ABSTRACT

EDGINGTON, CYNTHIA PAGE. Teachers' Uses of a Learning Trajectory to Support Attention to Students' Mathematical Thinking. (Under the direction of Dr. Paola Sztajn).

Teachers' ability to elicit and build on students' mathematical thinking during instruction is critical in order to support students' mathematical growth. An emerging hypothesis in the field is that the construct of a *learning trajectory* (LT) has the potential to support teachers in making sense of and using student thinking to improve teaching and learning. Early research on teachers' uses of LTs suggest LTs provide a framework for making instructional decisions and afford teachers with a means to focus on their students' mathematical thinking. However, little is known about the specific ways in which teachers use LTs throughout lesson planning, instruction, and assessment to focus on students' mathematical thinking.

This study investigated five second grade teachers' uses of a LT using a multi-case study design. The ways in which the participants used a LT throughout three teaching cycles of lesson planning, implementation, and assessment were examined. From this examination, an initial framework was developed to describe different levels of teachers' uses of LTs. The study's findings suggested that the LT supported teachers in specifying learning goals and anticipating levels of sophistication among expected strategies. The teachers used the LT to pay attention to the processes their students engaged in as they solved tasks and it supported the recognition of important mathematical ideas that surfaced during instruction. The variation among the ways the LT was used by teachers highlight key aspects to attend to as we continue to study the ways in which teachers use LTs to focus on students' mathematical thinking.

Teachers' Uses of a Learning Trajectory to Support Attention to
Students' Mathematical Thinking

by
Cynthia Page Edgington

A dissertation submitted to the Graduate Faculty of
North Carolina State University
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

Mathematics Education

Raleigh, North Carolina

2012

APPROVED BY:

Karen Hollebrands	Allison McCulloch
Roger Woodard	P. Holt Wilson

Paola Sztajn
Committee Chair

DEDICATION

To David, Zakary, and Emilie.

BIOGRAPHY

Cynthia Page Whitehouse was born November, 14, 1969 in Fair Haven, New Jersey. As an Army brat, Cyndi lived in many different parts of the country growing up. Upon graduating from Mater Dei High School in Santa Ana, California in 1988, Cyndi enrolled in North Carolina State University. She graduated Magna Cum Laude with a Bachelor of Science in Mathematics Education in 1992. Cyndi immediately began teaching high school mathematics in Wake County, NC.

Cyndi spent the majority of her early teaching career at Garner Senior High School where she taught Geometry, Algebra I and Pre-Calculus. In 1998, Cyndi married David Edgington, also an NCSU alumnus. She continued teaching in Garner as the lead Geometry teacher and class sponsor for the Class of 2000. At the end of the 1999-2000 school year, Cyndi decided to leave the classroom to stay home with her family. In 2005, she began working on her Master's Degree on a part-time basis and enrolled full-time two years later. After receiving her Master's Degree, Cyndi continued at North Carolina State University to pursue a Ph.D. in Mathematics Education. Throughout her studies, she worked as a teaching assistant and most recently as a research assistant on the "Learning Trajectory-Based Instruction" project.

ACKNOWLEDGEMENTS

My deepest thanks to:

My advisor and dissertation chair, Dr. Paola Sztajn for her time, feedback, continued support and guidance on this dissertation, and for providing the opportunity to work on the LTBI team.

Members of my committee: Dr. P. Holt Wilson and Dr. Allison McCulloch, for their leadership, thoughtfulness, and most importantly, their friendship; Dr. Karen Hollebrands and Dr. Roger Woodard, for their leadership, support and insight.

The faculty at NC State and beyond: Dr. Karen Keene, Dr. Hollylynne Lee, Dr. Gemma Mojica, and Dr. Erin Krupa for their encouragement and advice.

My fellow graduate students: My DSG: Dr. Krista Holstein, Dr. Charity Cayton, and Richelle Dietz; My LTBI partner, Marrielle Myers; Ayanna Franklin Perry, Zuhal Yilmaz, Nadia Monrose, Dr. Tina Starling, Marggie Gonzalez, Christine Taylor, Aaron Trocki, Sarah Wylie, and Reema Alnizami.

The teachers participating in my study: for welcoming me into their classrooms, and for their dedication to teaching.

My family: Mom, Dad, Andrea, Ben, Don, Beth, Felicia, Grace, Harrison, Ilene, Glenn, Mary, Kathy, Andy, Elizabeth, and Renee, for their constant love and support.

My friends, whom I am blessed to have: for listening, encouraging, helping, caring for me, and laughing with me, especially Monica, Kym, Sue, and Susie.

My husband and best friend, David: for sacrificing, supporting me, and always believing in me.

My amazing children, Zak and Emilie: for understanding, patience, and for always cheering me on.

Note: This report is based upon work supported by the National Science Foundation under grant number DRL-1008364. Any opinions, findings, and conclusions or recommendations expressed in this report are those of the author and do not necessarily reflect the views of the National Science Foundation.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER ONE

Attention to student thinking has been identified as a critical tool to initiate changes in teachers' knowledge for teaching and improvements in classroom instruction (Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996; Franke, Carpenter, Levi, & Fennema, 2001; Kazemi & Franke, 2004; Sherin & van Es, 2009). In their seminal work with teachers around children's mathematical thinking, Fennema and colleagues (1996) developed Cognitively Guided Instruction that provided teachers with a framework for listening to and understanding children's problem solving strategies and mathematical thinking. When given access to specific knowledge about children's problem solving, teachers were more likely to pay attention to their students' thinking processes, and they made sense of strategies they observed their students using (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989).

As researchers continued to consider the role of student thinking in teacher learning, Ball and Cohen (1999) suggested that teacher education should be closely connected to teachers' practice and their learning experiences authentic to what takes place in their classrooms (Garet, Porter, Desimone, Birman, & Yoon, 2001; Putnam & Borko, 2000; Webster-Wright, 2009). Professional development shifted to explore how teachers learn from purposefully designed professional learning tasks that incorporate practice-based artifacts (Ball & Cohen, 1999; Kazemi & Franke, 2004; Silver, Clark, Ghousseini, Charalambous, & Sealy, 2007). As such, research on professional development in mathematics education has examined the ways in which focusing on children's thinking through student work samples, classroom videos, and curriculum materials allow teachers to examine their own mathematical knowledge as well as their instruction (Kazemi & Franke, 2004; Sherin & van Es, 2009; Silver et al., 2007).

Student Thinking and Teaching

One way that research on student thinking is making progress is through the construct of a *learning trajectory*. The field of mathematics education hypothesized that learning trajectories have the potential to support teachers in making sense of and using student thinking to improve teaching and learning (Daro, Mosher, & Corcoran, 2011). Simon (1995) first used the term hypothetical learning trajectory to describe paths that learning might follow as children progress towards intended learning goals. Its hypothetical nature implies that no teacher or researcher knows exactly how learning will proceed for any one individual. As a result, teachers rely on their theoretical and experiential knowledge of students and content. Over time, teachers come to expect specific learning patterns, common strategies, and misconceptions based on previous experiences teaching particular topics.

More recently, trajectories have gained prominence as a bridge between research and practice and research has just begun to connect learning trajectories to curriculum development (Clements & Sarama, 2009), assessment design (Battista, 2004; Confrey, Maloney, Nguyen, Mojica, & Myers, 2009), and classroom instruction (Bardsley, 2006; Wilson, 2009). Learning trajectories (LTs) utilize research on student learning to describe probable pathways of learning over time. These empirically built trajectories differ from the sequence of topics typically used in instruction, which are most often based on the logic of the discipline (Corcoran, Mosher, & Rogat, 2009). LTs are not independent of instruction and are influenced by the interactions between instruction and students' prior knowledge. As such, there is no one correct pathway, but multiple ways to reach certain understandings, making trajectories approximations of expected learning. However, their empirical nature provides representations of highly probable steps along paths. Building off of Simon's (1995) hypothetical learning trajectory, Clements and Sarama (2009) maintained that, "a complete hypothetical learning trajectory includes: the learning goal, developmental progression of thinking and learning, and sequence of instructional tasks" (p. 84). They explained that integrating these three ingredients can alter what has been previously found in educational psychology "because it opens up new paths for learning and development" (p. 84).

As these definitions of LTs continue to be refined, researchers are working to empirically construct trajectories in different mathematical domains (Barrett, Clements, Klanderman, Pennisi & Polaki, 2006; Battista, 2004; Clements, Wilson, & Sarama, 2004; Confrey et al., 2009; Nguyen, 2010; Sherin & Fuson, 2005). The authors of the Common Core State Standards (2010) emphasized the use of research-based LTs in the development of the new standards and committed to using research and evidence of student learning to inform future revisions. These trajectories support the development of new standards as they identify significant clusters of concepts, connect students' thinking to those concepts, and offer foundations for describing the interim goals laid out in the content standards (Daro et al., 2011). Daro et al. (2011) stated that LTs serve "as a basis for informing teachers about the (sometimes wide) range of student understanding they are likely to encounter, and the kinds of pedagogical responses that are likely to help students move along" (p. 12). However, little is known about how teachers come to learn about LTs and appropriate them into their instruction.

As teachers increasingly attend to student thinking in lesson planning and instruction, researchers must consider the role of LTs in supporting teachers' complex work. Research has yet to address how teachers adjust their lesson planning and instruction when they have information about common misconceptions and important landmarks inherent in LTs, or how teachers use evidence of student thinking to inform future instruction in light of LTs. This study contributes to the research on teachers' uses of LTs to support attention to student thinking in lesson planning, instruction, and assessment.

Significance of the Study and Research Questions

Based on one particular mathematics learning trajectory, the Equipartitioning Learning Trajectory (EPLT, Confrey et al., 2009), the current study sought to examine the influence of LTs on teachers' instructional practices. More specifically, this study examined the ways in which teachers used a LT to plan and implement instruction on equipartitioning, and assess their students' knowledge to inform future lessons. From this examination, an initial framework was developed to describe different levels of teachers' uses of LTs. The study was designed to enhance the field's understanding of how a LT can support teachers' attention to student thinking during mathematics instruction.

The following research questions are addressed:

1. In what ways do teachers use the EPLT to plan a set of lessons on equipartitioning?

2. In what ways do teachers use the EPLT when implementing classroom instruction?
3. In what ways do teachers use the EPLT to assess what happens during instruction?
4. What factors mediate and moderate the ways in which teachers use the EPLT to engage in cycles of lesson planning, instruction, and assessment?

Overview of Methodological Approach

This study is part of a larger design study to examine the ways in which teachers learn about learning trajectories and use them in their classrooms. Design studies provide researchers with a means of studying learning in context and to generate and test theories through an iterative process (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). The study of learning is considered from both an individual, psychological perspective and a collective, social perspective. Cobb and colleagues (2003) argued that the two perspectives are not studied in separation, but that as one is brought to the forefront, the other serves as a situating context. As such, the current study focused on the individual teacher in the context of the larger design study.

To examine individual teachers' uses of a LT, the design of this study followed a case study design. The study is particularistic (Merriam, 1998) in that its focus is on the particular phenomenon of the influence of the EPLT throughout cycles of planning, instruction, and assessment for five elementary school teachers. Moreover, this is a multi-case study (Stake, 1995) bounded by individual teachers to serve as illustrations of the utility of LTs. The final product is interpretive (Merriam, 1998) in nature as it provides descriptions of teachers' uses over time of a learning trajectory in their mathematics instruction while also presenting conceptual categories to depict the ways in which teachers use LTs. Through systematic data analyses, I interpreted the participants' experiences to create categories that conceptualize the utilization of LTs to support attention to students' mathematical thinking. The end product emphasized cross-case analysis with the goal of proposing a framework of teachers' uses of LTs in their mathematics instruction.

A group of second grade teachers from one elementary school collaboratively planned and individually implemented a set of lessons on equipartitioning over the course of one semester. These teachers previously attended a year-long professional development program designed to teach them about the EPLT. They engaged in a series of teaching cycles that began with lesson planning, where the LT aided teachers in identifying learning goals and sub-concepts necessary to support intended learning goals. Teachers then implemented the designed lessons. Finally, teachers assessed their students' understanding by reflecting on the implemented lessons, examining evidence of student learning to inform future instruction.

The LT offered teachers insight into how to use student work to navigate productive classroom discourse. The study's findings showed that the LT supported teachers in specifying learning goals and anticipating levels of sophistication among expected strategies. The teachers used the LT to pay attention to the processes their students engaged in as they solved tasks and it supported the recognition of important mathematical ideas that surfaced during instruction. However, the teachers did not use the LT in the same ways or to the same degree, suggesting possible categories to differentiate the ways in which LTs can be used in instruction.

Outline of Dissertation

This dissertation is organized into five chapters. In Chapter One, I discuss the significance of the study and an introduction to its focus. In Chapter Two, I describe the theoretical perspective that undergirds the study, followed by a review of emerging literature on teachers' uses of LTs, relevant literature on lesson planning, instruction that is focused on student thinking, and formative assessment. Then, the conceptual framework used to guide this study is described. In Chapter Three, the research questions are refined to reflect the conceptual framework and to clearly focus the study. The methodological approach is justified and outlined in detail, describing the context, sample, sources of data, and methods of analysis used. In Chapter Four, I provide profiles of each participant, a detailed description across the cases, and analysis of the data to answer the research questions. Finally, in Chapter Five I build on the results of the cross-case analysis to present a framework of teachers' uses of LTs, along with limitations, implications, and areas of future research related to the study.

CHAPTER TWO

In this chapter, I present a review of relevant literature to situate the study. I begin by presenting the theoretical perspective that undergirds this research. Then, after briefly discussing the construct of learning trajectories, I review studies on teachers' uses of LTs for instruction, followed by literature on lesson planning, using student thinking in instruction, and assessment to inform future instruction. Finally, the conceptual framework used to guide this study is described.

Theoretical Perspectives

This study takes a social constructivist theoretical perspective on teaching and learning. In this section, I provide an overview of this perspective and highlight ideas that are relevant to this study. Among the multiple theoretical viewpoints present in mathematics education, constructivism is a perspective that has dominated the research community over the last several decades (Ernst, 2006). Rather than perceive learning as a passive activity where knowledge is transmitted from teacher to student, constructivism views learning as actively built from previously formed cognitive structures that reside within an individual.

Constructivism, grounded in the work of Piaget, views knowledge as individually constructed through an active assimilation of "reality into systems of transformations" (Piaget, 1970, p. 15). In this view, individuals construct schemes through their experiences that correspond to their reality, in essence creating models of reality. When an individual encounters a new problem or situation, the individual searches for internal stability or what Piaget (1975) called equilibration. Equilibration is attained by assimilation, where new ideas are incorporated into existing schemes, or by accommodation, where existing schemes are modified to fit new ideas. Piaget (1970) stated that “knowledge, then, is a system of transformations that become progressively adequate” (p. 15) and from which new structures are subsequently abstracted.

von Glasersfeld (1984) stated that an individual cannot construct an exact match of reality, but instead searches for viability, or the notion that as long as an idea serves to organize the experiential world, then it is useful. He stated, “Just as the environment places constraints on the living organisms and eliminates all variants that in some way transfers the limits within which they are possible or ‘viable’, so the experiential world… constitutes the testing ground for our ideas” (von Glasersfeld, 1984, p. 22). Despite the differences in views of how schemes change, either through equilibration or viability, constructivism views learning as constructed within the individual (Ernst, 1994).

What is missing from this perspective is the social and cultural nature of learning that is present in our world. With foundations in the work of Vygotsky, socio-cultural theory posits that learning is not individually constructed, but that it is a social process that occurs when individuals engage with others in meaning-making activities (Cobb, 2007). At the core of Vygotsky’s (1986) theory of learning is the relationship between thought and language that develops over time through socio-cultural interactions. He stated, “Unlike the development of instincts, thinking and behavior of adolescents are prompted not from within but from without, by the social milieu” (Vygotsky, 1986, p. 108). Opportunities to learn and explore concepts that are connected to children’s social and cultural ways of knowing allows for the emergence of conceptual formation and thinking. In addition, language serves as a bridge between concept formation and socio-cultural interactions by allowing a child to ask questions, communicate, and participate in social and cultural activities, first developing within the social realm, then developing within the individual. Vygotsky (1986) stated, "Any function in the child's cultural development appears twice or on two planes. First, it appears on the social plane, and then on the psychological plane. First, it appears between people as an interpsychological category, and then within the child as an intrapsychological category" (p. 163).

From both constructivist and socio-cultural perspectives, the role of the teacher and of instruction is critical to providing students with the types of opportunities to learn described above. Piaget (1970) marks the importance of problematic tasks that perturb schema so they may adapt. Vygotsky (1978) acknowledges the importance of instruction as building on students' informal experiences (prior knowledge) and then providing students with problem solving situations within his/her zone of proximal development. A key component from both perspectives is the function of the interaction between the individual with others around a cultural activity. For constructivism, the other creates cognitive dissonance in order for the learner to search for equilibration and learning is studied with a focus on adaptations to people's schema. For socio-cultural theory, learning is the process of internalizing knowledge from culture through interacting with the more knowledgeable other, and as such, learning is studied with a focus on changes in people's participation in established cultural practices.

Social constructivism evolved from both the constructivist and socio-cultural perspectives to provide a theory that acknowledged the view that "both social processes and individual sense-making have a central and important parts to play in the learning of mathematics" (Ernst, 1994, p. 64). First used by sociologists, social constructivism came about to account for the social construction of scientific concepts and mathematics while at the same time acknowledging the individual learning of ideas. Ernst (1991) stated that "there is an active construction of knowledge, typically concepts and hypotheses, on the basis of experiences and previous knowledge…Secondly, there is the essential role played by experience and interaction with the physical and social worlds" (p. 72). Moreover, the role of language, conversation, signs, and text are vital to teaching and learning (Ernst, 2006).

Cobb and colleagues (Cobb, 2007; Cobb & Yackel, 1996; Cobb, Yackel, & Wood, 1992) interpreted social constructivism as emergent or distributed cognition perspective. From this view, individual learning is studied as it occurs in a social context. Learning can be characterized as the process of self-organization, but this "can be seen to occur as students participate in and contribute to the development of practices established by a local community" (Cobb, 1995, p. 25). In this way, the social constructivist perspective takes into account "both individuals' socially situated knowing and the taken-as-shared ways of knowing constituted by these individuals" (Cobb, 1995, p. 26).

Cobb and Yackel (1996) argued that this view is not only appropriate to consider students' mathematical learning, but also teachers' socially situated work. That is to say that, teacher learning can be studied from both the perspective of the individual teacher and of the practices established by a group of teachers and how the individuals participate within the group. In the context of research, it is often the case that the individual learning is brought to the forefront while the social and cultural aspects serve as the background context for the individual.

For the current study, teachers had learned about the trajectory prior to the study, but this learning continued as they used the trajectory in their instruction. So, although this is a study about teachers' uses of LTs, they were also in the process of better understanding and learning about the LT as they used it. To study teaching and learning from a social constructivist perspective necessitates the study of the individual as well as the collective. Hence, in this study, the focus is on the individual teacher's use of a LT over time where the teachers' collectively work together to plan and reflect on their lessons served as the social and cultural context from which they worked. The teachers were studied individually in their classrooms and in the social context of their grade level planning meetings. Moreover, my participation in their planning meetings contributed to the meaning making that occurred for the teachers as they used a LT to plan and implement instruction on equipartitioning. Ernst (1994) emphasized that "the social domain impacts on the developing individual in some formative way, with the individual constructing (or appropriating) her meanings in response to experiences in social contexts" (p. 66). Hence, while this is a study of the individual teacher, it is critical to acknowledge the influence of the group on the individual's way of knowing.

Learning Trajectories

As the field of mathematics education explores methods to improve instruction and student learning, the construct of a learning trajectory (LT) has gained prominence as a foundation for standards, curriculum, assessment and instruction (Daro et al., 2011). While various definitions for learning trajectories exist within mathematics and science education (Battista, 2004; Clements & Sarama, 2009; Confrey et al., 2009; Corcoran et al., 2009; National Research Council, 2007), Maloney and Confrey (2010) suggested similarities in these definitions, including: 1) The assumption that students enter the classroom from different cognitive origins and that the trajectory assists in moving students to more sophisticated ideas, 2) the assumption that tasks and instruction are not independent of a learning trajectory, 3) learning trajectories consist of a sequence of goal understandings or levels that build upon each other from simple to complex, 4) learning trajectories are constructed from an empirical basis (e.g., a literature review, synthesis, or empirical research from working with students) but best guesses are sometimes used to fill in gaps, and 5) gaps in a learning trajectory are completed with additional empirical research and this process is iterative as it informs the original literature synthesis.

Often cited as the first use of the term learning trajectory in mathematics education literature, Simon (1995) defined a hypothetical learning trajectory to be, “The learning goals, the learning activities, and the thinking and learning in which students might engage” (p. 133). Later, the National Research Council (2007) promoted learning progressions in science education as “descriptions of the successively more sophisticated ways of thinking about a topic that can follow one another as children learn about and investigate a topic over a broad span of time" (p. 211). The 2009 NAEP Science Framework (National Assessment Governing Board, 2008) emphasized that these trajectories depend on the interaction between students' prior knowledge and instruction.

Clements and Sarama (2004) conceptualized LTs as "descriptions of children's thinking and learning in a specific mathematical domain and a related, conjectured route through a set of instructional tasks" (p. 83) calling attention to the inclusion of learning goals and instruction in the path of students' learning. The fact that LTs use empirical research on how students come to learn key mathematical concepts allows educators to consider new paths for learning other than what is laid out in traditional curricula.

Battista's (2004) interpretation of a LT consisted of levels of sophistication, beginning with informal, pre-instructional reasoning, and moving towards formal mathematical concepts. Along the way, students encounter plateaus, or key processes and conceptualizations required to reach formal understanding. In his study of cognition-based assessment, Battista used area and volume measurement tasks with third and fifth grade students in order to create assessments built from a research-based framework. He claimed that in order for teachers to support students' construction of mathematical understanding, they must be able to identify core mathematical ideas, know conceptual frameworks for understanding children's thinking, and utilize a valid set of assessment tasks. In his view, the learning trajectory provides teachers with information on children's thinking as well as a structure for assessment.

The DELTA research group defined a LT to be an "empirically-supported description of the ordered network of constructs a student encounters through instruction (i.e. activities, tasks, tools, forms of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time" (Confrey et al., 2009, p. 347). This definition highlights learning trajectories as identifying important aspects of a concept as it evolves over time, with the influence of instruction.

Although the initial hypothetical learning trajectory was meant to be a framework to help teachers think about how students' learning may evolve, it did not provide direct implications for teaching. Rather, its purpose was to emphasize "the importance of having a goal and a rationale for teaching decisions and the hypothetical nature of such thinking" (Simon, 1995, p. 136). Its hypothetical nature implied that no teacher or researcher knows exactly how learning will proceed. More precisely, teachers rely on expected tendencies from their theoretical and experiential knowledge of students and content. With the more recent work to empirically construct LTs, researchers are beginning to study the utility of LTs for instruction.

Learning trajectories and teaching. Few researchers have studied the implications of LTs for teachers (Bardsley, 2006; Clements, Sarama, Spitler, Lange, & Wolfe, 2011; McKool, 2009; Mojica, 2010; Wickstrom, Baek, Barrett, Cullen, & Tobias, 2012; Wilson, 2009). Bardsley (2006) conducted a case study of 14 pre-kindergarten teachers on their use of *Building Blocks* (Clements & Sarama, 2007), a curriculum developed around empirically supported learning trajectories on early- childhood mathematical concepts. She concluded that teachers' motivation for participating in the professional development influenced how they used the curriculum materials. Teachers who wanted classroom activities were more likely to focus on moving students through the levels. However, teachers who participated to learn better mathematics used the curriculum as a structure for making instructional decisions.

Continuing to study the *Building Blocks* curriculum, Clements and Sarama (2008) and colleagues (2011) reported on a large-scale professional development project to examine the effectiveness of the curriculum. Using a randomized trial design, the researchers studied 36 teachers in three groups: a treatment group of teachers using the trajectory-based curriculum, a comparison group of teachers who used a research-based curriculum, and a control group. Teachers in the treatment group and comparison group received the same amount of professional development and similar curricular materials. Their results showed that teachers in the *Building Blocks* treatment group developed richer mathematical classroom environments and were more responsive to their students' mathematical thinking. Moreover, children in the treatment group had significantly greater gains from pre- to post-tests compared to students in both the comparison group and the control classrooms. The authors argued that differences in the treatment group and comparison group can be attributed to differences in the teachers' knowledge of the learning trajectories that were foundational to the *Building Blocks* curriculum (Clements & Sarama, 2008).

McKool (2009) conducted a collaborative teaching experiment with one teacher who utilized a measurement learning trajectory in working with two fifth grade students. McKool reported that the professional development modeled around LTs supported the teacher in focusing on her students' mathematical thinking, improving her ability to assess her students' knowledge and more effectively use her own mathematical knowledge to consider instruction to address areas where her students struggled.

More recently, Wickstrom and colleagues (2012) reported a case study of a first grade teacher who participated in professional development on a learning trajectory for length measurement. Using Jacobs, Lamb, and Phillips (2010) framework of teacher noticing, the findings indicated the learning trajectory provided the teacher with language to describe student thinking, and supported her in attending to strategies students used as they engaged in length tasks. When the teacher provided evidence of students' strategies, she was more successful at interpreting student understanding; however, both acts of noticing were more difficult during classroom instruction than in individual student interview settings.

Wilson (2009) and Mojica (2010) conducted two related design studies on in-service and pre-service elementary teachers' use of the equipartitioning learning trajectory (EPLT) as a model of student thinking. In Mojica's study, 56 pre-service teachers participated in an elementary mathematics methods course that emphasized the EPLT as a way to process and organize students' thinking. Her results indicated that learning about the LT deepened the pre-service teachers' own knowledge of mathematics and they used the LT to consider students' mathematical thinking in instruction.

With respect to practicing teachers, Wilson (2009) reported the ways in which 33 teachers from two rural schools used a LT to organize and process students' mathematical thinking, and it's implication for instruction. Using a framework of describing, comparing, inferring and restructuring as a lens to understand how teachers construct models of student thinking, Wilson interpreted teachers' uses of the EPLT throughout a professional development experience and in classroom instruction. He stated, "As teachers use their own understandings of mathematics to make sense of students' behaviors and verbalizations, they are creating a theoretical model of students' thinking that allows them to relate and infer students' understandings of mathematics" (p. 38).

In the professional development context, Wilson reported that teachers used the EPLT to predict students' approaches to tasks and were more aware of students' verbalizations and behaviors and their implications for students' understandings. The LT provided the teachers in his study with a common language to talk about student thinking related to equipartitioning. Teachers began to describe student work with more detail (rather than simply right or wrong) and comparisons of students' work were made based on empirical evidence using what they knew theoretically from the trajectory as opposed to age or grade.

Additionally, Wilson found that curricular influences interfered with teachers' ability to incorporate equipartitioning tasks into their existing curriculum, and teachers used the EPLT to evaluate the relative difficulty of a task. In the classroom, he observed teachers using the EPLT to make sense of various approaches students took to solve equipartitioning tasks and to then use those approaches in instruction. For some teachers, the LT provided a framework for selecting and sequencing student work to afford all students with opportunities to refine their understanding of equipartitioning concepts during whole group discussions.

Wilson (2009) and Mojica (2010) concluded that teachers used the learning trajectory as a tool to organize student behaviors and cognition simultaneously to construct models of student learning. Pre-service teachers used the trajectory as a means to focus on student thinking. For in-service teachers, the trajectory provided a tool to unpack previously formed models to coordinate specific behaviors with cognition.

Summary. The studies reviewed provide evidence of the utility of LTs in focusing instruction on children's mathematical thinking and providing teachers with a framework for understanding their students' thinking. The current study builds on the existing knowledge base of teachers' uses of LTs to examine the ways in which teachers use a LT as they engage in cycles of lesson planning, whole class instruction, and assessment over time.

The Teaching Cycle

Wilson (2009) described instructional activity as planning, teaching, and assessment. In this study, I refer to iterations of the instructional activity as *the teaching cycle*. The separation of the teaching cycle into three phases creates a theoretical support to study each aspect of instructional activity. Planning relates to the decisions teachers make prior to instruction. While the terms instruction and teaching are often used interchangeably, for the purposes of this study, I use instruction to refer to the enactment of classroom teaching. Assessment refers to the analysis of what occurred during instruction. I recognize that in practice, planning, instruction and assessment do not necessarily occur separately, but attending to these activities independently allows for a closer examination of the ways in which teachers use a LT throughout the teaching cycle. Therefore, in this section I review literature related to each of the three phases of the cycle.

Lesson planning. Often considered a core routine of teaching, lesson planning refers to the time teachers spend preparing for instruction before students enter the classroom. Previous research on teacher planning focused on what teachers attend to during lesson planning, such as content, learning goals, or activities (McCutcheon, 1980; Yinger, 1980). Much of teachers' day-to-day planning is done mentally or as notes jotted down in a plan book with textbook pages listed (John, 2006; McCutcheon, 1980). With respect to how specific planning time is spent, a study by Decker and Ware (2001) revealed that very little time is "actually devoted to the tasks that enhance the form of teacher delivery of instruction" (p. 8) and that specific planning time during the school day is more likely to be spent doing other tasks such as photocopying, running errands, or interacting with other teachers.

A common practice in any teacher preparation program is to exercise the craft of writing detailed lesson plans (John, 2006; McCutcheon, 1980). However, such a craft is not always easy to teach or to learn. Since the 1950's, a model for planning that is commonly taught to preservice teachers and used universally is the rational model, or objectives-first model (McCutcheon, 1980; Yinger 1980; Zahorik, 1975). This model is expressed as a linear model where planning begins with formulating an objective that describes a desired student behavior. Next, the teacher selects learning activities, organizes the lesson around these activities, and finally chooses some form of assessment (John, 2006; Yinger, 1980; Zahorik, 1975). Implicit in this time period is the notion that teachers created their own objectives and activities for students as opposed to working from mandated standards or curricula.

Research on teacher planning showed that most teachers do not follow the dominant model of planning; that is, they do not specifically attend to behavioral objectives first, activities, and finally assessment (John, 2006; McCutcheon, 1980; Yinger, 1980; Zahorik, 1975). Studies on what teachers attend to in planning their lessons indicated that teachers focus on ideas such as content, activities or tasks, materials, textbooks, and routines, as well as students' needs and backgrounds (Clark & Lampert, 1986; Fernandez & Cannon, 2005; McCutcheon, 1980; Yinger, 1980; Zahorik, 1975). In his 1975 study of teacher planning, Zahorik surveyed 194 teachers from various fields to determine what teachers focused on most often in their lesson planning. The results of the study showed teachers attended to content more often than objectives. Also, although they were not always attended to first, activities were thought about by the majority of teachers in the study (Zahorik, 1975).

In a study of 12 elementary school teachers, McCutcheon (1980) found that teachers used their textbook as a main source for activities and depended heavily upon suggestions from the teachers' guide. In a later study, Brown (1988) examined the lesson planning practices of 12 middle school teachers in various content areas. She found that teachers relied heavily on curriculum materials, building their lessons off of objectives expressly stated in the curriculum resources.

