

GRAPHS IN WHICH EACH PAIR OF VERTICES IS
ADJACENT TO THE SAME NUMBER d OF OTHER VERTICES

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1. INTRODUCTION. A strongly regular graph may be defined as a finite regular graph with v vertices and of valence n_1 , such that each pair of adjacent vertices is adjacent to exactly p_{11}^1 other vertices, and each pair of non-adjacent vertices is adjacent to exactly p_{11}^2 vertices. This paper deals with some methods of constructing strongly regular graphs from other strongly regular graphs. In particular we consider graphs for which $p_{11}^1 = p_{11}^2 = d$, so that each pair of vertices (whether adjacent or not) is adjacent to exactly d other vertices. The problem of investigating such graphs has recently been proposed by Szekeres. Erdős, Renyi and Sós [9] have shown that if there is a graph G (not necessarily regular) with the property that every pair of vertices is adjacent to exactly one other vertex then G must consist of n triangles with a common vertex. G has thus $2n+1$ vertices which may be labeled by the integers $0, 1, \dots, 2n$ such that the edges are the pairs $(0, 1), (0, 2), \dots, (0, 2n), (1, 2), (3, 4), \dots, (2n-1, 2n)$. Also it has been shown by Erdős (in a private communication) that if $d > 1$, and every pair of vertices of a graph G is adjacent to exactly d other vertices, then G must be regular, and hence strongly regular. We shall denote such a graph by $G_2(d)$. If the number of vertices is v and the valence is

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n_1 , we say that v, n_1, d are the parameters of the $G_2(d)$ graph. We prove:

THEOREM (1.1). If $G_2(d)$ is a finite graph without loops or multiple edges, in which each pair of distinct vertices is adjacent to exactly d other vertices, $d \geq 2$, then $G_2(d)$ is regular of valence n_1 such that $v - 1 = n_1(n_1 - 1)/d$ where v is the number of vertices and there exists a positive integer m , such that

$$(i) \quad n_1 = d + m^2$$

$$(ii) \quad d/m \text{ is an integer, with the same parity as } v - 1 - m.$$

We then consider methods of obtaining such graphs. A number of examples can readily be obtained from known configurations, and graphs of certain partial geometrics [2], but to obtain others it is necessary to use the principle of switching which was first introduced by Seidel [14].

In particular we show the existence of $G_2(d)$ graphs with parameters

$$(i) \quad v = 4r^2, \quad n_1 = r(2r-1), \quad d = r(r-1),$$

$$(ii) \quad v = 4r^2, \quad n_1 = r(2r+1), \quad d = r(r+1),$$

$$(iii) \quad v = 4r^2 - 1, \quad n_1 = 2r^2, \quad d = r^2,$$

for all $r = 3^m \cdot 2^{m+n-1}$, where m and n are any non-negative integers, $(m, n) \neq (0, 0)$.

2. REGULARITY OF $G_2(d)$. The regularity of $G_2(d)$ was proved by Erdős, but a slightly different proof is included here for completeness.

Suppose $G_2(d)$ has v vertices, $1, 2, \dots, v$. Let S_i be the set of vertices adjacent to the vertex i , and let $S = \{S_1, S_2, \dots, S_v\}$ be the class of sets S_i . Then any two distinct sets of S must have

exactly d vertices in common, and conversely every pair of distinct vertices appears in exactly d distinct sets of S . Consider a particular set S_i of S . Let $|S_i| = n$. Let us count the number of symbols $\{(\ell, m), S_u\}$, (ℓ, m) being an unordered pair of vertices belonging to both S_i and S_u , where S_u belongs to S , $u \neq i$. We can choose (ℓ, m) belonging to S_i in $n_1(n_1-1)/2$ ways. Now each (ℓ, m) must occur in exactly $d-1$ of the sets of S other than S_i . Hence the number of symbols is $(d-1)(n_1-1)n_1/2$. Again since S_i intersects any S_u ($u \neq i$) in exactly d elements, for any fixed S_u we get $d(d-1)/2$ symbols. Hence the number of symbols is $(v-1)d(d-1)/2$. Thus for $d \geq 2$ we have $v-1 = n_1(n_1-1)/d$. This uniquely determines n_1 . Hence $G_2(d)$ is regular. This proves the first part of Theorem (1.1).

3. STRONGLY REGULAR GRAPHS. A finite connected graph G (without loops or multiple edges) is called strongly regular [1], if it is regular of valence n_1 , and any pair of adjacent edges is adjacent to exactly p_{11}^1 other vertices, and any pair of non-adjacent vertices is adjacent to exactly p_{11}^2 vertices. If v is the number of vertices, we shall call $(v, n_1, p_{11}^1, p_{11}^2)$ the parameters of the strongly regular graph G .

The complementary graph \bar{G} of G is regular of valence $n_2 = v - n_1 - 1$. Now let us call two vertices of G first associates if they are adjacent and second associates if they are non-adjacent. If the vertices ℓ, m of G are i -th associates let $p_{jk}^i(\ell, m)$ denote the number of vertices which are simultaneously j -th associates of ℓ and

k -th associates of m ($i, j, k = 1, 2$). Easy counting arguments show [3], [6] that if G is strongly regular with parameters $(v, n_1, p_{11}^1, p_{11}^2)$, then all the numbers $p_{jk}^i(\lambda, m)$ are constants independent of λ, m and that the following relations hold:

$$(3.1) \quad p_{12}^1 = p_{21}^1, \quad p_{12}^2 = p_{21}^2$$

$$(3.2) \quad p_{11}^1 + p_{12}^1 = n_1 - 1, \quad p_{21}^1 + p_{22}^1 = n_2, \quad p_{11}^2 + p_{12}^2 = n_1, \quad p_{21}^2 + p_{22}^2 = n_2 - 1,$$

$$(3.3) \quad n_1 p_{12}^1 = n_2 p_{11}^2, \quad n_1 p_{22}^1 = n_2 p_{12}^2.$$

It is clear that \bar{G} is strongly regular with parameters $(v, n_2, p_{22}^2, p_{22}^1)$. The graphs $G_2(d)$ which we want to investigate are strongly regular with parameters (v, n_1, d, d) .

