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Simple approximation to non-exceedance probability using third-moment

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ABSTRACT : I proposed a simple method which can evaluate a non-exceedance probability of the skewed distribution. It was shown that the proposed method was stable for the evaluation of the non-exceedance probability, and the result obtained by the proposed method was more close to the theoretical one than the other methods.

1 INTRODUCTION

In order to evaluate a structural failure probability of a nuclear reactor building in a Probabilistic Safety Assessment (PSA), the Second Moment method (SMM) is widely used, because of its simplicity. The SMM, however, can not deal with the skewness of the performance function, which considerably affects on the fragility evaluation. Thus, we developed the simple method which can evaluate the non-exceedance probability of the skewed distribution approximately.

In this paper, we show the outline of this method and compare its results with the theoretical results.

2 OUTLINE OF THE METHOD

The exact non-exceedance probability of the skewed distribution can be calculated, only when we have the detailed information about the shape or all the probabilistic moments of the distribution, i.e. the characteristic function. It is, however, difficult to obtain such information. Therefore, some assumption about the distribution is needed in order to evaluate the non-exceedance probability of the skewed distribution approximately.

In the proposed method, it is assumed that the non-exceedance probability is represented as the function of the skewness. Because the theoretical distribution, for example, log-normal, weibull, exponential etc., which we usually deal with, do not have a complicated shape, it is considered that the higher moments are not needed for the evaluation of the non-exceedance probability of such distribution.

For the above reason, after investigating the relationship between the non-exceedance probability of log-normal distribution and its skewness, because of its easy handling, we developed the non-exceedance probability evaluation method by simplifying and expanding this relationship as related below.

The skewness of the log-normal distribution, $E[X_{(2)}{}^3]$, is represented as follows.

$$E[X_{(2)}{}^3] \equiv \frac{E[(X-\mu)^3]}{\sigma^3} = \{2+\exp(\zeta^2)\}\sqrt{\exp(\zeta^2)-1} \qquad (1)$$

Where, μ and σ are the mean value and standard deviation, respectively, and ζ is the log-normal standard deviation. The subscript of $X_{(2)}$ denotes that the random variable, x, is normalized up to the second moment. Solving the above equation with respect of ζ, we have

$$\zeta = \sqrt{\ln\left[\left\{\frac{2}{\eta}\right\}^{2/3} + \left\{\frac{\eta}{2}\right\}^{2/3} - 1\right]} \tag{2}$$

where, $\eta = E[X_{(2)}{}^3] + \sqrt{4 + E[X_{(2)}{}^3]^2}$

On the other hand, the non-exceedance probability P_{non} for the log-normal distribution is defined as follows.

$$p_{non} = \Phi\{\beta(x)\} \tag{3}$$

where, $\Phi(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u} e^{-t^2/2}dt$, $\beta(x) = \frac{\ln(x/\lambda)}{\zeta}$,

λ : mean parameter of log-normal distribution

We can readily obtain the $\beta(x)$ for the normalized r.v., $x_{(2)}$, substituting $x = x_{(2)}\sigma + \mu$ in it.

$$\beta(x_{(2)}) = \frac{\zeta}{2} + \frac{\ln\left\{1 + x_{(2)}\sqrt{\exp(\zeta^2) - 1}\right\}}{\zeta} \tag{4}$$

Eqs.(2) to (4) mean the relationship between the skewness of the log-normal distribution and its non-exceedance probability. In other words, the non-exceedance probability for the normalized r.v. of log-normal distribution can be evaluated from the skewness alone using these equations.

Eqs.(2) and (4), however, are limited $E[X_{(2)}{}^3]>0$. Then, in order to make these equations possible to use for all the skewness, it is assumed that these equations in $E[X_{(2)}{}^3] \leqq 0$ are symmetric about the point (0,0) as shown in Figs.1 and 2. While the solid lines in Figs.1 and 2 show the Eqs.(2) and (4), respectively, the dotted and broken lines indicate the assumed equations in $E[X_{(2)}{}^3] \leqq 0$. Therefore, Eqs.(2) and (4) are rewritten for all the skewness as follows.