More recently, Superfine (2008) studied three teachers' lesson planning with respect to a specific mathematics curriculum. Participants taught sixth grade at the same school and were observed planning and enacting the same unit from a reform-based curriculum. Through the analysis of pre- and post-lesson interviews, the study revealed two difficulties the teachers had in planning: anticipating student work, misconceptions, and potential errors for a given task, and understanding the treatment of the content in the curriculum. She concluded that the conceptions teachers hold with respect to the teaching and learning of mathematics as well as years of experience mediated their management of these difficulties.

In an attempt to articulate high-quality mathematics instruction, Corey and colleagues (2010) studied conversations between seven Japanese student teachers and three cooperating teachers from one school. These conversations occurred during planning meetings that took place between the student teachers and their cooperating teachers, where the pairs met on average three times prior to teaching a lesson. The findings described six principles of high-quality instruction, with three of the principles directly related to lesson planning: an ideal lesson is guided by a set of explicit long and short term goals, a particular lesson is created with clear connections to previous and future lessons, and high-quality instruction requires anticipating student thinking in relation to the goals of the lesson. This study offers support for the importance of teachers' ability to choose learning goals and to consider students' approaches to intended instructional tasks in order to foster meaningful learning.

Summary. One may question what *should* be the focus of planning when instruction attends to students' mathematical thinking. Specifically with respect to mathematics lesson planning, Ainley, Pratt, and Hanson (2006) stated, “If teachers plan from tightly focused learning objectives, the tasks they set are likely to be unrewarding for the pupils, and mathematically impoverished. If teaching is planned around engaging tasks the pupils' activity may be far richer but it is likely to be less focused and learning may be difficult to assess” (p. 24). The research reviewed points to identifying learning goals and anticipating students' likely approaches to intended tasks as important foci of lesson planning that takes student thinking into account. Teachers must consider how to construct lessons that address specific learning goals and allow teachers to gather evidence of their students' understanding towards the chosen goals. Anticipating likely responses and misconceptions can prepare teachers to navigate the complex moments of instruction. Moreover, as student learning progresses overtime, teachers' consideration of learning goals allows them to consider how to build on students' current conceptions to reach the intended goals. This study aimed to examine the utility of a learning trajectory as a resource for identifying goals and anticipating student work as teachers plan a set of lessons over time.

Instruction. The second element in the teaching cycle is the enactment of teaching, or instruction. Educators have attempted to describe effective instruction and to link specific acts of instruction to student learning. It cannot be denied, however, that instruction is a complex task involving students, teachers, and the interactions that take place in classroom settings. In the past, instruction referred more often to teachers' visible actions and students' abilities to replicate those actions (Oser & Baeriswyl, 2001). Over time, educators began to acknowledge instruction as more than a one way relationship of teachers acting on students.

Instruction became not just about the visible interactions between the teacher and the learner but also the inner mental processes that take place as a result of those interactions. Cohen, Raudenbush, and Ball (2003) emphasized this bidirectional relationship and stated, "Instruction consists of interactions among teachers and students around content" (p. 22). Building on this interpretation, Hiebert and Grouws (2007) defined instruction as the "classroom interactions among teachers and students around content directed toward facilitating students' achievement of learning goals" (p. 372).

While a plethora of research exists on instruction, due to the nature of this study, I focus the review of literature on instruction that attends to student thinking. As research on children's learning progressed, researchers began to consider the implications of such research for instruction (Carpenter et al., 1989). The work of cognitive psychologists such as Piaget and Vygotsky allowed researchers in mathematics education to more clearly articulate what it means to teach and learn mathematics with understanding. In particular, learning must be built on prior knowledge and children must have experiences in communicating about and reflecting on mathematical ideas. In this view, instruction must be centered on students' construction of knowledge (Ball, 1993; Carpenter et al., 1989; Lampert, 2001) and student thinking must serve as launching points for the development of important mathematical concepts (Rasmussen & Marrongelle, 2006).

Through their research on learning, Bransford, Brown, and Cocking (2000) proposed that classroom environments must be learner-centered with an emphasis on formative assessment. In this way, teachers can "grasp the students' preconceptions, understand where the students are in the ‘developmental corridor’ from informal to formal thinking, and design instruction accordingly” (Bransford et al., 2000, p. 35). Instruction requires teachers to build on students’ prior knowledge and for ideas to be shared and communicated, making student thinking visible to the teacher and other students.

The most well-known research on instruction centered on student thinking is the Cognitively Guided Instruction (CGI) project. The initial purpose of the project was to examine the instruction of teachers who were provided with explicit knowledge derived from research on children’s thinking (Carpenter et al., 1989). One of the goals of the project was “to help the teachers understand the ways students intuitively solve problems, so that they can help students build on that knowledge” (Carpenter et al., 1996, p. 15). Through 80 hours of professional development over the course of 4 weeks, teachers learned about problem types and the strategies that children typically use to solve different problems through watching clinical interviews and conducting interviews with children solving addition and subtraction problems. The researchers also discussed with teachers broad principles of instruction centered on students’ active development of understanding and teachers had time to examine curricular materials and plan for the coming school year. The project teachers were studied through content assessments and classroom observations throughout the school year. Carpenter, Fennema and colleagues concluded that by listening to their students’ problem solving strategies and being equipped to make sense of the strategies their students were using, CGI teachers were better able to adapt their instruction to provide appropriate learning activities for their students (Carpenter et al., 1989; Fennema et al., 1996).

Franke, Carpenter, Levi, and Fennema (2001) continued to study teachers' use of student thinking four years after their participation in the CGI project. Through interviews and classroom observations of 22 elementary teachers from six schools who participated in the CGI study, the researchers sought to understand "what characterized and supported teachers' continued learning and growth" (Franke et al, 2001, p. 657). The researchers coded interview and observation data using a coding scheme that classified teachers' levels of engagement with children's mathematical thinking, and then identified themes related to talk about students' mathematical thinking, use of frameworks to organize knowledge of children's thinking, and engagement in creating their own knowledge of children's thinking. The results of their analyses showed that overall the participants remained engaged with children's mathematical thinking. In particular, one highly engaged teacher reported that by learning to focus on children's thinking, she not only created, but also built upon and extended frameworks she used to understand her students' thinking. She also saw her classroom as an interactive place for her to build her own knowledge about mathematics and student thinking.

Jacobs, Lamb, and Phillips (2010) used the construct of *professional noticing* in order to characterize the in-the-moment decision making that is foundational to mathematics instruction that promotes conceptual understanding. They define professional noticing as comprised of three component skills: attending to children's strategies, interpreting their thinking, and deciding how to respond on the basis of children's thinking. To investigate teachers' expertise at professional noticing, the researchers analyzed participants' written responses to prompts based on videos and written work of students engaged in mathematical tasks. Participants consisted of preservice elementary teachers and experienced practicing K-3 teachers who participated in CGI-based professional development on children's mathematical thinking for varying lengths of time (0 years, 2 years, and 4 years). They concluded that teachers with limited experience with children's mathematical thinking (preservice teachers and teachers yet to start the professional development) struggled to attend to students' thinking, while teachers with sustained experience with children's mathematical thinking (those who participated in the professional development for 2 or 4 years) consistently coordinating the three components of professional noticing. Thus, professional development can support teachers in learning to attend to students' mathematical thinking.

Few researchers have worked as teacher-researchers to study the role of student thinking in their own instruction (Ball, 1993; Lampert, 2001; Lubienski, 2000). Ball (1993) reported on her challenges in creating classroom practices that engaged students in meaningful mathematics and connected to their prior experiences. Her reflections upon her teaching revealed the necessity to understand the content through the eyes of her students and "consider the mathematics in relation to the children and the children in relation to the mathematics" (Ball, 1993, p. 394). In her work as a teacher-researcher using a reform curriculum, Lubienski (2000) described her pedagogy as consistent with standards-based instruction where students made their problem solving strategies public and connections between different mathematical ideas were emphasized.

In her book, *Teaching Problems and the Problems of Teaching*, Lampert (2001) presented a case study of her own mathematics teaching of a fifth grade class over the course of one year. Her goals were to use her own teaching using problem-based instruction to develop a model of practice. Her approach to instruction was to center discussions around students' thinking and for students to learn to justify their thinking in a public manner. She concluded with a model of practice that represents the interactions between the teacher, students, and the content, emphasizing the importance of the students' conceptions that they bring to the complex interactions.

Summary. While the research described above speaks to the value of attending to student thinking, knowledge about specific teacher practices that support students in making their thinking explicit and subsequently how teachers respond to such evidence of their students' understanding in the moments of instruction is only emerging (Franke et al., 2001). From previous research, we know that the information about common strategies and misconceptions highlighted in a LT can support teachers in recognizing these ideas among their students, make sense of these ideas, and in turn, helping students make connections to important mathematical ideas. The current study examined how a LT facilitated teachers' ability to elicit evidence of student thinking and to navigate whole class discussions.

Assessment. The final element in the teaching cycle is assessment. In the report, *Knowing What Students Know*, Pellegrino, Chudowsky, and Glaser (National Research Council, 2001) emphasized one of the main purposes of assessment as assisting in learning. Assessment to assist in learning involves not only quizzes, tests, and homework, but also the daily observations and interactions that teachers have with their students. These approaches to collect information about what students know are often referred to as formative assessments. Pellegrino et al. (2001) characterized assessment as the process of reasoning from evidence of student learning to make inferences about what students know. The authors argued that there are three key elements that underlie any assessment and described them as "a model of student *cognition* and learning in the domain, a set of beliefs about the kinds of *observations* that will provide evidence of students' competencies, and an *interpretation* process for making sense of the evidence" (Pellegrino et al., 2001, p. 44, emphasis in original).

Formative assessment is ongoing and happens not only during the moments of teaching where evidence is gathered, but also after instruction when teachers have time to reflect on their lessons in order to make decisions about future instruction. Reasoning from evidence of student learning takes time; teachers must stop and consider how the interactions that occurred during instruction inform what they will do next. For the purposes of this study, the focus is on formative assessment that occurs after instruction, in the moments in between lessons, where teachers interpret the evidence they gather during instruction about their students' learning to inform future instruction. While the majority of research on formative assessment focuses on the moments during instruction, reviewing this literature serves to better understand assessment that takes place in between instructional episodes.

In 1998, Black and Wiliam published an extensive review of over 200 empirical studies on the impact of formative assessment practices on student achievement. They conclude that while the effective use of formative assessment is positively correlated with student achievement gains, the ways in which teachers incorporate formative assessment practices varies and that “reform in this dimension will inevitably take a long time, and need continuing support from both practitioners and researchers” (Black & Wiliam, 1998, p. 37). While researchers have identified effective formative assessment strategies and studied professional development projects that facilitate teachers’ implementation of formative assessment into their classroom practices (Black, Harrison, Lee, Marshal, &Wiliam, 2004; Heritage, 2008; McManus, 2008; Wiliam, Lee, Harrison, & Black, 2004), little is known about teachers’ processes between lessons as they evaluate the learning that took place in prior lessons to inform future instruction.

Based on their initial literature review, Black and colleagues (2004) studied secondary science and mathematics teachers’ implementation of formative assessment practices. Twenty four teachers participated in the project intervention which included collaboration between the researchers and participants on how to employ specific formative assessment practices (questioning, comment-only marking, sharing criteria with learners, and student peer- and self-assessments). The researchers supported the teachers in deciding what and how they would attempt some of these strategies in their classrooms and collected evidence of participants’ formative assessment practices through classroom observations, interviews, and written reflections. An important result from the study was that as the teachers began to implement various formative assessment practices, they requested information on learning theories to help them make sense of the information they were learning about their students and to build models of student thinking.

Sato, Wei, and Darling-Hammond (2008) examined changes in 16 middle and high school math and science teachers' formative assessment practices over a three year period, where nine of the teachers engaged in the National Board Certification process, and the remaining seven served as a control group. Researchers collected and analyzed data packets for each participant consisting of twice-yearly classroom observations, collections of student work samples, bi-yearly teacher interviews, yearly student and teacher surveys, and final teacher interviews. The results indicated significant increases in formative assessment practices for National Board candidates over the control group, for which the researchers attributed to particular characteristics of the process such as writing prompts asking teachers to reflect on evidence of student learning and how they use this evidence to modify instruction, exposure to an explicit definition of formative assessment, and setting clear goals for teachers' practice.

Heritage, Kim, Vendlinski, and Herman (2009) conducted a study of teachers' knowledge of teaching mathematics in order to examine teachers' abilities to make inferences from evidence of student learning. Over 100 middle school mathematics teachers completed an online assessment where teachers reviewed student responses to various mathematical tasks and answered questions about key mathematical ideas in the task, evidence of student understanding, what feedback would be appropriate, and what might be next instructional steps based on the students' work. The results indicated when teachers are given assessment information, they can articulate key learning principles and evaluate students' understanding, but they are less successful in planning the next instructional steps based on the evidence of student learning.

In contrast, Wilson and Confrey (2011) argued that when provided with a model of student cognition in the form of a learning trajectory, teachers were able to observe and interpret student thinking in a more refined way. In a professional development setting, elementary teachers were able to make inferences about students' understandings from student work samples and then make decisions about next instructional steps based on the progression described in the LT. Similarly, Edgington, Sztajn, Wilson, and Confrey (2011) found that a LT supported teachers in analyzing student work beyond right or wrong to make conjectures about student understanding and to consider possible contexts and follow-up activities that would facilitate students' thinking.

Sikes (2008) conducted a case study of one elementary school's use of formative assessment and its impact on instruction. Through teacher interviews, planning meeting observations, and document analysis, she examined teachers' ability to make instructional decisions working in grade level teams based on data collected through specific formative tests. While the grade level team meetings gave the teachers opportunities to collaborate on analyzing data and to make instructional decisions for remediation and differentiated instruction, the lack of time and organizational structure impeded the teachers' use of the formative assessments.

Summary. While it is clear that teachers' use of assessment to assist in learning is an important component of the teaching cycle, how teachers reflect on evidence of student learning and use it to guide future lessons remains vague. When planning and instruction allow students to make their thinking public, how do teachers navigate future instruction based on what they learn about their students' thinking over time? Heritage (2007; 2008) claimed that learning trajectories are a key component of formative assessment as they provide teachers with short term goals in relation larger conceptual ideas, and help teachers "locate students' current learning status on the continuum along which students' are expected to progress" (Heritage, 2007, p. 142). This study aimed to examine the ways in which teachers used a LT to make sense of evidence of students' mathematical thinking and use it to inform future instruction.

Key Points from the Literature

This study was informed by several key points from the literature review. First, initial studies on teachers' uses of LTs indicate their potential utility to support teachers in focusing on students' mathematical thinking during instruction (Clements & Sarama, 2008; Clements et al., 2011; McKool, 2009; Wilson, 2009). Subsequent research can aid in understanding the specific ways in which teachers use trajectories over time as they plan for instruction, implement lessons, and assess their students' understanding.

With respect to each phase of the teaching cycle, the literature provides significant findings from which to focus the current study. Research on lesson planning indicated the importance of identifying learning goals and anticipating students' approaches to intended instructional tasks in relation to the learning goals (Corey et al., 2010; Superfine, 2008). Studies of classroom instruction emphasized the value of attending to students' mathematical thinking (Carpenter et al., 1989; Fennema et al., 1996; Franke et al., 2001) but further research is needed to understand more precisely how constructs such as LTs support teachers in this practice. Finally, formative assessment has been identified as a powerful tool to increase student achievement (Black & Wiliam, 1998) and LTs have been recognized as supporting teachers in evaluating student work in more refined ways and using that information to guide next instructional steps (Edgington et al., 2011; Heritage, 2007; Wilson, 2009). The current study aimed to extend the existing literature in consideration of teachers' uses of LTs to support attention to student thinking in lesson planning, instruction, and assessment.

Conceptual Framework

In order to examine each phase of the teaching cycle, the conceptual framework for this study draws upon the literature review as well as the work of Hiebert, Morris, Berk, and Jansen (2007), and that of Smith, Stein and colleagues (Smith & Stein, 2011; Stein, Engle, Smith, and Hughes, 2008). Hiebert et al. (2007) proposed a framework for competencies necessary to analyze teaching with the goal of improving on instruction. Smith and Stein (2011) presented a framework consisting of five practices to support productive mathematical discourse structured around students' responses to mathematical tasks. These two frameworks were chosen because of their emphasis on student thinking as a central feature of planning, instruction, and assessment and the ways in which they addressed the key points from the literature reviewed. In this section, I describe each of the above frameworks and the aspects of each framework that are used to inform the current study.

A framework for analyzing teaching. Hiebert and colleagues (2007) claimed "the core of teaching- interacting with students about the content- is not learned well through automatizing routines or even through acquiring expert strategies during a teacher preparation program. Rather, it is learned through continual and systematic analysis of teaching" (p. 49). A goal of their proposed framework was to examine how teachers can potentially develop disciplined inquiry into teaching; that is, to provide a foundation for teachers to generate knowledge that can be used to continually improve their teaching over time. They proposed four skills drawn from the daily routines of planning, instruction, and reflection on classroom lessons: 1) setting learning goals for students, 2) observing whether the goals are being achieved during the lesson, 3) specifying hypotheses for why the lesson did or did not work well, and 4) using the hypotheses to revise the lesson.

With the purpose of analyzing teaching, specifying learning goals is necessary in order to determine how instruction facilitated or inhibited student learning. Hiebert and colleagues (2007) contended that learning goals are the basis for gauging the effectiveness of particular instructional activities and for measuring evidence of student learning. The ability to unpack learning goals to specify necessary subconcepts can provide teachers with more detailed information with which to build a lesson. Teachers must be able to draw upon their subject matter knowledge to take apart larger goals as well as connect smaller sub-goals.

The second skill consists of collecting evidence of student learning to determine to what extent students achieve the intended learning goals. This requires that teachers shift their focus to attend to changes in student thinking. Gathering evidence of student learning is necessary to determine the effectiveness of teaching. More specifically, teachers need to be able to distinguish between student responses that are relevant to the learning goal and whether those responses are informative or not. Hiebert et al. (2007) recognized that unintended learning is likely to occur during instruction; however, as this type of learning is unplanned, teachers should focus on learning that is relevant to the goals in order to analyze the effectiveness of specific activities. Teachers must also be able to understand what student responses, both verbal and written, imply about students' thinking. This requires teachers to anticipate students' work and provide opportunities for students to make their learning visible.

The next skill requires teachers to take the evidence gathered during instruction to make conjectures about what aspects of instruction facilitated student learning (or not). Teachers use general knowledge of teaching and learning to evaluate how instruction contributed to or hindered the learning that took place. Hiebert and colleagues (2007) recognized that this entails a bit of skepticism as the connections between teaching and learning are quite complex. But, when hypotheses are centered on students' attainment of intended goals and include sufficient details about teaching and learning they are more likely to lead to improved teaching.

The fourth and final skill includes proposing improvements in teaching based on the hypotheses generated in skill three. Clarifying learning goals, collecting evidence of student learning, and making hypotheses about how instruction facilitated or hindered student learning leads to revisions that have the potential to improve instruction. Hiebert et al. (2007) emphasized that the revised lessons can be interpreted in light of larger learning goals (e.g. procedural fluency or conceptual understanding) as well as teachers' abilities to access students' thinking at key moments in a lesson. In their work, they consider that over time, year after year, teachers could improve instruction through continuous refinement of their lessons.

In summary, the framework proposed by Hiebert et al. (2007) builds upon the everyday practice of teaching to support teachers in developing disciplined inquiry (Dewey, 1929). These researchers supported the notion that taking on a research-oriented lens to examine teaching facilitates teachers' ability to learn from their own practice. The framework they proposed acknowledges the role of teachers' subject matter knowledge in the work of planning, implementing and assessing. Moreover, focusing on student thinking and making decisions based on individual students' learning can be an initial step to more equitable instruction. Knowing about the kinds of experiences and knowledge students bring to the classroom and why thinking may differ between students can assist a teacher in collecting evidence of student thinking.

Five practices to support productive mathematical discussions. In their book, *5 Practices for Orchestrating Productive Mathematical Discussions*, Smith and Stein (2011) presented a framework for supporting mathematical discourse based on student thinking while simultaneously advancing the mathematical learning of the whole class. They argued that the typical launch-explore-discuss lesson structure using cognitively demanding tasks often leads to “show and tell” discussions, lacking connections among students’ strategies to important mathematical ideas. The framework they proposed is intended to facilitate teachers’ ability to manage whole class discussions that honor students’ own thinking while at the same time move learning forward to important mathematical goals. The five practices are: 1) anticipate likely student responses to cognitively demanding tasks, 2) monitor students’ work on the tasks, 3) select particular students to share their strategies, 4) purposefully sequence student sharing, and 5) assist the class in making connections between students’ responses and mathematical ideas.

The first practice of anticipating students’ responses to instructional tasks involves considering how students might interpret a problem, the types of strategies and misconceptions that may come up, and how those strategies and misconceptions relate to the intended mathematical goals. Smith and Stein (2011) contend that this requires teachers to not only do the tasks themselves, but to also attempt to view the problem from the perspective of their students, inventing as many different solution strategies as they can. This entails knowing about their particular students’ mathematical skills and can be supported by teachers’ knowledge of research on similar tasks or common issues related to that particular topic. Anticipating students’ approaches to a task prior to instruction allows teachers to begin to think about how students’ work relates to the intended mathematical goals.

The second practice requires teachers to monitor and listen to their students as they are engaged in the mathematical tasks. This includes attending to the mathematics embedded in what students are saying and doing and making sense of students' mistakes or misconceptions. Teachers can ask probing questions that help them elicit their students' mathematical thinking and provide information to help teachers decide how to respond to students' approaches during whole class discussions.

In the next practice, teachers must select which students' ideas to share with the class. Through anticipating students' responses and then monitoring student activity, teachers can purposefully choose students to present their ideas, increasing the likelihood that important mathematical ideas will be highlighted and common misconceptions will be addressed. By asking particular students to share or carefully choosing volunteers, "the teacher remains in control of which students present their strategies, and therefore what the mathematical content of the discussion will likely be" (Stein et al., 2008, p. 328). Teachers can bring up important ideas themselves for students to consider or delay sharing certain ideas until the teacher or the class is more prepared to deal with them.

Once teachers have chosen which students' ideas to present, they must also consider how to sequence those ideas in order to "maximize the chances that their mathematical goals for the discussion will be achieved" (Stein et al., 2008, p. 329). Teachers may choose to build from least sophisticated to most sophisticated ideas in order to assist students in making sense of more complex conceptions. Alternatively, teachers may choose to begin with the most common strategy to engage a majority of students, or start with a common misconception so as to resolve that misunderstanding early on in the discussion. Regardless of the approach, teachers who purposefully select and sequence student work are more likely to engage students in more coherent discussions around the intended mathematical goals.

The final practice has teachers connecting students' approaches and conceptions to each other and to important mathematical ideas. With the ultimate goal of building on students' thinking to "develop powerful mathematical ideas" (Stein et al., 2008, p. 330), teachers can assist students in comparing and contrasting various approaches and in seeing how the same mathematics can be embedded in different strategies. Teachers can help students make connections between different representations, highlight new mathematical notation, or formalize generalizations. Doing so often requires students to reflect on their own ideas as well as communicate about the mathematical ideas present in the task and related discussion.

To summarize, the framework proposed by Smith and Stein (2011) allows teachers to build upon students' conceptions to develop important mathematical ideas. The authors recognized the importance of teachers creating a classroom culture that gives students' authority as thinkers and doers of mathematics while at the same time allows students to develop knowledge that is accountable to the discipline. The five practices described support teachers in developing whole class discussions that start with students' ideas and ultimately lead to the development of important mathematical understanding for the whole class.

Framework for analyzing the teaching cycle by attending to students' thinking. The structure for this study joins components of the Hiebert et al. (2007) and Smith and Stein

(2011) frameworks in relation to findings from the literature review to refine the teaching cycle and specify what it means to plan, instruct, and assess when teachers attend to student thinking, as described in Figure 1. The cycle begins with the planning phase. Teachers not only choose intended learning goals, but they decompose learning goals into smaller sub-concepts that comprise larger goals (Hiebert et al., 2007). In considering mathematical tasks proposed in a lesson, teachers use their own content knowledge as well as their knowledge of how students are likely to approach the task to anticipate students' responses and likely areas of difficulty. In this way, teachers can consider how students' responses, both correct and incorrect, can lead to the intended learning goals (Smith & Stein, 2011; Stein et al., 2008).

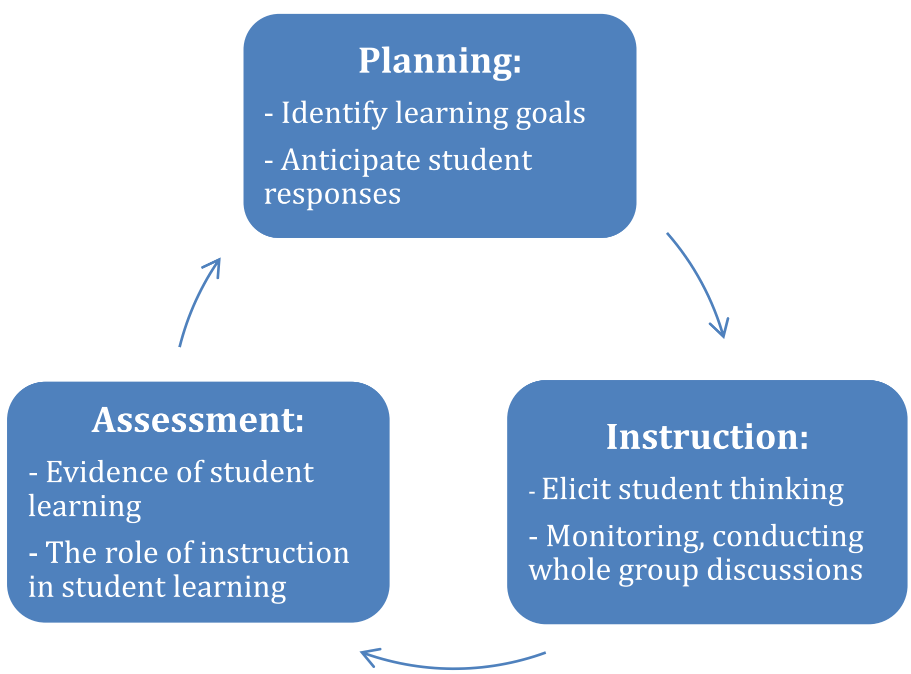


Figure 1. Analyzing the teaching cycle in terms of student thinking.

During instruction, teachers gather evidence of students' learning in order to determine if the learning goals have been met (Hiebert et al., 2007). This necessitates allowing students to make their learning visible and monitoring their work as they explore mathematical tasks. In addition, by carefully selecting and sequencing students' ideas to share in whole class discussions, teachers increase the likelihood that important mathematical ideas as well as common misconceptions are addressed (Smith & Stein, 2011).

Finally, once instruction is complete, teachers assess the evidence of student learning they have gathered to make decisions about future lessons. By comparing evidence of student learning to the intended learning goals, teachers can determine what aspects of their instruction helped or hindered their students' understandings (Hiebert et al., 2007). Once instruction has been evaluated, teachers consider new learning goals and instructional tasks that build on students current conceptions and move students to more complex mathematical understanding.

Attending to student thinking can support teachers as they engage in each phase of the teaching cycle. This attention allows teachers to acknowledge their students' current conceptions and design lessons that build on their prior knowledge. Furthermore, as teachers learn to listen to and understand their students' thinking, they can more explicitly connect students' conceptions to important mathematical ideas. As representations of student thinking, learning trajectories advance teachers ability to make sense of this evidence and use it to develop instruction that addresses their students' current conceptions and moves learning forward. Therefore, the LT is placed at the center of the teaching cycle (see Figure 2) and the current study examined the ways in which teachers used one particular LT, the equipartitioning LT, as they participated in iterations of planning, instruction, and assessment.

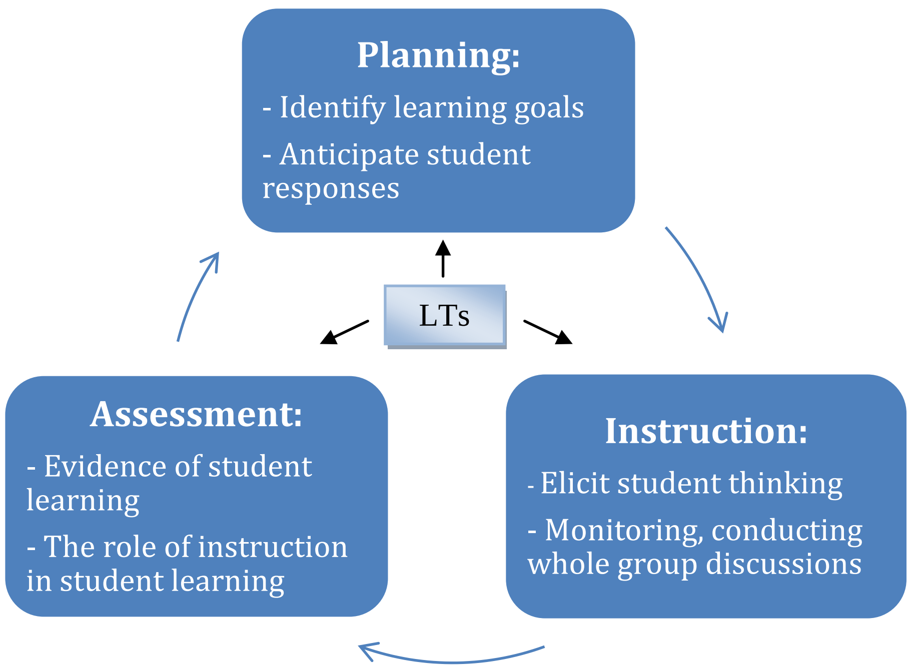


Figure 2. Placing LTs at the center of the teaching cycle.

Because LTs describe concepts from less formal to more sophisticated ideas, LTs can aid teachers in selecting appropriate learning goals and provide information about what sub-goals are associated with larger conceptual goals. LTs afford teachers with information about likely strategies, misconceptions and important milestones that teachers can then anticipate as they plan instructional activities. The knowledge of student learning inherent in LTs provides teachers with more detail as they compare evidence of student learning to learning goals, and gives them a repertoire of instructional moves based on the understandings their students' exhibit. LTs encourage teachers to focus on the process students engage in as they solve tasks, as opposed to the end product (Wilson, 2009), creating and providing feedback that moves students forward in their learning.

CHAPTER THREE

Refined Research Questions

The purpose of this study was to understand the role of LTs in supporting teachers' attention to student thinking as they engaged in cycles of planning, instruction, and assessment. In particular, the study investigated the ways in which five second grade teachers used the EPLT to plan and implement a sequence of lessons. Given the conceptual framework, the research questions are refined to specify components of planning, instruction, and assessment that attend to student thinking in light of the EPLT. In what ways do teachers use the EPLT to plan a set of lessons on equipartitioning?

   a. How do teachers use the EPLT to choose tasks, and to specify learning goals and subconcepts?
   b. How do teachers use the EPLT to anticipate students' strategies and approaches to intended instructional tasks?