4. PROOF OF THEOREM (1.1). The adjacency matrix of a finite graph G with v vertices is a $v \times v$ matrix $A = (a_{ij})$, $i, j = 1, 2, \dots, v$, such that $a_{ij} = 1$ if $i \neq j$ and i and j are adjacent, and $a_{ij} = 0$ otherwise. In particular $a_{ii} = 0$.

Let A be the adjacency matrix of a $G_2(d)$ graph with v vertices, and valence n_1 . Let J be the $v \times v$ matrix for which each element is unity. Then it is readily seen that

$$A^2 = (n_1 - d)I + dJ.$$

Hence

$$A^3 = (n_1 - d)A + n_1 dJ.$$

$$\therefore A^3 - n_1 A^2 - (n_1 - d)A + n_1(n_1 - d)I = 0.$$

Thus A has only three distinct eigen values n_1 and $\pm (n_1-d)^{1/2}$. Now from the regularity of $G_2(d)$ it follows that $A^* = A/n_1$ is a stochastic matrix, and since $G_2(d)$ is connected A^* is irreducible. It follows [8] that unity is a simple root of A^* , so that n_1 is a simple root of A . Let α_1, α_2 be the multiplicities of the roots $\theta_1 = (n_1-d)^{1/2}$ and $\theta_2 = -(n_1-d)^{1/2}$. Then

$$|A - I\theta| = (n_1-\theta)(\theta_1-\theta)^{\alpha_1}(\theta_2-\theta)^{\alpha_2}.$$

To determine α_1 and α_2 we note that

$$\text{Tr } I = 1 + \alpha_1 + \alpha_2 = v,$$

$$\text{Tr } A = n_1 + \alpha_1\theta_1 + \alpha_2\theta_2 = 0.$$

$$\therefore \alpha_1 = \frac{v-1}{2} + \frac{n_1}{2(n_1-d)^{1/2}}, \alpha_2 = \frac{v-1}{2} - \frac{n_1}{2(n_1-d)^{1/2}}.$$

Since the multiplicities are necessarily integral, there must exist an integer m such that $n_1 = d + m^2$. Also since

$$2\alpha_2 = (v-1) - (m + \frac{d}{m})$$

d/m must be integral, and the integers $v - 1 - m$ and d/m must have the same parity. This completes the proof of Theorem (1.1).

5. SOME EXAMPLES OF $G_2(d)$ GRAPHS.

(i) If we take $m = 1$, in Theorem (1.1), then $n_1 = d+1$, $v = d+2$.

Then $G_2(d)$ is obviously the complete graph with $d + 2$ vertices.

(ii) If we take $d = 2$, $m \neq 1$, then since d is divisible by m , the only possible value of m is 2. This gives $n_1 = 6$, $v = 16$. It is known that there are only two non-isomorphic strongly regular graphs with parameters $(16, 6, 2, 2)$, [14] and [15]. They are given as follows:

(a) Arrange the integers 1 through 16 corresponding to the vertices of the graph in a 4×4 scheme.

$$(5.1) \quad \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{array}$$

Then any two vertices are adjacent if and only if the corresponding integers are in the same row or same column.

(b) Arrange the integers corresponding to the vertices in a 4×4 scheme, and impose on it a cyclic Latin square. We thus obtain

$$(5.2) \quad \begin{array}{cccc} 1A & 2B & 3C & 4D \\ 5D & 6A & 7B & 8C \\ 9C & 10D & 11A & 12B \\ 13B & 14C & 15D & 16A \end{array}$$

Two vertices are adjacent if they do not occur in the same row or in the same column, and do not come together with the same letter of the cyclic square.

(iii) Consider the finite projective space $PG(3, 2)$ of three dimensions based on the Galois field of order 2. This space has 15 points and 35 lines. We may take the lines to correspond to the vertices of a graph where two vertices are adjacent if and only if the corresponding lines intersect. It is readily seen that each line is intersected by 18 lines,

and any pair of lines is intersected by 9 lines. Hence our graph is $G_2(d)$ with $v = 35$, $n_1 = 18$, $d = 9$.

6. $G_2(d)$ GRAPHS DERIVED FROM PARTIAL GEOMETRIES. A partial geometry (r, k, t) is a system of points and lines satisfying the following axioms:

- A1. Any two points are not incident with more than one line.
- A2. Each point is incident with r lines.
- A3. Each line is incident with k points.
- A4. If the point P is not incident with the line ℓ , there are exactly t lines through P intersecting ℓ .

The graph of a partial geometry (r, k, t) is obtained by taking the vertices of the graph to correspond to the points of the partial geometry, and taking two vertices to be adjacent if and only if they are incident with the same line of the geometry. Partial geometries have been discussed in [2]. It was shown that the graph of a partial geometry is strongly regular with parameters

$$(6.1) \quad v = k[(r-1)(k-1) + t]/t, \quad n_1 = r(k-1),$$

$$(6.2) \quad p_{11}^1 = (t-1)(r-1) + k-2, \quad p_{11}^2 = rt.$$

We will call it a geometric graph $G(r, k, t)$.

A strongly regular graph G with the parameters (6.1), (6.2) for some positive integral values of r, k, t is defined to be pseudo-geometric. Of course it may or not be the graph of a partial geometry. It will be called a pseudo-geometric graph $G(r, k, t)$. If for a

pseudo-geometric graph $G(r, k, t)$ the condition $k = r + t + 1$ is satisfied then $p_{11}^1 = p_{11}^2 = rt$ and we will have an example of the kind desired. Many examples of partial geometries are known satisfying the required condition.

(a) Consider an elliptic non-degenerate quadric in the finite projective space $PG(5, q)$ where q is a prime power. It is known [11], [13] that this quadric is ruled by straight lines called generators, but contains no planes. The number of points on the surface is $(q^3+1)(q^2+1)$. It was shown in [2], that the points and generators can be regarded as the points and lines of a partial geometry $(q^2+1, q+1, 1)$. The dual of this partial geometry is obtained by taking the points and lines of the dual to be the lines and points of the original geometry. Thus the dual partial geometry has the parameters $(q+1, q^2+1, 1)$. Another way to obtain a partial geometry with the same parameters is to take the surface $x_0^{q+1} + x_1^{q+1} + x_2^{q+1} + x_3^{q+1} = 0$ in $PG(3, q^2)$. It has been shown in [4], that this surface contains $(q^2+1)(q^3+1)$ points, and $(q+1)(q^3+1)$ generators which may be taken to be points and lines of a partial geometry $(q+1, q^2+1, 1)$. If we take $q = 2$, the condition $p_{11}^1 = p_{11}^2$ is satisfied. The graph of this partial geometry is strongly regular with parameters $(45, 12, 3, 3)$. Thus we get a $G_2(d)$ with $v = 45$, $n_1 = 12$, $d = 3$. Here $m = 3$.

