SOME COMMENTS ABOUT THE J_1 INTEGRAL CRITERION IN POST YIELD FRACTURE MECHANICS

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SUMMARY

Several criteria have been proposed for Post Yield Fracture Mechanics. One of the most interesting ones is the J_1 integral. When the behaviour of material is elastic (even non-linear) it can be shown that J_1 is not path dependent (for a straight crack without thermal stresses). For this reason, it may be considered that J_1 characterizes the crack tip singularity. Extension is easy to deformation-type elastic plastic material, but there is no proof of path independence for flow-type plastic material (incremental plasticity or creep). Experimental results are often given as a proof of J_1 criterion validity. But there is no experimental value of a contour integral and assumptions are made in the use of experimental results. The main assumption implies that the received mechanical work (strain energy) is not dependent of the loading history (is only dependent of mechanical state).

A general method to assess J_1 path dependence can be founded on the "defect vector" (or driving force) concept. The special density of defect is:

$$\overline{W^*} = \overline{\text{grad}\, W} - \bar{\bar{\sigma}}\, \text{grad}\, \bar{\bar{\varepsilon}}$$

(W = strain energy, $\bar{\bar{\sigma}}$ stress tensor, $\bar{\bar{\varepsilon}}$ strain tensor).

It can be shown that the resultant of defects included in a volume is the J integral on the surface surrounding the volume (and L for the moment). So the path independence condition is:

$$\frac{\partial W}{\partial X_1} = \sigma_{ij} \frac{\partial_{ij}}{\partial X_1}.$$

It is not proved that this condition is satisfied for incremental plasticity (or creep).

In order to have an empirical idea of the J_1 path independence, it is possible to make computations with finite elements method. Some results are given and it seems that no noticeable path dependence is seen with simple shapes and radial (proportional) loading. A few cases with complex way of loading are also studied.

I - INTRODUCTION

Several criteria have been proposed for Post Yield Fracture Mechanics. One of the most interesting ones is the J_1 integral proposed by RICE /1/ in 1968. An experimental way for the determination of the critical value of J_1 (the so called J_{1C}) has been proposed by BEGLEY and LANDES /2/ and practical application of the J_{1C} criterium has received considerable attention. Though several critical evaluations on PYFM criteria have been made in the past (see in particular the review by TURNER and BURDEKIN /3/), many features are not very clear and there is still some questions about them.

Initially, J_1 integral was only defined for non linear elastic materials

$$J_1 = \int_\Gamma \left(W dy - T_i \frac{\partial u_i}{\partial x} ds \right)$$

with W = elastic energy density, $T_i = \sigma_{ij} n_j$, σ_{ij} stress tensor, n_j external normal, u_i displacement and Γ integration path around the crack tip (see fig. 1).

When the four following requirements are satisfied :

- elastic (linear or non linear) behaviour of material
- straight crack along x axis
- material homogeneity
- no thermal stresses (in fact no transient thermal stresses), two important rules can be shown.

First rule : J_1 is path-independent, that is to say the J_1 value is not dependent on the choice of the path Γ surrounding the crack tip.

Second rule : if P is the total potential energy of the system (cracked structure and loading device) and a is the crack length (along x axis), then $J_1 = -dP/da$.

On the basis of these two rules, J_1 can be used in non linear elastic fracture mechanics. From a first point of view it may be considered that, because of its path independence, the J_1 integral characterizes the crack tip singularity as K_I does in LEFM. From the second point of view J_1 is the potential energy rate needed for application of the GRIFFITH stability criterion like G_1 in LEFM.

The two points of view lead to the following rule : Crack propagation begins when J_1 reaches a critical value J_{1C}.

II - EXTENSION IN PYFM

Extension of the J_1 criterion from Non Linear Elastic Fracture Mechanics to Post Yield Fracture Mechanics is needed for practical use. But this extension is not obvious because the preceeding rules are based on "potential energy".

In elastic problems, the strain energy density W is a well defined function of the local state of strain (if the material is homogeneous) and P is the system potential energy which is only function of the state. But for plastic materials, there is no mechanical potential energy and the internal energy (in the thermodynamic sense) is quite different from the received mechanical work. Any child knows the mechanical difference between rubber and marshmallow.