2. In what ways do teachers use the EPLT when implementing classroom instruction?
   a. How do teachers use the EPLT when monitoring students' progress as they work on instructional tasks?
   b. How do teachers use the EPLT during whole classroom discussions?
3. In what ways do teachers use the EPLT to assess their students' understanding?
   a. How do teachers use the EPLT to determine what counts as evidence of student knowledge?

   b. How do teachers use the EPLT to make conjectures about aspects of instruction that helped or hindered student learning?

4. What factors mediate and moderate the ways in which teachers use the EPLT to engage in cycles of lesson planning, instruction, and assessment?

I recognize that planning, instruction, and assessment include many important components not addressed in these research questions, but for the purposes of this study, I chose to attend to certain aspects of teaching that focus on student thinking, as specified in the research questions. These aspects emerged from the research literature and guided the research I conducted. This study utilized a qualitative case study design as part of a larger design study. In this chapter, I describe the design of the study, its context and sample, as well as the study's data sources, data collection and data analysis processes. Validity and reliability issues are included as well as a subjectivity statement and ethical considerations.

Context for the Study

The current study was part of a larger NSF-funded research project entitled Learning Trajectory Based Instruction (LTBI). The overall goals of the LTBI project were to examine the ways in which teachers come to learn about and use learning trajectories in their mathematics instruction and to conceptualize a model of instruction that is centered on students' learning trajectories. In the first year of LTBI, the project engaged teachers from one elementary school in learning the equipartitioning learning trajectory (EPLT). Although the research reported in this study took place in the second year of the LTBI project, to make available information about the context of the case studies, the following sections provide details about equipartitioning as the content of the professional development, the LTBI professional development itself, and the larger LTBI research.

Equipartitioning. Rational number reasoning has been identified as foundational in moving towards advanced mathematical understanding (Confrey & Scarano, 1995: Lamon, 2007; Post, Cramer, Behr, Lesh, & Harel 1993). Recognizing this, Confrey and colleagues (Confrey, 1988; Confrey & Lachance, 2000; Confrey et al., 2009) explored the conception of splitting, also known as partitioning or equipartitioning. In particular, Confrey defined equipartitioning to be the cognitive behaviors that have the goal of producing equal-sized groups (from collections) or equal-sized parts (from continuous wholes), or equal-sized combinations of wholes and parts, as typically encountered by children in constructing 'fair shares' for each of a set of individuals (Confrey et al., 2009).

Confrey (1988) hypothesized that, as multiplication is the reversal of division, and in particular, repeated addition is the inverse of quotitive division, there exist a model of multiplication that is related to partitive division. Confrey et al. (2009) asserted that foundations of rational number reasoning are related to the more basic act of equipartitioning and that the evolution of the concept throughout elementary and middle school is necessary for a robust understanding of rational number. As another facet of the multiplicative conceptual field, equipartitioning can generate concepts of ratio, rate, and fraction in addition to multiplication and division.

Based on a synthesis of the research and clinical interviews, Confrey and the DELTA research team developed the *Equipartitioning Learning Trajectory* (EPLT) that describes how children use their informal knowledge of fair sharing as a resource to build an understanding of partitive division that unifies ratio reasoning and fractions (Confrey, 2012; Confrey, Rupe, Maloney, & Nguyen, in review). The full trajectory can be found in Figure 3.

Equipartitioning Learning Trajectory Matrix (grades K-8) Task Parameters→	Proficiency Levels	**A** Collections	**B** 2-split (Rect/Circle)	**C** 2^n split (Rect)	**D** 2^n split (Circle)	**E** Even split (Rect)	**F** Odd split (Rect)	**G** Even split (Circle)	**H** Odd split (Circle)	**I** Arbitrary integer split	**J** $p = n + 1$; $p = n$-1	**K** p is odd, and $n = 2^i$	**L** $p >> n$, p close to n	**M** all p, all n (integers)
16	*Generalize*: a among $b = a/b$													
15	*Distributive property*, multiple wholes													
14	*Direct-, Inverse- and Co-variation*													
13	*Compositions* of splits, multiple wholes													
12	Equipartition *multiple wholes*													
11	Assert *Continuity principle*													
10	*Transitivity* arguments													
9	Redistribution of shares (*quantitative*)													
8	Factor-based changes (*quantitative*)													
7	*Compositions* of splits; *factor-pairs*													
6	*Qualitative compensation*													
5	*Re-assemble*: n times as much													
4	*Name* a share w.r.t. the referent unit													
3	*Justify* the results of equipartitioning													
2	*Equipartition single wholes*													
1	*Equipartition Collections*													

Figure 3. The equipartitioning learning trajectory. Adapted from Confrey (2012) and Confrey et al. (in review).

The EPLT begins with experiences of fairly sharing collections of items or single wholes. As students enact strategies to complete these tasks, they gain proficiency in mathematical reasoning practices like justification and naming and begin to develop understandings of fundamental mathematical properties that later influence the ways that they fairly share multiple wholes (Confrey, Maloney, Wilson, & Nguyen, 2010). The trajectory describes how these strategies, practices, and properties ultimately unify as a generalization of partitive division that relates ratio reasoning and fractions. Important to the trajectory are not only the levels of sophistication of reasoning but parameters associated with the tasks, including the number of wholes and number of sharers. Beginning with equipartitioning collections, the next task parameters address equipartitioning single wholes (circles and rectangles), building on primitive splits such as halves and fourths, to eventually include arbitrary integer splits. The upper levels of the trajectory address tasks that involve multiple wholes and multiple sharers when the number of wholes is both less than and greater than the number of sharers.

The LTBI professional development. Learning trajectory-based instruction (LTBI) is a research project with a strong professional development component for elementary school teachers. Funded by the National Science Foundation, LTBI is a three year project and, in its first year, partnered with one elementary school in a mid-sized suburban school district in the Southeastern United States. Twenty-two kindergarten through fifth grade teachers chose to participate in the project and each teacher received a stipend. Through 60 hours of face-to-face professional development which took place at the school site, these teachers learned about the EPLT.

The professional development was designed over a 12 month period and began with a 30-hour intensive summer institute in which participants engaged in professional learning tasks on equipartitioning and different aspects of the EPLT, including video analysis of students working through equipartitioning tasks, videos of classroom instruction, analysis of students' written work, and curricular connections. By the end of the summer institute, the various ideas participants experienced were formalized through a presentation of Confrey and colleagues' EPLT, with an emphasis on the first twelve levels. Some of the professional learning tasks were designed to allow teachers to make connections to existing curricula.

Throughout the school year, teachers met with project leaders monthly for two hours after school to continue to build their knowledge of the learning trajectory and to try out tasks that incorporated equipartitioning concepts in their classrooms. During these meetings they engaged in activities such as analyzing student assessments and watching video clips of participants' equipartitioning lessons. The professional development concluded with a two-day follow-up summer institute.

The LTBI research. The research conducted around the LTBI professional development followed a design study methodology. Design studies, developed from the teaching experiment tradition and are conducted as a way to provide researchers with a means to study learning in context, allowing for the development of theories about the complexities of teaching and learning (Cobb, Confrey, diSessa, Lehrer, and Schauble, 2003;

Confrey, 2006). Built from a social constructivist perspective, design studies consider learning as "both a psychological process of the individual and the social process of the group" (Simon, 2000, p. 337). Design studies are often iterative with conjectures continually being modified and tested. Analysis is ongoing, with the final analysis resulting in robust theory (Cobb et al., 2003).

In year one of the LTBI work, researchers collected video recordings of all professional development meetings, audio recordings of teachers' small group discussions during these meetings, pre- and post- content assessment data, and one classroom observation. The data were used to examine how teachers learned about the EPLT during the professional development. The present study took place during the second year of the LTBI project and focused on how selected teachers used the EPLT in their classrooms. This study used information on participants that was available from year one to select case teachers, as described later in this chapter. However, all data collected and analyzed for the current study presented here were not used in the larger design research.

Researchers suggest there is a need to use "paradigm cases" of the individual along with inquiry around the collective to generate and modify the theoretical constructs used in design studies (Cobb, 2000; Simon, 2000). Therefore, in relation to the larger LTBI project, the current study added to the work of the project through the use of case study methodology "to identify paradigmatic situations in which to develop a theoretical account that can address issues relevant to the mathematics education research and/or teaching communities" (Simon, 2000, p. 343). The details of the study are discussed next.

Design of the Study

Creswell (2007) stated that qualitative research methods are used when one seeks "a *complex*, detailed understanding of the issue" (p. 40, emphasis in original). While traditional scientific approaches to research seek to test hypotheses or find causal relationships, the goal of qualitative research is to describe and make sense of a phenomenon in its natural setting from the view of the participants (Creswell, 2007). Data are physically obtained through the researcher, allowing a more complete view of the context to be considered, including the complexities inherent in human behaviors and interactions (Merriam, 1998).

This study sought to understand how teachers who participated in a year-long project to learn about the EPLT used the construct of a LT as they engaged in cycles of planning, instruction, and assessment. A qualitative approach was appropriate in order to understand the meaning the participants created with respect to their use of the EPLT in their mathematics instruction. Participants were observed and interviewed in order to gather evidence of their understanding and use of a LT throughout the teaching cycle.

There are distinctions between case study as the research process, the unit of analysis, and the end product (Creswell, 2007; Merriam, 1998; Stake, 1995; Yin, 2003). The design of this study followed a case study design while the end product consisted of a multi-case report with an eye toward theorizing (Yin, 2003). Yin (2003) defined case study as a process "that investigates a contemporary phenomenon within its real-life context, especially when the boundaries between phenomenon and context are not clearly evident" (p. 13). Because the contextual conditions of teaching are highly pertinent to the ways in which teachers' work, case study methodology is appropriate to study how teachers use a LT in their teaching.

Merriam (1998) further characterized case studies as particularistic in that the value of the case is based on what it can reveal about a particular phenomenon, situation, or event. Case studies allow the researcher to uncover and examine significant interactions that are characteristic of the phenomenon under study as well as provide concrete and contextual knowledge as evident in the end product (Merriam, 1998).

Additionally, case studies present thick, rich descriptions that "illuminate the reader's understanding" (Merriam, 1998, p. 30) of a given situation or event. Such an approach in this study highlighted the utility of LTs to support teachers in attending to student thinking throughout the teaching cycle. The final descriptions of the cases revealed the complexities inherent in teachers' uses of a LT in mathematics instruction. These descriptive data were then used to develop conceptual categories of teachers' uses of LTs in instruction. A purpose of this study was not only to describe how teachers use a LT in instruction, but to also create a framework that builds towards an initial theory of teaching based on learning trajectories. As such, the emphasis of the final product was on the cross-case analysis and the framework constructed from this analysis.

Definition of the case. Case studies provide description and analysis of a bounded system (Merriam, 1998; Miles & Huberman, 1994; Stake, 1995). The current study was bounded by the individual participants who provided examples of the larger phenomenon under study. Creswell (2007) identified this type of case study as a collective, or multiple, case study. In the case of this research, the issue of the use of LTs throughout instructional activity was the focus of the study, with individual teachers serving as illustrations of this construct. Yin (2003) proposed that a multiple case study design allows the researcher to replicate procedures for each participant in order to increase generalizability across the cases. Cross-case analysis increases the strength of multiple cases and allows for the emergence of generalizations across the cases.

The final product for this research consisted of a cross-case analysis constructed from descriptions of each participant's use of the EPLT throughout three iterations of the teaching cycle. Initially, an interpretive approach (Merriam, 1998) was utilized to provide thick images of each teacher's use of the learning trajectory through successive iterations of the teaching cycle. According to Schwandt (1998), an interpretive approach affords insight into "the complex world of lived experience from the point of view of those who live it" (p. 118). Through systematic data analyses, I interpreted the participants' experiences across points in time to create categories that conceptualize the utility of LTs. Then, a cross-case analysis was conducted in order to examine the range of cases in this study. Data was examined across the cases to look for categorical aggregation to develop initial themes. Details of this process are described in subsequent sections. Examining factors that influenced the teachers' uses of the LT across the cases allowed for local generalizations about the utility of LTs for lesson planning, instruction, and assessment.

Sample. The sample for the current study was a subset of participants from the LTBI project. Project participants were offered the opportunity to continue working with the research team in some respect in the following school year. The second grade team, consisting of five teachers, expressed an interest in developing a set of equipartitioning lessons based on the EPLT. These teachers volunteered and, as such, their participation indicated a willingness to explore the utility of the EPLT as a tool for planning, instruction, and assessment, justifying their selection as a purposeful sample for this study.

The five selected teachers were previously identified by the project research team as highly engaged with the LTBI professional development, which increased the possibility of actually observing the phenomenon of interest, that is, the use of the LT in the classroom. At the same time, these teachers varied in their mathematical knowledge for teaching, years of experience, and models of instruction, which created interesting variation for the investigation. The fact that the teachers met regularly to plan and discuss their mathematics instruction in a professional learning community setting made an argument to choose all five teachers. The use of the same curriculum and the selection of similar tasks for implementation made the connection between teachers' uses of the LT and their curriculum (Wilson, 2009) in certain ways uniform across the cases.

In summary, the selected sample had a high probability of producing cases that were both information-rich and varied, offering in-depth details on the teachers' uses of the EPLT as they engaged in cycles of planning, instruction, and assessment. Each teacher was given a pseudonym used to report findings of the study in order to maintain confidentiality. The five participants are: Lara, Ellen, Bianca, Tracy, and Emma.

Data Collection and Analysis

Data collection. In order to capture each participant's use of the EPLT, this study engaged participants in three iterations of the teaching cycle focused on instruction related to equipartitioning. The initially designed protocol called for data collection to begin with a two-hour grade level planning meeting before the start of the school year when teachers would plan their first lesson on equipartitioning. Then, individual classroom observations of each teacher teaching this lesson would take place within the first month of school. Teachers were asked to make any necessary modifications for their students and document these changes on a pre-lesson questionnaire. The protocol then called for individual teacher interviews to take place within one to three days of the observation to discuss the lesson, evidence of student learning, and how instruction helped or hindered student learning. This would complete one teaching cycle and the cycle would be repeated two more times.

The initial intent of observing multiple cycles of planning and instruction were to triangulate data from multiple sources over time. Merriam (1998) stated that long-term observations in qualitative research increases validity of the findings. Findings can be verified or refuted across multiple points in time, or changes can be observed as participants engage with the phenomenon over time. Observing the participants in the current study on three separate occasions supported the research in several ways. First, it allowed for initial analysis to inform future cycles of data collection. For example, after the first cycle, the notion of "moving beyond the goals of the lesson" appeared in two teachers' instruction. This allowed me to inquire about when teachers felt compelled to explore students' ideas not related to the goals of the lesson and to look for future instances of this phenomenon. Second, it allowed me to observe changes in the participants' uses of the LT over time. Although I had not anticipated that teachers' practice would change, observing their instruction over a three month period provided space for these findings that a different research design may not have supported. Finally, multiple observations allowed me to refine the initial findings that resulted in the themes and categories that will be discussed in Chapter four.

Data collection proceeded as stated in the designed protocol with few modifications. The first planning meeting took place before the start of the school year and although the participants discussed general learning goals for the first lesson, they did not articulate specific goals, but instead collaboratively designed a lesson that they could all implement within the first few weeks of school. Due to scheduling restraints, the teachers were unable to teach the initial lesson until the sixth and seventh weeks of school. At this time, each teacher individually taught the lesson, making any necessary modifications for her particular students. All but one of the participants completed pre-lesson questionnaires for each of the three lessons. One teacher did not complete a pre-lesson questionnaire for any of the lessons.

Post-lesson interviews took place the same day as the observation in some instances, and one, two, or three days after the lesson in other cases, and the interviews lasted anywhere from 15 to 45 minutes. During the second and third planning meetings, teachers reflected on their lessons with each other, and planned the second and third lessons in the sequence. The teachers were unable to complete the design of the second lesson during the two hour time frame, so we met again a week later during the teachers' regularly scheduled math planning time to finish planning the second lesson. With the exception of the pre-lesson questionnaire, data were collected from all participants throughout the teaching cycle. A timeline of data collection procedures can be found in Figure 4.

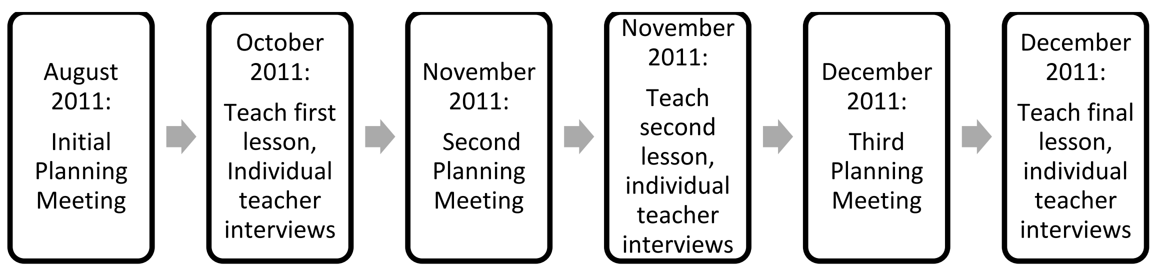


Figure 4. Data collection time line

Data sources. As with other qualitative approaches, the strength in case study methodology is that is relies on thick descriptions of the phenomenon obtained from multiple sources (Creswell, 2007; Merriam, 1998). The primary sources of data for this study were transcripts of grade level planning meetings, copies of teachers' pre-lesson questionnaires, videos of classroom observations of teachers' instruction, and transcripts of teacher interviews. Secondary sources include lesson planning documents, field notes, and teacher profile documents.

Teacher profile documents were initially created using existing data from the LTBI project. Data from the LTBI project included written reflections by the teachers and one classroom observation of an equipartitioning lesson. At the end of the LTBI professional development, participants completed two post-assessments related to mathematical knowledge for teaching: an assessment of equipartitioning (DELTA-T) designed by researchers at North Carolina State University, and the University of Michigan's Learning Mathematics for Teaching (LMT) assessment for rational number reasoning grades 4-8 (Hill & Ball, 2004). The post-test scores were used to characterize the teachers' math knowledge for teaching at the end of the professional development. Teachers also completed written reflections at the conclusion of the LTBI professional development about their views on how children learn mathematical ideas, the meaning of learning trajectories and the EPLT, and their implications for teaching.

In addition to this existing data, I asked each participant to complete a beliefs instrument prior to the first lesson planning meeting and used the results to expand the initial profile to characterize each teachers' mathematics instruction. Teachers completed the *Teachers' Beliefs about Mathematics and Mathematics Teaching* instrument (Campbell et al., 2011; Clark et al., 2011). According to Campbell et al. (2011), this instrument measures teacher beliefs related to mathematics instruction and the role of students and teachers in the mathematics classroom. The instrument (found in Appendix A) aided in investigating factors that arbitrated the teachers' uses of the EPLT throughout the teaching cycle.

As teachers engaged in iterations of the teaching cycle, data collection paralleled these cycles. Each data cycle began with a grade level lesson planning meeting, followed by individual classroom observations, and concluded with individual post-lesson interviews. Grade level planning meetings were video and audio recorded. Field notes were recorded during the grade level planning meetings. Prior to each lesson, four of the five participants completed a pre-lesson questionnaire (see Appendix B) to provide information about the teachers' learning goals and any adaptations they may have made to the lesson plan. The pre-lesson questionnaire consisted of open ended questions in order to acquire specific information from each participant and was collected prior to the lesson observation either through email or as a hard copy.

Observations took place in each participant's classroom during the regularly scheduled math instructional time and were video recorded. Observations were used to seek evidence of the teacher's use of the EPLT as they monitored their students' progress on tasks, and as they structured whole class discussions. They were video recorded using a blue tooth microphone and video camera in order to capture dialogue between the teacher and students during whole group and small group classroom interactions. Field notes were recorded during classroom observations and samples of student work were collected when available. In order to manage classroom observations, teachers were asked to stagger their lessons so that each teacher's lesson was taught on a different day. An observation protocol was used during each classroom observation (see Appendix C).

Following each lesson, a semi-structured interview was conducted with the participant to discuss the teacher's perceptions of what learning took place as well as evidence of that learning, and how the teacher used that evidence to inform future learning goals. The post-lesson interviews took place within one to three days after a lesson, and on two occasions took place immediately following the lesson. Interviews were audio recorded and the protocol can be found in Appendix D. These interviews were semi-structured in order to collect similar data from each participant but also allowed for the researcher to respond to the individual participants' views that emerged through data collection (Merriam, 1998). Table 1 summarizes the research questions and relevant sources of data.

Table 1

Research Questions and Data Sources

Research Question	**Sources of Data**
1. In what ways do teachers use the EPLT to plan a set of lessons on equipartitioning? a) Choosing tasks and selecting learning goals b) Anticipating student responses	• Transcripts and field notes from grade level lesson planning meetings • Pre-lesson questionnaires • Transcripts of post-lesson interviews
2. In what ways do teachers use the EPLT when implementing classroom instruction? a) Monitoring students' work b) During whole group discussions	• Videos and field notes from classroom observations • Transcripts of post-lesson interviews • Student work
3. In what ways do teachers use the EPLT to assess their students' understanding? a) Determine what counts as evidence of student learning b) Impact of instruction on student learning	• Transcripts of post-lesson interviews • Transcripts and field notes from grade level lesson planning meetings
4. What factors mediate the ways in which teachers use the EPLT to engage in cycles of planning, implementation, and assessment?	• Transcripts and field notes from grade level lesson planning meetings • Videos and field notes from classroom observations • Transcripts of post-lesson interviews • Teacher profile documents

Data analysis. For this study, I used the qualitative data analysis software, Atlas.ti (2012) to organize and analyze data. Merriam (1998) stated that coding takes place at two levels- first to identify information about the data and then to interpret concepts that emerge from analysis. For this study, the first level of coding used *a priori* codes from the conceptual framework as well as open coding using a constant comparative method (Strauss & Corbin, 1998). The use of pre-identified codes provided structure from the conceptual framework to help guide data analysis, while open coding allowed for the emergence of unanticipated ideas. As new codes were identified, I went back to the data to check for further instances of these emerging ideas. Examples of these procedures are provided below.

The grade level planning meetings and individual teacher interviews were transcribed verbatim by a paid transcriptionist. Video recorded lessons were viewed to identify critical moments. According to Powel, Francisco, and Maher (2003), a method for video analysis follows seven nonlinear, interactive phases. The phases are as follows: viewing the video attentively, describing the video data, identifying critical events, transcribing, coding, constructing a storyline, and composing a narrative. This model was adapted by the researcher to analyze the video recorded lessons to identify critical moments in which an opportunity existed to draw upon specifics of the learning trajectory or elicit evidence of student learning. The way in which critical moments were identified is described in detail in a subsequent section.

Evidence was collected from grade level meetings, teacher interviews, classroom observations, pre-lesson questionnaires and lesson planning documents. Evidence from the lesson planning meetings and pre-lesson questionnaires were used to examine the ways in which teachers use the EPLT to select learning goals and anticipate students' responses.

Classroom observations and post-lesson interviews were analyzed for the ways in which teachers use the EPLT to elicit evidence of student thinking as students engaged with the task and during whole group discussions. Evidence from post-lesson interviews and lesson planning meetings were considered to determine the ways in which teachers use the EPLT to reflect on the impact of instruction on student learning, evaluate evidence of student learning, and how they used the EPLT to inform future instruction. Analyses of the first three research questions were used to provide information about the factors that mediated and moderated the teachers' uses of the EPLT throughout the teaching cycle. Informal data analysis in the form of memos and a research journal was kept throughout data collection to inform future data collection. Next, I describe the coding process in detail.

Coding of lesson planning meetings. First, data were organized into what Patton (1990) referred to as a case record; all data for one participant was gathered for analysis. Since all five participants participated in the planning meetings, transcripts from these meetings were analyzed separately. That is, for each case, the participation of the particular teacher was analyzed with the other teachers serving as the context for that case. Analysis began by coding the transcripts of the grade level planning meetings using codes identified *a priori* based on the conceptual framework for the study. Table 2 lists the initial codes along with a definition and example for each code. The codes and definitions for task and learning goal, anticipating, monitoring, selecting and sequencing, and connecting were adapted from codebooks developed by the LTBI research team (Wilson, Sztajn, Edgington, & DeCuir-Gunby, 2012).

Table 2

Initial Codes, Definitions, and Examples

Code	Definition	Example
Task and learning goal	Teachers' description of, reasoning for the use of, and/or directions given for (a) specific task(s) used during the lesson including the mathematical goals	*Students will fairly share a collection of 36 objects among 2 people; students will practice naming part of a collection.*
Anticipating	Teachers' purposeful expectations for how students will engage with the task, including the approach they might take, the misconceptions they might bring, and the relationships to previous learning.	*I think one difficulty might be naming the shares…I know this is higher up on the trajectory and my students haven't had many experiences naming.*
Monitoring	Teacher's interactions with individuals or small groups of students before whole group discussion that help students make progress on the task, or make students' thinking visible to the teacher.	*Teacher observes students share counters and questions them about their strategy and how they named one share.*
Eliciting evidence of student thinking	Actions by the teacher to listen to students' mathematical thinking around the proposed task. This includes questioning and making connections to individuals, small groups, and whole class discussions.	*T: So how many did you have?* *S1: 16* *T: How many did you have?* *S2: 16 and 20* *T: So what are you thinking?* *S2: Not a share fair.* *T: Not a share fair? Why not?* *S2: Because I have more than him and he has less than me.*
Selecting and sequencing	Teachers' identification and ordering of particular approaches generated by students (or introduced by the teacher) for whole group discussion.	*I will first select a student who has not shared "halves" of one rectangle fairly to remind students that you need to have equal sizes. Then I'd like to show correct half of the small rectangle and then compare it to half of the large rectangle.*
Connecting	During whole group discussion, when the teacher provides opportunity for students to make connections between various mathematical ideas or strategies.	*T: Would we call this one half?* *S1: One out of 8.* *T: One out of 8? Do you guys agree? Isn't that what Emma said she was going to get?* *(teacher explicitly connects one student's idea to another)*

Table 2 Continued

Evidence of student learning	Statements teachers make about what they think their students learned related to the intended learning goal(s) *and* evidence they use to back up their claims.	*They were sharing for each other and they cut it in half and then they cut it in half again and they each got two and she said, "we got quadruples." And I said, "Well, did you get quadruples?" She said, "No, we have two of the quadruples." And she was the first person to use language "something of a unit".*
Instruction that helps student learning	Teachers' hypotheses about connections between instruction and promoting student learning, including particular moments in a lesson as well as general student learning connected to the whole lesson.	*I'm glad that we started at the carpet and that we kind of tied it in with some real world. We didn't use the wrapping paper today, but I'm going to let them wrap a little present. You know, sort of trying to make it real world. But then we tied it back in at the end and we made a generalization.*
Instruction that hinders student learning	Teachers' hypotheses about connections between instruction and obstructing student learning, including particular moments in a lesson as well as general student learning connected to the whole lesson.	*I know my kids got hung up on the language of the task. They just had a hard time processing what it meant to have to re-share with two more people.*

In addition to the pre-determined codes, I used open coding to capture emerging ideas not included in the initial codes. For example, during the first planning meeting, issues of when to teach equipartitioning concepts surfaced in the teachers' discussions. I coded this as "curricular connections" and began to look for further instances when the teachers discussed the EPLT in relation to other concepts in their curriculum, or difficulties they perceived in fitting equipartitioning in with their existing curriculum. Codes that resulted from the open coding can be found in Table 3. When new codes were identified, I returned to previously coded data to look for further instances of the new code. Codes were applied to chunks of the transcripts from the lesson planning meetings according to idea units that consisted of dialogue about one particular idea. After the initial coding of the lesson planning meetings, I coded each individual case that consisted of the teacher's pre-lesson questionnaires, observations, and interviews in chronological order.

Table 3

Codes Resulting from the Open Coding Process

Code	Definition	Example
Connect to curriculum	Comments teachers make about connections between equipartitioning and their curriculum, pacing guides, or textbooks	*It would have been nice if this was around a different time in our curriculum because I kind of feel like I did that kind of assessment, but then it got dropped in favor of other stuff that we had to do. And um, I think one thing I don't feel good about is not having it been on the tail of a unit that was a little more closely aligned.*
Formative assessment	Comments teachers make about how to structure a lesson in order to assess student learning, or the purpose of a lesson as assessment versus teaching	*And I think also because we're trying to do a lot with one thing, we're trying to gather, you know maximize all of the data we can collect from [the lesson], which you know, maybe we don't need to do that.*
Listening to students	When teachers talk about listening to their students thinking, or instances when teachers recognize how they paid attention to their students' thinking and used that to guide their instruction	*Because in my mind, that was like a whole other lesson. And I was still kind of committed to, "okay, and I want to talk to you about what happens when we go from two to four, what do you notice?" Like, "are we getting more or are we getting less?" But then, to me it just seemed too important for them – for the naming piece. I kind of realized, "no this is really important"*
Obstacles	Comments teachers make about obstacles that make it difficult to engage in teaching and learning of equipartitioning	*I think what stood out for me is still just the – the way that it's really hard to manage kids being at different paces because I had so many that finished the first part, but I did want to stop and finish the discussion before we moved on to the second part and so that was a little tricky.*
Instruction	When teachers make references to instruction, how to group students, or teaching versus assessment	*I would think the first one just see if they can name but not use it to teach naming. And just do the fair sharing and then the....I think that's too much. I mean I think some of them are going to be able to do it, but...we can always say it but not specifically teach it that time.*

Coding of individual cases. I began coding each case by reading the pre-lesson questionnaire, watching the classroom video and reading the post-lesson interview transcript in their entirety for the first of the three lessons. Then, I coded the pre-lesson questionnaire using the initial codes previously described and looked for emerging ideas through open coding. Next, I viewed the classroom video to identify critical moments. Using Atlas.ti, segments of the video were chunked and categorized as “monitoring” or “whole group discussion”. These comprised the critical moments. Monitoring units were defined as moments after the launch of the lesson when the teacher interacted with individuals or small groups of students before whole group discussion that helped students make progress on the task, or made students' thinking visible to the teacher (Wilson et al., 2012). As the teacher moved to monitor a new student or group of students, a new monitoring unit was created.

Whole group discussion units were identified as portions of the whole group discussion part of the lesson when the teacher or student shared a strategy or approach to the task, or engaged one another in discussions about: (a) a student’s mathematical idea; (b) explicit connections of one mathematical idea or approach to another; or (c) advancing the learning goals of the lesson (Wilson et al., 2012). As the discussion changed to a different student’s strategy or explanation, or to a new mathematical idea, a new whole group discussion unit was created. Critical moments were then transcribed to include all dialogue between the teacher and students as well as a description of any actions that were visible on the video recordings.

Once critical moments were identified and transcribed, I applied open coding and coded using the pre-determined codes. Then, I coded the post-lesson interview transcript in the same way the lesson planning meetings were coded. When new codes emerged, I went back through previously coded data to check for further evidence of the new codes. This process was repeated for the second and third lessons, then for each of the remaining cases.

I created memos throughout the coding process to develop emerging themes and categories. For example, I created a memo after one of the teachers' first lessons to describe how she structured the whole group discussion and de-emphasized the process students' engaged in to create fair shares as she concluded her lesson. I returned to this memo after her second and third lessons to note changes the teacher made in how she had students' share their strategies and what she focused on during the discussions. In the final analysis, this helped me articulate changes the teacher made over the course of the three lessons as well as the role of focusing on the process as an emerging theme.