$$\begin{cases} \zeta = \sqrt{\ln\left[\left\{\frac{2}{\eta}\right\}^{2/3} + \left\{\frac{\eta}{2}\right\}^{2/3} - 1\right]} , & (E[X_{(2)}{}^3] > 0) \\ \zeta = -\sqrt{\ln\left[\left\{\frac{2}{\eta}\right\}^{2/3} + \left\{\frac{\eta}{2}\right\}^{2/3} - 1\right]} , & (E[X_{(2)}{}^3] \le 0) \end{cases} \tag{5}$$

$$\begin{cases} \beta(x_{(2)}) = \frac{\zeta}{2} + \frac{\ln\left\{1 + x_{(2)}\sqrt{\exp(\zeta^2) - 1}\right\}}{\zeta} , & (E[X_{(2)}{}^3] > 0) \\ \beta(x_{(2)}) = x_{(2)} , & (E[X_{(2)}{}^3] = 0) \\ \beta(x_{(2)}) = \frac{\zeta}{2} + \frac{\ln\left\{1 - x_{(2)}\sqrt{\exp(\zeta^2) - 1}\right\}}{\zeta} , & (E[X_{(2)}{}^3] < 0) \end{cases} \tag{6}$$

Note that each above equation is continuous at $E[X_{(2)}{}^3]=0$. In fact, the gradient of the both equations of Eq.(5) at $E[X_{(2)}{}^3]=0$ is 1/3 as follows.

$$\lim_{E[X_{(2)}{}^3] \to 0} \frac{\partial}{\partial E[X_{(2)}{}^3]}\left[\pm\sqrt{\ln\left[\left\{\frac{2}{\eta}\right\}^{2/3} + \left\{\frac{\eta}{2}\right\}^{2/3} - 1\right]}\right] = \frac{1}{3} \tag{7}$$

On the other hand, the upper and lower equations of Eq.(6) is coincide with the middle equation when $E[X_{(2)}{}^3]=0$, because

$$\lim_{E[X_{(2)}{}^3]\to 0}\left[\frac{\zeta}{2}+\frac{\ln\left\{1\pm x_{(2)}\sqrt{\exp(\zeta^2)-1}\right\}}{\zeta}\right]=x_{(2)} \tag{8}$$

Now, we can evaluate the non-exceedance probability considering the skewness of the distribution using above equations. There is, however, a few problems. One is that these equations are somewhat complicate as an application. Another is that the normalized r.v., $x_{(2)}$, is bounded by the following equations, because of Eq.(6).

$$\begin{aligned} x_{(2)} &> -\frac{1}{\sqrt{\exp(\zeta^2)-1}}, \quad \left(E[X_{(2)}{}^3]>0\right)\\ x_{(2)} &< \frac{1}{\sqrt{\exp(\zeta^2)-1}}, \quad \left(E[X_{(2)}{}^3]<0\right) \end{aligned} \tag{9}$$

In order to simplify and expand the applicability of Eqs.(5) and (6), we modify them as related below.

At first, referring to the form of the solid and dotted lines in Fig.1, it is assumed that ζ is in proportion to the arctangent of the skewness. So as to expand the applicability of the third moment method, it is better to estimate the absolute value of ζ low, because of Eq.(9). Therefore, we decided the ratio of ζ to the arctangent of the skewness, such that it is coincide with the gradient of Eq.(5) at (0,0), i.e. 1/3 (see Eq.(7)). As the result, instead of Eq.(5), we obtain the following approximated equation which is indicated as the chained line in Fig.1.

$$\zeta=\frac{1}{3}\tan^{-1}\left(E[X_{(2)}{}^3]\right) \tag{10}$$

Next, assuming that the absolute value of ζ is smaller than about 1, we substitute the following approximation into Eq.(6).

$$\left.\begin{aligned} &\text{when } \zeta\geq 0,\ \sqrt{\exp(\zeta^2)-1}\\ &\text{when } \zeta\leq 0,\ -\sqrt{\exp(\zeta^2)-1} \end{aligned}\right\}\cong\zeta \tag{11}$$

As the results, we obtain the following equations instead of Eq.(6).

$$\begin{cases} \beta(x_{(2)})=\dfrac{\zeta}{2}+\dfrac{\ln\{1+\zeta\cdot x_{(2)}\}}{\zeta}, & \left(E[X_{(2)}{}^3]\neq 0\right)\\ \beta(x_{(2)})=x_{(2)}, & \left(E[X_{(2)}{}^3]=0\right) \end{cases} \tag{12}$$

Where, $\beta(x_{(2)})$ is assumed to be $+\infty$ for $E[X_{(2)}{}^3]>0$ and $-\infty$ for $E[X_{(2)}{}^3]<0$, if $1+\zeta\cdot x_{(2)}<0$.