RICE'S proposal for extension defines W, not as the elastic energy density, but as the "strain energy" (density), that is, the received work density. This proposal is justified in /4/ for a deformation-type elastic plastic material, where the strain energy is a function of strain state, that is to say for a non linear elastic material. But real material behaviour is rather incremental-type plasticity (like Prandtl-Reuss law) than deformation-type plasticity. So the question is : "Is J_{1C} a good crack-initiation criterion for real plastic material ?" It is believed that the answer is "yes" when there is no unloading.

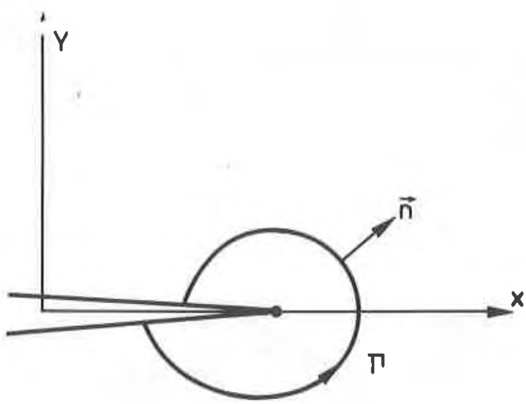

Fig 1

III - USE OF EXPERIMENT

Experimental tests results are often given as a proof of this assumption /2/ but as noticed by NEALE & TOWNLEY /5/ there is not experimental value of a contour integral. But it is possible to know L (δ), the load-deflexion curve during a monotonic loading (load L versus deflexion δ). From this result the received mechanical work P is given versus the deflexion δ and the value of $\frac{dP}{da}$ (a = crack length) is taken as the J_1 value.

Various methods have been proposed in order to determine dP/da from the L (δ) curve /6/ /7/ /8/.

It must be pointed out that very important assumptions are made in the use of experimental results. The goal is to know the variation δP of mechanical work P due to a virtual variation δa of the crack length a, under the condition of constant load L (or constant deflexion δ in some cases). What is really tested are samples with various cracks lengths under monotonic loading. There is not physical reason, when the crack extends by δa under the constant load L, for the final (virtual) deflexion to be the same as the one of the a + δa cracked sample loaded up to L and it is not sure that the mechanical work difference is the same (see fig. 2).

From a theoretical point of view, such an assumption implies that the received mechanical work is a well defined function of the geometrical and mechanical state, and is not dependent of the loading history. In other words, in using experimental test results, it is assumed that mechanical work is a potential energy (i.e. it is assumed that the material is non linear elastic).

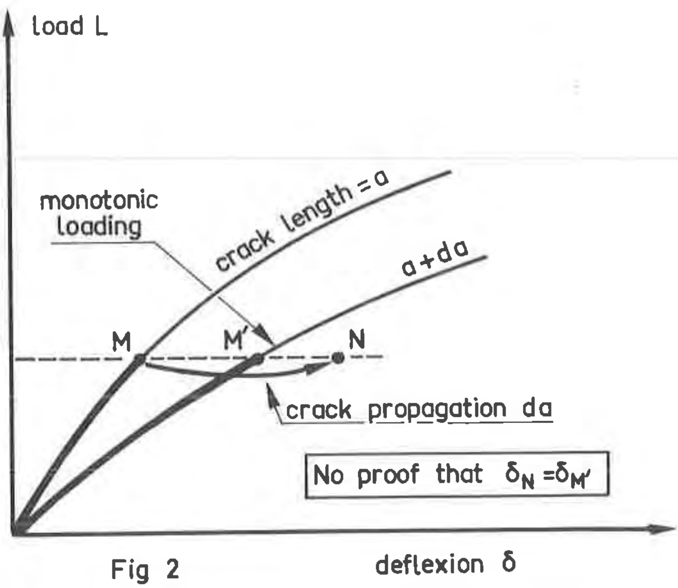

Fig 2

IV - IMPORTANCE OF THE J_1 PATH-INDEPENDENCE QUESTION

Experimental validation of the J_{1C} criterion is not possible without theoretical hypotheses based on the energetic meaning of J_1.

Another way of extension is based on the path independence of the J_1 integral. Roughly speaking, if J_1 is not path dependent, it characterizes the tip singularity of the crack, therefore the material damage, at the crack tip. Therefore J_1 is a good criterion for material damage and crack initiation.