With-in and cross-case analysis. Once all data were coded, I completed a with-in case analysis by answering each research question using detailed descriptions for each case and its context. Instances occurred where I used my knowledge of the teachers' experience with the professional development and the culture of their classrooms to make inferences about their uses of the trajectory. For example, the concept of "qualitative compensation" (level 6 in the EPLT; the idea that as the number of sharers changes, the size of the shares change inversely) was addressed during the professional development. The teachers recognized the value of asking students "if more (or less) people share, what happens to the size of the share?" and incorporated the question into tasks they developed to use with their students. Therefore, when this idea came up in participants' lessons or in conversations during interviews or planning meetings, I inferred that teachers' were directly drawing upon the EPLT.

Once the with-in case analyses were completed, a cross-case comparison (Merriam, 1998; Miles & Huberman, 1994) was conducted to facilitate clarification about the teachers' uses of the EPLT. I began to look across the cases to create categorical aggregation and establish patterns. For example, four of the cases provided evidence of coordination between proficiency levels and task parameters of the EPLT to calibrate tasks to fit the needs of their students and the mathematical goals. Because the teachers varied in how they used the EPLT in this way, I used this idea, along with others, to categorically describe the ways teachers use LTs to choose tasks and specify learning goals. The final product emphasized the cross-case comparison with the goal of proposing a framework for teachers' uses of LTs to focus on student thinking in their mathematics instruction.

A constant-comparative method was used in order to confirm and build more refined categories (Merriam, 1998; Strauss & Corbin, 1998). For example, the idea of "moving beyond the goals of the lesson" first surfaced in two teachers' lessons. I went back to the data to search for further evidence and to unpack the issues teachers faced as they considered new mathematical ideas that emerged that were not directly related to the goals of the lesson. Similar ideas were found during the planning phase and as teachers monitored their students' work on tasks. Thus, the ability to use the LT to anticipate, recognize, and address important mathematical ideas beyond the goals of a lesson became a central theme and an identifying characteristic of the resulting framework. The cross-case analysis was organized by research question. Within each research question, data from across the cases were organized by themes that emerged from the analysis.

Validity and Reliability

In order to learn from research conducted in the field of education, it is important that studies be rigorous and valid (Merriam, 1998). While goals of research in the physical sciences may include generalizations and the creation of laws, research in social sciences such as education are often more action-oriented and explanatory in order to impact practice in meaningful ways. Regardless of the intentions, validity and reliability "are concerns that can be approached through careful attention to a study's conceptualization and the way in which data were collected, analyzed, and interpreted, and the way in which the findings are presented" (Merriam, 1998, p. 199-200).

Validity. Often referred to as trustworthiness or credibility of a study, validity deals with how well research findings match reality (Creswell, 2007). Merriam (1998) suggested strategies to enhance the validity of a study that allow for stronger congruence between the participants' construction of reality and the researcher's interpretation of this reality. These strategies include 1) triangulation of data (using multiple sources of data), 2) member checks, 3) long-term observations, 4) peer examination, 5) collaboration with participants, and 6) disclosing the researcher's bias. This study employed triangulation, member checking, long-term observations, peer examination, and disclosure through a subjectivity statement in order to increase validity.

Various forms of data were collected in the context of the phenomenon under study in order to triangulate the participants' perceptions and interpretations of the use of a learning trajectory throughout the teaching cycle. As such, the end product includes quotes from the teachers' pre-lesson questionnaires, planning meeting transcripts, interview transcripts, and portions of classroom transcripts to help describe and understand the teachers' uses of the EPLT. Moreover, because multiple teaching cycles were observed over the course of a semester, repeated observations of the same phenomenon were conducted, increasing validity of the findings (Merriam, 1998).

As each with-in case analysis was completed, I shared with participants descriptions of their uses of the LT during planning, instruction, and assessment with excerpts from interviews and classroom observations as supporting data. I met with each participant as a form of member checking to solicit their views of the credibility of my interpretations. At the same time, I met bi-weekly with my doctoral committee chairperson for "peer debriefing sessions" (Creswell, 2007, p. 209) to discuss questions about the research process, data analysis, and interpretations. Once the cross-case analysis was completed, I met with a peer doctoral student who reviewed the findings as an external check of the accuracy of my interpretations.

Reliability. Reliability is addressed in the sense that the research is dependable and consistent, as opposed to the traditional meaning of reliability as the ability to replicate findings (Lincoln & Guba, 1985; Merriam, 1998). Merriam (1998) pointed out that because human behavior is seldom static, qualitative research is less concerned with the ability to replicate findings and more concerned that the results make sense given the data presented. She suggested strategies such as triangulation and audit trails that allow the reader to authenticate the results of a given study. For the current study, details of data collection and analysis were disclosed. Triangulation of data was used to strengthen the consistency of results compared to the data collected. Memos were used to create an audit trail in order to authenticate the research methods and findings (Merriam, 1998).

Subjectivity Statement

Perspectives. As a former high school mathematics teacher, I began my graduate work as a way to update and expand my knowledge of teaching. My coursework exposed me to organizations such as the National Council of Teachers of Mathematics and the view of teaching and learning mathematics as shifting away from focusing solely on computational accuracy and rote memorization to a focus on developing a deeper understanding of important mathematical ideas. Through research assistantships, I also gained experience in working with teachers at the elementary level and became aware of the issues and challenges they face as they teach children mathematics. Although I have limited experience teaching at the elementary level, I believe this separation allows me to closely examine issues related to the teaching and learning of early mathematical concepts.

As a mathematics educator, I hold a reform-oriented approach to teaching mathematics. Mathematics is not a subject that can be “dropped” into the minds of children, but one that must begin with children's prior knowledge and informal experiences. With the guidance of a knowledgeable other and experiences that allow them to explore connections and various representations in social settings, children can refine their informal understandings to build increasingly sophisticated conceptions. Furthermore, I do not believe it is productive to think in terms of what children *cannot* do, but instruction should instead focus on what children *can* do and build from there. The same philosophy applies to teachers. Work with teachers should focus on supporting them in making sense of their students' understandings and in finding ways to provide all of their students' opportunities to engage in mathematics in meaningful ways.

While it is true that many teachers choose to teach at the elementary level because they believe their knowledge of mathematics is weak, a deep understanding of the concepts taught at the elementary level is still necessary. Although their content knowledge may be lacking, most teachers have a strong understanding of the mathematics their students can do and how they might approach various mathematical tasks. That is to say that, with experience, teachers know that children more readily learn how to count by fives and tens, but they struggle with multi-digit subtraction, for example. Along with enrichment in content knowledge, elementary teachers also need support in making sense of the strategies and conceptions they observe in their students on a daily basis. Therefore, my emphasis in this study was not to examine teachers' mathematical knowledge per se, but instead to use children's mathematics to engage the participants in thinking about what their students know and how to move them to more sophisticated conceptions over time.

The role of the researcher. Qualitative research is characterized by the study of a social phenomenon in the natural setting from the participants' perspective. Thus, it is important to consider the relationship between the researcher and the participants. Patton (1990) noted that "the challenge is to combine participation and observation so as to become capable of understanding the program as an insider while describing the program for outsiders" (p. 207). Researchers must balance a certain amount of participation to gain the confidence of the participants and to experience the phenomenon first hand, but at the same time stay suitably removed in order to observe and analyze.

At the onset of this study, I had worked with the participants through the LTBI project for one year. One may argue that because my teaching experience has been at the high school level, the participants viewed me as an authority on mathematical knowledge, causing feelings of inadequacy or that my role was to judge teachers' mathematical knowledge and teaching. However, I argue that because my experiences at the elementary level were limited, my approach to this study was that there was much for me to learn from these teachers about their expertise in the classroom as well as how they came to understand and use the construct of a learning trajectory. A goal of the project was to communicate this with the participants in order to engage the teachers in honest and thoughtful discussions surrounding their practice. My intention was to reduce the teachers' view of my presence as evaluative and more as seeking understanding. In this way, I hoped to see a more accurate picture of the participants' classroom practices over time with respect to the EPLT.

Additionally, a goal of the project was not to identify what teachers do not know or do not do, but to build on their existing knowledge and practice to examine the role of the EPLT as they engaged in cycles of lesson planning, instruction, and assessment. Although I do not have the same teaching experience as the participants, my knowledge of the learning trajectory allowed me to be a resource for the participants as they engaged in planning and instruction using the EPLT. During the grade level planning meetings, my role was mainly as an observer, but I also served as a resource or knowledgeable other about the EPLT when requested and when I considered my input would be helpful. The teachers often looked to me to clarify language from the trajectory or to explain complex ideas, such as factor based change. After the second and third planning meetings, but prior to their instruction, I shared with the teachers documents that described two of the proficiency levels: Naming, and Transitivity. These documents consisted of detailed descriptions of the mathematical understanding characterized by the level and examples of student thinking related to the level (Wilson, Edgington, & Confrey, 2010). I did not tell teachers what I thought they should teach or instructional tasks to use, but I inquired into decisions they made to help them clarify their thinking and justify their choices.

During classroom observations, my role was to observe teachers' instruction, not to co-teach or coach during instruction. During the post-lesson interviews, I used the interview process to encourage the teachers to reflect on their instruction, to clarify their thinking regarding the instructional decisions they made, and to evaluate evidence of student learning. I also engaged in dialogue with the teachers as they consider evidence of their students learning to choose future learning goals. My goal was to build rapport with the teachers in order to hold open discussions about not only successful teaching moments but also difficulties they experienced in their instruction.

Merriam (1998) emphasized the need to consider how the observer/researcher affects what is being observed. The nature of interactions between researcher and participants in qualitative research may bring about changes in both parties. The issue is not that changes occur, but that the researcher identify and account for such changes. In the current study, teachers' uses of the EPLT did change over time, partly attributed to my role in the lesson planning meetings and through the interview process. Asking questions specific to how teachers were monitoring or how they chose students' solutions to share brought certain issues to the teachers' attention that they may not have considered before. Additionally, over time teachers and students became more comfortable with my presence in the classroom, allowing for a more natural learning environment.

Ethical Issues

Consent was obtained from the North Carolina State University Institutional Review Board (IRB) to conduct this study. Each participant signed a legal consent form before they participate in this study. Parents or guardians of students from each participating teacher were asked to sign legal consent forms as well. Students who opted out of the study were seated outside of view of the video camera during observed lessons. There were no risks anticipated for participation in this study and all participants and their students were given pseudonyms for presentations of the written or audio data. Participants also received a stipend for their involvement in the study. All data were stored on the researcher's personal computer. Written documents were securely stored in the researcher's office in Poe Hall at North Carolina State University. Back up files were kept on an external hard drive and securely stored.

Chapter Summary

This chapter detailed the context and design of the study. This is a multi-case study consisting of five teachers who participated in the LTBI professional development. Data were collected and analyzed using qualitative research methods in order to answer the research questions. Chapter Four follows with the findings and analysis across the cases.

CHAPTER FOUR

In this chapter, the cross-case analysis is presented to answer each of the six refined research questions, and the final question related to issues that mediate/moderate teachers' uses of LTs. I begin by providing a description of each of the grade level lesson planning meetings in order to describe the instructional tasks the teachers chose to use and to provide context for how each teacher implemented the tasks. These descriptions illuminate the nature of the tasks, issues the teachers discussed as a group, and provide a baseline for which to compare how the teachers' modified the lessons to fit their individual classrooms. Next, I present profiles of each individual case to characterize each teacher's instruction and the role the teachers played in the planning meetings. These profiles provide a brief overview of each case teacher's instruction and offer an understanding of each teacher's perspective about the LT.

After the planning meeting descriptions and teacher profiles, each research question will be addressed. Within each question, findings are organized by themes that emerged from data analysis. When the emphasis is on the cross-case analysis, Yin (2003) described a written product applied to multi-case studies:

> In this situation, there may be *no* separate chapters or sections devoted to the individual cases. Rather, your entire report may consist of the cross-case analysis, whether purely descriptive or also covering explanatory topics. In such a report, each chapter or section would be devoted to a separate cross-case issue, and the

information from the individual cases would be dispersed throughout each chapter or section. (p. 148, emphasis in original)

Analysis began with the development of each case description and the examination of categories by case. To answer the research questions, I looked across cases. Because the emphasis is on the cross-case analysis, cases are not presented individually, but instead information from the cases is distributed throughout each section.

Lesson Planning Meetings and Instructional Tasks

Second grade teachers who participated in this research met regularly to discuss their mathematics instruction; these meetings took place throughout the year. Therefore, when it came to planning for the three EPLT lessons, details of the lesson planning took place in similar grade level planning meetings scheduled just for the lessons they would use for the research. Although not part of the individual cases, the planning meetings are described in what follows to provide context for the teacher's individual instruction. Each individual teacher's role in the lesson planning meetings is characterized in the teacher profiles in the following sections.

Lesson planning meeting #1. I met with the teachers for the first planning meeting prior to the start of the school year. The teachers agreed during this meeting that it made sense to start instruction on equipartitioning by using a task related to sharing a collection, since it is low on the trajectory. They discussed using the task to address the three equipartitioning criteria (i.e., the correct number of shares, shares of the correct size, and using the entire whole/collection), anticipating that they would start the school year with students who had a variety of experiences and held different conceptions related to equipartitioning.

The teachers decided to use a task they used the previous year during the LTBI professional development that engaged students in sharing 24 counters among two, four, and three friends. The task asked students to explain how they shared the counters, what they would name each person's share, and to predict what would happen to the size of the share when more or less friends shared the counters. Students recorded their work on the worksheet (see Appendix E). Some of the teachers discussed how the lesson could be used as an opportunity to strengthen their students' number sense and to teach socio-mathematical norms of how to record and explain mathematical thinking. It was important for the teachers that the task addressed multiple levels of the trajectory so they could get a sense of what conceptions their students held at the beginning of the school year, even though they may not explicitly attend to each level during the lesson. They did not talk about specific learning goals they had for the lesson, however, some of the teachers mentioned using the lesson to address naming and qualitative compensation, in addition to addressing the three equipartitioning criteria.

Using the EPLT to consider connections between equipartitioning collections and the second grade curriculum came up during this first meeting. Part of their curriculum during the first half of the school year was to develop students' facility with doubles facts as a strategy for addition and subtraction. A few of the teachers noticed and brought up the idea that students could potentially use doubles facts as a strategy to share collections between two people and as a way to generally strengthen students' number sense.

Lesson planning meeting #2. The second lesson planning meeting took place after school once all the teachers had taught the first equipartitioning lesson. The teachers spent the first half of this planning meeting sharing their observations from the first lesson. All of the teachers thought their students were successful on the task and agreed that in general, their students struggled to consider mathematical names for the shares they created. The teachers discussed various routines they developed for their classes that incorporated equipartitioning concepts, including daily story problems that used equipartitioning contexts as well as estimation activities that utilized the idea of sharing a collection in half.

As the teachers considered possible follow up activities for the second lesson, they struggled to articulate where on the trajectory they wanted to focus. Because they considered that equipartitioning was only explicitly addressed in their curriculum in relation to naming fractional parts, some of the teachers found it difficult to conceptualize how equipartitioning concepts related to the curriculum they were teaching at that time. Others suggested that since they were currently working on developing doubles facts, which was an idea that came up in everyone's classroom during the first equipartitioning lesson, perhaps this idea could be further developed in conjunction with equipartitioning concepts.

Teachers considered using a similar activity to the first lesson but with a different sized collection, possibly focusing on qualitative compensation or reallocation (levels 8 and 9 respectively in the trajectory). Since they were unable to decide on a task by the end of the two-hour planning meeting, they agreed to continue the discussion during their regularly scheduled weekly mathematics planning time. During this second meeting, which I attended, they chose to focus on naming, but to keep the task parameter low: sharing a whole and collection for two. The lesson they developed (see Appendix F) began by having students explore how to share two different sized rectangles for two and how each share can be named "half", but of a different sized rectangle. Then, students explored sharing small collections of six, eight, and ten counters that were arranged in arrays for two friends and named the resulting share, as shown on the student worksheet. Through discussion, the goal was for students to understand that each share can be named as "half" but of different sized collections.

Lesson planning meeting #3. Similarly to the second lesson planning meeting, the teachers spent the first part of this meeting reflecting and sharing observations of their instruction from the second equipartitioning lesson. Overall, the teachers believed their students were beginning to understand the concept of "half" as one of two equal shares, and as a way to name the resulting share when equipartitioning for two. When they moved to discuss what they might teach for their third lesson, several ideas surfaced.

First, one of the teachers suggested a possible follow up lesson would be to do a similar activity as the second lesson with different numbers of counters and without using the array structure in order to further develop the idea of naming in relation to the whole collection. Another teacher also suggested doing a similar activity but to move beyond sharing for two to include sharing for four and three. They discussed how to potentially connect this lesson with other topics they were teaching at the time such as congruent shapes and area. Because they were moving into geometry topics, they decided a lesson that focused on sharing wholes would be more in line with their curriculum. They decided to use a task from the LTBI professional development that they called “the wrapping paper task” (see Appendix G). The task used the context of fairly sharing holiday wrapping paper and could be adapted to address a number of proficiency levels and task parameters. Later, on their own, the teachers modified the task in different ways and created worksheets for their students to record their work (see Appendices H, I and J).

Summary. In general, the teachers used the lesson planning meetings to choose instructional tasks. Although they mentioned specific goals at times, they did not use the meeting to discuss in detail or come to consensus on particular learning goals they wanted to address for each of the lessons. At the time of the study, it was unclear whether this was a normal practice of their common planning time. Table 4 presents a summary of the three tasks the teachers chose to use and analysis of potential proficiency levels and task parameters that the tasks could address depending on implementation. The ways in which each teacher adapted the tasks for her particular classroom will be discussed in the profiles of each case.

Table 4

Tasks Descriptions with Potential Proficiency Levels and Task Parameters(see Figure 3)

Task Description	Proficiency Levels	Task Parameters
Lesson #1: Fairly share a collection of 24 counters for 2, 4, and then 3 friends. Explain, and name the resulting shares.	Sharing Collections Justification Naming Reassembly Qualitative Compensation Factor Based Change Reallocation	Sharing collections for • 2-splits • 2^n-splits • odd splits
Lesson #2: Fairly share a rectangle and small collections for 2, name the resulting share.	Sharing Collections Sharing Wholes Justification Naming Reassembly Qualitative Compensation	Sharing wholes (rectangles) for • 2 splits Sharing collections for • 2-splits
Lesson #3: Fairly share a rectangular piece of wrapping paper for different numbers of equal-sized gifts.	Sharing Wholes Justification Naming Reassembly Qualitative Compensation Composition of Splits with Multiple Methods Transitivity	Sharing wholes (rectangles) and collections for • 2-splits • 2^n-splits • even splits • odd splits

Individual Teacher Profiles

In this section, I present profiles of each individual case. I originally created the profiles based on teacher reflections, one classroom observation and assessment data collected in the larger LTBI project at the end of the professional development. Then, I augmented the data set with a beliefs instrument I gave to case teachers, planning meeting notes and transcripts, classroom observations, and interview transcripts. Table 5 summarizes the mathematical knowledge for teaching post-assessment test scores from the LTBI project.

Table 5

Summary of Mathematical Knowledge for Teaching Post-test Scores from the LTBI Project

Assessment	Mean (n=22)	St. dev	**Lara**	**Ellen**	**Bianca**	**Tracy**	**Emma**
DELTA-T	32	7.24	31	34	37	40	37
LMT (scaled score)	.19	.92	.-.51	-.65	.63	.47	.63

I used these profiles to first characterize each teacher's mathematics instruction. In addition, based on the classroom observations from this study, I provide in the profiles a concise description of what took place in each teacher's classroom for each of the three lessons. Finally, the profiles also include a synopsis of the teacher's role in the grade level planning meetings.

The profiles are presented in the following order: Lara, Ellen, Bianca, Tracy, and Emma. I chose to order the presentation from teachers whose use of the trajectory was less developed, to teachers whose use of the trajectory was more sophisticated so as to highlight differences between the teachers. The order of the last three teachers is arbitrary as these three cases were more similar in their uses of the LT than the first two.

Lara. At the time of the study, Lara, a white female, was in her eighth year of teaching, and taught second grade for six years. She earned an undergraduate degree in elementary education as a second career and during the time of the study, she was working towards certification to teach academically and intellectually gifted students. She scored below average compared to other teachers in the study on both post-assessments of mathematics knowledge for teaching, scoring a 31 on the DELTA-T and a -.51 scaled score on the LMT.

Lara viewed learning trajectories as "developmental stages" that students move through "as they gain skills in mathematics with increasing complexity." She valued listening to students' thinking and allowing opportunities for exploration and discovery, often without explicit mathematical goals in mind. In her reflection, she described a student who violated one of the three equipartitioning criteria as not having the correct answer, "but he would have understood the problem from another point of view." It is unclear if Lara saw this as problematic in the student's mathematical thinking or as an acceptable solution to the problem.

Mathematics instruction. Although Lara valued exploration and discovery as a way to understand what her students were thinking, she did not consider these as learning opportunities. She believed the best way to teach students mathematics was to model how to solve one kind of problem at a time and then provide practice until mastery is achieved. She thought her role as teacher was to teach content that was on the state mathematics assessments and to help students complete classroom activities. Her mathematics lessons often consisted of her modeling procedures for her students, providing practice, and then allowing her students to play "mathematics games" to develop number sense and place value concepts. Lara often used the expression “normalize anxiety” when she talked about her students and her mathematics instruction. It was common for her to ask what was easy or difficult about an activity as a way to “normalize” students’ feelings about what she perceived as difficult concepts for her students.

Lara’s participation in the lesson planning meetings. Lara played a secondary role in the lesson planning meetings, mainly listening to her colleagues’ ideas. Overall, she struggled with making connections between equipartitioning and the curriculum. One can characterize Lara’s understanding of the LT as tool for assessment rather than an instructional framework. For example, while some of the teachers used the first lesson to facilitate students’ understanding of doubles and to address number sense and organizational issues, Lara did not see these connections, as evidenced in this quote from the second lesson planning meeting:

> It would have been nice if this was around a different time in our curriculum because I kind of feel like I did that kind of assessment, but then it got dropped in favor of other stuff that we had to do. And um, I think one thing I don't feel good about is not having it been on the tail of a unit that was a little more closely aligned.

After the teachers discussed focusing the second lesson on naming when sharing for two, she stated, “I'm just kind of wondering, we're talking about a concept of half and of course that's related to fractions and stuff, geometry, and I'm wondering if, if some of this maybe be, better be couched closer to the fractions unit.” Even when her colleagues suggested tasks, she struggled to see connections to what she was currently teaching or that the lessons on equipartitioning would help strengthen her students' developing fraction concepts.

During the lesson planning meetings, Lara willingly shared her thoughts on her instruction and engaged in conversations about how her lessons went. However, when she discussed difficulties she thought her students had, she was often challenged by Tracy, as shown in this dialogue from the second planning meeting:

> Tracy: I'm going to back up a little...what was the challenge for them? I mean, you thought the number was too big?
>
> Lara: Oh, I think it was just some of the language and thinking through that [problem] number six, that was just...and going back to "Would the person's share be greater or fewer than it was before?"
>
> Tracy: So if you had just asked ok, so if we had 24, what were they? Counters, they were counters right?
>
> Others: Mm-hmm.
>
> Tracy: Counters, and three people were sharing them...
>
> Lara: Mm-hmm. Yeah, if this were reworded you could just like, yeah.
>
> Tracy: So it's not the task, it was just the language?
>
> Lara: I think that it's the language.
>
> Tracy: Ok.
>
> Lara: Because I had a lot of kids, it just seemed to be a bit of a maze, you know just the language seems like a maze for a lot of my kids. There's a group, I read it

> to them, they're just, "what?" They didn't understand it. It seemed convoluted to them.

With Tracy's prompting, Lara expressed that her students' struggles with the task were not related to the mathematics, but mostly due to the way the questions in the tasks were worded. Lara more often engaged in conversations about pedagogy and less often about the mathematics.

Lesson #1. For the first lesson, Lara began by modeling how to share six items between two people by dealing six markers to two students. Then, her students worked in pairs to complete the task. As students completed the worksheet, she walked around to help students read the questions and stay on task. After most students had completed the worksheet she held a whole group discussion on the carpet. Lara began by asking her students what was difficult about the task. When a student commented that going from sharing for four to sharing for three was difficult, Lara worked through this part of the task with her students, modeling reallocation with counters on the carpet. The students agreed that when sharing for four, each person should get six counters.

Lesson #2. Lara used the second task to introduce the term "half" as students explored how to fairly share two different sized rectangles for two. She first had students fold two different sized rectangles for two and they discussed how each share can be called "half". Then, students worked through the worksheet on sharing the various collections for two. When Lara found that many of her students were finishing the activity quickly, she gave them larger numbers to share (e.g. collections of 26 and 50 to share for two). In the whole group discussion, Lara had students share their answers for each problem, focused on naming the share as a count, but did not emphasize referring back to the whole ("each person's share is 3" as opposed to "each person's share is 3 out of 6") or naming each share as "half".

Lesson #3. In the third lesson, Lara's students equipartitioned a rectangle for four, eight, three, and six equal-sized gifts in multiple ways, named each share, and she asked students to consider the equivalence of non-congruent shares when sharing for six (see Appendix H). Students worked individually on the task. During the whole group discussion, she asked for volunteers to share their strategies. At one point, a student shared a solution that created six unequal pieces and did not use the entire rectangle. Although Lara knew this was not a correct method, she did not use the incorrect solution as a means to build students' understanding of the three equipartitioning criteria. She instead allowed her students to consider and accept it as a viable solution.

Ellen. Ellen, a white female, was in her 6^{th} year teaching, with all six years spent teaching second grade. She obtained her teaching certification after working for several years as a teacher's assistant at the same school where she taught. At the end of the LTBI professional development, she scored 34 out of 40 on the DELTA-T and scored below the mean with a scaled score of -.65 on the LMT. It was difficult to characterize Ellen's beliefs about mathematics instruction and learning trajectories for two reasons. First, Ellen chose not to complete the written reflection about her understandings of the EPLT and learning trajectories in general at the end of the LTBI professional development. Second, her responses on the beliefs instrument were often contradictory. For example, Ellen agreed with the statement that much of mathematics must be accepted as true and remembered, but disagreed with the statement that learning mathematics requires a good memory. In addition, she agreed with the statement that students learn best by paying attention to what their teacher demonstrates and then practicing, but disagreed with the statement that mathematics should be taught by first modeling, then providing practice.

Mathematics instruction. Ellen's mathematics instruction was typically structured around centers where students were grouped together by ability and moved through different activities that focused on the same mathematical idea. When Ellen did utilize whole group instruction, she often introduced the topic, modeled problems for her students, and then had students practice either independently or in small groups.

Ellen's participation in the lesson planning meetings. Like Lara, Ellen played a peripheral role in the lesson planning meetings. She saw her colleagues as more competent teachers and was therefore comfortable following their lead. There is no evidence that she disagreed with the decisions the other teachers made and she ultimately implemented the lessons as designed with few modifications. In the second and third planning meetings, Ellen engaged in conversation about her lessons, sharing difficulties as well as successes she had in her instruction.

Lesson #1. For the first lesson, Ellen chose to only use the first part of the task where her students worked in pairs to share the collection between two friends, explain their strategy, and name the resulting share. The questions Ellen posed to her students as they worked on the task focused on determining students' strategies for equipartitioning and how they knew they had in fact created fair shares. During the whole group discussion, Ellen asked for volunteers to share their strategies for sharing for two in no particular order. The class discussed what they named each share, naming the shares as "12", "equal", or "fair share", and Ellen continued to question her students until a student named the share as "half". Ellen concluded the lesson by talking with her students about what was difficult and easy about the task and summarized some of the dealing strategies her students used.

Lesson #2. In Ellen's lesson, students worked in pairs to complete the task. Students shared strategies for equipartitioning two different sized rectangles and agreed that each share could be called "half". Students then completed the rest of the activity to share the small collections for two and name the resulting share. Ellen asked for volunteers to share their strategies for sharing the collections, focusing more on how they shared the counters and less on how they named each share. Ellen used the whole group discussion to focus on naming the shares as "3 out of 6", "4 out of 8", and "5 out of 10" but did not connect back to naming the shares as "half" of each collection.

Lesson #3. For the third lesson, Ellen chose two, four and six as task parameters for sharing the rectangular wrapping paper (see Appendix I for the student worksheet). Students worked individually to equipartition for all three splits. Then, she had various students share their strategies in no particular order. Ellen used the whole group discussion to highlight different strategies for sharing a rectangle such as using all vertical or horizontal cuts and using diagonal cuts, and introduced naming the shares using fraction notation. In all of her lessons, it was important for Ellen to have as many students share their strategies as possible and each time she asked for volunteers in no particular order to present their ideas. The focus of her lessons remained lower on the trajectory, mainly emphasizing strategies for sharing or one particular way to name the shares (e.g. as a fraction, or as a ratio, but not both).

Often, opportunities arose during Ellen's lessons to engage in deeper mathematical discussions. However, she did not pursue them. For example, in the second lesson she did not pursue connections between naming shares as a ratio and a fraction. In the third lesson, she did not elaborate on the idea of equivalence of non-congruent shares and did not question when an incorrect split for six was shared by a student.

Bianca. At the time of the study, Bianca was in her fifth year teaching, with all five years spent teaching second grade. She is a Hispanic female with an undergraduate degree in elementary education. She believed children learn mathematics "through experiences they have both in and outside the classroom. They learn through social interactions with peers and adults." She saw learning as a fluid process where "students can 'move' in their understanding by increasing the complexity of a problem or the sophistication of the skill. Learning is a process. Children do not arrive in their learning, but are constantly growing." Bianca's mathematical content knowledge measured above average compared to other teachers in the LTBI project at the end of the professional development. On the post-test measures, she scored above the averages with a 37 out of a possible 40 on the DELTA-T and .62 on the LMT for rational number reasoning.

The LTBI professional development experience learning about the EPLT was powerful for Bianca. In her reflection, she commented, "This group has helped me feel more confident in the release of power to my students as they work through their math problems. For example, through our work I have experienced firsthand watching kids struggle through a task but come out successful." As Bianca learned more about the strategies and misconceptions highlighted in the LT, she was able to recognize these in her students and make connections to the conceptions her students held regarding equipartitioning. This knowledge gave her confidence that her students' learning would progress by providing them with high-demand tasks and instruction that utilized appropriate scaffolding and meaningful mathematical discussions.

For Bianca, the idea of learning trajectories resonated with her experiences in working with children. She stated, "This idea that students are on a continuum of learning where there are key stopping points and also major misconceptions is how I've begun to look at my students across the spectrum of learning, not just with equipartitioning."

Mathematics instruction. Mathematics instruction in Bianca's classroom could be characterized as student-centered. She believed that mathematics instruction should focus on non-routine problems that can be solved in a variety of ways. She considered the teacher's role was to question students' thinking, allow students with the opportunity to struggle with mathematics, and to highlight multiple approaches to solving problems through discussions. Bianca expected her students to communicate with each other about mathematical ideas and believed that her students could solve novel problems without explicitly being told what to do.

