1) = p\alpha$, $P(X_i = 1 | X_{i-1} = 0) = p\beta$, $0 < \alpha$, $\beta < 1$, $i = 2, 3, \cdots$. Assume further that α and β are both known. We are interested in the estimation of $1/p$ once again under this set-up. The nomenclature carries through *except* for the fact that the probability of an accessible point or a boundary point $(x, y) = r$ included in a path from the origin is now entirely *different* from $k(r)\, p^y q^x$ (unless, of course, $\alpha = \beta = 1$) where $k(r)$ is the number of paths from the origin to the point r. Our approach to this problem is very similar to that for other inference problems studied under this model by Rustagi (1975). We will consider two specific sampling plans and carry out our investigation.

4.2. Estimation under inverse binomial sampling plan. Consider the inverse binomial sampling plan $S(m)$ whose boundary points are $\{(0, m), (1, m), \cdots,\}$ i.e., all points on the m^{th} parallel line. Let $\alpha = (Y, m)$ be a boundary point. Then $N = Y + m$ represents the total number of trials needed to hit the boundary point α. Let the random variable R denote the total number of *runs* of *successes* in N trials. [Note that moves along X-direction are denoted by Y). Then, following Rustagi (1975), we write

$$\text{i)}\quad P(X_1 = x_1, R = r, Y = y) = \binom{m-1}{r-1}\binom{y-1}{r-x_1-1}(p\alpha)^{m-r}(1-pa)^{r-1}(p\beta)^{r-x_1}(1-p\beta)^{y-r+x_1}p^{x_1}(1-p)^{1-x_1}$$

$$x_1 = 0, 1;\ r = 1, 2, \ldots, m;\ y = 0, 1, 2, \ldots$$

ii) $N = Z_1 + Z_2 + \cdots + Z_m$ where

$$P(Z_1 = z_1) = \begin{cases} p \text{ for } z_1 = 1 \\ (1-p)(1-p\beta)^{z_1 - 2} p\beta \text{ for } z_1 \geq 2 \end{cases}$$

and $Z_2, \cdots, Z_m$ are iid random variables with

$$P(Z_i = z) = \begin{cases} p\alpha \quad \text{for } z = 1 \\ (1-p\alpha)\ p\beta\ (1-p\beta)^{z-2} \quad \text{for } z \geq 2 \end{cases}$$

iii) $P(X_1 = x_1, R = r) = \binom{m-1}{r-1}(p)^{x_1}(1-p)^{1-x_1}(p\alpha)^{m-r}(1-p\alpha)^{r-1}$

iv) $P(Y = y | R = r, X_1 = x_1) = \begin{cases} 1 \quad \text{for } y = 0 \quad \text{if} \quad (r, x_1) = (1, 1) \\ \binom{y-1}{r-x_1-1}(p\beta)^{r-x_1}(1-p\beta)^{y-r+x_1} \text{ for } y \geq r-x_1 \text{ if } (r, x_1) \neq (1,1) \end{cases}$

v) $E(N) = m + (p\beta)^{-1}\{m(1-\alpha p) + p(\alpha-1)\} = \frac{m}{p\beta} + \{m - \frac{1+\alpha(m-1)}{\beta}\}$

Clearly (v) enables one to form an estimator for $1/p$. We, however, want to examine how far the construction principle in Gupta (1967) applies to this set-up (Vide (c)(i) in Section 2). Naturally, the plan $S(m)$ is closed for every $m = 1, 2, \cdots$. Consider then the plan $S(m+1)$ as the 'shifted' form of $S(m)$. We may now develop the following:

$$1 = \sum_{x_1=0}^{1} \sum_{r=1}^{m+1} P(X_1 = x_1, R = r | m+1)$$

$$= \sum_{x_1=0}^{1} \sum_{r=1}^{m} P(X_1 = x_1, R = r | m+1) + P(X_1 = 0, R = m+1 | m+1)$$

$$+ P(X_1 = 1, R = m+1 | m+1)$$

$$= \sum_{x_1=0}^{1} \sum_{r=1}^{m} \binom{m}{r-1}(p\alpha)^{m+1-r}(1-p\alpha)^{r-1} p^{x_1}(1-p)^{1-x_1} + (1-p\alpha)^m$$

$$= \sum_{x_1=0}^{1} \sum_{r=1}^{m} P(X_1 = x_1, R = r | m) \cdot \left\{\binom{m}{r-1} / \binom{m-1}{r-1}\right\} \cdot p\alpha + (1-p\alpha)^m$$