From this point of view the only important question is "Is J_1 really path independent for cracked elastic-plastic material ? There is no known answer when the material is an incremental-type elastic plastic one. Nevertheless it may be hoped that J_1 is almost path independent in practical cases.

It would be useful to have a general rule for the J_1 path independence, without reference to the material constitutive equation. For straight cracks, this condition can be translated into "Under what condition does J_1 vanish for all closed paths bounding a solid region ? (for anyone material)". One method to find this condition can be derived from the ESHELBY theory on Energy Momentum Tensor /9/, but with noticeable change in order to avoid any assumption on material behaviour /10/.

V – GENERAL METHOD TO ASSESS J_1 PATH DEPENDENCE

The method which can be called "defect vector theory" or "driving force theory" is based on the comparison between the real space variation of the received work W (so-called strain energy) around a point and the fictitious variation due only to strain space variation /11/.

The expression of the defect vector or driving force (space density) is

$$\overline{W}^{x} = \overline{\text{grad } W} - \overline{\overline{\delta}} \overline{\overline{\varepsilon}}$$

or in index notations $W_K^X = W,k - \delta'_{ij}\, \varepsilon_{ij}$

(δ_{ij} and ε_{ij} stress and strain tensor, W strain energy and W,k partial derivation of W relative to x_k).

There exists a defect vector (or driving force) when the space variation of the strain energy is not given by the space variation of strain. The above relation gives the expression of the space density of this vector. It is possible to define also the surface density (surface singularity) and the line density (line singularity like crack tip)...

With this definition, it is easy to show /11/ /12/ that the surface integral $\vec{J}$ on the surface S is the resultant of all the defect vectors included in the volume V limited by S.

$$J_i = \int_S (W n_i - \delta_{jk} n_k u_{j,i})\, ds$$

$$\vec{J} = \int_V \vec{W}^*\, dv$$

Beside the "force resultant" $\vec{J}$, there is the "moment resultant" L given at the point O by :

$$\vec{L} = \int_V (\overline{OM} \wedge \overline{W}^*)\, dv$$

which is one of the integrals introduced by KNOWLES and STERNBERG /12/ /13/ which are not as popular as the first component of vector $\vec{J}$.

Now it is possible to have the condition for surface independence of $\vec{J}$. The volume integral $\int_V \overline{W}^* dv$ must be equal to zero for any volume, then the condition is $\overline{W}^* = 0$.

$$\text{or } W_{,k} = \sigma_{ij} \, \varepsilon_{ij,k}$$

which can be written in different ways /11/ /12.

If the condition is required at each instant of the loading, its becomes

$$\sigma_{ij,k} \; \varepsilon_{ij,t} = \sigma_{ij,t} \; \varepsilon_{ij,k}$$

which can be expressed in the space time configuration (x_i, t).

Remark 1 : No assumption has been made about the material constitutive equation in that analysis. When the relations between stress and strain (and time in case of creep) are chosen, it is possible to know W, ε_{ij}, σ_{ij} at each point at any time (during the loading). Then the $\overline{W}^*$ field is known and the surface dependence of the $\vec{J}$ integral can be computed.

Remark 2 : There is a defect vector surface density on the body boundary. Its expression is

$$-\overline{Y}_i = Wn_i - \overline{X}_j u_{j,i}$$

which is reduced to $\overline{Y}_i = -W\overline{n}_i$ where no external forces are applied (free boundaries). The body is in static equilibrium between the "external forces" W^*_i and $\overline{Y}_i$ (volume and surface forces).

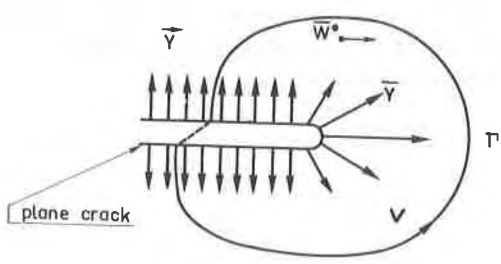

Fig 3

$\overline{Y}$ boundary driving force

$\overline{W}^*$ volume driving force

$J_1 = \int_V (\overline{Y}\, ds + \overline{W}^*\, dv)$

VI - J_1 PATH INDEPENDENCE CONDITION

If we return to the J_1 approach in Fracture Mechanics, the preceeding method can be applied. This particular case concerns a two dimensional problem and a body with a (plane) crack in the xz plan.