It was common for Bianca to structure her mathematics time using centers where students were grouped based on pre-assessments and moved through different activities that focused on the same mathematical concept. Bianca often had a parent volunteer work with a small group of students while she worked with another group, and other students worked independently at the remaining centers. Through our work together, Bianca was reminded of the value of engaging students in whole group discussions. At the final grade level planning meeting she stated,

> I will say that doing this lesson was kind of a smack in the face to me about how I've been structuring my math instruction…I do like rotations so I can work with those kids in small groups and get through a lot of them and give them [help], you know? And I don't do a lot of large group talking and sharing. Even in the beginning in my, you know in my warm ups. I was having kids come up to the board and we were doing "today's number" but I just noticed that I wasn't giving kids enough time to share their own strategies and explanations. And I realized after this lesson because I was like, that was, they didn't do a very good job I felt like of listening to one another…I haven't done a good job of facilitating that as the medium for how you really learn mathematics. It's been a lot of "you do the center and you do a paper at your desk, then you work with [me]. Then you rotate around…And I'm like, you know, really having kids validify what they're saying with examples and an argument

> of some sort. So this activity definitely made me think about, I don't think large group instruction is always the worst thing. Because sometimes I think, "Oh, I need to differentiate because my kids are so here." But, I felt like that lesson, while I don't think behavior management wise, it was the greatest thing you'd ever watch in the world, it was a really rich lesson and a lot of great content. And it wasn't awful. So it really made me pause and think about how I want to start structuring math some more.

Bianca's participation in the lesson planning meetings. Bianca played a prominent role in the lesson planning meetings, often initiating conversations or offering suggestions related to the tasks that they chose. In the first planning meeting, Bianca suggested starting with collections and was the first to bring up the idea of addressing the three equipartitioning criteria with her students in the first lesson. She was interested in knowing if her students knew the three equipartitioning criteria, stating "I would think you would start with, 'okay, do they have the three criteria?' kind of thing with collections. I would think that would be a good place to start. But I don't...I imagine it's not going to be the first time they're exposed to something like that." She recognized that although they may have had instruction related to equipartitioning concepts, starting lower on the trajectory would give her a good indication of the concepts her students had already mastered and where she might eventually target her instruction.

When the group had difficulty articulating a task for the second lesson, Bianca proposed the idea of addressing naming, but keeping the task parameter lower so the focus would be on naming, and not on the strategies students used to equipartition. Moreover, Bianca suggested addressing naming with a whole prior to addressing naming with a collection, as evidenced in her statements from the second planning meeting:

> I know this goes against what the trajectory is and I know it's obviously really well researched. But with naming, for some reason I feel like naming a collection is more difficult than naming a whole…even by second grade I feel like they've heard the word "half". But if I said you know, "I have six, you have six." You know, or "We had twelve, I have six, you have six." I wouldn't think to call that half, I would think to call it "I have six." And so being able to name in relation to the whole with a collection is probably, that would be harder for them...But I guess my question is, what if we went the opposite sequence?

She utilized her knowledge of the trajectory along with her experiences teaching second grade to design a lesson that could potentially scaffold students' ability to flexibly name the resulting share of equipartitioning a collection.

In the third planning meeting, Bianca, along with the other teachers, brainstormed potential tasks for the third lesson. She agreed with the other teachers that addressing equipartitioning rectangles in connection to their unit on geometry made sense. In all of the meetings, Bianca was willing to share insights about her own instruction and engaged in conversations with her colleagues about ideas that surfaced in their mathematics instruction.

Lesson #1. For the first lesson, Bianca adapted the task by increasing the size of the collection to 36 counters instead of 24. She previously taught a lesson using a collection of

24 and based on her students' performance, she wanted to see how they would approach a larger collection. Her goals for the lesson were to share a collection among two people, practice naming a share, and recognizing the fact that as more people share the collection, the size of the share becomes smaller (qualitative compensation). She began by having her students work in pairs to share the collection of 36 counters among two friends, explain their strategy, and name the resulting share. Bianca paired students based on strategies she observed them using from a previous fair sharing activity. About half way through the lesson, Bianca called her students back together to discuss their progress thus far. She had several students share their strategies for equipartitioning the collection for two, such as dealing by ones and dealing by composite units, and they also predicted what would happen to each person's share if more people shared the counters. Next, she combined pairs of students to form groups of four and each group completed the next part of the activity to share the counters fairly among four friends and to name the resulting share. At the end of the lesson, the class discussed how they named each share and summarized what it means to share fairly.

Lesson #2. For the second lesson, Bianca began by having her students work in pairs to fold or cut different sized rectangles for two. She discussed with them how they could name each share as "half" of the different sized rectangles. Students then completed the second part of the activity, sharing small collections for two. Bianca led a whole group discussion focusing on the idea that when sharing for two, the result is always "one-half" of the original whole or collection. She and her students made connections between the methods for equipartitioning the rectangles and the collections, focusing on naming the shares as “half”, “3 out of 6”, “4 out of 8”, and “5 out of 10”.

Lesson #3. In the final lesson, Bianca asked her students to equipartition a rectangle for 2, 4, and 8 equal-sized gifts in multiple ways, and to name the resulting shares (see Appendix I for the student worksheet). She chose students to share strategies in increasing order of sophistication, beginning with common splits such as horizontal and vertical cuts, then moving to cuts that produced non-congruent parts and utilized composition of splits. Bianca emphasized making connections between strategies for equipartitioning and naming the resulting share. For example, when sharing for two, Bianca discussed with her students how one of two equal shares can be named as “one-half” or “1 out of 2”, and that two of four equal shares can also be named as “one-half” or “2 out of 4”. She also used the whole group discussion to address a misconception highlighted in the EPLT, that the number of cuts is equal to the number of desired shares.

Tracy. Tracy is a white female with 18 years teaching experience, 14 years teaching second grade. At the time of the study, she had recently completed a master’s degree in education with a concentration in mathematics, grades K-8 from a large, research-intensive institution. Tracy recognized that children come to school with a wealth of mathematical knowledge and it was her job to “create opportunities for students to deepen their understanding and/or correct misconceptions.” She valued using real-world contexts and providing a learning environment that “promotes risk taking, conjecturing, proving, reflecting, and generalizing.” At the end of the LTBI professional development, Tracy scored a perfect score (40) on the DELTA-T and a scaled score of .47 which was above average on the LMT measure of rational number reasoning.

Tracy's experiences in the classroom and in her graduate coursework provided her with a broad knowledge base for how children approach various mathematical concepts and she was familiar with reform-oriented ideas such as promoting conceptual understanding and creating a learning environment where students communicate and justify their mathematical ideas. Her experiences in learning about the EPLT afforded her with the language to describe students' mathematical conceptions to a greater level of specificity than she did prior to LTBI. In one of her interviews, she stated that before her participation in the LTBI professional development,

> I [was] pretty good at noticing what kids do and how it's different from each other, but still not necessarily understanding that one represents a more sophisticated thought. For some things, maybe…but I just had never thought of that before. I think whenever we did fair shares, it was "did they get it, or didn't they get it?" You know, maybe observing how they did it, but not really reflecting and analyzing and giving much thought to how they do it.

At the end of the LTBI professional development, Tracy commented that for her, LTs represent "a framework of essential understanding of a particular concept or area of math that builds in sophistication. Students may not all follow the exact course, but tend to hit certain landmarks before others (not exactly linear). Understandings are largely dependent upon experiences and not just developmental readiness." While this idea of "levels of sophistication" was not new to her, she indicated that the EPLT provided her with more specificity about the mathematical ideas her students were using. In this excerpt from a post lesson interview, she stated,

> We did look at always sequencing levels of sophistication, never with equipartitioning, more with problem solving and number sense. But, I knew what to be looking for. The flags were up for expecting a real simple level from certain kids or from first level of understanding from certain kids. You don't always know who is going to reflect that, but I knew what that would be. And I sort of had in my mind what those could be, you know, just doing things [dealing] singly, using those number facts.

Mathematics instruction. In her classroom, Tracy encouraged students to explain their thinking, making ideas available to others in the class. She used novel problems and highlighted multiple solution approaches in whole classroom discussions. Tracy was comfortable allowing her students to struggle with mathematics and believed part of a teachers' role was to ask questions that helped students make sense of the mathematics. In addition, she stated that as a teacher, "I need to try to analyze what a student <u>does</u> know/understand and not just whether she can or can't do something." Tracy encouraged students to work together and share ideas.

Tracy's participation in the lesson planning meetings. Similar to Bianca, Tracy played a prominent role in the lesson planning meetings, often suggesting ideas or voicing her opinion about potential tasks they were discussing. As the most experienced teacher on the team, the other teachers looked to Tracy for leadership and guidance and Tracy was comfortable in this role. Some of the teachers asked her for advice on what language to use when writing word problems or how to address early ideas of multiplication that often come up in second grade. In the first meeting, Tracy agreed with Bianca that it made sense that the first lesson address equipartitioning collections and also brought up the idea of using this lesson to discuss with her students how to communicate and organize their mathematical ideas both verbally and in writing.

Tracy was adept at making connections between equipartitioning and the second grade curriculum. She discussed students' use of doubles facts in relation to equipartitioning collections and saw these lessons as a way to strengthen her students' number sense. For example, in the first planning meeting she stated:

> Well, if you're going to go from sharing between two to among four, you know if you're going back and forth, one is the inverse of the other. And it's easier for them too I think, well I don't remember. They're either going to make, or recognize an amount as being perhaps double or half of what they had before, or to start to get them to think about how it relates to the whole group, whole collection, back to the referent. Starting to train them to look, zoom in and back and always be aware of that.

Tracy also knew from the EPLT that students often justify fair shares when equipartitioning rectangular wholes using measurement ideas and saw this as a connection to their unit on geometric figures and area that they would address later on in the year.

Tracy shared lessons with her colleagues that she used in the past to address ideas related to naming a share with respect to the referent unit and also routines such as estimation activities that she used in her classroom. The estimation activity that she used with her students incorporated the idea of "half", where she asked her students to estimate a set of objects in a jar. They would count out "about half" and use that to revise their estimates, then count the whole collection and evaluate their estimates. She also shared several lesson ideas that focused on the concept of half and the importance of the referent unit. In this excerpt from the second lesson planning meeting, she suggested a sequence of addressing naming with collections in the second lesson, then moving to naming with wholes for the third lesson:

> I mean, I can be lead to do naming. And halving and doubling and maybe just even exploring, just out of curiosity, because it would be a nice segue into doing composite wholes afterwards. Like the wrapping paper and naming that. But also thinking ok, how far can we get them with half? How quickly can we get that and then [understanding] maybe what is half of a half, because some of mine talked about that that day [in the first lesson].

In the third planning meeting, Tracy pushed her colleagues to consider moving on to sharing wholes because she felt like her students were competent with sharing collections and wholes were more in line with the geometry unit they were getting ready to start. She respectfully challenged her colleagues by asking them questions related to their instruction, encouraging them to articulate issues they had in teaching the equipartitioning lessons.

Lesson #1. During the first lesson, Tracy's students worked in pairs to complete the task of sharing the collection of 24 counters among two friends, explain their strategies and name the resulting share. Her goals for the lesson were to assess her students' knowledge of equipartitioning, focusing on the three criteria, and to emphasize how to record and justify their thinking. As students worked on the task, Tracy emphasized with her students how to record the process they used to fairly share the counters, not just the resulting shares. Like Bianca, she stopped after her students completed the first portion of the activity to discuss their strategies and names when sharing for two. Tracy purposefully chose students to share their strategies in increasing order of sophistication (dealing by ones, dealing by composite units, and using number facts) and she summarized the different names students used, such as "equal", "12", and "half", by writing them up on the board. She also asked her students to consider what happened to the size of the share when they moved from sharing for two to sharing for four (qualitative compensation). Students then completed the activity for sharing among four friends. She concluded the lesson with a whole group discussion where students shared strategies and how they named the resulting share when sharing for four as "6" or "one quarter".

Lesson #2. Tracy had students work in pairs for the second lesson and focused on justifying fair shares when equipartitioning the rectangle for two as well as naming when sharing for two. She used the first part of the activity to introduce names such as "half", "1 out of 2" and "2 out of 4". Students then worked together to share the collections and consider names for the resulting shares. She used the whole group discussion at the end to introduce fraction notation and summarize how students named their shares as “3 out of 6 or 3/6” , “4 out of 8 or 4/8” and “5 out of 10 or 5/10” and how each instance can also be named as “one-half” of the different sized collections.

Lesson #3. Using the wrapping paper task (see Appendix I for the student worksheet), Tracy had her students share a rectangle for two and four equal-sized gifts, focusing on multiple methods for equipartitioning, justifying fair shares, and transitivity. Students worked with partners to share for two, then Tracy led a whole group discussion, sharing strategies from less sophisticated (vertical and horizontal cuts) to more sophisticated (diagonal cuts). She had students prove that a share from the diagonal cut for two was equivalent to a share from the vertical cut for two by decomposing the triangular share to match the rectangular share. Then, students worked to share for four, followed by a whole group discussion. Again, Tracy had students share strategies in increasing order of sophistication, including a discussion about the equivalence of non-congruent parts produced by two diagonal cuts. Tracy also discussed naming the resulting shares using fraction notation of ½ and ¼.

Emma. Emma, a white female with an undergraduate degree in elementary education, was in her 6th year of teaching, all of which was teaching second grade. Emma believed that children learn mathematics best through experience and through social interactions. At the end of the LTBI professional development, she stated

> I really enjoy when my students share and explain their thinking in math [because] there is a huge social component to learning and kids want to listen to their peers and

> learn from how they approach a task. Kids also learn and develop ideas when they have the opportunity to struggle with a task. Frequently they are able to make connections and experience success, especially when they have opportunities to talk through their thinking.

Emma scored above average compared to other teachers in the LTBI project on the post-test measures of mathematical knowledge for teaching, scoring 39 out of 40 on the DELTA-T and a scaled score of .63 on the LMT.

The LTBI professional development provided Emma with the opportunity to strengthen her mathematical content knowledge, particularly with making connections between various areas of the curriculum she taught including fair sharing, fractions, and multiplication and division. She thought of a learning trajectory as “a path that learners move along as they acquire new knowledge through experiences, conversation, and reflection” and used this idea “to differentiate where [my students] are along the trajectory. It has been helpful for me as I plan and design tasks for my kids and reflect on my teaching and my students’ work.” It was important for Emma to know how children learn and approach problems in order to inform her instruction. She stated, “I have to think about the tasks and parameters I set for my kids carefully and use what my kids say and do to inform my instruction and decide whose strategy to share. This year I have changed the way I respond to incorrect thinking and the conversations we have about student work.”

Mathematics instruction. Emma’s mathematics instruction most often consisted of a “mini-lesson” where she facilitated discussions about important mathematical ideas.

Students then worked in small groups based on ability on an activity related to the mini-lesson as well as daily story problems that utilized Cognitively Guided Instruction (Carpenter, Fennema, Franke, Levi, & Empson, 1999) problem types on addition, subtraction, and place value concepts. While the research goals were to develop three lessons on equipartitioning, Emma struggled with the fact that these lessons were in isolation and that she did not have the time and flexibility with in her pacing guide to revisit concepts over several days. During the second planning meeting, she stated,

> What I have a hard time with is that it seems like we're going to do this in isolation, almost, and then move onto something else. And so I like working on something and coming back to it, like having some solid time set aside where this is what I'm working on for a week or two. And then revisiting through morning work or through a warm up and stuff like that and coming back to it throughout the year.

One solution for her was to frequently include an equipartitioning problem into the daily story problems that she used with her students. One or two days a week, Emma deviated from this structure where she engaged students in a novel mathematical task and then facilitated discussions where students shared their strategies and approaches. She saw her role as "that of facilitator and that I get to have my kids share their thinking with each other."

Emma's participation in the lesson planning meetings. Like Tracy and Bianca, Emma played a primary role in the lesson planning meetings. In the first meeting, Emma agreed with Tracy that the first task was an important opportunity to talk with her students about how to organize their thinking. She suggested they modify the wording of the original task to encourage students to explain their thinking process and not just record their answer. She also brought up the idea of using the first lesson to address the concept of qualitative compensation, the notion that as the number of people sharing changes, the size of a share changes inversely. In addition, Emma appeared to be the “recorder” for the team, often creating the task documents and sharing them with her colleagues.

Because Emma incorporated equipartitioning concepts into her daily story problems and other routines, she often took time during the lesson planning meetings to share these ideas with her colleagues. Emma explained the daily story problems on addition, subtraction and place value concepts that she gave her students and how she incorporated equipartitioning problems engaging students in determining fair shares of collections among different numbers of people. She also shared how she used the structure of her lessons to help facilitate her students’ learning. For example, in the second meeting, she shared how she structured her lesson to allow her students to work on the task, and then engaged in whole group discussions after each part of the task:

> But I wanted to be able to go around and check in on everybody but then actually come back together each time just because I felt like it was a management piece that I could bring them back. Reign them in, set up like "This is what I want you to do next" and then send them out.

Similarly to Tracy, Emma was thoughtful about how equipartitioning concepts related to the curriculum they were teaching at the time. In the third lesson planning meeting, she suggested that a lesson addressing rectangles could connect to their unit on geometry:

> Geometry is coming I think that would be important to think about…doing something with geometry that uses equipartitioning. Because I was thinking a little bit about when you were talking about not only when are we doing it, but when you asked us about the trajectory, I was thinking about what kind of shape. If we were doing thirds or fourths, I was thinking rectangles would be the best starting point.

Lesson #1. Emma's students worked in pairs for the first lesson and focused on strategies for sharing the collection for two and four friends. She structured her lesson in the same way that Tracy and Bianca did, stopping half way through to have students share strategies for sharing for two. At this point, Emma used the whole group discussion to share various strategies, explicitly addressing more efficient strategies (such as dealing by composite units or using number facts) as well as strategies that relied on organizing counters into arrays. In her post-lesson interview, she stated,

> Like in thinking about it a little bit more because I didn't want to just have the number fact first. And so then I was, like, "so that's one way - so let's take a look at another way." And so the little girl that I had come up kind of anticipating that she would have dealt by ones just from the work that I've seen on story problems and then Joe, I think, came up and did it by threes… So, that was really interesting, but I did want to, like - my thinking was I did want to, like, point out to them because some of them were all over the place when I was walking around. Just in completing number one even, that it really helps to organize your work.

Emma also asked students to share different ways they named the resulting share, such as “12”, “12 out of 24”, and “1 out of 2 shares”. Students returned to work with their partners for the second part of the activity, sharing the counters among four friends. Emma concluded the lesson by having students share strategies for sharing the collection among four friends. Like Tracy, she used this lesson to help her students consider how to organize and show their thinking on paper.

Lesson #2. Emma began with an estimation activity where she engaged her students in considering the relationship between the size of “half” of a collection and the size of the entire collection. Then, students shared two different sized rectangles for two and discussed how each share was “half” of the original whole. Students individually completed the rest of the activity sharing the three different sized collections for two. Then, Emma purposefully chose student work to share with a focus on naming the share from less sophisticated ways (a count: 3) to more sophisticated ways (one-half of the collection). Emma used the whole group discussion to highlight the various ways students named the resulting shares as a count, a ratio, and a fraction.

Lesson #3. In the third lesson, Emma had students equipartition the rectangular wrapping paper for two and four equal-sized gifts in multiple ways and name the resulting shares (see Appendix J for the student worksheet). Students first explored sharing the rectangle for two, and then Emma purposefully chose students to share in increasing order of sophistication (horizontal and vertical cuts, then composition of splits and diagonal cuts), similar to Tracy and Bianca. During the whole group discussion, Emma focused on strategies for equipartitioning the rectangle and introduced fraction notation. Next, students explored sharing the rectangle for four. Because she encouraged her students to find multiple ways to equipartition, she found her students were struggling with the idea that non-congruent shares were equivalent (for example, using two diagonal splits to create fourths on a rectangle). Therefore, she used the lesson to engage her students in exploring how to justify fair shares as well as proving the equivalence of non-congruent shares when sharing for four.

Summary. In this section, I presented brief descriptions of each participant in an attempt to characterize their instruction, their role in the lesson planning meetings, and to describe how they implemented each of the three equipartitioning lessons. These profiles provide the reader with a sense of how the teachers viewed learning trajectories and the nature of their instruction so that the emphasis for the remainder of the chapter will be on answering the research questions across the cases.

Cross-case Analysis

In this section, I present findings from the cross-case analysis in order to answer the research questions. The findings are organized by research question and within each question, by themes. Quotes from lesson planning meetings, pre-lesson questionnaires, and individual interviews as well as excerpts from classroom observation transcripts are included as evidence of the themes. However, not all teachers are reflected in each theme. Because the teachers varied in how they used the trajectory, some teachers provided evidence of certain themes while others did not. For example, while Ellen, Bianca, Tracy, and Emma all anticipated likely approaches to tasks, only Tracy and Emma anticipated levels of sophistication among the expected approaches.

First, I present findings from the grade-level planning meetings, pre-lesson questionnaires, and individual interviews to offer evidence of how the teachers used the EPLT during the lesson planning process to choose instructional tasks, identify goals and subconcepts, and to anticipate students' approaches to intended instructional tasks. Next, I present findings from the classroom observations and individual interviews to discuss how the teachers used the EPLT during instruction as they monitored their students' progress on the tasks, and as they engaged in whole group discussions with their students. Third, I present findings from the grade-level planning meetings and individual interviews to explain how the teachers used the EPLT to consider evidence of student learning and how their instruction helped or hindered their students' learning. Finally, I present findings from across the data to answer the fourth research question, which addressed factors that mediated and/or moderated the teachers' uses of the EPLT as they engaged in cycles of planning, instruction, and assessment.

Question 1: Lesson Planning. Beginning with first phase of the teaching cycle, this section addresses findings related to how the participants used the EPLT for lesson planning. As they planned, teachers not only choose instructional tasks, but they also attended to specific learning goals and subconcepts in relation to longer term goals. In this section, I offer evidence for the first research question: *In what ways do teachers use the EPLT to plan a set of lessons on equipartitioning?* I provide details of how the LT supported the teachers in the planning phases of the three lessons. First, I present findings on using the LT to choose instructional tasks and learning goals, followed by findings on using the LT to anticipate students' likely approaches to intended tasks.

Q1a: How do teachers use the EPLT to choose tasks and specify learning goals? Themes that emerged related to tasks and learning goals are as follows: a) considering the purpose of the task, b) attending to short and long term goals, c) coordination of proficiency levels and task parameters, d) coordination among proficiency levels, and e) considering a range of instructional moves. The teachers utilized the EPLT to choose instructional tasks that spanned multiple proficiency levels, thereby providing entry points for students with a variety of levels of proficiency with equipartitioning concepts. The tasks supported multiple approaches, utilized various representations, and each teacher implemented the tasks with the expectation that students would communicate with each other about mathematical ideas. The tasks were considered high demand in that they were not routine and asked students to consider the mathematics beyond general procedures (Smith & Stein, 2011), although not all teachers kept the demand of the tasks high as they were implemented.

Considering the purpose of the task. The participants were not required to teach using the same tasks, but all five teachers agreed to use the same tasks and then adapted them slightly either in form or in presentation to fit their individual teaching styles and students. The teachers varied in their purposes for using the tasks. While they all saw the first task in part as an informal assessment to determine what their students knew about equipartitioning collections, Emma and Tracy both used the lesson as an opportunity to teach their students how to organize their work and explain their thinking beyond reporting a numeric answer. During the first lesson planning meeting, Tracy commented, “I think also when we think about the beginning of the year and we're trying to teach the tools of being a mathematician and the organizational skills that are necessary, we need to emphasize that. And sort of just gather, what are the strategies that they're using and model or have other kids model in a more systematic way for divvying things up.”

Based on the results of this lesson, four of the participants chose to focus on the long term goal of developing the concept of naming fractional parts, in line with the second grade curriculum. In the second planning meeting, Bianca and Tracy agreed that naming was an important concept to address:

> Bianca: I feel as if the naming is the hardest part…I think it would be great to have some particular lessons to start really pulling that out of kids and then help scaffolding them with that. Because when we teach fractions explicitly, I feel like they get to the wholes and they get to the actual sharing of things. But I feel as if we'd be doing our kids a disservice if we didn't hit on what they are most needing. Which I, from my class, I definitely think the naming thing.
>
> Tracy: Yeah, well, I’m thinking - I’m always thinking of how difficult naming is. And I - if you know, they’re - to me the biggest hope of getting them to name is understanding half and if, you know, by the end of second grade they can get [what] “half,” is with a collection, and maybe more than that later during the year, I’d be

happy. Because it is such a difficult concept. Not just count them, to give the number a count - to refer to the whole.

However, one participant, Lara, saw all three of the tasks as various forms of formative assessment to find out what her students knew about equipartitioning. Lara did not identify specific learning goals for any of the lessons. During the second planning meeting, the teachers discussed how they structured the first lesson:

Tracy: So are you saying, did you just give them this? And have them read, you didn't...

Lara: Yes. I didn't stop...

I: They kind of went at their own pace.

Lara: Because I guess I thought the benefit was I wasn't polluting by interjecting too much. I didn't want to, well maybe that's not a good way to put it. If I say something I could skew their thinking and I didn't want to give them any help. This is like raw, I just wanted to see what they can do on their own.

Tracy: Yeah, I understand that for assessment purposes...

Lara: Mm-hmm.

Tracy: But I guess I'm seeing this as an instructional task.

Lara: I guess I was looking at this as assessment like, what do they know if I don't say anything? What can they do on their own?

After the third lesson in her post lesson interview, Lara related her experiences in the professional development learning about the EPLT to how she structured her lessons:

Lara: Yeah. You know, I guess part of my hesitation was how much should I interfere? And I think -- I think that was my biggest hang-up, you know, with this whole thing was how much do I – how much do I teach along the way? To me, I guess it was about discovery and maybe it was because of our training was discovery. My experience of our training was discovery.

I: Okay. Do you mean in the professional development?

Lara: Yeah, which was good.

I: Mm-hmm.

Lara: That part was really important for me to have what I call "beginner's mind." You know, like, to put myself in a position where –

I: That makes sense. Yeah.

Lara: Like a child, where, "what the heck do you do with this?" You know, and so I think it's really important as a teacher to go back to discover – a true discovery mode and I suppose I was carrying that into the lesson. And I really want – since I hadn't done anything with this – except we had done equipartitioning, so it's not like it was an entirely alien concept. But, I guess I was bringing forth the open discovery and I was really hesitant – in fact, I didn't really, um, interf – to me it was like interfering with their thinking. I think that was the way I was really looking at it. I don't know if interfering was really the right word. It was really like, interjecting too much and then skewing what they did to be more what I wanted them to do eventually. So, I guess I was letting them have a lot of open rein on this. And then, what I would do – if we were doing a whole unit on this would be to pull some of these out and say, "hmm," well, probably I think I would go back to the paper and say, "okay, so cut it just like you drew it. Are there equal parts?" Or something like [that]. I'm not exactly sure what I want to do with this…

I: So, do you feel like your intention, really, was just to give them the experience of trying out different strategies and just kind of using it as an assessment to see what they do know or are --

Lara: I think that's really --

I: --capable of?

Lara: --that was more -- I mean now that I'm talking about it I think that was more my deep -- my deep reasoning. You know? Is just to kind of see what would happen with this and I think it - you know, it goes again back to the training because I thought it was so interesting for me to have to experience that. And really it was given to me in an open-ended way.

Lara struggled with making connections between equipartitioning and the second grade curriculum and did not see the three lessons as instructional, as opportunities to strengthen her students' number sense, or as opportunities to build a foundation for fractional concepts. Her model of instruction was something different from having students engage in a novel task and then discussing students' ideas to bring forth important mathematical concepts.

Attending to short and long term goals. In contrast to Lara, the other four teachers specified goals related to equipartitioning for each of the three lessons. They were able to use the EPLT to consider short term goals in relation to the long term goal of naming fractional parts. For example, for the second lesson, Bianca hypothesized that naming would be easier with a whole so she suggested starting with sharing a rectangle for two and naming the resulting share to help scaffold students' ability to name the resulting share from equipartitioning a collection. She also recognized from the first lesson that students readily made connections to doubles facts, so that could potentially also scaffold students' ability to name 2-splits. She suggested,

> What if we went, this is, I'm just throwing this out there, this could be, you know. But what if we went to wholes and just worked on halving to see if a name came out of that? And then we went back to doubles with collections and see, saw if the, you know if the vernacular, if the vocabulary came out with a whole, if they would transfer it then to collections.

In addition, the teachers decided to keep the size of the collections smaller in the second lesson in order to provide opportunities for students to use their knowledge of doubles facts to determine the size of the share. In this way, they were attempting to scaffold students to move from dealing strategies to the more sophisticated strategy of using number facts. During the second planning meeting, Tracy stated, "Well I do like how it sort of scaffolds their thinking in a visual sense. So it's connecting the collections with the whole.

And where as we don't have to, and it is very complicated for second graders to be able to name collections, half is you know, you can get them to half."

Emma outlined specific goals for the second lesson and reported in her pre-lesson questionnaire that her goals were: 1) recognizing that "half" is one of two equal sized shares of a whole, 2) recognizing the importance of naming a half in relation the whole, 3) seeing connections between sharing a whole and sharing a collection, and 4) seeing that there is more than one way to name a share. The teachers used the trajectory to identify naming as an important goal and to integrate equipartitioning concepts with their curriculum.

Coordinating between proficiency levels and task parameters. Ellen, Bianca, Tracy, and Emma coordinated the proficiency levels of the EPLT with the task parameters to calibrate tasks so they were appropriate for their students and the mathematical goals they had chosen. For example, Ellen decided for the third lesson to have her students explore various ways to share a rectangular whole for different task parameters. She knew from the trajectory that odd splits were harder than even splits, but she also considered repeated halving to be too easy for her students. Consequently, she chose to have students explore sharing for two, four, and six, with an emphasis on justifying fair shares and naming the resulting shares using fractional names. Her specific goals were "To be able to understand why each piece of paper is the same size, to be able to name each piece of wrapping paper ½, ¼, 1/6, possibly 1/3." When asked why she chose two, four, and six, she stated,

> Because we had done it before and I felt comfortable with it. I actually thought about doing two, four, and eight, but I didn't want them to just halve, and halve, and halve.

> And I felt like they might halve, and halve, and halve. So I wanted to see how they would do the sixths. And I didn't want to get them into odd yet. I'd rather have them get real comfortable for halves, fourths, sixths and then we'll talk about third.

Similarly, for the third lesson, Bianca attended to the task parameters as a way to address naming with her students. Her specific learning goals were for students to "share a whole fairly for 2, 4, and 8 people. Students will focus on how they might name the share in relation to the whole, for example each person got 'one of 8 pieces'." She also considered that because her focus was on naming, which is higher in the trajectory, that keeping the task parameters lower would allow her students to focus more easily on the name rather than on the strategy for equipartitioning, as evidenced in this interview excerpt:

> I: Why did you choose two, four, and eight?
>
> Bianca: Yeah, so I wanted to keep with repeated halving just knowing the trajectory. You know, I know that that's easier and since naming is a little bit harder, I didn't want the sharing to be as diff-too difficult for them.
>
> I: I see.
>
> Bianca: I wanted them to be able to feel successful sharing so that they could focus on what do we call what we've just shared.