$$\Rightarrow 1/p = \sum_{x_1=0}^{1} \sum_{r=1}^{m} P(X_1 = x_1, R = r | m) \cdot \left\{ \frac{m\alpha}{m-r+1} \right\} + (1-p\alpha)^{m-1}(1/p - \alpha)$$

$$= \sum_{x_1=0}^{1} \sum_{r=1}^{m} \left\{\frac{m\alpha}{m-r+1} \quad I_{R=m}(1/p - \alpha)\right\} P(X_1 = x_1, R = r | m)$$

where $I_{R=m}$ is the indicator function for the event '$R=m$'. Now note that for any $m \geq 2$, $m-x_1 > 0$ and, hence, $E\left(\frac{Y}{m-x_1} \middle| R=m, X_1=x_1\right) = 1/p\beta$. Also, for $m=1$, $E(Y|R=1, X_1=0) = 1/p\beta$. This leads us to the following estimator for $1/p$:

$$e^* = \frac{m\alpha}{m-r+1} + I_{R=m}\left(\frac{\beta Y}{m-x_1} - \alpha\right) \text{ when } m \geq 2$$

$$e^* = \begin{cases} \alpha & \text{for } r=x_1=1 \\ \alpha + I_{R=1}(\beta Y - \alpha) & \text{for } (r, x_1) \neq (1,1) \end{cases} \text{ when } m=1 .$$

We will denote this e^* by $e^*(\alpha, \beta, m, r, x_1, y)$. In the particular case of $\alpha=\beta=1$, we know that Y is the complete sufficient statistic and, hence, we get $E(e^*|Y=y) = 1+y/m$ as we expect. It may be noted that in this case

$$P(R=r, X_1=x_1|Y=y) = \binom{m-1}{r-1}\binom{y-1}{r-x_1-1}\Big/\binom{m+y+1}{m-1}, \quad r=1,2,\cdots,m; \quad x_1=0,1 .$$

4.3 Estimation under 'two-way inverse binomial' sampling plan.

Next we consider the following plan which we call a two-way inverse binomial plan.

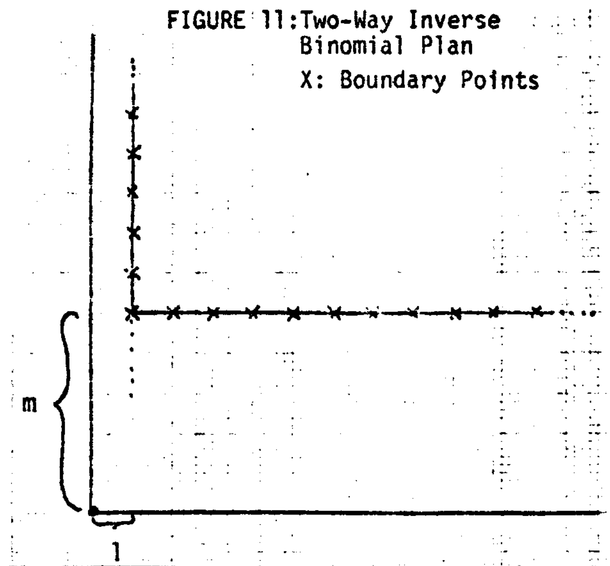


FIGURE 11: Two-Way Inverse Binomial Plan
X: Boundary Points

For $m=1$, our estimator for $1/p$ would be: $e^* = 1 + I_{X_1=0} \cdot (\beta y)$. This means that $e^*((1,1)) = 1$ if the path is $\uparrow\rightarrow$ and $= 1+\beta$ if the path is $\rightarrow\uparrow$. For other boundary points, it is unique. For any other $m \geq 2$, the estimator is given by

$e^{**} = 1 + I_{X_1=0}\, e^*(\alpha, \beta, m, r, x_2, y-1)$ for $y \geq 1$. Here in e^*, x_2 relates to X_2 having a distribution $P(X_2=0|X_1=0) = 1-p\beta = 1-P(X_2=1|X_1=0)$.
Once again, we will have $E(e^{**}) = 1 + q \cdot E(e^*|X_1=0) = 1 + q/p = 1/p$.