(b) A net of degree r and order k is a system of undefined points and lines together with an incidence relation subject to the following axioms (i) Each line is incident with k points, $k \geq 1$ (ii) The lines of the net can be partitioned into r disjoint, non-empty "parallel classes"

such that each point of the net is incident with exactly one line of each class, (iii) Given two lines belonging to distinct classes there is exactly one point of the net which is incident with both lines. Then it readily follows that the number of points is k^2 , the number of lines is rk , which fall into r parallel classes of k lines each. It is well known that the existence of a net of degree r and order k is equivalent to the existence of $(r-2)$ mutually orthogonal Latin squares of order k . If we superpose the Latin squares then each cell contains $r-2$ symbols, belonging in order to the different Latin squares. The k^2 cells can now be identified with the k^2 points of the net. Points belonging to the same row give one set of k parallel lines. Points belonging to the same column give another set of k parallel lines. Cells which contain the same symbol of the i -th Latin square give a set of parallel lines for each value of i ($i = 1, 2, \dots, r-2$). We thus get r classes of parallel lines.

The points and lines of a net of degree r and order k satisfy the axioms of a partial geometry with parameters $(r, k, r-1)$. The graph of this partial geometry is strongly regular with parameters.

$$(6.3) \quad v = k^2, n_1 = r(k-1),$$

$$(6.4) \quad p_{11}^1 = (r-2)(r-1) + (k-2), p_{11}^2 = r(r-1).$$

Such a graph will be called a net graph $L_r(k)$. Any strongly regular graph (not necessarily the graph of a net) with the parameters (6.3), (6.4) will be called a pseudo net graph, $L_r(k)$. The condition $k = r + t + 1$ reduces to $k = 2r$. Thus a pseudo net graph $L_r(2r)$ is a $g_2(d)$ graph with $v = 4r^2$, $n_1 = r(2r-1)$, $d = r(r-1)$.

We can thus get $G_2(d)$ graphs for all values of r for which there exist $r-2$ mutually orthogonal Latin squares of order $2r$. This is always true if $r = 2^m$ where m is a non-negative integer. We can therefore obtain a $G_2(d)$ graph with $v = 2^{2m+2}$, $n_1 = 2^m(2^{m+1}-1)$, $d = 2^m(2^m-1)$. The example (ii)(a) of paragraph 6, is a special case of this. Again since there exists a Latin square of order 6, we get a $G_2(d)$ graph with $v = 36$, $n_1 = 15$, $d = 6$. Also the existence of 5 mutually orthogonal squares of order 12 is known [5]. By taking 4 mutually orthogonal squares of order 12, we can get a $G_2(d)$ graph with $v = 144$, $n_1 = 66$, $d = 30$.

(c) A balanced incomplete block design (BIB) is an arrangement of a set of v_0 objects or treatments in b_0 sets or blocks, such that (i) each block contains k_0 distinct treatments (ii) each treatment is contained in r_0 blocks (iii) each pair of distinct treatments is contained in λ_0 blocks. We say that the design has parameters $(v_0, b_0, r_0, k_0, \lambda_0)$. The dual of a design is defined as a new design whose treatments and blocks are in $(1, 1)$ correspondence with the blocks and treatments of the original design, and incidence is preserved (a block and treatment are incident if the treatment is contained in the block and non-incident otherwise). It is known [2] that the dual of a BIB design with $\lambda_0 = 1$ can be regarded as a partial geometry (r, k, r) where $r = k_0$, and $k = r_0$, the lines of the partial geometry being the blocks of the dual design.

Such a dual design (or partial geometry) may be called a linked block design (or geometry). The corresponding strongly regular graph has the parameters

$$(6.5) \quad v = k(kr-k+1)/r, \quad n_1 = r(k-1),$$

$$(6.6) \quad p_{11}^1 = (k-2) + (r-1)^2, \quad p_{11}^2 = r^2.$$

It will be called a linked block graph and will be denoted by $LB_r(k)$. Any strongly regular graph (not necessarily the graph of a linked block design) will be called a pseudo linked block graph $LB_r(k)$. The condition $k = r + t + 1$ now reduces to $k = 2r + 1$. Thus a pseudo linked block graph $LB_r(2r+1)$ is a $G_2(d)$ graph with $v = 4r^2 - 1$, $n_1 = 2r^2$, $d = r^2$. Now BIB designs with parameters $v_0 = 2^{m-1}(2^m - 1)$, $b_0 = 2^{2m-1}$, $r_0 = 2^{m+1}$, $k_0 = 2^{m-1}$, $\lambda_0 = 1$ are known [7] for every integral value of m . We can therefore get a corresponding $G_2(d)$ graph with $v = 2^{2m-1}$, $n_1 = 2^{2m-1}$, $d = 2^{2m-2}$ for all integral m . Also BIB designs with parameters

$$(6.7) \quad v_0 = r(2r-1), \quad b_0 = 4r^2 - 1, \quad r_0 = 2r+1, \quad k_0 = r, \quad \lambda_0 = 1$$

are known for values of $k_0 = 2, 3, 4, 5$ and 7 [1], [12]. Hence the corresponding $G_2(d)$ graphs with parameters $v = 4r^2 - 1$, $n_1 = 2r^2$, $d = r^2$ can be constructed for $r = 2, 3, 4, 5$, and 7 .

We shall now show that for an infinity of values of r pseudo net graphs $L_r(2r)$, and pseudo linked block graphs $LB_r(2r+1)$ can be obtained even when $r-2$ orthogonal Latin squares of order $2r$, or a BIB design with parameters (6.7) is unknown. However to do this we first need the concept of Seidel equivalence of strongly regular graphs, and some theorems relating to this equivalence. As a preliminary to this we give some definitions and notations in the next paragraph.