In this way, Bianca and Ellen used the task parameters to adjust their lessons to the appropriate level of difficulty for their students.

Coordinating among proficiency levels. Emma not only coordinated task parameters and proficiency levels as she considered her learning goals, she also moved flexibly between proficiency levels. For example, in her third lesson, Emma's goals were to address naming a share when equipartitioning a rectangle for two, four, and eight, and also qualitative compensation. However, she found as her students progressed through the lesson, that in order for her students to more fully understand naming, she needed to address the idea of the equivalence of non-congruent shares. This resulted in her students' exploring ideas of justification and transitivity instead of qualitative compensation. Her use of a task that spanned multiple levels of the trajectory and her ability to see connections between various levels allowed her the flexibility to listen to her students during the lesson and address related concepts as she deemed necessary to move her students' learning forward. After her lesson, she stated,

> But I think what I found was interesting about that was that sometimes the conversation didn't go exactly the way that you expected it to, or they would surprise you with something. And then you were planning on doing one thing but then you could go into something else. I kind of liked that because I think that we do need to have focus, like an objective for a lesson. But if you're seeing 'ok the kids are really getting that part and they're going into this', I like that there is that freedom there.

Considering a range of instructional moves. Some of the teachers also used the EPLT to consider a range of potential instructional moves in their planning based on their students' understandings. For example, after the second lesson, Bianca recognized that she could repeat the task by either changing the task parameter to sharing for four, or by changing the size of the collection but still focus on naming, as shown in this dialogue from the second post-lesson interview:

Bianca: Because I feel like some kids, I would break them off and we would start talking about collection - like, naming of collections when it's not just half, maybe.

I: Alright.

Bianca: You know, or maybe doing the same kind of lesson but with fourths.

I: Okay.

Bianca: Like, exact same thing where we take [a rectangle] and we share for four people and then we take a collection and you share for four people and what can we call it. You know, maybe we could go that route. Or I would go a completely different route for the kids who are, like, I'm not really sure if they got halves. Let's work on sharing wholes and naming them and seeing, you know, and can we call the same - you know, if we had the same size. You know, exploring that more deeply. So, I feel at this point, it could go - there's, like, a couple different ways you could go. I could do something, like, really similar again, which is what I tried to do with the last lesson.

I: Right.

Bianca: But, like, let's--

I: With different numbers.

> Bianca: --do it, like, increase the numbers. Now, seeing what I did last time, increasing the numbers, I wouldn't do it too much more, but I could - especially if it was structured this way - I wonder if we did [collections of] twelve, fourteen, sixteen.

After the third lesson, both Bianca and Emma considered a possible follow-up activity would be to change from sharing a rectangle to sharing a circle. In her post-lesson interview, Emma stated, "It would be good to do some more multiple methods. I mean, we could also try it with a circle. And see how that would vary - couldn't do wrapping paper, but, coming up with something else. Snowmen, or I don't know, cookies." Similarly, Bianca commented during the post lesson interview after the third lesson:

> Well, I think - I mean I would like to see - I would probably do something the same, maybe with circles. And still focus on naming because we've kind of gotten there. But I now we're taking it down a - I would still do two, four and eight, but let's do circles and see can we still name them, but are our shares - but, like, at the side be like, "okay, what's going to happen when we get a circle? Can we share it?" Because here, they were successful sharing it and so they could be successful naming it, alright so now we're just going to take a circle and we're going to try to share it fairly.

Both teachers knew from the trajectory that equipartitioning circles is more difficult than equipartitioning rectangles and since their students were successful with strategies and naming using rectangles, a possible move would be to explore strategies and naming using circles while keeping the task parameters the same (2^n-splits). The trajectory supported the teachers in considering both horizontal and vertical movement along the trajectory to refine and push their students' understanding related to equipartitioning.

Summary. The ways in which the teachers used the trajectory to choose tasks ranged from considering tasks for assessment purposes only to coordinating both proficiency levels and task parameters and coordinating among proficiency levels to choose tasks appropriate for their students. When tasks were used for assessment purposes only, no learning goals were specified. When learning goals were specified, teachers maintained a focus on one or two goals, but at the same time use the EPLT to recognize other ideas that surfaced during the lessons. The trajectory also supported some of the teachers in considering a range of potential instructional moves during planning that was based on their students' learning.

Q1b: How do teachers use the EPLT to anticipate students' strategies and approaches to intended instructional tasks? The participants varied in their use of the EPLT to anticipate how students would approach the intended instructional tasks. Because Lara did not complete any of the pre-lesson questionnaires, there is no evidence that she used the EPLT to engage in this practice, or that she anticipated at all how her students might approach the tasks she used. The themes that emerged are: a) identifying specific strategies and misconceptions, b) attending to levels of sophistication, and c) going beyond the goals of the lesson.

Identifying specific strategies and misconceptions. The other four teachers did provide evidence of anticipation, using the EPLT in different ways. Ellen used the EPLT to anticipate some of the strategies her students would use to equipartition collections and wholes as well as how they might name the resulting shares, but she often underestimated what her students could do. For example, for the third lesson, she anticipated that students would use vertical and horizontal cuts to create halves and fourths, and that they would unintentionally create eighths by repeated halving when trying to share for six. She did not anticipate that students would use diagonal cuts, which gave her pause during her lesson. After the lesson, she stated,

> I thought it was very interesting when they did the diagonal. And when – as I was teaching it and we were talking about it and I was asking who they would – to explain it, [how] it was 'equal pieces', I was in my head going, 'how am I going to explain to them?' So, I'm just going to let them give their simple explanation and just go from there because I didn't want to take the time to cut and those kinds of things to do that.

Had she anticipated the variety of approaches students often use when equipartitioning a rectangle (e.g. using diagonal cuts), she could have been more prepared to discuss how to justify the equivalence of shares produced by diagonal cuts, including a common misconception that diagonal cuts can be used to create six equal-sized pieces. Perhaps she would have considered her own mathematical thinking about using diagonals to create six unequal sized pieces and addressed this with her students when it came up in her lesson.

Tracy also used the EPLT to anticipate common strategies. In the first pre-lesson questionnaire, she anticipated specific strategies that she expected her students to use:

> To determine if the shares are fair some students will organize the red and yellow chips in rows and visually compare them. Others will compare numerically by counting each set, particularly if the chips fall relatively evenly. To share, some students will deal systematically one at a time while others may give a few at a time to each person, then distribute the remaining chips one at a time.

Tracy's anticipations connected to how she planned to sequence student work during the whole group discussion. She expected to see a range of dealing strategies which allowed her to consider how to highlight different strategies during the whole group discussion to make the mathematics available to all of her students and to encourage more efficient strategies.

Bianca used the EPLT in a general way to consider what might be difficult for her students by noting on her pre-lesson questionnaire, "I think one difficulty will definitely be naming the shares. I know that it is a more difficult task on the learning trajectory and they haven't had many experiences doing so." She also used it to expect specific behaviors from her students such as using dealing strategies along with number facts and doubles facts as strategies to determine fair shares of collections. For the third lesson, because she purposefully chose powers of two, she predicted that her students would use a repeated halving strategy, saying, "I hope that a few of them notice the repeated halving and give their own language and explanation as we go from 2 to 4 to 8." Bianca also used the EPLT to expect different mathematical names such as "one out of four, or one part of the four whole parts, or one part out of eight parts etc." Bianca used the trajectory as a tool to consider how her students would approach the tasks, thereby allowing her to be better prepared to connect their strategies with her goals for the lessons.

Attending to levels of sophistication. Tracy, as an experienced teacher, recognized the different strategies students used to share collections, but learning the EPLT gave her more precise language with which to describe her students' anticipated behaviors. In her pre-lesson questionnaire, she used specific language from the EPLT to describe how she thought her students would approach the first task:

> I imagine most students will not name a share mathematically, though a few may do so with prompting. I do expect that the vast majority of my students will be able to create fair shares…I anticipate the greatest difference among students to be in how they go about equipartitioning. Some will be able to reallocate chips when going from sharing between two and sharing among four by simply halving each person's share to make two new shares. Others may need to reassemble the collection and begin dealing from one all over again.

This specificity supported Tracy in recognizing levels of sophistication among the strategies her students might use, such as using a composition of splits ("halving each person's share to create two new shares") or needing to reassemble and re-deal. She also used the trajectory to recognize difficulties her students might have, such as naming a share without using the referent unit and proving the equivalence of non-congruent shares on a rectangle. Because the EPLT helped her anticipate these difficulties, she was more prepared to address them when they arose during her instruction.

As did Tracy, Emma used the EPLT to anticipate levels of sophistication among the approaches she expected her students to use. For the second lesson, Emma anticipated that her students would be familiar with the word “half” and would be comfortable naming shares from collections using a count, but they may struggle to see the connections between naming one of two shares of a rectangle as “half” and naming one of two shares of a collection as “half”. She used the EPLT to think of the levels of sophistication among the various names she expected her students to use (e.g. “3”, “3 out of 6”, “one-half”) and this guided how she intended to share students’ ideas during the whole group discussion.

Going beyond the goals of the lesson. For the third lesson, Emma anticipated that the task might bring up ideas related to the equivalence of non-congruent shares, and was prepared to discuss these ideas when they did in fact arise. In her pre-lesson questionnaire, she wrote, “I think they will easily generalize ‘half’ for familiar representations, but not necessarily for less common ones. There may be some discussions about transitivity. I think naming fourths and eighths will be more difficult.” Her ability to anticipate ideas not directly related to the goals of her lesson supported her in listening to her students’ thinking and using that to guide her instruction “in the moment”. In fact, Emma found the EPLT most useful as a tool to anticipate how her students might approach a given task. In her third post-lesson interview, she stated, “I knew a lot of the time that most of my children were not necessarily going to already be *on* the level -- but then I would see some things [during the lesson]. And so [the trajectory] was good for, like, helping me to review, like, ‘okay, these are some things that I might expect.’”

Summary. Teachers' uses of the EPLT to anticipate ranged from not anticipating at all, to using the trajectory to expect levels of sophistication among the strategies their students could use as well as anticipating mathematical ideas beyond the goals of the lesson. The teachers that did anticipate how their students would approach the intended tasks used the EPLT to expect common strategies and difficulties highlighted in the trajectory. For Ellen, who underestimated what her students would do, the EPLT was useful to gauge if a task would be easy or difficult for her students. Tracy, Emma, and Bianca anticipated levels of sophistication among the strategies they expected, allowing them the opportunity to consider how these approaches related to the mathematical concepts imbedded in the tasks, and supported them in orchestrating whole class discussions.

Question 2: Instruction. The second phase of the teaching cycle is instruction and includes monitoring students as they work through instructional tasks and orchestrating whole group discussions about the important mathematical ideas embedded in the task. In this section, I present findings related to the teachers' uses of the EPLT during instruction, focusing on how they monitored their students' work, and how they facilitated whole group discussions. This section addresses the second research question: *In what ways do teachers use the EPLT when implementing instruction?*

Q2a: How do teachers use the EPLT when monitoring students' progress as they work on instructional tasks? As with anticipating, the teachers' uses of the EPLT as they monitored their students' progress on the tasks varied from not using it at all to using it as a tool to support attention to the processes students were using as they engaged in the task and how those processes related to the goals of the lesson. Some of the teachers' monitoring changed over the course of the three lessons. Themes that emerged related to monitoring include: a) not using the trajectory, b) focusing on the process, c) recognizing common misconceptions, d) attending to levels of sophistication, and e) connecting to the goals of the lesson.

Not using the trajectory. Lara used the monitoring phase of her lessons to make sure her students were on task and completing the activity. During the first lesson, she spent time reading the direction to students, providing game pieces to represent people when students were sharing the counters among four and three people. In her post-lesson interview after the lesson she stated that she was mainly looking for "kind of a social thing - could they work together? Could they work this out, were they following the directions…But when it got to the second part of the activity, the language was tough for a lot of my kids. 'You and your partner join another two people and share the counters fairly.' They seemed to get all kind of confused." Because of this confusion, she felt like she needed to take time to explain the directions to students. Whereas other teachers broke the lesson up to have discussions in between parts of the task, Lara let her students explore and work through the activity on their own. She mainly facilitated their ability to complete the task, not checking on their understanding or how they were arriving at their answers. Lara monitored similarly during the second and third lessons. In these ways, she did not use the EPLT as she monitored her students' progress on the three tasks.

Focusing on the process. The other teachers used the monitoring phase of their lessons to elicit evidence of their students' thinking, either by observing their strategies or probing students to explain their thinking. In this way, they were focusing on the processes by which students' arrived at their answers as opposed to only attending to the final answer. For example, Emma paid attention to the strategies her students were using, asking questions such as "What did you do to make it fair?" and "What were you thinking about when you did that?"

Similarly, Ellen used the monitoring phase of her first lesson to determine students' understanding of what it meant to create fair shares and how they knew they had created fair shares. She asked questions such as "How do you know you have fair shares?" or "How could we show a fair share?" By the third lesson, Ellen used the EPLT to monitor for "simple to more complicated" strategies to share during the whole group discussion. Although she wanted students to volunteer to share their approaches, she had some idea of what students had done by observing them working on the task. Over the course of the three lessons, Ellen shifted from monitoring for general strategies to recognizing levels of sophistication among the mathematical strategies her students were using.

Overall, Tracy used the monitoring phase to listen to her students' thinking related to each tasks. In the first lesson, Tracy monitored her students' work to determine the strategies they were using for sharing the counters and to also encourage them to record their thinking, not just the answer, which was a goal she had for the lesson. She asked questions such as "How do you know?" and "How could you record what you did?" to not only note the processes her students were using for herself, but to have them focus on their strategies themselves and not just on writing the final answer. In her post-lesson interview after the first lesson she commented:

> Well, I tried to get around to see, because short of that it's hard to know what they're doing. I had to follow up and ask them questions. You know, "how did you get this answer? How did you know that each share would be 6?" And really have them try to break it down to sort of back up and explain it. At least to me, even if it never made it to paper.

Recognizing common misconceptions. As teachers attended to the processes their students were using, they were also able to use the EPLT to recognize and address common misconceptions highlighted in the trajectory. In the first lesson, Bianca mainly attended to her students' progress on the task ensuring students had the correct number of counters and were following directions. By the second and third lessons, she attended more so to the ways in which her students' were engaging with the mathematics. In the third lesson, Bianca monitored for misconceptions she knew from the trajectory that were highlighted in the professional development, recognizing when her students were using the same number of cuts to produce the number of desired parts, as evidenced in this transcript from her third lesson:

> Bianca: So where did you guys get these numbers from? Talk to me about this.

S1[1]: One wrap paper, we can make it in four and across and one like that and lines like that.

Bianca: And lines like that? How many, how many people get a piece of paper in this one?

S1: Four.

Bianca: Yeah?

S1: No, one.

Bianca: What do you mean, no? Each person gets one?

S2: There's one-two-three-four-five.

Bianca: So this one...uh-oh.

S2: You made too many.

Bianca: I think I know what you did, Ella. How many lines did you make?

S1: Four.

Bianca: You made four lines. But when you make four lines, how many [parts]...?

S2: Five [parts].

After her lesson, she spoke of the value of asking her students questions to inform her orchestration of the whole group discussion. It was through asking questions and listening to her students' thinking that she was able to identify misconceptions she knew were common in the trajectory and subsequently address them in the whole group discussion.

---

[1] S1, S2, etc will be used to identify students within a transcript excerpt. Note that within an excerpt, S1 refers to the same student. In the next excerpt, the notation will start over from S1, so that S1 in one excerpt is not necessarily the same student as S1 in a subsequent excerpt.

> I think I asked some good questions when I was going around…when I was able to ask them questions about what they were thinking, and 'what do you call it?' and 'what does each person get?' And like, Ella and I had a really great conversation when she drew the eight lines. You know, like we had that whole unders - like, kind of - she's not fully there but she's starting to get, like, 'if I draw one less line that's going to be how many [parts] I get'.

During her third lesson, Tracy observed her students exploring different ways to equipartition a rectangle for four. From discussions during the professional development, she knew from the EPLT that students have difficulty justifying fair shares when two diagonals are drawn to create four shares. In this excerpt from her lesson, she noticed a group of students using this strategy to she stopped to question them:

Tracy: Are those equal shares? Fair shares?

S1: They look like it.

Tracy: They look like it, she says. But is she right? Is there a way to figure it out?

S1: Maybe cut out the sections.

Tracy: You want me to try? What if we did this? Let me get a ruler for you. [Tracy gets a note card, scissors, and a ruler. She draws the diagonals on the note card and cuts out the four resulting non-congruent triangles for the students to use]

Tracy encouraged her students to consider the equivalence of the shares produced by the diagonals and after some time, the students were able to prove the triangles were equivalent by cutting them apart and transposing them. Later, this was an important "aha" moment for many of the students in her class as they visually justified the equivalence of the triangles.

Attending to levels of sophistication. Teachers also used the EPLT during the monitoring phase of their lessons to pay attention to levels of sophistication among the strategies their students were using. In the second and third lessons, Emma used the monitoring phase of her lessons to be purposeful about students' work that she chose to share during the whole group discussion. In the second lesson, she made notes about what strategies her students were using and how they were naming shares, and then purposefully chose students' work to share. In the third lesson, she again monitored with a focus on the process, asking students to explain their thinking, exploring ideas related to naming shares and transitivity. In particular, after the third lesson she stated:

> I was - well, for the first part I was always looking for just different ways of how they shared it and also different kids. And so I was looking for just different ways and then when I [presented] them I tried to start with just the benchmarking strategies and then tried to show some other ones afterwards, like, "oh, look at this. Isn't this neat?" I was also, you know, looking at how they named it… when they did a diagonal this way and a diagonal that way I was, like, "well, why aren't these the same," and so they were, like, "well, this one the line is going from the top to the bottom, this one's going from the top from left to right, right to left." And I was, like, "but, are they the same or are they different?" And they were like, "well, they actually would be the

same if you flipped them around," and so I was trying to get to that again, like, "are these equivalent to each other?"

In this way, she attended to the levels of sophistication of strategies for sharing a whole, recognizing common strategies such as vertical and horizontal splits as well as conceptions related to the equivalence of shares that came up during the lesson.

Ellen also paid attention to the differences in the strategies her students were using as they engaged with the tasks. By the third lesson, she indicated on her pre-lesson questionnaire that she intended "to monitor for 'simple to more complicated' strategies" to share during the whole group discussion. Although she wanted students to volunteer to share their approaches, she has some idea of what students had done by observing them working on the task. This was a shift from her initial purposes for monitoring in general to observe strategies for equipartitioning.

Connecting to the goals of the lesson. In addition to focusing on the process, teachers were able to use the monitoring phase to ask students questions specifically related to the mathematical goals of the lesson. In the second and third lessons, Bianca's questions during the monitoring phase were geared towards her learning goal of helping her students understand naming a share using fractional names such as "half" and as ratios such as "one out of four pieces". For example, in the second lesson, she asked her students "What would you call what you get?" and scaffolded students to name their share as half:

Bianca: What do you call what you get?

S1: We call it a fair share.

Bianca: Ok, what did you do?

S1: Well, so we cut it in half.

Bianca: Ok, you cut it in half.

S1: [inaudible]

Bianca: It's just for the two of you, right? So you just said you cut it in half. So what could you call your part? Could you call it half?

S1: Yeah.

Bianca: Could you say you got half?

S1: I got half and he got half.

Bianca's goal for her students was to learn that when sharing for two, each person's share is "half", and she also understood that naming is higher up on the trajectory and as a mathematical practice, would require more explicit instruction. She used the monitoring phase to observe how her students were sharing the rectangles and counters for two and then scaffolded them to call each share as half, but also recognizing each share is half of a different amount. These questions often pushed students to consider aspects of the task they may not have attended to otherwise. In this way, she prepared the ground work for the whole group discussion about naming shares as "half" with respect to the referent unit (half of 6, half of 8, half of the big rectangle, etc).

Tracy's questions as she monitored were also focused towards the goals of the lesson. For example, during the second lesson she asked questions such as "How much does each friend get?" and "What is another name you could call each share?" to encourage her students to think of naming the share from the collections beyond a count. In her post-lesson interview, she stated that she was looking for accuracy and,

> Just, you know, to see if they could think of other names for things. Because again, I knew that they were dividing these fairly…And then really, could they recognize – it wasn't, I guess until the end that we came back again to the idea of something can be called a half and a half and yet not equal because the wholes are not the same to begin with. The totals are not the same. That's what needs to be connected more.

Summary. The use of the trajectory to monitor students' work ranged from not using it at all and only focusing on monitoring to ensure completion of the task, to monitoring with a focus on the process of equipartitioning and not just the arriving at "the answer." When teachers focused on the process, they were able to identify misconceptions highlighted in the trajectory and consider levels of sophistication in the strategies their students were using in preparation for the whole group discussions. Teachers used the monitoring phase to gear students' thinking to the goals of the lesson through questioning and scaffolding.

Q2b: How do teachers use the EPLT during whole group discussions? The teachers' uses of the EPLT during the whole group discussion portion of their lessons were closely related to how they used the EPLT during the monitoring phase. Teachers that focused on the processes, misconceptions, and levels of sophistication as they monitored carried that focus into their discussions with the entire class. Teachers varied in how explicitly they built upon students' ideas to help their students develop more sophisticated conceptions, with some teachers sharing all strategies without a particular organization to the order and other sharing strategies very purposefully. Some teachers applied the EPLT to make connections between various levels of the trajectory in order to strengthen the mathematical ideas they were developing. The following themes emerged with respect to whole group discussions: a) not using the trajectory, b) focusing on the process, c) attending to levels of sophistication, d) addressing misconceptions, e) making connections between proficiency levels, and f) moving beyond the goals of the lesson.

Not using the trajectory. Without instructional goals for her EPLT lessons, Lara did not use the LT to structure whole group discussions or to address particular aspects of the trajectory through discussions with her students. She often used this part of her lesson to "normalize" students' feelings about mathematics, asking students what was easy or difficult about the task. By doing so, she believed she was making her classroom a place where students felt safe to talk about perceived difficulties. She tended to ask volunteers to share their answers or thinking about particular parts of the task, but maintained a focus on the answer. For example, in the second lesson, Lara first had students discuss how they shared different sized rectangular pieces of paper for two. She showed the class three different ways they folded the paper and the class agreed one share from each could be called "half". For the second part of the activity, students shared answers to the question "How many counters does each friend get?" for each collection of six, eight and ten counters. When all students answered as a count (e.g. 3, 4, or 5), Lara did not press her students to name in other ways, such as "3 out of 6 counters" or "half", discontinuing the focus on naming she began to develop at the beginning of the lesson. In this interview excerpt, she discussed what connections she thought her students made:

I: Do you think that they made any connection between sharing the counters and to sharing the rectangles that you did in the beginning?

Lara: That's a good question. I'm not sure. I guess I should have brought it back to that and I didn't really do that.

I: Well, what - what do you think, I mean?

Lara: That's a good question because the medium is so different, I think - I mean sometimes kids will surprise you and it's a whole different universe.

I: Uh-huh.

Lara: And the rules don't necessarily apply, so I wouldn't put money on that one.

I: That they would make the connection --

Lara: I think some kids do understand the concept of half.

I: --between the two things?

Lara: I would hope.

I: Uh-huh.

Lara: I would hope. I think that the concept of half and fair shares, I would hope would transfer. But, I suppose I didn't really make a point of making that connection, which I guess I should have. The - yeah. I'm not sure. I was just kind of hoping that that was the case.

Because Lara did not have a good sense of where she wanted to go with the lesson or facility with different ways to name resulting shares from collections or wholes, she did not press her students to make connections between naming a share of a rectangle for two as "half" and naming a share from different sized collections for two as "half". Moreover, in the third lesson, Lara allowed her students to explore an incorrect strategy of not using the whole rectangle without ever addressing this misconception. When asked about this after the lesson, she stated,

> I thought, well, sure you could cut this part off if you need to have six parts. I mean, hey that's what we do, we jerry rig in life, too, right? You kind of just make it happen. So, jerry rigging is a good strategy in life, you know, when you have to be creative and do - if you're looking for a desired result, there are different ways to get there, right? And we are encouraging creativity in kids. You know, "how else could you do it?" "Well, you could cut that off if you don't need it." Right? So, I don't want to disvalue that. So, at any rate, I think that that's - I guess that's really where I was aiming at and I think that really the discovery is incredibly important.

Although Lara valued discovery, the mathematical ideas that she wanted her students to discover remained unclear.

Focusing on the process. Teachers that did have particular goals for the lesson and who focused on the process during the monitoring phase carried that into the whole group discussions. For Ellen, this allowed her to provide opportunities for her students to make connections in different ways. In the second lesson, she had various students share their strategies for equipartitioning the collections. As students shared their strategies, she asked questions such as “What did you do to figure that out?” and “How do you know it’s a fair share?” She also encouraged her students to name the shares not only as half, but also as a ratio of “3 out of 6 counters” or “5 out of 10 counters”. She used discussions at the end of the lesson to summarize different ways to name a share:

Ellen: When we share with two people, we can call each share what?

S1: Equal.

Ellen: An equal. And if we put it in two pieces, it's called what?

S2: Half.

Ellen: Half, ok. And if we share 4 counters, what would be a half share or an equal share? If we share 4 counters?

S3: Two.

Ellen: What’s another way to say that?

S3: An equal share of two.

Ellen: What's another way to say that?

S4: Two out of four.

Ellen: Two out of four. Each person gets two out of four.

As another example, Tracy’s goal of emphasizing how students could record their thinking facilitated a focus on the process both during the monitoring phase and the whole group discussions. In her first lesson, she encouraged students to explain the process they used to fairly share the counters, as evidenced in this excerpt from the whole group discussion:

Tracy: You all did different things to make them fair. So I would like Reba's group, would you tell the class how you shared your chips? So they knew they had 24 counters because they counted them all. And what did you do?

S1: We did, I gave Denise two and I put two here and I gave Denise [motions with her hands] until we got to twelve.

Tracy: Ok, so this is what they did. So Reba said what they did was, they went like this "two for you, two for me, two for you, two for me" [gestures with two fingers on each side of her body]. And they kept doing that until there were none left. And then, how did you make sure? Because sometimes Denise was taking two while Reba was taking two and it was sort of overlapping and I was like, "Oh, no! Are they sure it's fair?" But they did something to verify. They did something to double check. What did you do?

S1: We counted it.

Tracy: They counted it again. And they discovered that Reba has--

S1: 12.

Tracy: 12

S2: And 12.

Tracy: And Denise has 12. And that's exactly what they're writing on their sheet. They're not just showing the picture of 12 and 12 because that doesn't tell me how they got 12 and 12. So they wrote how they did it. "I gave Denise two, I gave two to myself. I gave Denise two, I gave two to myself and so on until they were gone. Then we counted each group and we both had, we each had 12.

After the student explained and showed her approach, Tracy repeated it and emphasized what the students could write on their paper to show exactly what they did.

Emma also emphasized the process during whole group discussions as students' shared their strategies. For example, in her first lesson, she used magnetic counters on a white board for students to explain how they created fair shares from a collection. In this excerpt, she encouraged students to show and explain how they equipartitioned a collection of 24 counters between 2 friends:

Emma: Can anyone show me a way to do that? Donald, what do you think?

[Student comes up to board and starts dealing magnetic counters by ones]

Emma: What is Donald doing? Who can describe what Donald doing? Beth, what do you see Donald doing right now?

S1: He's putting one on that side and one on that side.

Emma: He's putting one on one side and one on the other side. Ok, let's look and see how many are here [counts the counters in one group]....12. And how many are over there, Donald?

S2: 12.

Emma: Donald was dealing. He was passing them out, one for you, one for me, one for you, one for me. Thumbs up if you've ever dealt before, if you've ever been a dealer? Ok. Is there another way that we could deal them?

As students shared, she encouraged other students to describe the processes they observed in order to provide opportunities for students to make connections among strategies. She also described what students were doing in her own words, using specific vocabulary she wanted to formalize with her students.

Attending to levels of sophistication. Teachers that monitored for levels of sophistication used this to inform how they structured their whole group discussions. Bianca, Tracy, and Emma purposefully selected and sequenced students' work during the whole group discussion in order to build mathematical ideas from less sophisticated to more sophisticated. In this example, Tracy used the EPLT to consider how to sequence student work for the whole group discussion of how to share 24 counters between two friends, highlighting the process of sharing as opposed to merely stating the correct answer. In this portion of her lesson, students had just shared how equipartition the counters by dealing two at a time. Next, she chose another group to share their strategy:

Tracy: Kanye and Jose did something similar, but it wasn't exactly the same. Kanye, would tell the class what you did?

S1: Jose gave me four and he gave him four. And then he gave me four again and he gave himself four again and me four and him four.

Tracy: You did that three times, didn't you? Four for Kanye, four for yourself, four for Kanye, four for yourself. And I asked Jose, I said, "Why four? Why did you give four?" And you know what he said? Why do you think he gave four to start with? Just take a guess. Why do you think? Tell us Jose.

S1: Because it's much faster.

Tracy: It's faster. It's faster. And that Jose, he doesn't like to waste time. Okay? He's like "Wow, I see there's a lot here so I might as well not just start with one-one, one-one. He wasn't even happy doing two-two, two-two. He went straight for four-four, four-four, four-four. And it worked. Okay. Who did it a different way?

She purposefully had students' share increasingly sophisticated strategies, beginning with dealing by twos, then dealing by fours, and then using number facts to determine a fair share. During the post-lesson interview, she stated:

Tracy: I think I tried to go from the simplest way of sharing, you know, just sharing one by one and then the children who did it sort of as a composite unit, not the whole thing. You know, four, four, four. And then what was remaining.

I: So you had the students share that dealt by ones, dealt by twos, by fours and then-

Tracy: Mm-hmm.

I: So what were you --

Tracy: I was trying to get them the sense that there are more efficient ways to do it. And, you know, yes you can-there's more than one way to solve the problem so that it doesn't matter. They should all feel good that they all got it. Yet, you know, maybe you want to try a new way the next time. You know, I don't think I was explicit in that but that's where we'll head towards in sharing different strategies.

Similarly, Emma emphasized the efficiency of various strategies during the whole group discussion in her first lesson. Here is an excerpt from her first lesson where a student brings up the idea of efficiency:

Emma: What is Joe doing? Who can describe for me what Joe is doing? What is Joe doing? Diana?

S1: He's giving his partner and him three each at a time. So he doesn't have to spend most of his time doing it and he doesn't care about which color it is, he's just caring about how many they each get.

Emma: That was really interesting what you said about "he's not spending most of his time doing it." What do you mean?

S1: Like, the more, if there's...if you had...it's really just, it doesn't really matter about the colors as long as you have, you give some counters for your partner. And he's taking 3 at a time and if he takes 3 at a time, he won't be doing "one-one, two-two" and spend most of his time doing it. And he spends less time doing it then he'll have more time to do something else.

Emma: Ok, so that was really interesting because the way that Joe did it, did he do one at a time like Donald did?

Students: No.

Emma: He did 3 at a time. Was he still dealing?

Students: Yes.