It is easy to see that the above condition is reduced to "no W^*_1 component along Ox in the material, but at the crack tip". This can be written /12/.

$$\frac{\partial W}{\partial x} = \sigma_{ij} \frac{\partial \varepsilon_{ij}}{\partial x}$$

This is the condition for $J_1 = 0$ on all closed paths bounding a solid region, at the time T only. For any time we have

$$\frac{\partial \sigma_{ij}}{\partial t} \frac{\partial \varepsilon_{ij}}{\partial x} = \frac{\partial \sigma_{ij}}{\partial x} \frac{\partial \varepsilon_{ij}}{\partial t}$$

But we are interested by paths surrounding the crack tip and this condition must be completed. For these paths there is not only the defect vector density W*, but also the "boundary defect" $\overline{Y}$. (see fig. 3).

If there are not external forces on the crack, $\vec{Y} = W\vec{n}$ and $\vec{Y}$ is perpendicular to the crack surface. For a plane crack along the yoz axis, there is no $\overline{Y}$ component in the x direction, but a the crack tip where there is a point defect (equal to J_1).

Then for a plane crack along the x axis the only condition for path independence is the above condition, corresponding with $J_1 = 0$ along all closed paths not surrounding the crack tip.

VII - ENERGY VARIATION AND J_1 PATH INDEPENDENCE

When all the defects (volumetric, superficial, etc) included in a volume V bounded by the surface S move in a translation $\delta\vec{w}$, the work variation inside this volume δP is given by $\delta P = - \vec{J}\delta\vec{w}$ (see /12/).

If we consider a translation δa in the ox axis direction, we have $\delta P = - J_1 \ \delta a$ for the work variation inside the part of the body surrounded by the path Γ. As we have seen above, J is the sum of the crack tip defect and the volume defects W^*, then we can know the work variation for a translation δa of all the defects of the body, but not for the crack propagation δa above. For that, it is necessary that the W^*_1 sum be zero, that is to say that J_1 be path independent.

If J_1 is path independent the work variation δP_c for a crack propagation δa is given by

$$\boxed{\delta P_c = - J_1 \ \delta a}$$

If J_1 is path dependent a surface integral is needed to take account of the volumetric defects translation /12/ /15/.

VIII - USE OF COMPUTATION RESULTS BY FINITE ELEMENTS METHOD

It can be seen that if $\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}$ (elasticity) and W is only strain dependent (homogeneity) the condition is always satisfied and therefore J_1 is path independent. Then it is sufficient that the material be elastic, but it is not necessary.

When there are plastic or visco-plastic deformations, it is not possible to prove the condition is satisfied. From a mathematical point of view it is not satisfied when the strain is incremental. Nevertheless it may be hoped that in practical cases the structure behaviour is close to a non linear elastic one and therefore J_1 almost path independent.

In order to have some appreciation about this expectation, several computations have been made with Finite Elements Method /8/ /13/. Results are not very clear and there is some discrepancy between peoples conclusions. The computed J_1 values are not greatly dependent on the path, but it is difficult to assess wether variations are real or only due to numerical approximation. Someone suggests that J_1 integral is not path independent for paths passing through the plastic area.

It must be remembered that only few sample shapes have been considered, all having simple geometrical features. Moreover the loading path is always "radial", load components increasing proportionally with the time.

Nevertheless this method is able to give some indications on J_1 path dependence for flow plastic (or viscous) materials. In Saclay a J_1 computation program is in progress. Calculations are made with the CEASEMT system in which the PASTEL module enables calculation of J_1 simultaneously along several contours /15/. The plasticity model is incremental with the VON MISES' surface yield and a normal flow /14/ /15/ /16/. Results concerning two cases are given below.

IX - SQUARE PLATE IN PLASTICITY

A 2x2mm square plate with a crack (length 2a=1mm) is loaded by an uniform stress S on sides parallel to the crack (see fig. 4). The material characteristics are YOUNG'S Modulus E = 206,800 MPa, POISSON's ratio ν= 0,3 Yield stress σ_y = 310MPa, no hardening. The analysis was performed in plane strain with the mesh shown in fig. 5. For each of the 15 increasing values of the load S, J_1 was computed on 15 paths. Were also computed the portion Je due to elastic strain energy, that due to plastic strain energy (J_p) and that due to the forces on the path (J_F). As an example the results are given in Table 1 for S = 185 MPa. On Figure 6 the curve

$$F = \frac{1}{S} \sqrt{\frac{JE}{\pi a(1-\nu)^2}} \quad \text{versus is shown.}$$

It can be seen that there is no noticeable path dependence.