7. NOTATIONS AND DEFINITIONS. If M is a 0, 1 matrix we shall denote by \bar{M} the matrix obtained from M by interchanging zeros and ones. In particular M may be a vector.

For any 0, 1 vector, $\alpha = (a_1, a_2, \dots, a_n)$ we shall denote by $w(\alpha)$ the number of non-zero coordinates in α and call this number the weight of α . The weight of the i -th row of M will be denoted by $w_i[M]$. Clearly,

$$(7.1) \quad w(\alpha) + w(\bar{\alpha}) = n, \quad w_i[M] + w_i[\bar{M}] = n$$

where M is an $m \times n$ matrix.

Given two 0, 1 vectors $\alpha = (a_1, a_2, \dots, a_n)$, $\beta = (b_1, b_2, \dots, b_n)$ the Hamming distance $\delta(\alpha, \beta)$ between α and β is defined to be the number of coordinates in which α and β disagree. Clearly

$$(7.2) \quad \delta(\bar{\alpha}, \bar{\beta}) = \delta(\alpha, \beta), \quad \delta(\alpha, \bar{\beta}) = \delta(\bar{\alpha}, \beta) = n - \delta(\alpha, \beta).$$

Again if $(\alpha \cdot \beta)$ denotes the scalar product of α and β

$$(7.3) \quad \delta(\alpha, \beta) = w(\alpha) + w(\beta) - 2(\alpha \cdot \beta).$$

We shall denote by $\sigma_{ij}[M]$ the scalar product of the i -th and j -th rows of M , and by $\delta_{ij}[M]$ the Hamming distance between the i -th and j -th rows of M . Then

$$(7.4) \quad \delta_{ij}[M] = w_i[M] + w_j[M] - 2\sigma_{ij}[M].$$

Again if M_1 and M_2 are two 0, 1 matrices each with the same number of rows, but with possibly different number of columns, then

$$(7.5) \quad w_i[M_1, M_2] = w_i[M_1] + w_i[M_2], \quad \delta_{ij}[M_1, M_2] = \delta_{ij}[M_1] + \delta_{ij}[M_2] .$$

8. SEIDEL EQUIVALENCE OF STRONGLY REGULAR GRAPHS. Let G be a strongly regular graph with parameters

$$v, n_1, p_{11}^1, p_{11}^2 .$$

We can obtain another graph G^* from it by the following process: Let the set of vertices V of G be divided into disjoint subsets V_1 and V_2 , $V = V_1 \cup V_2$. G^* has the same set of vertices as G . Two vertices of G^* both of which belong to V_1 or to V_2 are adjacent or non-adjacent in G^* according as they are adjacent or non-adjacent in G . Two vertices of G^* one of which belongs to V_1 and the other to V_2 are adjacent in G^* if they are non-adjacent in G , and non-adjacent in G^* if they are adjacent in G . Then G^* may be said to be derived from G by complementation with respect to V_1 and V_2 . If G^* is strongly regular it is defined to be Seidel equivalent to G , or more briefly S -equivalent to G .

Let $|V_1| = v_1$, $|V_2| = v_2$, then $v = v_1 + v_2$. In writing down the adjacency matrix of G , we may take the first v_1 rows (columns) to correspond to the vertices in V_1 and the last v_2 rows (columns) to correspond to the vertices in V_2 . Then we can write the adjacency matrix of G as

$$(8.1) \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{11} and A_{22} are square matrices of order v_1 and v_2 respectively, A_{12} is a $v_1 \times v_2$ matrix, and $A_{21} = A_{12}'$. Then clearly the adjacency matrix of G^* is

$$(8.2) \quad A^* = \begin{bmatrix} A_{11} & \bar{A}_{12} \\ \bar{A}_{21} & A_{22} \end{bmatrix}$$

We wish to investigate the conditions under which G^* is strongly regular and therefore by definition Seidel equivalent to G . Let

$$1 \leq i \leq v_1, \quad 1 \leq j \leq v_2.$$

$$w_i[\bar{A}_{12}] = v_2 - w_i[A_{12}], \quad w_j[\bar{A}_{21}] = v_1 - w_j[A_{21}].$$

Again since G is regular and of degree n_1 ,

$$w_i[A_{12}] = n_1 - w_i[A_{11}], \quad w_j[A_{21}] = n_1 - w_j[A_{22}].$$

Hence for $1 \leq i \leq v_1$,

$$\begin{aligned} w_i[A^*] &= w_i[A_{11}] + w_i[\bar{A}_{12}] \\ &= w_i[A_{11}] + v_2 - w_i[A_{12}] \\ &= 2w_i[A_{11}] + v_2 - n_1. \end{aligned}$$

and for $1 \leq j \leq v_2$,

$$\begin{aligned} w_{v_1+j}[A^*] &= w_j[\bar{A}_{21}] + w_j[A_{22}] \\ &= v_1 - w_j[A_{21}] + w_j[A_{22}] \\ &= v_1 - n_1 + 2w_j[A_{22}]. \end{aligned}$$

If G^* is regular then $w_i[A^*]$ is independent of i . Hence each row of A_{11} has the same weight, say w_1 , i.e., $w_i[A_{11}] = w_1$. Similarly $w_{v_1+j}[A^*]$ is independent of j . Hence each row of A_{22} has the same weight, say w_2 , i.e., $w_j[A_{22}] = w_2$. Moreover $w_i[A^*] = w_{v_1+j}[A^*]$.

Hence

$$(8.3) \quad w_1 - w_2 = (v_1 - v_2)/2 .$$

It is also readily seen that these conditions are sufficient for the regularity of G^* . Hence we state:

For G^* to be regular it is necessary and sufficient that each row of A_{11} has the same weight w_1 , each row of A_{22} has the same weight w_2 and $w_1 - w_2 = (v_1 - v_2)/2$.

Then the degree n_1^* of G^* is given by

$$(8.4) \quad n_1^* = 2w_1 + v_2 - n_1 = 2w_2 + v_1 - n_1 .$$

Note that $w_i[A_{11}] = w_1$, means that each vertex of G belonging to V_1 is adjacent to w_1 vertices in V_1 , and $n_1 - w_1$ vertices in V_2 . In the same way $w_j[A_{22}] = w_2$ means that each vertex of G belonging to V_2 is adjacent to w_2 vertices in V_2 and to $n_1 - w_2$ vertices in V_1 .