Emma: Yeah. Every time he put 3 over here, he put 3 over here. He was giving them out but he was giving them out a little bit faster. Thumbs up if you've ever done that when you were doing a game? Where instead of giving everybody, like if you've ever had cards before and you were trying to pass out cards and it was a big deck of cards, have you ever dealt more than one at a time?

Students: Yes.

Emma: You can also do that.

Later in the discussion, she chose a student who had used a more sophisticated strategy of using doubles facts to determine each share when equipartitioning 24 counters among 4 friends:

Emma: Robbie is going to use his big meeting voice and I want you to listen because you can learn from him. He can be a teacher too. So let's listen to what he has to say. What were you doing, Robbie? What were you thinking about?

S1: I was thinking about the doubles chart over there.

Emma: Oh, he was using the doubles chart. How were you using this doubles chart to help you?

S1: Because the one that I knew said 12 for the equal was 6 and 6 [Emma shows the doubles chart to the class and the student points to 6+6=12] so I split the counters in half. [Student writes on board 6+6=12, 6+6=12] And I have four sixes so I know everybody has 6 counters.

Emma: Because there's how many people?

S1: Four.

Emma: Four people. So because he had four sixes and four people, did he know that it was a fair share?

S1: Yeah.

Emma: Because when he put the 12 and 12 back together what did that make? Everybody, 12 plus 12 made...

Students: 24

Emma: 24. And then 6 plus 6 plus 6 plus 6 also made 24. Does that make sense?

In this way, she was providing her students with the opportunity to consider a variety of approaches from dealing by ones, dealing by a composite unit, and using number facts in order to strengthen her students' understanding of equipartitioning.

Bianca also attended to levels of sophistication during her whole group discussions. During the third lesson, Bianca had students share strategies for equipartitioning a rectangle first for two, then four, then eight, stopping after each split for discussion. Each time, she attempted to build students' understanding by using progressively more sophisticated approaches. For example, when discussing the 2-split, she first had a student share who equipartitioned for two using one horizontal cut and then a student who used one vertical cut. Then, she showed a student who equipartitioned for two by creating four equal sized pieces and appropriating two pieces per person. She used a similar sequence when they discussed equipartitioning for four. In her post-lesson interview, she justified this choice:

> I started with half because I wanted us to get back to when you name something you can - there are special words to talk about it…when we talk about a person's share there is a special way to call it. Like, we call this a half, or we call that one out of two pieces. And we showed it - because it was like, okay we've been here, we've done this, but let's all get back together on the same page. So, we stopped because I wanted to show how we shared and how we can talk about it and then I picked someone who showed fourths to kind of give everybody a little bit of a, um, a buffer. If you don't know how to make four, like, we just kind of showed you. We talked about it as half, but, like, here you go. And I was, like, I did the same thing with Jill and the eighths because I wanted them… I really wanted them to make sure everybody felt like they could be successful in doing it so that then they could call it something appropriate. So that's why I picked those were my last ones before we went to the next one.

By sequencing strategies in this order, Bianca was attempting to scaffold students' ability to equipartition for larger numbers by first building on 2-splits, then building on 4-splits.

Addressing misconceptions. Teachers who focused on the process of equipartitioning also used the whole group discussion to address known misconceptions from the EPLT. In particular, the misconception that the number of cuts equals the number of desired parts was discussed frequently in the LTBI professional development. In this example from her third lesson, Bianca uses what she learned as she monitored one particular group of students to highlight this common misconception for the entire class.

Bianca: Now, Ella noticed something really interesting when she was over here. Can you talk about how you were cutting them and the lines that you were making and what you were noticing? Let's look at this one that she did to make eight pieces.

S1: I know that makes 8, so I did lines instead of doing that [diagonals].

Bianca: She did [parallel] lines. But when Ella and I were doing it together, at first, how many lines did you make?

S1: One. One at a time.

Bianca: Ok. She made one at a time and she went one-two-three-four-five-six-seven. And then you did--

S1: Eight.

Bianca: Eight lines. She did eight lines and she noticed...Ella noticed something really interesting. She said when you make these lines, she thought that she should make eight lines to make eight pieces.

S1: Instead of seven.

Bianca: But she noticed that seven lines was what really made eight pieces. And she tried to do that again. One-two-three-four-five-six-seven. She kind of ran out of space so she made it a little bit bigger. And I saw her do that because she knew it was going to take seven lines. She knew that seven lines was going to make eight pieces. So Ella do you think if I wanted to make a cake with 10 pieces, how many lines do you think you might make?

S1: [pause] Nine.

The same misconception surfaced in Ellen's third lesson and because Ellen encouraged her students to explain their thinking as they shared, she was able to address the misconception with the whole class. This excerpt from her whole group discussion shows how she chose to attend to this idea:

Ellen: Tania, how did you do yours?

S1: [Student has drawn three parallel cuts to create four equal sized pieces] I didn't really want to cut it the way Anna did so I did four lines- three lines down because if you did four lines it would make five.

Ellen: Hmm. She's saying she did three lines down because if she did four lines it would make five pieces. So she cut her whole rectangle into three lines and how many pieces do you have?

S1: Four.

Ellen: Four. Ok.

Although it is unclear if Ellen anticipated this misconception or if she knew ahead of time that the student would make this particular observation as she shared, she knew from the trajectory that this was a common mistake and worth mentioning.

Making connections between proficiency levels. Teachers who focused on the process also used the EPLT during whole group discussions to allow students opportunities to make connections between proficiency levels of the trajectory. Bianca and Tracy both used the whole group discussion in the second lesson to help students make connections between the strategies they used for equipartitioning and naming the share using a referent unit. For example, Tracy first had students share that folded a rectangle into two equal shares and named it as “half” and “1 out of 2”. Next, she had a student share a strategy of folding the rectangle into four equal shares and naming a share as “2 out of 4”. At the end of the lesson, she used the whole group discussion to summarize how they named the shares as “3 out of 6”, “4 out of 8”, “5 out of 10”, the fractional notation for each, and that each time they also called the shares “a half”. Here, she asks students to compare the ways that they named the shares:

Tracy: How are those problems similar? What's the same about them? Eliza?

S1: They're all a half.

Tracy: They're all a half. Well, isn't that confusing?

S2: No, it's not.

Tracy: Why not?

S2: They're all half of their number.

Tracy: They're all half of *their* number. What is she talking about?

Students: Their number!

S3: The total.

Tracy: Of the total. They're half of the--so you can't really think about half without referring back to the total. Is that what you're saying? Half just doesn't exist by itself, does it?

S4: You need an equal amount and--

Tracy: You need to know what the whole is to begin with. Otherwise, you'll never know what someone is really talking about, right?

Tracy used the ways that students equipartitioned the rectangles and named the resulting shares in the beginning of the lesson to connect the ways they named the various collections with naming a share as "half" of a particular whole or collection.

Similarly, in the third lesson, Bianca had students share that had equipartitioned a rectangle for two in different ways but both named the share as "half". She used this sequence to introduce fraction notation for "1/2", then had a student share that created four equal-sized parts. In this excerpt from her lesson, the student sharing has created four equal sized parts when sharing for two and has written on her paper " $\frac{2}{1}$ ":

Bianca: Look at what Emma did here.

S1: I did it cross way that way each person can have two.

Bianca: Ok, so she said each person gets two. And it's interesting, she said two, and remember, we just said Linda had a one, a line, out of two [$\frac{1}{2}$]? So she should put two, out of how many all together?

S1: Four.

Bianca: Yeah, so you want to get your pencil and we can switch that? So each person gets two out of four pieces. And that's still a half. Do you agree that that's still a half?

Students: Yes.

Bianca: If they get two, is that still fair, they still get a half?

Students: Yes.

Bianca: Ok.

Bianca used this opportunity to connect the strategy to the name (creating four parts when sharing for 2 and naming a share as “2/4”) and also to connect to the idea that this can also be called “half” of the rectangle.

Moving beyond the goals of the lesson. Often, teachers’ uses of the EPLT provided them with flexibility to recognize and address ideas that developed through the course of a lesson even if the ideas were not directly related to the expressed goals of the lesson. For instance, in the second lesson, Bianca’s students shared the rectangle for two in multiple ways and some students were not convinced that non-congruent parts were equivalent shares. While this was not a goal of the lesson, Bianca was able to address it and after her lesson, she commented,

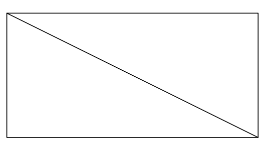
> But, yeah, I thought that conversation with Emma was, like, ‘no, but wait’ - I mean she literally said, ‘If it’s half of a square and this whole [has] half and this is a half, they both make a square so that half is the same as that half.’ And I was, like, ‘Yes, exactly!’ And probably, you know, that went over the heads of 50% of the kids and

probably the other 50%, like, they heard it and that will be somewhere for when they do fractions later on.

While Bianca chose not to explore the idea of equivalence of non-congruent shares with her students at that time, she was aware of the conceptions her students were beginning to develop related to this mathematical idea.

For Emma, this flexibility allowed her to realize that, in the third lesson, her students were having difficulties naming shares as “one-half” because they were struggling with different methods for sharing a rectangle for two. Her initial thought was to table this idea for a future lesson. However, when the idea came up again when students used diagonals splits to share a rectangle for four, she chose to explore the idea of equivalence of non-congruent parts with her students. In this excerpt, a student used two diagonal cuts to equipartition a rectangle for four. Emma uses this strategy to explore the idea of equivalence of non-congruent shares:

Emma: Diana's made a diagonal cut [shows index card with one diagonal drawn] and that cut it into how many parts?



Students: Two.

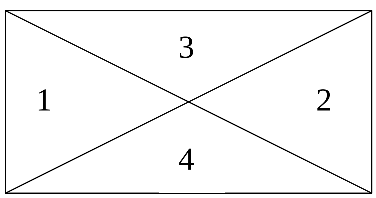
Emma: Two. So right now it's cut in...

S1: Half.

Emma: Half, where there's 1 out of 2 shares for each present. But how many presents were there?

Students: Four.

Emma: Four, so she decided she was going to make another diagonal cut. And so I really liked the way that she was using some adjectives to describe it. She said that these two [1, 2] were kind of short and fat, but these two [3, 4] were long and...



S2: Thin.

Emma: Skinny, yeah, and thin. So the long and skinny ones were equal to each other but the short and fat ones were equal...

S2: To each other.

Emma: To each other. But the long and skinny wasn't equal to the short and fat. Sam?

S3: Sometimes because that is fat and that is thin, or if it does have something like shape that stretch, stretch it out. Because that is fat, if you stretch it out then it becomes thin.

Emma: So he said if something is fat you can stretch it out to make it thin.

S3: And make it long too.

Emma: And make it long. So would they be equal then if we did that?

S3: Yes.

Emma: So do you think that it would be possible that these could be equal to each other even though one's long and skinny and one is short and fat?

S3: Yeah.

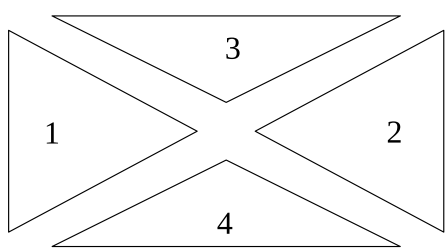
Emma: Is there a way that we could find out? Do you think we could do anything?

S4: You cut it.

Emma: So, what if I cut out each part?

S4: Yeah.

Emma: I've got Diana's pieces up on the sticky board [has cut index card along diagonals to create 4 equivalent, non-congruent triangles].



Emma: She said these two pieces [3, 4] were the same. Why did she say these two pieces were the same? What was her reasoning again, Nate?

S5: Because she was trying to make the same size for the first group and the second group.

Emma: So these two are the long and skinny and then the second group [1, 2], are they the same?

S5: Yes.

Emma: Yeah. Because these were the ones she said were the short and fat. So let's take a look at them and let's compare the short and fat one with a long and skinny one. When I take these and I put them and I kind of compare them with each other, do you notice anything when I compare them with each other? [T takes triangles 1 and 3 and overlays them] Alex, what do you notice?

S6: That that's a big one and a skinny one.

Emma: Yeah, it's longer. I'm wondering something. Just when I look at this, I almost see...I wonder what would happen if I cut this one in half again. [T cuts triangle 1 in half] Now, together this still makes a--it's going to be what, Donald?

S7: It's going to be the same.

Emma: What do you mean it's going to be the same, can you come show me?

S7: Because if you cut these two of them into half, it's going to be the same, but it's going to be little.

Emma: What do you mean? It's going to be the same as the long and skinny-er, the long and skinny one? So I just cut the short and fat one and Donald is kind of moving the pieces around...

S4: Yeah!

Emma: Like a puzzle. That was a good comparison. And when he turns them--

Students: Ahh!

S7: But it's just little.

Emma: Can I take a look? [takes pieces to show the whole group] I might not have cut it exactly, you know, because it was the index card. But look what happened [shows triangles overlaid]. Is the long and skinny one the same actually, as the short and fat one?

Students: Yes.

Emma: Yes. Here's why. When I did this [draws rectangle on white board], I cut this in half [draws one diagonal]. Do you agree that these pieces are equal to each other?

Students: Yes.

Emma: Okay. When I did this one [draws second diagonal], I cut in half again. Now Ms. M didn't do the best job drawing, but did we create 4 parts?

Students: Yes.

Emma: Is this 1 out of 4 shares [shares from diagonal cuts]?

Students: Yes.

Emma: Yes. So did I cut my whole into 4 parts?

Students: Yes.

Emma: So this is 1 out of 4 shares. Even though they aren’t exactly, they don't look visually the same, did I cut it into 4 equal parts?

Students: Yes.

Emma: Yes. So this is 1 out of 4 shares [shows triangular pieces] and because it's 1 out of 4, even though they don't look the same they still are the same. Wasn't that cool the way Donald figured that out and put it together?

Students: Yes.

Emma: Because now do they look like they're equal to each other?

Students: Yes.

Emma: So when I put it back into my long as skinny and short and fat, are they still equal even though they don't look equal?

Students: Yes.

Emma: Yes.

Emma continued on to explore the idea that one share from the diagonal splits for four is equivalent to one share from a “window pane” split for four. As she attended to her students’ thinking during the lesson, she was able to integrate multiple proficiency levels of the EPLT to connect her intended learning goals (naming) to other important concepts from the trajectory (multiple methods and transitivity). After the lesson, she stated,

> It is cool to see what they think of but I wanted the focus to be more on the naming and ironically, too I wanted to talk about, the compensation, the qualitative compensation and we didn’t really get into that. I mean, in a way we did when we were talking about, “well, okay, when we cut it again, instead of getting three parts, we got four because we folded it in half.” But we didn’t really get to talk about that but I - it was so funny that I wasn’t planning on it to be a transitivity lesson but then I

just kept on - I was like, they keep hitting this wall where they don't realize - and the whole thing I want them to understand is that half is one out of two equal parts of a whole. A fourth is one [of four] equal shares of a whole. You know, and so I was, like, "I think I do need to just stop and have this discussion and prove to them."

Summary. When teachers used the EPLT during their whole group discussions, they continued to focus on the processes their students were using as they engaged in equipartitioning tasks, not just on the answers. This attention allowed some of the teachers to highlight levels of sophistication among the ideas their students were using as well as address common misconceptions. Some of the teachers were able to provide opportunities for their students to make connections between various levels of the EPLT, strengthening their understanding of equipartitioning concepts.

Question 3: Assessment. This section addresses the third research question: *In what ways do teachers use the EPLT to assess their students' understanding?* In the third phase of the teaching cycle, teachers consider the evidence of what their students know to inform future instruction. During the individual interviews and lesson planning meetings, the teachers also considered how their instruction helped or hindered their students' learning. In this section, I present how the teachers used the EPLT to determine what counts as evidence of student knowledge and the role of their instruction in that learning.

Q3a: How do teachers use the EPLT to determine what counts as evidence of student knowledge? The ability to determine what their students may or may not know after instruction was closely connected to how the teachers used the trajectory in their planning and instruction. Lara, who did not use the trajectory the same ways as the other teachers in her planning and instruction, mainly considered her students knowledge in relation to their ability to complete the tasks. The other teachers who focused on the process by which their students' arrived at their answers used this as evidence of students' understandings. Themes that emerged related to evidence of student knowledge include: a) not using the trajectory, b) in relation to big ideas from the trajectory, c) as evident in the process, d) beyond the goals of the lesson, and e) along a continuum of understanding.

Not using the trajectory. Lara did not have specific pre-determined goals for her lessons and she therefore did not have a gauge to compare what her students may have learned. Overall, she considered her students' successful if they completed the task, which matched her purposes for monitoring during instruction. For the second lesson, Lara felt like her students were successful because they easily completed the activity. In her post lesson interview, she remarked, "They all got the task done though. It was not a hard task. Most of the kids breezed through." For the third lesson, Lara used her students' drawings and folded papers to determine if they were successful with sharing wholes for various numbers. In this excerpt from the post-lesson interview, she commented that she used the lesson to "get a lot of information" from her students:

I: So, when you say you got a lot of information, what kinds of information did you get from your kids and how did you get that information?

Lara: Looking at their drawings.

I: Uh-huh.

Lara: Some of them appear to have drawn it close enough for equal size parts. Close enough. Or, at least I think they're on the right track. At least they get, "okay, you're supposed to have enough spaces here for eight people." Some of them, if they've got, like, a diagonal cut or maybe vertical cuts, it's not gonna -well, I guess that one would have worked out if he drew it a little better. Some of these are kind of weird, but - I think they would need to do this more carefully on pre-cut pieces of paper that - where it was going to work out and they were able to show what they were thinking. But in here, just kind of throwing away the part that you don't need and having, you know, six nice little boxes, but then a whole hunk you're not using. We would have to talk about using the whole.

I: Mm-hmm. So what is that? What do you think the student does know?

Lara: Um, what do you mean? They're not using the whole. They're just using parts of it. I guess we would have to talk about how you use a whole - that if you're given a whole, use it unless you're told its okay not to…So, and I guess that that's - I guess that's - I'm not sure if that's something you want to say, or not. It's like, how much do you wants kids to - how much do you want to have them just do what they do and then use it as a point of instruction for the next time.

Despite being asked what the student's work showed about what he or she understood about equipartitioning, Lara only focused on the fact that the student did not use the whole. She did not express any thoughts about what students potentially understand about equipartitioning and was unsure how that should be addressed in instruction. With little evidence that she used the EPLT to structure her lessons, it was difficult for Lara to use the EPTL to make conjectures about what her students knew or to articulate evidence that her students learned anything about equipartitioning.

In relation to big ideas from the trajectory. Some of the teachers used the trajectory to speak more generally about their students' knowledge. Bianca reasoned that, in the first lesson, because of the size of the collection, her students struggled with naming the shares which is higher in the trajectory. After her lesson, she noted,

> I don't think naming really - I don't hit on naming very much. Especially because I was noticing just the mechanics of sharing was such a struggle, that I wasn't going to, you know, like, on the trajectory we're pushing up harder here with the numbers then I'm not going to go higher with the naming. You know? Because it was like that was too much. Although we did talk about it because some people said, "I call it a fair share." One person said, "I call it 6," or "I call it 18." You know? Like, they'd just call it the number. And at this point, just developmentally and where they are with just the experience that they've had with it. I know that naming is going to have to be its own entity.

In this way, she coordinated the task parameters and proficiency levels of the EPLT to generalize about her students' understanding of naming and used that to inform her instruction.

As evident in the process. Teachers used the EPLT to make sense of their observations of students' engaged in the process of working through the tasks as evidence of what their students' knew. For example, as Ellen considered her students knowledge, she focused on the processes of creating fair shares across all three of her lessons. In her post-lesson interview after the first lesson, she stated, "they all understood how to deal out one to each other and felt comfortable doing that. They all did it, I mean, several of them did it different ways and that one girl Ally did it like this (shows simultaneous dealing by ones with her hands)." She utilized the EPLT to notice when students dealt by ones, enacted simultaneous dealing, or used number facts to determine fair shares of collections.

Similarly, as Tracy considered evidence of her students' knowledge across the three lessons, she attended to the processes her students used as they engaged with the activities as opposed to arriving at the correct answer. After the first lesson she stated,

> I had wanted - my plan was, my intention was to sort of steer them in a direction for recording, for documenting so that I - I didn't want them to just share their, record their results, but to - and I guess it is asking a lot - but to somehow explain, show the process. And the reason was because I can't get to everybody. So it was just an aid for me later in analyzing their work or their understanding and knowing that I'll never get to every group. I think they did better with the first part, just showing how they shared between two. Because some of them said, and I think I had to scaffold that for them, or prompt them, I'd say, "Okay, now you just told me that, so write that." And I tried to hit as many [as I could] and then when we shared out, I made sure to

explain, highlight the process. And the different ways that kids did it. So that helped them record if they weren't finished recording. Then when they got to sharing among four, they pretty much just wrote the result. You know, I couldn't get them to capture that.

Part of her reasoning for encouraging her students to record their processes for creating fair shares during the first lesson was so she could later go back and analyze what students had done to further understanding their thinking, knowing she would not be able to get to every group during the lesson.

Beyond the goals of the lesson. Teachers used the EPLT to recognize equipartitioning concepts students were using even if they were not part of the explicit goals for the lesson. For instance, in Bianca's third lesson, students shared the rectangle for four in multiple ways and some students recognized the equivalence of non-congruent parts. While this was not a goal of the lesson, Bianca recognized it and addressed it when students brought it up in their small groups. When asked about the idea of transitivity in her post-lesson interview, she commented:

I: Right. And I guess [transitivity] didn't come up, but you didn't share it, so I don't know if any kids did it -- but the - if these are the same?

Bianca: Oh. It did come up --

I: It did come up? Okay.

Bianca: In the conversation as I was walking around.

I: Okay.

Bianca: Actually, Jenny was one - she talked about this and I was saying which one would she want and she said, “Well, this one of kind of bigger.” She was like, “Well, this one is kind of bigger this way, but this one is longer, so it’s fine. It’s the same.” And Henry said the exact same thing, too.

As another example, when Ellen’s students were sharing the wrapping paper for 6 equal-sized gifts, a student alluded to qualitative compensation, commenting that the gifts “must be really small.” Ellen was able to address this with her students and in her post-lesson interview, she noted, “They learned that as the more gifts you need to wrap with the same amount of wrapping paper, the smaller the pieces of paper. Even with Ally and her simple—even if she maybe didn’t say ‘the pieces get smaller,’ she knew they got smaller by her comments.”

Tracy was also aware of student responses that were informative of their thinking, but not what she had anticipated or related to what she intended for the lesson. For example, in the second lesson, when sharing for two, a student folded the rectangle to create four equal pieces. The students recognized that each person would get two out of the four equal-sized parts, but Tracy did not intend to talk about fourths in the lesson. She questioned her students about how they would name each person’s share and got responses such as “two quarters” and “2 out of 4”. Despite the fact that her students’ did not have the vocabulary to name the share as “two-fourths”, she was able to use the learning they did exhibit to build toward her goal of naming shares of different collections as half. In her post-lesson interview she commented,

> I was pleased when Angie, of all people, folded it because I remember hearing her say, “Oh, I want to fold it the other way.” I remember hearing that because I hadn’t notice that she’d already folded it and I was like, “Ooh.” You know, I don’t want to deal with fourths yet. But she said, with a little bit of guidance, but not a whole lot, I mean, I didn’t totally put the words in her mouth, she said, somewhere alluded to two out of four. You know? And that was great. To be able to right away say, “Ah, two out of four is still half.” And get that going.

Although she had not anticipated addressing fourths in the lesson, she used the student’s response to deepen her students’ understanding of “half”.

Along a continuum of understanding. Emma’s use of the trajectory supported her in considering her students’ knowledge as progressing from less to more sophisticated. After her third lesson, as she talked about her students’ understanding of the equivalence of non-congruent shares, she stated,

> Still, I think that some of them are starting to see - at kind of still a working level - like, they weren’t necessarily at the “well, this is half of the whole [rectangle] --you know, this is half of the whole [rectangle] so even though they don’t look the same, because they’re both half.” They weren’t at that level for the most part, but they were starting to see, “oh, I can cut up the triangle and move the shapes around to make another figure.” So I think they were seeing a little bit of equivalence, but not, like, all the way. And I think the naming was definitely coming along. You know, I don't think anything was named, “Billy Bob,” or anything like that. And a lot of them

were using the word, "half," instead of, like, naming it as a count. And I think that's, like - or, like, "one piece," or something like that.

In this way, she regarded her students' knowledge related to naming and transitivity not as fixed (they either know it or they don't) but along a continuum of understanding. The ability to recognize conceptions her students held related to equipartitioning supported her in providing learning opportunities that facilitated students' movement to more sophisticated conceptions.

Summary. Teachers' uses of the EPLT to determine and evaluate evidence of student knowledge ranged from not using it at all to using it to understand the processes students were using as evidence of their knowledge. By encouraging students to show their work and explain their thinking, student thinking became more visible in some of the case teachers' classrooms. Those that used the trajectory subsequently made sense of their students' work, recognizing equipartitioning concepts that were not the goals of the lesson, and regarding student knowledge along a continuum of understanding.

Q3b: How do teachers use the EPLT to make conjectures about aspects of instruction that helped or hindered student learning? Reflecting on their own instruction to pinpoint facets that impacted student learning was difficult for these teachers. Evidence from the lesson planning meetings and individual interviews provided information about the participants' models of instruction and connections to the goals of the Learning Trajectory Based Instruction professional development, such as listening to students and using students' ideas during instruction, but teachers did not explicitly draw up on the EPLT to consider aspects of teaching that helped or hindered their students' learning. Themes that emerged are: a) extraneous issues, b) making student thinking visible, and c) questioning.

Extraneous issues. The teachers recognized that extraneous issues, not related to mathematics, often hindered their students' learning such as classroom management or students being distracted while at their desks. Lara attributed aspects of her instruction that hindered students' progress on the tasks to the nature of the tasks themselves. For example, during the second planning meeting she commented that her students struggled with the first lesson due to the vocabulary used in the task:

> Then in question 5 you had to go back and it was involving another question as well. So it was pretty complicated. A lot of kids have issues with vocabulary. Whether it's ESL or socio-economic they have issues with vocabulary. And then just taking words you know and kind of twisting them around a little bit where you have to really follow sequentially what this all means. The words might be simple, but the concept is not simple. And so I think for a lot of kids vocabulary is tough on a couple of different levels. And we have a lot of kids who fall into either category, either ESL with language, or socio-economic with a language issue. So...a lot of sophisticated language is not used around them so it's not natural for them.

In this way, not only did she not use the EPLT to consider her instruction, but implied that the reason her students struggled with any parts of the task was due to the language, not because of their past mathematical experiences, or how she engaged with them around the task.

Making student thinking visible. Ellen, Bianca, Tracy, and Emma found that allowing students to share their ideas helped to facilitate student learning. While this in general relates to good teaching, the teachers in this study, through the process of student sharing, were able to make connections between ideas from the trajectory in their lessons, and also highlight levels of sophistication among ideas presented in their classrooms.

For example, Ellen, Tracy, Emma, and Bianca regarded the structure of their lessons as a way to help their students' learning. For Ellen, the way she structured her lessons changed over the course of the three lessons. During the first lesson, Ellen had students share their ideas verbally, but she did not have anyone demonstrate their strategies in front of the class. Later, she realized this hindered her students' learning as noted in the post-lesson interview excerpt:

> One of the things I probably should've done is have them come up and maybe under the [document camera], showing what they did. I should have - that's what I would do a little different next time…I should have done that yesterday instead of just having talk. Because so many kids in here need visual.

Based on this experience, Ellen structured the second and third lessons such that students had the opportunity to show their work and explain their thinking using the document camera. This facilitated her ability to orchestrate discussions that allowed her students opportunities to make connections between the strategies they were using and naming the resulting shares.

Tracy, Emma, and Bianca, because of their focus on the process and not just the correct answer, implemented their lessons in such a way as to provide stopping points where they could highlight various strategies for sharing the collection and wholes and how students could record their thinking. Part of Tracy and Emma's purposes for sharing strategies from least sophisticated to more sophisticated, was to provide space for students to build on their own and others' conceptions to consider more efficient ways to solve equipartitioning tasks. During the second lesson planning meeting Tracy commented, "And I guess I would say for myself similarly to Emma, I knew the management was going to be tough and I had to have sort of stopping points. And I wanted the kids, you know after the first part, I wanted to be able to share so they would see each other's ideas." In this way, Tracy was affording her students with opportunities to consider their strategies in relation to others, to build sophistication in sharing strategies, and ways to name shares.

These teachers recognized that opportunities for students to share their ideas helped all students learn. In her second post-lesson interview, Bianca commented,

> I mean, kids were trying to explaining their thinking and I was trying to paraphrase or help them restate what they said. I think there were sometimes where I had to point things out, but I think there were a lot of times kids were making observations and I was restating them. I think that's really helpful when kids are the ones, you know, taking us where we need to go.

The structure of the EPLT lessons reminded Bianca of the power of using whole group discussions to facilitate her students' learning.

Questioning. The EPLT also informed the types of questions the teachers asked and the teachers saw their questioning as an aspect of instruction that helped their students learn.

During the first lesson, as her students were working through the task, Ellen believed she asked good questions that helped her understand the strategies students were using, and how students' knew they had created fair shares. She asked general probing questions such as "how did you figure that out?" and "how do you know it's a fair share?" In her post-lesson interview she stated, "And there are several that are not able to articulate that, but they're some ways able to show me what they're drawing."

Bianca expressed that, by asking specific probing questions during the monitoring phase she was better able to understand her students' thinking, which then informed her whole group discussion. For example, when Bianca noticed a student create unequal pieces in the third lesson, she asked "Which piece would you want? Is that a fair share?" to help the student consider the size of the pieces he created.

Summary. While the teachers did not explicitly draw upon the EPLT to consider aspects of their instruction that facilitated their students' learning, the fact that they valued having students share ideas, used students' ideas throughout their lessons to make connections among levels in the trajectory, and highlighted more sophisticated strategies to encourage efficiency, shows that the EPLT did play a role in how some of the teachers structured their lessons. Because she considered the three lessons as more of an assessment, Lara was not focused on the idea that the lessons could be used to further her students' understanding of equipartitioning concepts such as naming shares as a fraction as the other teachers did.

Question 4: What factors moderate and mediate teachers' uses of the EPLT to engage in cycles of planning, instruction, and assessment? Evidence across the data indicated four areas that moderated or mediated the teachers' uses of the trajectory throughout the teaching cycle: a) mathematical knowledge for teaching, b) beliefs, c) the currently adopted curriculum, and d) engagement with the *Five Practices for Orchestrating Productive Mathematical Discussions* (Stein & Smith, 2011). I also include an "other" theme to address other characteristics of the participants that influenced their uses of the LT. Each theme is discussed in the following sections.

Mathematical knowledge for teaching. As discussed earlier, the participants completed assessments of their mathematical knowledge for teaching (MKT) at the conclusion of the LTBI professional development the summer prior to the current study (see Table 5 for a summary of scores). Recall that Bianca, Tracy, and Emma scored above the mean on both the DELTA-T assessment of knowledge of equipartitioning and the LMT assessment of knowledge of rational number reasoning. Ellen scored above the mean on the DELTA-T but below the mean on the LMT, and Lara scored below the mean on both assessments.