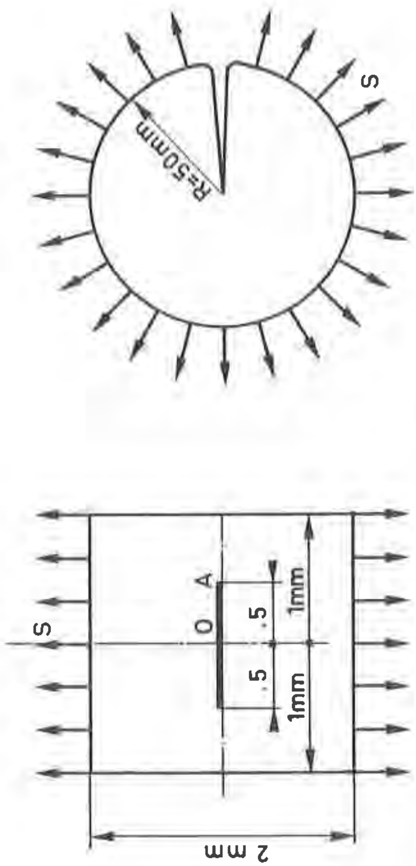

Fig.4

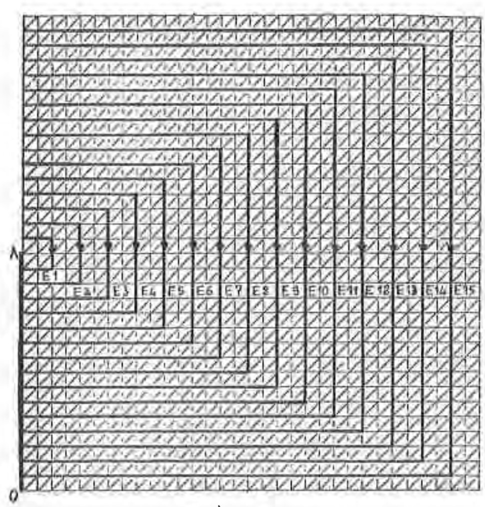

FIG. 5

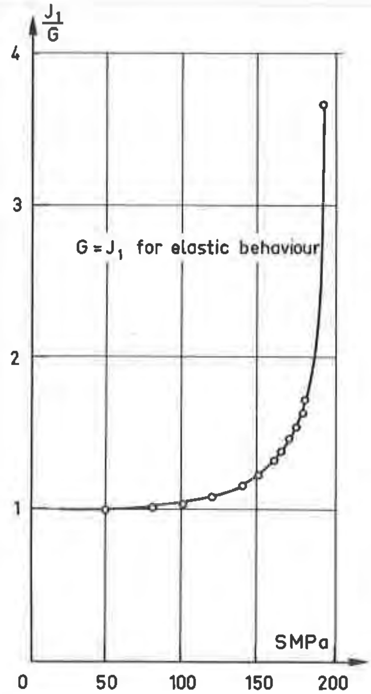

Fig 6

J_1 VERSUS APPLIED LOAD (SQUARE PLATE)