Now supposing G^* to be regular and of degree n_1^* let us investigate the further conditions which must be satisfied in order to make G^* strongly regular.

Let $x_1, x_2, \dots, x_{v_1}, x_{v_1+1}, \dots, x_v$ be the vertices of G , the first v_1 belonging to V_1 and the last v_2 belonging to V_2 , where $v = v_1 + v_2$. Now from (7.4)

$$(8.5) \quad 2\sigma_{ij}[A] = w_i[A] + w_j[A] - \delta_{ij}[A] = 2n_1 - \delta_{ij}[A],$$

$$(8.6) \quad 2\sigma_{ij}[A^*] = w_i[A^*] + w_j[A^*] - \delta_{ij}[A^*] = 2n_1^* - \delta_{ij}[A^*].$$

If x_i and x_j both belong to V_1 or both belong to V_2 , i.e., $1 \leq i, j \leq v_1$ or $v_1 + 1 \leq i, j \leq v$, we call it case I. From (7.2) and (7.5)

$$\delta_{ij}[A] = \delta_{ij}[A^*].$$

Hence from (8.5) and (8.6) it follows that in case I

$$\sigma_{ij}[A^*] = n_1^* - n_1 + \sigma_{ij}[A].$$

Again if x_i belongs to V_1 and x_j belongs to V_2 , i.e., $1 \leq i \leq v_1$, $v_1 + 1 \leq j \leq v_2$ we will call it case II. Then from (7.2) and (7.5)

$$\delta_{ij}[A] = v - \delta_{ij}[A^*].$$

Hence from (8.5) and (8.6) it follows that in case II

$$\sigma_{ij}[A^*] = n_1^* + n_1 - \sigma_{ij}[A] - \frac{v}{2}.$$

Case I can be sub-divided into cases I(a) and I(b) according as x_i and x_j are adjacent or non-adjacent in G . Similarly case II can be sub-divided into cases II(a) and II(b) according as x_i and x_j are adjacent or non-adjacent in G . Then x_i and x_j will be adjacent in G^* in cases I(a) and II(b) and will be non-adjacent in G^* in cases I(b) and II(a). Now $\sigma_{ij}[A] = p_{11}^1$ or p_{11}^2 according as x_i and x_j are adjacent or non-adjacent in G . Hence

$$\sigma_{ij}[A^*] = n_1^* - n_1 + p_{11}^1 \quad \text{in case I(a),}$$

$$\sigma_{ij}[A^*] = n_1^* - n_1 + p_{11}^2 \quad \text{in case I(b),}$$

$$\sigma_{ij}[A^*] = n_1^* + n_1 - p_{11}^1 - \frac{v}{2} \quad \text{in case II(a),}$$

$$\sigma_{ij}[A^*] = n_1^* + n_1 - p_{11}^2 - \frac{v}{2} \quad \text{in case II(b).}$$

The necessary and sufficient condition for G^* to be strongly regular, in addition to the condition for regularity already obtained, is that

$\sigma_{ij}[A^*]$ has a fixed value say p_{11}^{1*} in cases I(a) and II(b), and similarly another fixed value say p_{11}^{2*} in cases I(b) and II(a). It follows that the parameters of G must satisfy the condition

$$(8.7) \quad p_{11}^1 + p_{11}^2 = 2n_1 - \frac{v}{2} .$$

When this condition is satisfied the parameters p_{11}^{1*} and p_{11}^{2*} of G^* are given by

$$p_{11}^{1*} = n_1^* - n_1 + p_{11}^1, \quad p_{11}^{2*} = n_1^* - n_1 + p_{11}^2 .$$

We can thus state the following theorem:

THEOREM (8.1). Let G be a strongly regular graph with parameters

$$v, n_1, p_{11}^1, p_{11}^2 .$$

If the vertices of G are divided into two disjoint subsets V_1 and V_2 , where $|V_1| = v_1$, $|V_2| = v_2$, then the necessary and sufficient conditions for the graph G^* derived from G by complementation with respect to V_1, V_2 to be strongly regular are

(a) In G each vertex in V_1 is adjacent to w_1 vertices in V_1 (and therefore $n_1 - w_1$ vertices in V_2); also each vertex in V_2 is adjacent to w_2 vertices in V_2 (and therefore to $n_1 - w_2$ vertices in V_1), where

$$w_1 - w_2 = (v_1 - v_2)/2 .$$

$$(b) \quad p_{11}^1 + p_{11}^2 = 2n_1 - \frac{v}{2} .$$

When these conditions are satisfied the parameters of G^* are given by

$$(8.8) \quad v^* = v , n_1^* = 2w_1 + v_2 - n_1 = 2w_2 + v_1 - n_1 ,$$

$$(8.9) \quad p_{11}^{1*} = n_1^* - n_1 + p_{11}^1 , p_{11}^{2*} = n_1^* - n_1 + p_{11}^2 .$$

If the graph G^* is required to have the same parameters as G , then $n_1^* = n_1$. This automatically ensures that $p_{11}^{1*} = p_{11}^1$ and $p_{11}^{2*} = p_{11}^2$. Also $n_1 - w_1 = v_2/2$, $n_1 - w_2 = v_1/2$, i.e., in G each vertex of V_1 is adjacent to exactly half the vertices in V_2 , and each vertex in V_2 is adjacent to exactly half the vertices in V_1 .

We therefore have

THEOREM (8.2). Let G be a strongly regular graph with parameters

$$v, n_1, p_{11}^1, p_{11}^2 .$$

If the vertices of G are divided into two disjoint subsets V_1 and V_2 , then the necessary and sufficient conditions for the graph G^* derived from G by complementation with respect to V_1 and V_2 , to be strongly regular with the same parameters as G are

(a) In G_1 each vertex in V_1 is adjacent to exactly half the vertices in V_2 , and each vertex in V_2 is adjacent to exactly half the vertices in V_1 .

$$(b) \quad p_{11}^1 + p_{11}^2 = 2n_1 - \frac{v}{2}.$$

9. $G_2(d)$ GRAPHS OF THE PSEUDO NET AND NEGATIVE LATIN SQUARE TYPES.