3 COMPARISONS WITH THE OTHER METHODS

Substituting the skewness into Eq.(10) and using Eqs.(12) and (3), we can evaluate the non-exceedance probability of the skewed distribution for the normalized r.v.. In this section, the results by the developed method are compared with those by the other methods, which are the higher-order moment standardization technique (HOMST) by Ono et al. (1989) and the Edgeworth's series expansion method modified by Murotsu et al. (1981).

The non-exceedance probability is evaluated by HOMST as follows.

$$P_{non} = \Phi\{(c_2 + c_3 x_{(2)} + c_4 x_{(2)}{}^2)/c_1\} \tag{13}$$

where, $c_1 = 2E[X_{(2)}{}^3]^2 - 3E[X_{(2)}{}^4] + 3$, $c_2 = -E[X_{(2)}{}^3]$, $c_3 = 3 - 3E[X_{(2)}{}^4]$,
$c_4 = E[X_{(2)}{}^3]$, $E[X_{(2)}{}^4]$: Kurtosis

On the other hand, the original Edgeworth's series expansion method gives the non-exceedance probability using the following equation.

$$P_{non} = G(x_{(2)}) \tag{14}$$

where, $G(x_{(2)}) = \Phi(x_{(2)})\{1 - E[X_{(2)}{}^3](x_{(2)}{}^2 - 1)/6 - (E[X_{(2)}{}^4] - 3)x_{(2)}(x_{(2)}{}^2 - 3)/24$
$- E[X_{(2)}{}^3]^2 x_{(2)}(x_{(2)}{}^4 - 10x_{(2)}{}^2 + 15)/72\}$

Eq.(14), however, yields the negative values when $x_{(2)}$ is small. Murotsu et al. modified the above equation in order to remedy its defect, i.e.,

$$P_{non} = \frac{G(-1.5)}{\Phi(-1.5)}\Phi(x_{(2)}) \text{ , for } x_{(2)} < -1.5 \text{ and } (E[X_{(2)}{}^3] > 0 \text{ or } E[X_{(2)}{}^4] < 3) \tag{15}$$
$$P_{non} = G(x_{(2)}) \text{ , for the other case}$$

It is noted that the above methods need the skewness and kurtosis as represented by Eqs.(13) and (14), while the proposed method requires only the skewness.

Fig.3 shows the comparisons of the non-exceedance probability for (a) log-normal distribution, (b) Beta distribution and (c) weibull distribution, whose probabilistic density function is shown in Table 1, respectively.

Fig.3(a) shows that the proposed method is close to the theoretical results in the case of log-normal distribution, because the proposed method is based on the relationship between the non-exceedance probability of log-normal distribution and its skewness. On the other hand, the HOMST uses quadratic transformation function, so that the non-exceedance probability does not keep to increase for the small normalized r.v. when the skewness has the negative value (see the left figure of Fig.3(b) and right of 3(c)). The non-exceedance probability by the modified Edgeworth's series expansion method has discontinuous form. As a whole, compared with the other methods, it is considered that the proposed method is stable for the evaluation of the non-exceedance probability and gives the non-exceedance probability which is close to the theoretical one as shown in Fig.3.

4 CONCLUSIONS

I have proposed the simple method which can evaluate the non-exceedance probability of the skewed distribution using the skewness alone. Through the numerical example, it was shown that this method had few defect than the other methods.

Using this method, we can easily obtain the structural failure probability considering the skewness. Thus, it will be promised to be the alternative of the Second Moment method in a Probabilistic Safety Assessment for a nuclear reactor building.

REFERENCES

Ono, T. et al. 1989. System reliability using high-order moments. *5th international conference on structural safety and reliability*: 959-966

Murotsu, Y. et al. 1981. Fourth-order moment approximation to reliability of non-linear structure. *6th SMiRT, M12/5*