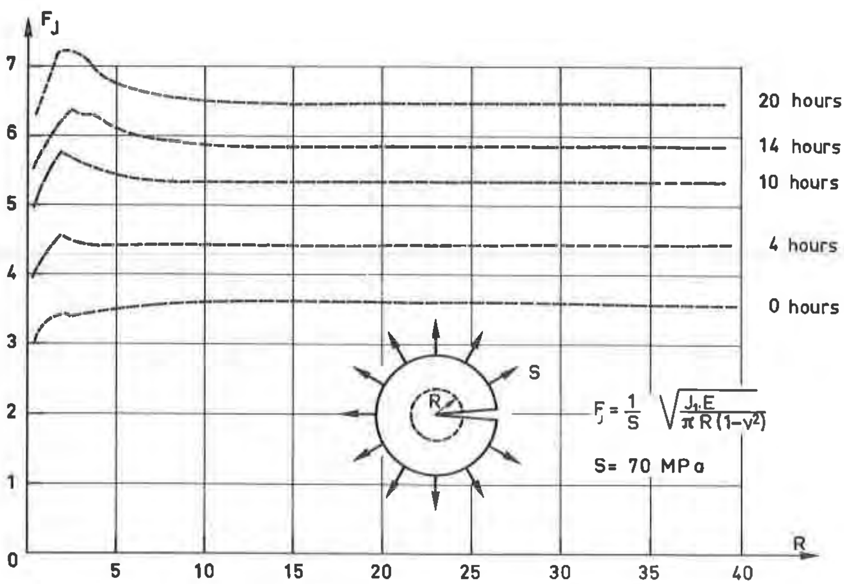

Fig 7 J_1 DEPENDENCE ON PATH RADIUS AND TIME (effect of creep)

X - CIRCULAR PLATE WITH PLASTICITY AND CREEP

A thick circular plate (radius 50mm) with a radial crack (along x axis) is loaded with an uniform stress S on its circular boundary (fig. 4). The material characteristics are

Young's Modulus 200 000 MPa
Poisson's ratio 0,3
Yield stress (no hardening) 200 MPa
Creep law $\mathring{\varepsilon} = 3{,}547\ 10^{-9}\ \sigma$
($\mathring{\varepsilon}$ strain rate hour^{-1}, σ stress MPa).

The load is applied at initial time (value S = 70 MPa) and there is plastic deformation at first. Then creep occurs and strain and stress fields change with time.

As an example of results, table II gives the values of J_1, J_e, J_p, J_F and F after 24 hours of loading for 17 circular paths. On Figure 7 are the curves F versus path radius for differents holding times.

There is not important path dependence.

CONCLUSION

If J_1 is path independent, it can be considered as a good criterion for crack initiation. There is no proof that J_1 is path independent in general cases. In the particular case where the material is homogeneous and non-linear plastic, J_1 is not path dependent. In the use of experimental results it is assumed that the mechanical work is a potential energy, that is to say that the material is elastic. In practical cases, it may be hoped that the real material behaviour is close to a non linear elastic one. Computations seem to show that there is no noticeable J_1 path-dependence. But computations and tests have been made on simple specimens under radial loading.

It seems that the J_1 criterion is a good engineering tool, but not theoretically proved. So some doubts may be raised when it is used for intricate components with complex loading history.

TABLE I

Values of J_1 as a function of the path for

S = 185 MPa

Path	J_e	J_p	J_k	J	F_J
E1	21,94269	47,57885	395,3121	618,3389	1,6167
E2	72,56514	78,44835	241,0944	651,1183	1,6590
E3	95,39873	85,40796	237,7422	686,0815	1,7030
E4	116,8801	90,20854	230,1949	696,0885	1,7154
E5	139,0969	93,98199	229,7872	706,1758	1,7277
E6	162,1676	97,49022	238,6404	710,7148	1,7333
E7	187,1811	99,77941	258,5909	713,8895	1,7372
E8	211,0149	93,52403	289,3743	716,1115	1,7399
E9	228,7590	84,42559	323,0511	717,6181	1,7417
E10	241,1316	74,25306	355,8748	718,4059	1,7426
E11	247,9207	62,04478	387,2128	718,6188	1,7429
E12	247,6554	48,31504	417,3847	718,5753	1,7428
E13	240,1627	31,72009	446,2875	718,1703	1,7424
E14	225,2344	19,00052	473,2213	717,4563	1,7415
E15	204,1574	15,89420	496,3753	716,4269	1,7402

Unit J N/m

TABLE II

Values of J_1 versus the path-radius, 24 hours after loading

R_{mm}	J_e	J_p	J_F	J_1	F	λ
0,510	-0,0084	-1,3386	17,0469	15,700	6,695	587,4
1,020	-0,1083	-1,7194	18,1711	17,343	7,037	617,4
1,529	-0,2408	-1,7859	21,4807	19,454	7,453	653,9
2,039	-0,1110	-1,6786	22,6356	20,846	7,715	676,9
2,549	0,2362	-1,7644	22,4827	20,954	7,735	678,6
3,059	0,3620	-1,8246	22,4675	21,005	7,745	679,4
3,718	0,4761	-1,5553	20,9685	19,889	7,536	661,2
4,570	0,5521	-1,2839	19,4611	18,730	7,313	641,6
5,673	0,6139	-1,0898	18,5007	18,025	7,174	629,4
7,099	0,6629	-0,9108	17,6862	17,438	7,056	619,1
8,942	0,6726	-0,7611	17,1179	17,029	6,973	611,8
11,327	0,6232	-0,5677	16,6639	16,719	6,909	606,2
14,410	0,5129	-0,2924	16,3204	16,541	6,872	602,9
18,398	0,3903	0,1659	15,8771	16,433	6,850	601,0
23,555	0,2799	0,8437	15,3211	16,445	6,852	610,2
30,223	0,2562	1,8344	14,4061	16,497	6,863	602,1
38,847	0,3278	3,1313	13,1954	16,654	6,896	605,0