We can define after Mesner [10] a negative Latin square graph $NL_r(k)$ as a strongly regular graph with parameters,

$$v = k^2, \quad n_1 = r(k+1),$$

$$p_{11}^1 = (r+1)(r+2) - (k+2), \quad p_{11}^2 = r(r+1).$$

If $k = 2r$, then $p_{11}^1 = p_{11}^2 = r(r-1)$. Hence a negative Latin square graph $NL_r(2r)$ is a $G_2(d)$ graph with parameters $v = 4r^2$, $n_1 = r(2r+1)$, $d = r(r+1)$.

(a) We shall now show that a negative Latin square graph $NL_r(2r)$ can be derived from a net graph $L_r(2r)$. A net graph $L_r(2r)$ is the graph of a net of degree r and order $2r$. Take any class C of parallel lines in the net, and divide them into groups of r lines each. Let V_1 be the set of vertices corresponding to the $2r^2$ points on lines of the first group, and V_2 be the set of vertices corresponding to the $2r^2$ points on the lines of the second group. If P is a point on a line ℓ of the first group, then the vertex x corresponding to P is adjacent to the $2r-1$ vertices corresponding to the other points on ℓ . Also through P there pass $r-1$ lines other than ℓ

(one belonging to each of the parallel classes other than C). Each of these lines intersects each line of the first group (other than ℓ) in a single point. The vertices corresponding to these $(r-1)^2$ intersections are adjacent to the vertex x . We thus get $w_1 = r^2$ vertices in V_1 adjacent to x . It is clear that these are all the vertices in V_1 adjacent to x . Similarly each vertex in V_2 is adjacent to exactly $w_2 = r^2$ vertices in V_2 . Again for the net graph $L_r(2r)$, $2n_1 - (v/2) = p_{11}^1 + p_{11}^2 = 2r(r-1)$. Thus the condition (b) of Theorem (8.1) is satisfied. Hence by complementation with respect to V_1 and V_2 we obtain a strongly regular graph with parameters

$$v^* = 4r^2, n_1^* = r(2r+1),$$

$$p_{11}^{1*} = r(r+1) = p_{11}^{2*}.$$

This is a negative Latin square graph $NL_r(2r)$ by definition.

We thus have

THEOREM (9.1). A negative Latin square graph $NL_r(2r)$ exists whenever a net graph $L_r(2r)$ exists. In particular a negative Latin square graph $NL_r(2r)$ exists for $r = 3, 6$ and 2^{n-1} where $n \geq 1$.

(b) A balanced incomplete block design with parameters

$(v_0, b_0, r_0, k_0, \lambda_0)$ is said to be symmetric if $v_0 = b_0$, and $r_0 = k_0$. We shall then write the parameters as (v_0, k_0, λ_0) .

The incidence matrix N of the design is defined to be a

$v_0 \times v_0$ matrix $[n_{ij}]$ where $n_{ij} = 1$ or 0 according as the treatment i occurs or does not occur in the block j . The

adjacency matrix A of a $G_2(d)$ graph with parameters v, n_1, d is the incidence matrix of a symmetric BIB design with parameters (v, n_1, d) . But the incidence matrix N of a symmetric BIB design (v_0, k_0, λ_0) is not always the adjacency matrix of a $G_2(d)$ graph. For this the necessary and sufficient conditions are $n_{ii} = 0$, $n_{ij} = n_{ji}$. Thus N must be symmetric and the main diagonal must consist of zeroes. When these conditions are satisfied we shall say that the incidence matrix is of type I.

Let J denote the $v_0 \times v_0$ matrix all of whose elements are unity. If N is the incidence matrix of a symmetric BIB design (v_0, k_0, λ_0) , and $\bar{N} = J - N$ then \bar{N} is the incidence matrix of a symmetric BIB design $(v_0, v_0 - k_0, v_0 - 2k_0 + \lambda_0)$. If N is of type I, then \bar{N} is symmetric, and each element of the main diagonal is unity. The incidence matrix of a symmetric BIB will be defined to be of type II when it satisfies these conditions.

Consider a class Ω of symmetric BIB designs (v_0, k_0, λ_0) for which the condition $v_0/4 = k_0 - \lambda_0$ is satisfied. Then the following result which we state in the form of a lemma is known [16].

LEMMA (9.1). If N_1 and N_2 are the incidence matrices of two symmetric BIB designs (v_1, k_1, λ_1) and (v_2, k_2, λ_2) belonging to the class Ω , then

$$N = N_1 \times N_2 + \bar{N}_1 \times \bar{N}_2$$

is the incidence matrix of a symmetric BIB design (v_0, k_0, λ_0)

belonging to Ω where

$$v_0 = v_1 v_2, \quad k_0 = k_1 k_2 + (v_1 - k_1)(v_2 - k_2), \quad \lambda_0 = k_0 - (v_0/4),$$

and \times denotes the Kronecker product.

We also note that if N_1 is of type II and N_2 is of type I, then N is of type I.

If the adjacency matrix of a $G_2(d)$ graph with parameters v, n_1, d is the incidence matrix of a BIB design of the class Ω , then $v = 4(n_1 - d)$. Since $v - 1 = n_1(n_1 - 1)/d$, it follows that $n_1 = \frac{1}{2} [(4d+1) \pm (4d+1)^{1/2}]$, which shows that $n_1 = r(2r+1)$ or $n_1 = r(2r-1)$ for some positive integer r . Thus $G_2(d)$ must be either a pseudo net graph $L_r(2r)$ or a negative Latin square graph $L_r(2r)$.

THEOREM (9.2). The existence of pseudo net graphs $L_{r_1}(2r_1)$ and $L_{r_2}(2r_2)$ implies the existence of a pseudo net graph $L_r(2r)$ with $r = 2r_1 r_2$.