Further, for $\alpha = \beta = 1$, $E(e^{**}|Y=y) = \begin{cases} 1 \text{ for any point } (x,1) \text{ with } x \geq m+1 \\ 1 \text{ for the point } (m,1) \text{ reached by any path with first trial resulting in S} \end{cases}$

and $E(e^{**}|Y=y) = \begin{cases} 2 \text{ for the point } (m,1) \text{ reached with the first move as F} \\ 1+y/m \text{ for the point } (m,y),\ y \geq 2\,. \end{cases}$

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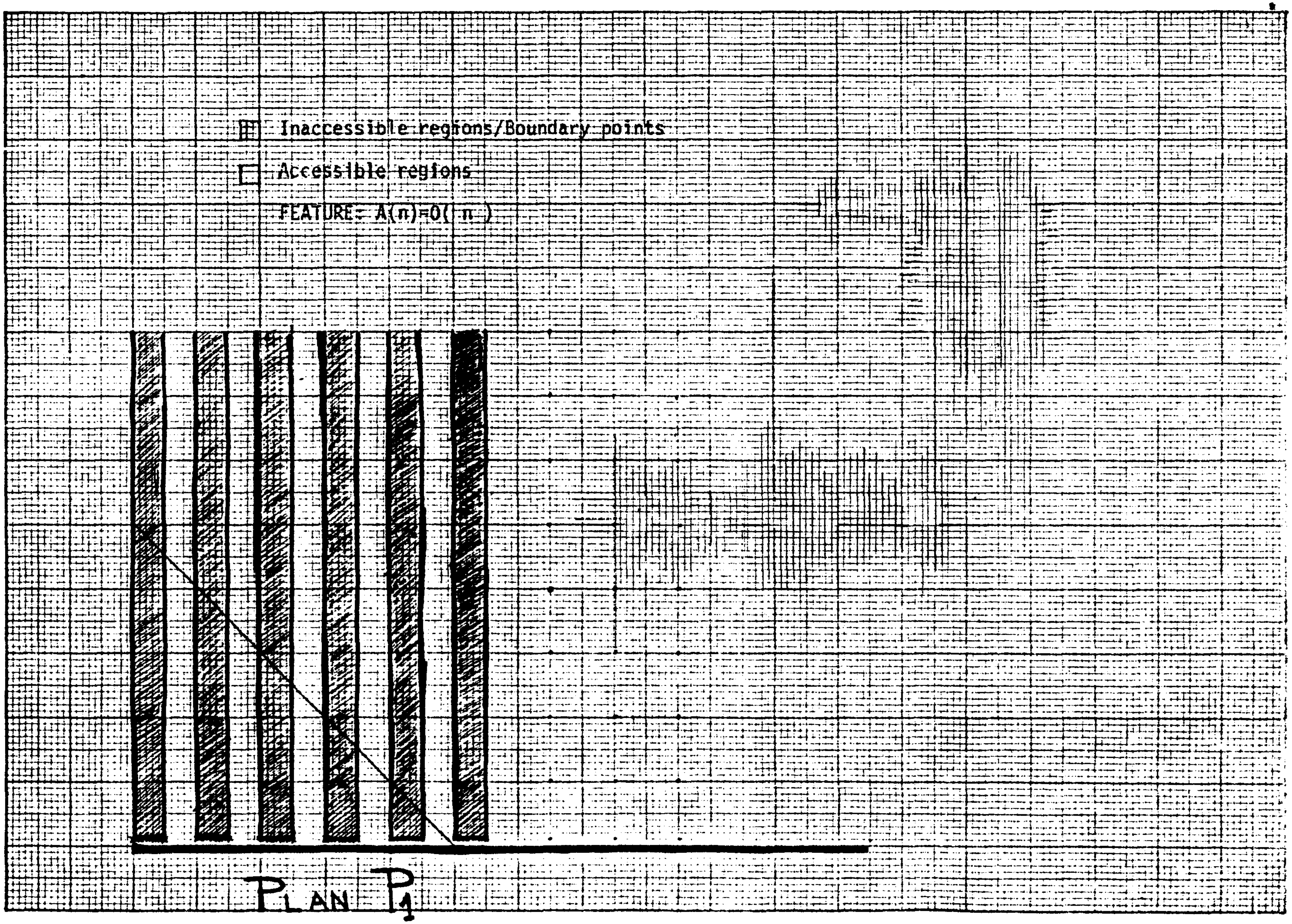
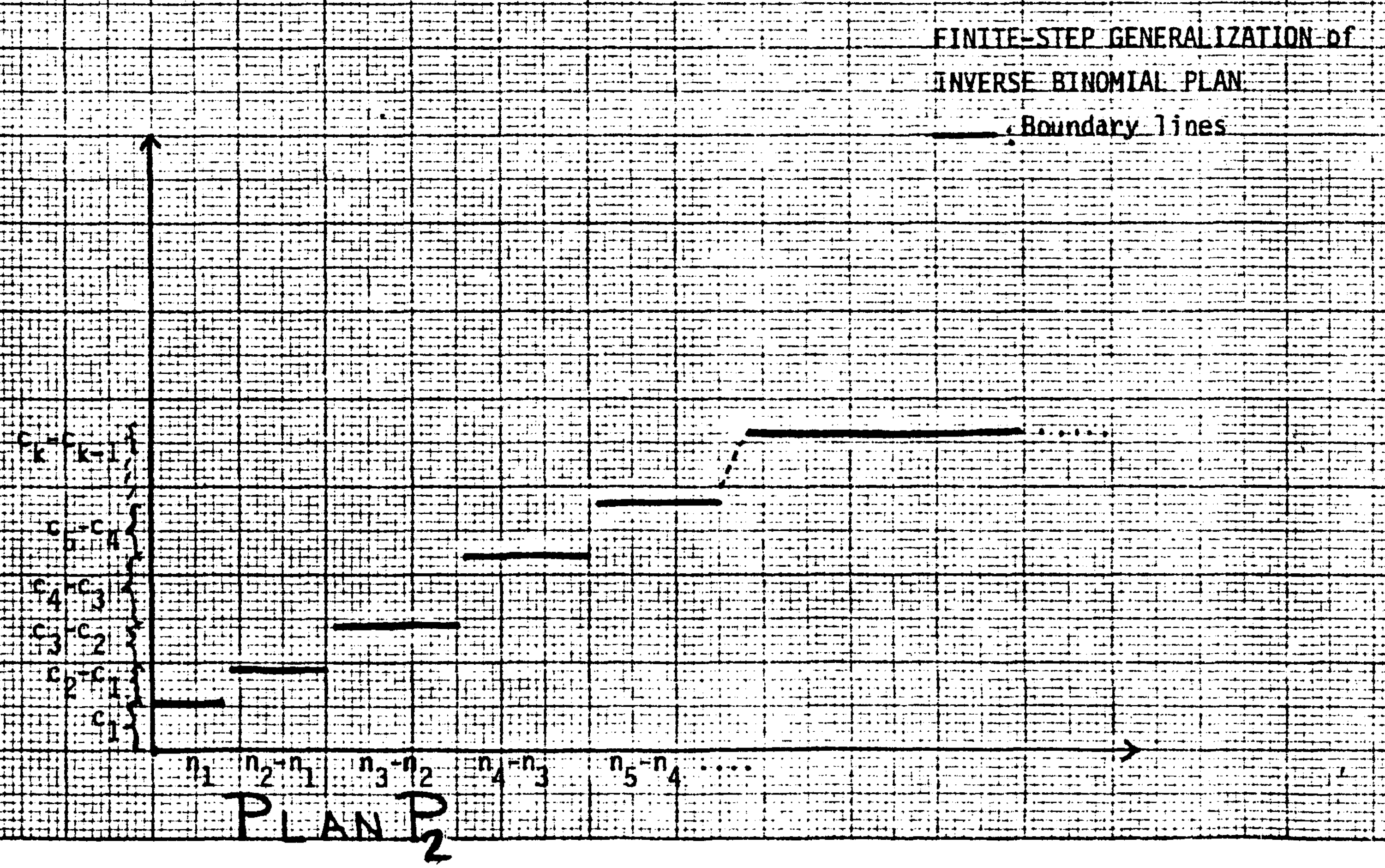

Inaccessible regions/Boundary points
Accessible regions
FEATURE: A(n)=0(n)
PLAN P1


FINITE-STEP GENERALIZATION of
INVERSE BINOMIAL PLAN
Boundary lines
$c_k - c_{k-1}$
$c_5 - c_4$
$c_4 - c_3$
$c_3 - c_2$
$c_2 - c_1$
c_1
n_1
$n_2 - n_1$
$n_3 - n_2$
$n_4 - n_3$
$n_5 - n_4$ ....
PLAN P_2