R path radius mm $J_1 = J_e + J_p + J_F$ $(10^4 N/m)$

$$F = \frac{1}{S} \quad \frac{J.E.}{\pi a(1-\nu)^2}$$

REFERENCES

/1/ J.R. RICE - In Fracture (ed. H. LIEBOWITZ) -
Academic Press (1968) 191-311

/2/ BEGLEY J.A. and LANDES J.D.- In Fracture Toughness ASTM
STP 514 - American Society for Testing of Materials, 1972

/3/ C.E. TURNER, F.M. BURDEKIN - "Review of current status of
yielding fracture mechanics" - Atomic Energy Review
1974, 12, n° 8

/4/ J.R. RICE
"A path independent integral and the approximate analysis
of strain concentration by notckes and cracks"
J. of Appl. Mech., June 1968

/5/ "B.K. NEALE and C.H. TOWNLEY"
An unified approach to elastic-plastic fracture mechanics
RD/B/N 3358 Central Elasticity Generating Board, May 1975

/6/ A. PELISSIER - TANON
"Fracture toughness measurement from the load deflexion
curve in elastic-plastic conditions"
Advanced course on fracture mechanics, ISPRA Octob. 1975

/7/ J.R. RICE, P.C. PARIS, J.G. MERKLE
"Some further results of J. integral analysis and estimates"
ASTM STP 536 American Society for testing of materials,
1973.

/8/ J.D. SUMPTER and C.E. TURNER
"Fracture analysis in areas of high nominal strain"
2nd Inc. Conf. on Pressure Vessel Technology, Oct. 1973,
San Antonio.

/9/ J.D. ESHELBY
"The energy momentum tensor in continuum mechanics" in
Inelastic Behaviour of Solids (Kanninen ed.)
Mc Graw-Hill, New-York, 1970

/10/ R. ROCHE
"Analyses de mécanique de la rupture par éléments finis -
Sur la signification des intégrales J et analogues"
Note CEA N-1807 - Centre d'Etudes Nucléaires de Saclay,
Juillet 1975

/19/ M. SARTHE and al. "Utilisation de la méthode des éléments finis au calcul de l'intégrale J_1. Etude élastoplastique de pièces fissurées".
Note CEA (to be published) CEN Saclay.

/11/ R. ROCHE
"Dépendance vis-à-vis de la surface d'intégration des intégrales caractéristiques de la mécanique de la rupture"
Note CEA N-1829 - Centre d'Etudes Nucléaires de Saclay, Novembre 1975.

/12/ R. ROCHE
"Quelques commentaires sur l'utilisation de l'intégrale J"
(with English translation) Note CEA-N-1933
Centre d'Etudes Nucléaires de Saclay 1976.

/13/ J.K. KNOWLES and E. STERBERG
"On a class of conservation laws in linearized and finite elasto-statics"
Archive for rational mechanics and analysis, Vol. 44, n° 3, 1972.

/14/ B. BUDIANSKY and J.R. RICE
"Conservation laws and energy release rates"
J. of Appl. Mech., March 1973

/15/ J.M. BOISSENOT et M. DUBOIS (CETIM - Senlis - France)
Private communication.

/16/ B.K. NEALE "Elastic plastic analysis of cracked bodies using the J integral method"
RD/B/3253/Central Electricity Generating Board 1975.

/17/ R. ROCHE "Use of the Calculation of the J. Integral"
Paper to be presented at the Fourth International Conference on Fracture. Waterloo June 1977.

/18/ F. JEANPIERRE and al. "CEA System of Finite Element Computer Programs - Use for inelastic analysis" Specialists Meeting on Hight temperature Design Technology of LMFBR. Champion Avril 1976 - Edit. IAEA - Vienna.