Let N_1 be the adjacency matrix of the pseudo net graph $L_{r_1}(2r_1)$ and N_2 the adjacency matrix of the pseudo net graph $L_{r_2}(2r_2)$. Then N_1 is the incidence matrix of a symmetric BIB $[4r_1^2, r_1(2r_1-1), r_1(r_1-1)]$ and \bar{N}_2 is the adjacency matrix of a symmetric BIB $[4r_2^2, r_2(2r_2+1), r_2(r_2+1)]$, where N_1 is of type I and \bar{N}_2 is of type II. From Lemma (9.1)

$$N = \bar{N}_2 \times N_1 + N_2 \times \bar{N}_1$$

is the adjacency matrix of a symmetric BIB (v_0, k_0, λ_0) , belonging to Ω , where

$$v_0 = 16r_1^2 r_2^2, \quad k_0 = 2r_1 r_2 (4r_1 r_2 - 1),$$

$$\lambda_0 = (2r_1 r_2 - 1).$$

Since N is of type I, it follows that it is the adjacency matrix of a pseudo net graph $L_r(2r)$, where $r = 2r_1r_2$.

THEOREM (9.3). A pseudo net graph $L_r(2r)$ exists for all $r = 3^m \cdot 2^{m+n-1}$, where m and n are non-negative integers, $(m,n) \neq (0,0)$.

A net graph is also a pseudo net graph. Hence a pseudo net graph $L_r(2r)$ exists for $r = 2^{n-1}$, $n \geq 1$. Hence the theorem is true for $m = 0$, $n \geq 1$. Again a net graph $L_r(2r)$ exists for $r = 3 \cdot 2^0$. Assuming the existence of a pseudo net graph $L_r(2r)$ for $r = 3^{m-1} \cdot 2^{m-2}$, the existence of a pseudo net graph $L_r(2r)$ for $r = 3^m \cdot 2^{m-1}$ follows from Theorem (9.2) by choosing $r_1 = 3^{m-1} \cdot 2^{m-2}$, $r_2 = 3$. Hence by induction a pseudo net graph $L_r(2r)$ exists for $r = 3^m \cdot 2^{m-1}$. Thus the theorem holds for $m \geq 1$, $n = 0$.

Finally the existence of the pseudo net graph $L_r(2r)$ for $r = 3^m \cdot 2^{m+n-1}$ follows from Theorem (9.2) by choosing $r_1 = 3^m \cdot 2^{m-1}$ and $r_2 = 2^{n-1}$, where $m \geq 1$, $n \geq 1$. Hence the theorem holds for $m \geq 1$, $n \geq 1$.

COROLLARY. $G_2(d)$ graphs with parameters $v = 4r^2$, $n_1 = r(2r-1)$, $d = r(r-1)$ exist for all $r = 3^m \cdot 2^{m+n-1}$ where m and n are non-negative integers and $(m, n) \neq (0, 0)$.

THEOREM (9.4). The existence of a pseudo net graph $L_{r_1}(2r_1)$ and a negative Latin square graph $NL_{r_2}(2r_2)$ implies the existence of a negative Latin square graph $NL_r(2r)$, where $r = 2r_1r_2$.

Let N_1 be the adjacency matrix of the pseudo net graph $L_{r_1}(2r_1)$ and N_2 the adjacency matrix of a negative Latin square graph $NL_{r_2}(2r_2)$. Then $\overline{N_2}$ is the incidence matrix of a symmetric BIB design $[4r_2^2, r_2(2r_2-1), r_2(r_2-1)]$ and is of type II, where N_1 is as in

the proof of Theorem (9.2). Hence from Lemma (9.1)

$$N = \bar{N}_2 \times N_1 + N_2 \times \bar{N}_1$$

is the adjacency matrix of a symmetric BIB (v_0, k_0, λ_0) belonging to the class Ω , where

$$v_0 = 16r_1^2 r_2^2, k_0 = 2r_1 r_2 (4r_1 r_2 + 1), \lambda_0 = (2r_1 r_2 + 1).$$

Therefore N is of type I and is the adjacency matrix of a negative Latin square graph $NL_r(2r)$, where $r = 2r_1 r_2$.

THEOREM (9.5). A negative Latin square graph $NL_r(2r)$ exists for all $r = 3^m \cdot 2^{m+n-1}$, where m and n are non-negative integers, $(m, n) \neq (0, 0)$.

The existence of a negative Latin square graph $NL_r(2r)$ for $r = 2^{n-1}$, $n \geq 1$ has already been proved in Theorem (9.1).

Hence the required theorem holds for $m = 0$, $n \geq 1$.

Again a pseudo net graph $L_r(2r)$ exists for $r = 3^{m-1} \cdot 2^{m-2}$, $m \geq 2$ by Theorem (9.2), and a negative Latin square graph $NL_r(2r)$ exists for $r = 3$ by Theorem (9.1). Putting $r_1 = 3^{m-1} \cdot 2^{m-2}$, $r_2 = 3$ in Theorem (9.4), we get the existence of a negative Latin square graph $NL_r(2r)$ for $r = 3^m \cdot 2^{m-1}$. Hence the theorem holds for $m \geq 1$, $n = 0$.

Finally taking $r_1 = 3^m \cdot 2^{m-1}$, $m \geq 1$ and $r = 2^{n-1}$, $n \geq 1$ in Theorem (9.4), we get the existence of a negative Latin square graph $NL_r(2r)$ for $r = 3^m \cdot 2^{m+n-1}$. Hence the theorem holds for $m \geq 1$, $n \geq 1$.

COROLLARY. $G_2(d)$ graphs with parameters $v = 4r^2$, $n_1 = r(2r+1)$, $d = r(r+1)$ exist for all $r = 3^m \cdot 2^{m+n-1}$ where m and n are non-negative integers and $(m, n) \neq (0, 0)$.

10. DESCENDANT OF A STRONGLY REGULAR GRAPH. Let G be a strongly regular graph with parameters

$$v, n_1, p_{11}^1, p_{11}^2$$

and vertex set

$$x_0, x_1, \dots, x_{n_1}, x_{n_1+1}, \dots, x_{v-1}$$

where x_1, x_2, \dots, x_{n_1} are adjacent to x_0 , and the remaining $n_2 = v - n_1 - 1$ vertices $x_{n_1+1}, \dots, x_{v-1}$ are non-adjacent to x_0 . Let G_0 be the subgraph of G obtained by deleting the vertex x_0 and the edges incident with x_0 . Then the adjacency matrix of G can

be written as

$$(10.1) \quad A = \begin{bmatrix} 0 & \underline{1}' & \underline{0}' \\ \underline{1} & A_{11} & A_{12} \\ \underline{0} & A_{21} & A_{22} \end{bmatrix}$$

where $\underline{1}'$ is a row-vector of order n_1 with all its elements unity, and $\underline{0}'$ is a row-vector of order n_2 with all its elements 0.