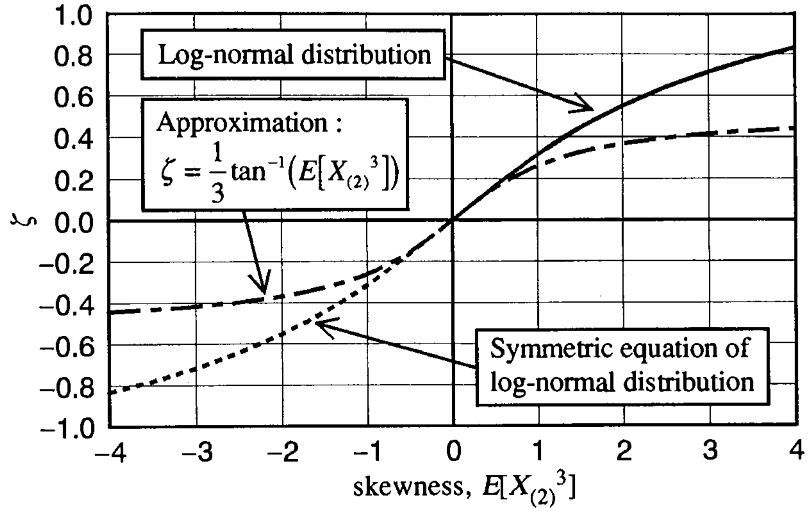


Fig.1 Relationship between the skewness and ζ

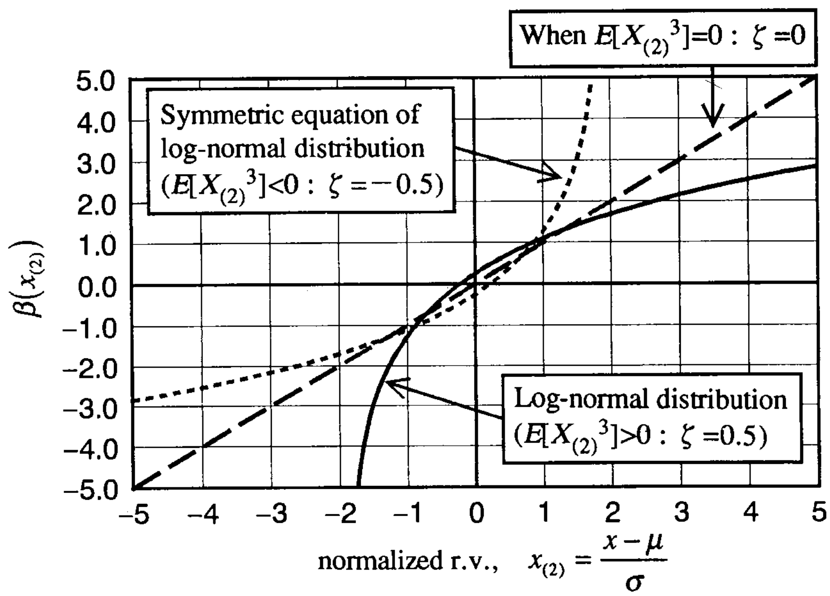
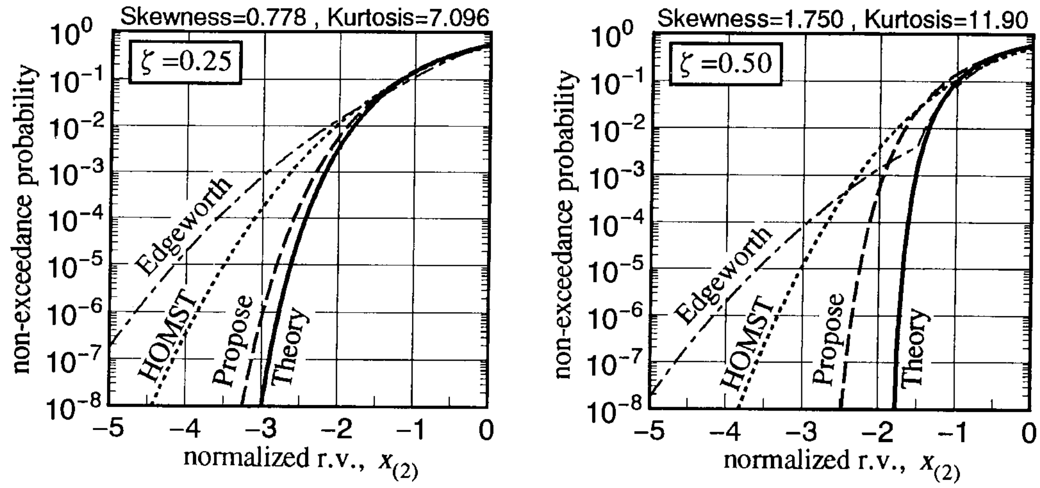


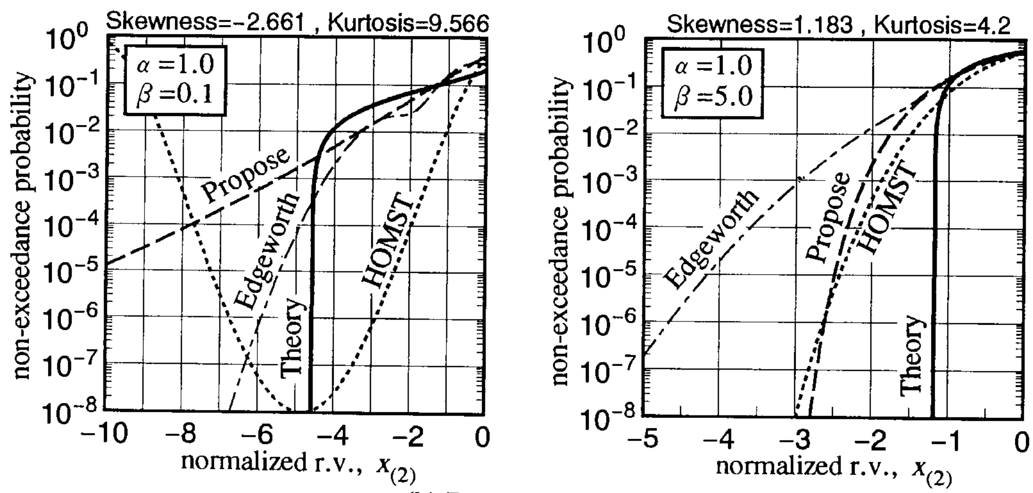
Fig.2 Relationship between the normalized random variable and $\beta(x_{(2)})$

Table 1 Probabilistic density function

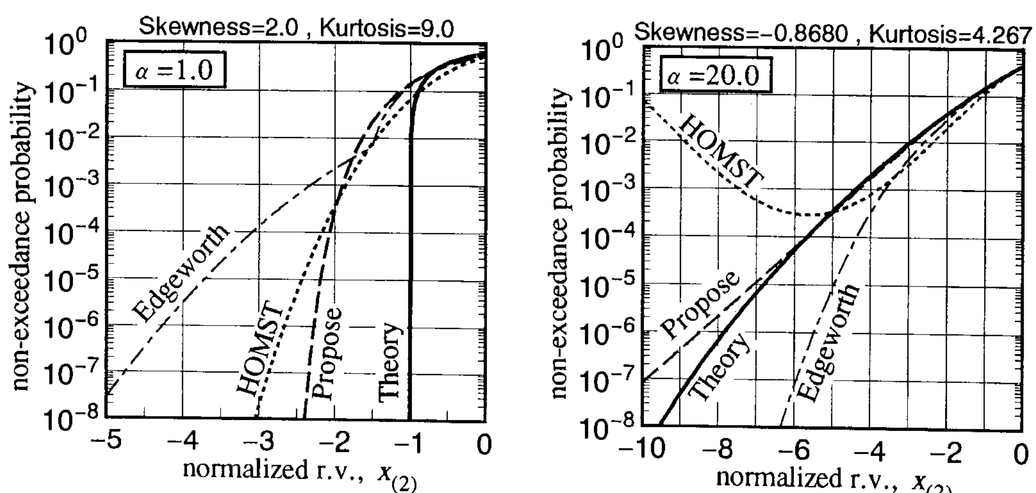
Probabilistic Distribution	Probabilistic density function	Note
Log-normal ($0<x$)	$\frac{1}{\sqrt{2\pi}\zeta x}\exp\left\{-\frac{1}{2}\left(\frac{\ln x-\ln\lambda}{\zeta}\right)^2\right\}$	λ :mean parameter ζ :logarithmic standard deviation
Beta ($\alpha<x<\beta$)	$\frac{(1-x)^{\beta-1}x^{\alpha-1}}{B(\alpha,\beta)}$	α :lower limit, β :upper limit B(a,b):Beta function
Weibull ($0<x$)	$\frac{\alpha x^{\alpha-1}}{\beta^{\alpha}\exp\{(x/\beta)^{\alpha}\}}$	α , β :parameter



(a) Log-normal distribution



(b) Beta distribution



(c) Weibull distribution

Fig.3 Comparisons of non-exceedance probability