The adjacency matrix of G_0 is

$$(10.2) \quad A_0 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

Let V_1 be the set of vertices x_1, x_2, \dots, x_{n_1} and V_2 the set of vertices x_{n_1+1}, \dots, x_n . Let G_* be obtained from G_0 by completion with respect to V_1 and V_2 , i.e., the vertices of G_* are the same as those of G_0 . Two vertices both belonging to V_1 or V_2 are adjacent or non-adjacent in G_* according as they are adjacent or non-adjacent in G_0 . Two vertices one of which belongs to V_1 and the other to V_2 are adjacent in G_* if they are non-adjacent in G_0 and are non-adjacent in G_* if they are adjacent in G_0 . Given G and the vertex x_0 , G_* is completely determined. We may define G_* to be the descendent of G with respect to the vertex x_0 . The adjacency matrix of G_* is

$$(10.3) \quad A_* = \begin{bmatrix} A_{11} & \bar{A}_{12} \\ \bar{A}_{21} & A_{22} \end{bmatrix} .$$

We wish to investigate the conditions under which G_* is strongly regular. Since G is strongly regular with parameters $v, n_1, p_{11}^1, p_{11}^2$, the weight of any row of A is n_1 and the scalar product of any two rows is p_{11}^1 or p_{11}^2 according as the rows correspond to a pair of adjacent or non-adjacent vertices. Hence for $1 \leq i \leq n_1, 1 \leq j \leq n_2 = v - n_1 - 1$,

$$w_i(A_{11}) = p_{11}^1, \quad 1 + w_i(A_{11}) + w_i(A_{12}) = n_1 .$$

$$\therefore w_i(A_{12}) = n_1 - 1 - p_{11}^1 .$$

$$w_j(A_{21}) = p_{11}^2, \quad w_j(A_{21}) + w_j(A_{22}) = n_1 .$$

$$\therefore w_j(A_{22}) = n_1 - p_{11}^2 .$$

Again from (7.1)

$$w_i(\bar{A}_{12}) = n_2 - w_i(A_{12}) = n_2 - n_1 + 1 + p_{11}^1 ,$$

$$w_j(\bar{A}_{21}) = n_1 - w_j(A_{21}) = n_1 - p_{11}^2 .$$

$$\begin{aligned} w_i(A_*) &= w_i(A_{11}) + w_i(\bar{A}_{12}) \\ &= 2p_{11}^1 + n_2 - n_1 + 1 . \end{aligned}$$

$$\begin{aligned} w_{n_1+j}(A_*) &= w_j(\bar{A}_{21}) + w_j(A_{22}) \\ &= 2n_1 - 2p_{11}^2 . \end{aligned}$$

Hence necessary and sufficient condition for G_* to be regular is that

$$2p_{11}^1 + n_2 - n_1 + 1 = 2n_1 - 2p_{11}^2$$

or

$$(10.4) \quad p_{11}^1 + p_{11}^2 = 2n_1 - \frac{v}{2} .$$

This is the same as the condition (b) of Theorem (8.1) . When this condition is satisfied we can prove as in Theorem (8.1) that G_* is strongly regular with parameters

$$(10.5) \quad v_* = v - 1 , n_{1*} = 2n_1 - 2p_{11}^2 ,$$

$$p_{11*}^1 = n_{1*} - n_1 + p_{11}^1 , p_{11*}^2 = n_{1*} - n_1 + p_{11}^2 .$$

THEOREM (10.1) If G is a strongly regular graph with parameters $v, n_1, p_{11}^1, p_{11}^2$, the necessary and sufficient condition for the descendant G_* of G (with respect to any vertex x_0) to be strongly regular is

$$p_{11}^1 + p_{11}^2 = 2n_1 - \frac{v}{2}.$$

When this condition is satisfied the parameters of G_* are given by (10.5).

The condition (8.7) or (10.4) which appears in Theorems (8.1), (8.2) and (10.1) will be called the switching condition.

11. $G_2(d)$ GRAPHS DERIVABLE AS DESCENDANTS. Consider a strongly regular graph G with parameters $v, n_1, p_{11}^1, p_{11}^2$, for which the switching condition (10.4) is satisfied. Then its descendant G_* has the parameters (10.5). If G_* is a $G_2(d_*)$ graph, $p_{11*}^1 = p_{11*}^2$. Hence $p_{11}^1 = p_{11}^2$. Thus G itself must be a $G_2(d)$ graph, for which $p_{11}^1 = p_{11}^2 = d$. Substituting in (10.4) we have $v = 4(n_1 - d)$. It follows as in paragraph 9(b), that G must either be a pseudo net $L_r(2r)$ graph or a negative Latin square $L_r(2r)$ graph. We are therefore lead to studying descendants of such graphs.

THEOREM (11.1). The descendant of a pseudo net graph $L_r(2r)$ or a negative Latin square graph $L_r(2r)$ is a pseudo linked block graph $LB_r(2r+1)$.

A pseudo linked block graph $LB_r(2r+1)$ exists for all $r = 3^m \cdot 2^{m+n-1}$ where m and n are non-negative integers $(m, n) \neq (0, 0)$.

Let the parameters of a pseudo net graph $L_r(2r)$ be $v = 4r^2$, $n_1 = r(2r-1)$, $p_{11}^1 = p_{11}^2 = r(r-1)$. The switching condition (10.4) is satisfied and the parameters of the descendant graph G_* given by (10.5) are

$$(11.1) \quad v_* = 4r^2 - 1, \quad n_{1*} = 2r^2, \quad p_{11*}^1 = p_{11*}^2 = r^2.$$

Thus G_* is by definition a pseudo linked block graph $LB_r(2r+1)$.

It can be proved exactly in the same way that the descendant of a negative Latin square graph $L_r(2r)$, has precisely the parameters (11.1).

COROLLARY. $G_2(d)$ graphs with parameters $v = 4r^2 - 1$, $n_1 = 2r^2$, $d = r^2$ exist for all $r = 3^m \cdot 2^{m+n-1}$ where m and n are non-negative integers $(m, n) \neq (0, 0)$.

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