Some of the differences in the ways in which the teachers used the EPLT as they planned and implemented lessons on equipartitioning may be attributed to their MKT. For example, Ellen and Lara, who both had lower MKT scores, accepted incorrect answers in their class discussions without recognizing or addressing the ideas in productive ways. In fact, Lara considered misconceptions her students held as valid and "creative" in a real-life context without attending to the mathematics behind the misconceptions. In Ellen's case, because she held less sophisticated conceptions herself, she often underestimated what she thought her students would do and did not feel prepared to help her students deepen their understanding during her lessons.

On the other hand, the teachers with higher MKT scores were more likely to address misconceptions in productive ways during their lessons. Emma, Tracy, and Bianca were more likely to use language from the EPLT to talk about the tasks during the lesson planning meetings as well as to describe their students' mathematical work. It is perhaps this facility with the EPLT that supported them in participating in more of leadership roles during the lesson planning meetings, where as Ellen and Lara did not participate in the same ways. Therefore, for some teachers their knowledge of equipartitioning and rational numbers supported their use of the EPLT while for others, it was restrictive.

Beliefs. Recall that the participants in this study completed the *Teachers' Beliefs about Mathematics and Mathematics Teaching* instrument (Campbell et al., 2011; Clark et al., 2011). Tracy, Emma, and Bianca's responses indicated they were more attentive to students' mathematical thinking in their instruction and their beliefs were in line with the goals of the LTBI project. Ellen's responses were inconclusive and Lara's responses indicated less of an orientation to students' mathematical thinking. Those teachers who used the EPLT more proficiently held beliefs compatible with the LTBI project and teaching with a focus on student thinking. While an in depth analysis is beyond the scope of this study, it makes sense that beliefs about mathematics and mathematics teaching would influence use of a learning trajectory to focus on students' mathematical thinking.

Curriculum. The participants did not use their district adopted curriculum on a regular basis. The district provided them with a pacing guide that more generally determined the content these teachers taught at any given time. Despite these constraints, the participants in this study were much more likely to use a variety of resources in their instruction and used the pacing guide for general direction as opposed to dictating what they taught from day to day. Hence, the curriculum did not interfere with the teachers' uses of the EPLT in the same way that Wilson (2009) observed.

However, not all of the teachers in this study were able to make connections between equipartitioning and the mathematical topics they were teaching at the time of the study. Lara interpreted much of the EPLT as fractional concepts and since those ideas were not addressed until the end of the school year, she struggled to see connections between the tasks they were using and the mathematical concepts they were teaching at that time.

Other teachers such as Tracy, Bianca, and Emma recognized connections between sharing collections for two and doubles facts, which was a skill they worked on with their students at the time of the study. Bianca and Emma used vocabulary from their geometry unit to talk about congruency and properties of rectangles during the third lesson. Moreover, Emma incorporated equipartitioning concepts into her students' mathematical work on a daily basis. She considered their development of equipartitioning concepts equally important as their development of addition, subtraction, and place value concepts. Although not as regularly as Emma, Ellen occasionally engaged her students in tasks related to equipartitioning concepts during "morning work." Although they did not address factions until later in the school year, these teachers saw value in developing foundational concepts with their students over time.

Use of the Five Practices framework. During the LTBI professional development, the participants were introduced to the practices of anticipating, monitoring, and selecting and sequencing student work. In the current study, the teachers were asked to specify learning goals, anticipate students' approaches to intended tasks, what they would look for as they monitored, and to consider how they might select and sequence student work for whole class discussions. The *Five Practices for Orchestrating Productive Mathematical Discussions* (Smith & Stein, 2011), as part of the conceptual framework, framed how data were collected and analyzed, and hence, one would expect a relationship between the participants' uses of the EPLT and the five practices. However, a synergy between these two constructs emerged through data analysis that warrants further examination.

Not all participants engaged in these practices. Lara did not complete the pre-lesson questionnaires which contained questions related to each practice. Although she did monitor her students work, there is no evidence that she drew upon the EPLT as she did so, nor was she purposeful about student sharing of ideas during whole group discussions. Not only did she not engage in the five practices, she did not draw upon the EPLT to plan her lessons, during instruction, or in assessing her students' knowledge.

The other teachers did engage with the five practices and drew upon the EPLT as they did so, to varying degrees. These teachers presented a reciprocal relationship between the five practices and the trajectory. In one respect, the five practices drew their attention to different aspects of the trajectory. As the teachers considered learning goals, they used the trajectory to identify specific concepts they wanted to develop in relation to the long term goal of naming fractional parts. As they anticipating students' approaches, they attended to the descriptions of students' mathematics that are inherent in the trajectory. Similarly, as they monitored students' progress and used students' ideas during whole group discussions, they focused on levels of sophistication and misconceptions that are highlighted in the trajectory.

From an opposite perspective, the details about specific learning goals, levels of sophistication among students' conceptions, and common strategies and misconceptions detailed in the trajectory supported the teachers in engaging in the five practices to the extent that Smith and Stein (2011) intended. The trajectory supported the teachers in choosing open tasks that addressed multiple proficiency levels, providing space for students with a range of conceptions to engage with the task. Details about student strategies and misconceptions from the LT brought specificity to the ways in which the teachers monitored their students' work. This specificity also afforded the teachers with a framework to draw upon as they considered how to sequence students' ideas during whole group discussions to allow for opportunities to make important mathematical connections. The nature of the descriptions of students' thinking from the trajectory enabled the teachers to attend to their students' mathematical thinking, and use their ideas to move learning towards key mathematical ideas. In this way, teachers' engagement with the *Five Practices* moderated their uses of the EPLT.

Other. Because the participants in this study varied in their years of teaching experience and their paths to teaching, I considered whether these two factors may have influenced their uses of the EPLT. Tracy, the most experienced teacher, and Bianca and Emma, the least experienced teachers, used the EPLT in more sophisticated ways that Ellen or Lara. Therefore, it appears teachers' experience was not a mediating factor to teachers uses of the LT. Moreover, Ellen and Lara both came in to teaching in non-traditional ways, where as Tracy, Emma, and Bianca all obtained certification through a traditional teacher education program. Both years of experience and path to teaching would need further investigation beyond the scope of this study.

Summary. To explain the differences in the ways that the participants engaged with the EPLT through the cycles of planning, instruction, and assessment, the data were examined to identify factors that influenced the teachers' uses of the EPLT. Mathematical knowledge for teaching, curriculum, beliefs, and application of the *Five Practices* emerged across the cases as influencing the ways in which the teachers used the EPLT. Higher MKT, beliefs compatible with the LTBI professional development, and engagement with the *Five Practices* supported teachers' uses of the EPLT.

Chapter Summary

In this chapter, I presented profiles of each individual case and findings across the cases to answer the research questions. The teacher profiles provided details about the individual teachers and set a context for their uses of the EPLT through three cycles of lesson planning, instruction, and assessment. The bulk of this chapter focused on the cross-case analysis. Teachers' application of the EPLT varied from not using it all, to supporting attention to students' mathematical thinking in productive ways. The trajectory supported some of the teachers in choosing learning goals, attending to levels of sophistication in their students' work, and focusing on the process as a way to facilitate student learning.

CHAPTER FIVE

The purpose of this study was to examine the ways in which teachers used one particular learning trajectory to plan and implement instruction, and assess their students' knowledge to inform future lessons. Such an examination is aimed at enhancing the field's understanding of how a LT can support teachers' attention to student thinking during mathematics instruction. Franke and colleagues (2001) argued that a focus on student thinking is a powerful mechanism to engage teachers in generative growth. That is, when individuals learn with understanding, they are more likely to add to that understanding and use it to navigate novel situations. As teachers learn to focus on children's thinking, they can create, build upon, and extend frameworks to understand students' thinking, using the classroom as an interactive place to build knowledge about students' mathematics as well as their own mathematical understanding. Learning trajectories, as models of student thinking, can support teachers in using students' mathematics in instruction. The information about students' mathematical thinking inherent in LTs provides teachers with a framework to not only predict how students might approach a task, but to make connections between students' mathematical work and the conceptions they might hold. With this knowledge, teachers tailor instruction that starts with students' current conceptions and builds towards more advanced mathematical ideas.

Early work on teacher's uses of LTs indicated that while some teachers saw trajectories as "check lists" of ideas their students should learn, others used them as a structure for making instructional decisions (Bardsley, 2006; Wilson, 2009). Clements and colleagues (2011) showed that teachers who used curriculum built from empirically developed trajectories were more responsive to students' mathematical thinking and had more productive mathematical classroom environments.

LTs are useful for supporting teachers' attention to students' mathematical thinking (McKool, 2010; Wilson, 2009) and LTs can strengthen teachers' mathematics knowledge for teaching (Mojica, 2010; Wilson, 2009). They sensitize teachers to the processes students use to arrive at their answers, moving beyond seeing student work as merely right or wrong (Edgington, et al., 2011; Wilson, 2009).

These initial findings demonstrate the potential of LTs for instruction. However, the connection between LTs and instruction does not happen without challenges. Teachers struggle to respond to student thinking during the moments of instruction (Wickstrom et al., 2012) and curricular materials can interfere with teachers' ability to choose tasks aligned with LTs (Wilson, 2009).

The research reported in this study refines these initial connections between teachers' uses of LTs and teaching. The analysis presented in chapter four provides detailed evidence of the utility of the EPLT in relation to each of the three phases of the teaching cycle, that is, for lesson planning, instruction, and assessment. As described, the EPLT provided a structure to support teachers in choosing open tasks that addressed important mathematical goals related to equipartitioning. The teachers used the EPLT to anticipate, recognize and address common strategies as well as misconceptions in their lesson planning and during instruction. For some teachers, the EPLT presented a foundation to allow for a focus on the processes students used to solve problems, helping them to gather evidence of student thinking and then use that evidence to guide whole group discussions. These emerging understandings, in conjunction with previous research, support the formulation of an initial framework of teachers' uses of LTs across six critical aspects of teaching as identified in the conceptual framework: tasks and learning goals, anticipating, monitoring, whole group discussion, evidence of student knowledge, and the role of instruction.

A Framework for Teachers' Uses of LTs

With a focus on theorizing, a goal of this multi-case study was to conduct cross-case analysis to consider the various ways in which teachers use LTs in these six aspects of instruction. Thus, in this concluding chapter, I propose a framework for teachers' uses of LTs as a first step towards a theory of teaching based on learning trajectories. Schoenfeld (2011) articulated the differences between a theory and a framework, stating that the former explains "how and why things work the way they do and it allows for explanations and even predictions of behavior" (p. 4), and the latter points to important information to examine and the potential impact of such information. Using Schoenfeld's interpretations, what I present in the following sections is a framework to understand variation in the ways that teachers use LTs. The categories do not predict or explain why teachers use LTs in particular ways, but begin to highlight key issues related to the use of LTs for planning, instruction, and assessment. Based on the similarities and differences I observed across the five cases considered in this study, I use the terms *emergent user, initial user,* and *proficient user* to describe different levels of teachers' uses of LTs throughout each phase of the teaching cycle. The following sections are organized by phases of the teaching cycle. Within each phase, teachers' uses of the EPLT are briefly reviewed, followed by a more general description of the different levels of teachers' uses of LTs. The chapter concludes with limitations of the study as well as implications for future research.

Lesson planning. Previous research recognized the importance of identifying specific learning goals and anticipating students' approaches to tasks during the lesson planning process (Corey et al., 2010; Superfine, 2008). Identifying and unpacking learning goals to specify necessary subconcepts can provide teachers with more detailed information with which to build a lesson (Hiebert et al., 2007). In addition, the selection of open tasks that provide students with opportunities to engage with the mathematics and discuss their solutions forms the foundation for rich classroom discussions (Smith & Stein, 2011). Teachers use LTs to choose tasks, specify learning goals, and anticipate students' approaches in a variety of ways.

Choosing tasks and learning goals. Teachers in this study coordinated both task parameters and proficiency levels of the EPLT to calibrate tasks to meet the needs of their students; others had difficulty relating equipartitioning to the mathematics they were teaching at the time. The teachers worked collaboratively to choose tasks that spanned multiple proficiency levels, and in doing so, supported the engagement of students with a variety of zones of proximal development.

Despite the fact that these teachers regularly planned together, each participant interpreted the agreed upon tasks in her own way, choosing a variety of learning goals. One teacher maintained the purpose of each lesson as assessment. Some of the teachers used the EPLT to specify short term goals of equipartitioning collections and single wholes in relation to the longer term goal of naming resulting shares. Those teachers with a stronger knowledge of the EPLT maintained a focus on specific learning goals, simultaneously aware of the potential to connect to other important ideas from the trajectory that might surface during their lessons.

Emergent user. When the goal of a task is for general purposes, teachers choose tasks that are open ended and address multiple levels of the trajectory for the purpose of determining where students are in the progression of their learning. As such, teachers are able to gather evidence of students' understandings in relation to general ideas associated with the trajectory. However, without the identification of specific learning goals, teachers' focus remains at the general level. Students may experience highly engaging tasks, but learning is likely to be difficult to identify and articulate (Ainley et al., 2006).

Initial user. As teachers shift from the general use of tasks to using tasks for instruction, the learning trajectory supports them in choosing short term goals in relation to long term goals (Heritage, 2008). Open ended tasks that address multiple proficiency levels are used by teachers to focus on specific mathematical ideas. The emergent user often views levels of the trajectory as isolated concepts, and may subsequently struggle to make connections among the levels of trajectory addressed in a given task. Teachers at this level are likely unaware of the potential to connect to other ideas from the trajectory that may surface in a lesson. Without this awareness, it is more difficult to build off students' ideas that emerge during a lesson that are not directly related to the chosen learning goals.

Proficient user. Teachers who use LTs efficiently select open ended tasks, choose specific learning goals, and use the LT to consider connections to other mathematical concepts that may emerge during a lesson. The coordination among proficiency levels supports teachers in listening to students' mathematical ideas and using these ideas during instruction to further their students' learning toward long term goals highlighted in the LT. The proficient user draws upon aspects of the trajectory (in the case of the EPLT, task parameters and proficiency levels) to adjust tasks to fit the needs of his/her students, and to consider a range of next-instructional-steps based on his/her students' understanding.

Anticipating. Smith and Stein (2011) claimed that anticipating students' approaches to a task prior to instruction permits teachers to begin to think about how students' work relates to the intended mathematical goals. The cases in the current study allowed me to further refine the specific ways in which the trajectory was useful for anticipating prior to whole class instruction and how language supported this practice. Teachers ranged from not anticipating, to using the LT to anticipate levels of sophistication among known strategies highlighted in the LT. The language provided by the LT supported some of the teachers in giving specificity to how they expected their students to approach equipartitioning tasks.

Emergent user. Teachers primarily consider if a task is accessible to students, or whether it will be easy or difficult. For teachers who do not set clear goals, there is no need to anticipate likely student approaches and in general. Thus, the emergent users do not use learning trajectories to consider students' approaches in relation to the mathematics and LTs are not used as a tool for anticipation.

Initial user. When teachers begin to use the learning trajectory to consider how students might approach a task, they use the LT for information about likely strategies and misconceptions. Initial users are aware of some common strategies that their students might use as well as a few difficulties that can be expected because they are highlighted in the trajectory. Teachers at this level are only beginning to make connections between what they anticipate and the mathematical goals for the lesson and often underestimate what students can do.

Proficient user. Proficient users draw upon the LT to not only anticipate known strategies and misconceptions, but they regard these ideas with respect to the mathematical goals of the lesson and the long term goals detailed in the LT. They use the LT to consider a range of student approaches and often use language from the LT to give specificity to their descriptions of student behaviors and related understandings. Teachers can effectively use the LT to judge levels of sophistication among the strategies they expect students to use. This type of anticipating supports teachers in making sense of students' ideas that emerge as students engage with a task, preparing them to use and build off students' ideas to expand and extend students' mathematical knowledge.

Instruction. Research on instruction highlights the value of attending to student thinking during instruction (Carpenter et al., 1989; Fennema et al., 1996), but little is known about specific teacher practices that support this attention (Franke et al., 2001). Early research on teachers' uses of LTs indicated LTs support teachers in focusing on the process (Wilson, 2009) and providing a framework to understand students' mathematical thinking (Clements et al., 2011; Wickstrom et al., 2012; Wilson, 2009). Collecting evidence of student thinking through monitoring and during whole group discussions provides teachers with spaces to make student thinking visible (Smith & Stein, 2011) and use the LT to make sense of students' ideas.

Monitoring. Teachers' uses of LTs when monitoring varied across teachers, including changes in purposes for monitoring over time. Teachers shifted from keeping students on task to focusing on the process by which students' arrived at their answers. This attention to the process supported teachers in observing students strategies or probing students to explain their thinking.

Focusing on the process allows teachers to attend to information in the trajectory about common misconceptions and relative sophistication among expected strategies. Teachers identified misconceptions highlighted in the trajectory in their students' work and considered levels of sophistication in the conceptions their students' held. The EPLT was also used during the monitoring phase to gear students' thinking to the goals of the lesson through questioning and scaffolding.

Emergent user. Teachers' main objectives are to ensure that students remain on task, are working well together, and can complete the task in the time allotted. They maintain holistic descriptions of students' work as right or wrong and attend to student's answers as they work on tasks. They have yet to focus on the processes by which students arrive at their answers as indicative of their understanding.

Initial user. As teachers begin to focus on the process as opposed to the answer, teachers use the LT to support their attention to specific strategies and misconceptions highlighted in the LT as they observe their students working through a task. Initial users often elicit evidence of their students thinking as they monitor by asking probing questions of individuals and small groups of students.

Proficient user. Proficient users of the LT focus on the processes students use to reach their answers. Teachers at this level recognize levels of sophistication among known strategies highlighted in the LT. They use the monitoring phase of their lesson to help students make connections between the strategies they are using, clarify misconceptions they may have, and move them towards the mathematical goals of the lesson by asking probing questions and providing scaffolding. In addition, proficient users recognize other mathematical ideas that may be surfacing that are not necessarily the goal of the lesson and make decisions as to whether to pursue these ideas with individual students or with the whole class.

Whole group discussion. Teachers' uses of LTs during whole group discussions are closely related to their uses of LTs as they monitor. Learning trajectories support teachers in relating students' approaches to what they may be thinking, facilitating the selection of student work to highlight during whole group discussions. The cases in the current study provide detail for how the LT was a useful tool to select and sequence student work during whole group discussions. Teachers' uses ranged from not using the LT at all to using it to continue a focus on the processes students were using to emphasize important mathematical ideas.

Similarly to the monitoring phase, teachers that focused on the process utilized information from the EPLT to highlight misconceptions and attended to levels of sophistication in the students' ideas they chose to share during whole group discussions. While some teachers shared students' approaches in no particular order, others carefully sequenced students' work to bring forth noteworthy ideas and to make connections to the mathematical goals of the lessons.

Emergent user. As in the monitoring phase, teachers continue to focus on correct answers during whole group discussions. The emergent user often calls on students to share their ideas during whole group discussions and the focus is on answers only or as a way for all students to have a chance to participate by sharing their ideas, regardless of the strategies they used. They are interested in hearing students' ideas but without specific goals for the lesson, these ideas are often shared and then left, without consideration to the underlying mathematical concepts.

Initial user. The initial user draws on the LT during whole group discussions to focus on known strategies and misconceptions that are a part of the LT. Again, a focus on the processes that students engage in as they solve tasks supports teachers' ability to use the LT in this manner. The initial user does not consistently provide opportunities for students to make connections among strategies or between strategies and mathematical goals. Lack of attention to the ways in which students' ideas relate to underlying mathematical concepts inhibits teachers from using the LT to structure whole group discussions to "develop powerful mathematical ideas" (Stein et al., 2008, p. 330).

Proficient user. Teachers purposefully share students' ideas during whole group discussions with levels of sophistication in mind. The proficient user provides opportunities for students to make explicit connections between strategies and important mathematical ideas imbedded in the LT and addresses common misconceptions when appropriate. Teachers recognize other important mathematical concepts that are not necessarily the goals of the lesson and know when to allow students to explore these ideas in productive ways.

Assessment. Pellegrino et al. (2001) characterized assessment as the process of reasoning from evidence of student learning to make inferences about what students know. Hiebert and colleagues (2007) argued that teachers must collect evidence of student knowledge at key points during a lesson in order to gauge students' progress towards intended learning goals. LTs have been recognized as supporting teachers in evaluating student work beyond "right or wrong" and using that information to guide next instructional steps (Edgington et al., 2011; Heritage, 2007; Wilson, 2009).

Evidence of student knowledge. Teachers used the EPLT to make sense of their observations of students' working through equipartitioning tasks as evidence of students' knowledge. The LT provided teachers with language to describe students' work in more nuanced ways. Sensitivity to certain strategies and misconceptions allowed teachers to pay attention to the specific means by which students arrived at their answers and then use those observations as evidence of student knowledge. The EPLT was used to recognize student knowledge of equipartitioning concepts that were not directly related to the goals of the lesson, providing teachers with a broader sense of students' equipartitioning conceptions.

Emergent user. Emergent users often consider a task successful if all students are able to complete it or evaluate a task as easy or difficult as opposed to valuing the ways in which a task can reveal student thinking. Without specific learning goals, teachers at this level do not use the LT to consider what counts as evidence of student knowledge. When teachers do listen to student thinking, they do not use the LT to help them make sense of students' approaches or understandings.

Initial user. As teachers begin to focus on the processes students engage in as they work through tasks, they make use of the LT to understand their observations. While the initial user struggles to make sense of student work during instruction, they are more adept at interpreting student understanding after the fact. In their attempt to make sense of student work, teachers draw upon the LT to provide language with which to describe students' work and to make connections between the strategies students are using and understandings they hold. In this way, the process is the evidence of student knowledge.

Proficient user. The proficient user not only uses the LT to focus on the process as evidence of student knowledge, but they connect that evidence back to the learning goals for the lesson. In this way, the LT supports teachers to look for more detailed and specific evidence of particular learning goals in order to capture student thinking. The proficient user is aware of important information from the LT that may surface during a lesson and uses the LT to consider the range of student understandings that are present in a classroom.

Role of instruction. As explained in the previous chapter, the teachers in the current study did not explicitly draw upon the EPLT as they considered aspects of their instruction that facilitated their students' learning. However, the trajectory did support some of the teachers in how they structured their lessons to afford a greater focus on student thinking, and they saw this as positively impacting their students' learning. These teachers valued having students share ideas, used students' ideas throughout their lessons to make connections among levels in the trajectory, and highlighted more sophisticated strategies to encourage efficiency. Teachers drew upon these types of student-centered practices to make conjectures about their instruction that helped student learning.

Emergent user. The emergent user perceives his or her role as observer, allowing students the opportunity to discover ideas on their own without interjections or interference from the teacher. Teachers at this level view tasks associated with the LT for assessment purposes only, and in relation to general mathematical ideas. The emergent user does not draw upon the LT to consider the effects of teaching on student learning.

Initial user. Teachers begin to see connections between the structure of lessons and student learning. As teachers provide students with opportunities to share and explain their thinking, teachers at this level start to notice key ideas from the LT in their students' work. The initial user questions students to encourage them to explain their thinking and then uses the LT to make sense of students' responses, but does not use the LT to inform more specific questions related to the mathematical ideas present in the LT.

Proficient user. Proficient users make conjectures about instruction based on the visibility of student thinking in a lesson. Teachers make connections between levels of the trajectory in his or her instruction and highlight levels of sophistication among ideas presented in the classroom. Teachers at this level draw upon the LT as they structure whole class discussions in order to provide space for students to build on their own and others' conceptions to consider more proficient ways to solve tasks. The proficient user utilizes the LT to inform questions they pose to students in order to learn more about student thinking, to move student's thinking forward, and to inform whole group discussions.

Summary. Building off of previous research, the cases in this study were analyzed to examine themes around teachers' uses of LTs to plan, implement and assess mathematics instruction on equipartitioning. This interpretation was used to conceptualize different levels of the ways in which teachers use LTs throughout the teaching cycle. The emergent user has a limited view of LTs as an instructional tool and subsequently does not use LTs to choose learning goals, anticipate students' approaches to intended tasks, or during whole class discussions. The initial user begins to make connections between ideas highlighted in the LT and the processes by which students' arrive at their answers. The proficient user flexibly coordinates their knowledge of the LT with learning goals, expectations of how students engage with a task, and students' strategies and ideas that emerge during instruction to facilitate students' movement towards more sophisticated mathematical conceptions.

Table 6 provides a summary of the findings as an initial framework of teachers' uses of LTs. The framework suggests features to attend to as we examine teachers' uses of LTs in their teaching. The key issues highlighted in the framework support movement towards theorizing the ways in which teachers use LTs with some explanatory and predictive power. Understanding whether a teacher is emergent, initial, or proficient can aid in predicting teacher moves and ultimately inform the ways in which LTs are presented to teachers and support teacher learning of LTs.

Table 6

Framework for Teachers' Uses of LTs

		Emergent User	Initial User	Proficient User
Planning	**Task and Learning Goals**	• Selects open ended tasks • Tasks are used for assessment purposes only	• Selects open tasks • Chooses short term goals in relation to long term mathematical goals detailed in the LT	• Selects open tasks • Chooses short term goals in relation to long term mathematical goals detailed in the LT • Coordinates among proficiency levels in the LT to adjust tasks based on students' understanding
	Anticipating	• No anticipation in relation to the LT	• Anticipates likely strategies and misconceptions from the LT	• Anticipates likely strategies and misconceptions from the LT • Relates the anticipated strategies and misconceptions to learning goals detailed in the LT • Anticipates levels of sophistication among students' approaches as highlighted in the LT
Instruction	**Monitoring**	• Monitors for holistic descriptions of students' work as right or wrong	• Focus on students' solution processes • Monitors for known strategies and misconceptions from the LT • Elicits evidence of student thinking by asking probing questions	• Focus on students' solution process • Recognizes levels of sophistication among strategies from the LT • Helps students make connections to move towards the goals of the lesson • Asks probing questions to elicit evidence of knowledge • Recognizes important mathematical ideas from the LT beyond the goals of the lesson
	Whole group discussion	• Focus on student work as right or wrong • Ideas are often shared without consideration to underlying concepts	• Focus on process related to known strategies and misconceptions • Inconsistently provides opportunities to make connections among approaches and important mathematical concepts	• Focus on process in relation to known strategies and misconceptions from the LT, with levels of sophistication in mind • Provides explicit opportunities to make connections among approaches and important mathematical ideas from the LT
Assessment	**Evidence of student knowledge**	• Success of a task based on students' ability to complete the task	• Uses the process as evidence of knowledge • Uses language from the LT to describe students' mathematical work	• Uses the process as evidence of knowledge in relation to goals for the lesson • Uses the LT to consider a range of student knowledge present in a classroom
	Role of Instruction	• Views the LT for assessment purposes only	• Begins to notice key ideas from the LT as opportunities are provided for students to share their thinking	• Draws upon the LT to structure whole class discussions to provide opportunities for students to consider more sophisticated conceptions • Uses the LT to inform specific questions geared to make student thinking visible and move student thinking forward

Limitations

There are limitations to the study. First, the design of the study to collect three distinct classroom observations spread over time represented a challenge for the teachers; thus, perhaps deviating them from their usual teaching. Participating teachers were accustomed to developing a concept with consecutive lessons over the course of one to two weeks, so teaching three discrete lessons on equipartitioning over time was difficult for them. Hence, the development of ideas across the three lessons may not have been as rich as if the teachers taught a distinct unit on equipartitioning. The fact that the concepts were not developed in a typical way for these teachers may have influenced how they used the LT.

Second, my role in the research process may have influenced the teachers' behaviors during the lesson planning meetings, classroom observations, and interviews. I attempted to convey to the teachers that my intentions were not to judge their instruction or to serve as an instructional coach; however, it is possible they perceived my presence in a different manner. It is possible the observed lessons were not typical of the teachers' day-to-day mathematics instruction, but instead the type of instruction the teachers perceived I wanted to see. Moreover, the teachers' responses on the pre-lesson questionnaires and in the post-lesson interviews could have been skewed by their attempt to answer in the ways they thought I wanted them to, as opposed to honestly and accurately.

Because this is a dissertation, the study was bounded by time and by the fact that I was the sole researcher. As such, classroom observations had to be staggered in order for all teachers to be observed and observations could only take place over a four month period in total. The participants had previously attended 60 hours of professional development on one particular LT, so the results are bounded by this selection feature of the participants. Findings may differ with more or less professional development or with a different trajectory. The EPLT was organized in a particular way and other LTs may lead to other discussions for emergent, initial and proficient users.

Finally, these results are only an initial step towards a theory of teaching based on learning trajectories. These findings were gained through the cross-case analysis of five teachers and this should be taken in to consideration. Further work is needed to refine the initial framework and to empirically test these levels of teachers' uses of LTs.

Implications

This study was part of the larger Learning Trajectory Based Instruction project to understand how teachers come to use learning trajectories in their mathematics instruction. The results indicated the EPLT was useful for the participating teachers in this study throughout each phase of the teaching cycle, supporting their attention to student thinking as they chose tasks and learning goals, monitored their students' work and conducted whole group discussions, and in assessing their students' knowledge and the role of their instruction. These findings have implications for teachers, teacher educators, and researchers.

Teachers. With the adoption of the Common Core State Standards (CCSS, 2010) that are built on empirically developed trajectories, teachers need time and support to learn about the various trajectories that comprise these standards. Curricular materials that are aligned with learning trajectories and the Standards can support teachers in making sense of the LTs and integrating them into instructional practices. Time is required to thoughtfully consider instructional tasks in light of LTs, to anticipate students' approaches to tasks, and to reflect on instruction and student learning. Teachers should be supported to work collaboratively with others to take into account the role of trajectories in their work.

Mathematics teacher educators. Mathematics teacher educators and designers of professional development must continue to consider the role of trajectories in teacher development. Teacher educators must consider ways to present trajectories to teachers in such a way that they are accessible and usable. Professional developers can use students' mathematics in the form of LTs to engage teachers in considering their own math knowledge for teaching in tandem with student-centered frameworks for instruction such as the *Five Practices for Productive Discussions* to support teachers' attention to students' mathematical thinking in ways that are meaningful for instruction.

More specifically, the framework resulting from this study can be used as a guide for teacher educators to consider how to support teachers from moving from emergent, to initial, and ultimately to proficient users of LTs. The framework provides key indicators to examine in order to locate a teacher's development of his or her use of LTs in instruction, such as identifying learning goals, focusing on the process, or anticipating levels of sophistication. Teacher educators and instructional coaches can specifically address these key issues through professional development as they consider ways to support teachers to move to more sophisticated uses of LTs. The framework can be used by mathematics teacher educators to anticipate how prospective and novice teachers might use LTs and to design learning opportunities that address key components of the framework. For example, given a group of teachers with limited knowledge of learning trajectories, professional learning tasks might focus on moving teachers from initial users to emergent users by engaging teachers in tasks that provide them with opportunities to focus on the process students' use as they engage in mathematical tasks. As teachers learn to make sense of students' strategies, professional learning tasks can then encourage teachers to consider levels of sophistication among students' approaches and have teachers consider possible ways to organize a whole group instruction around a set of student work.

Future research. Researchers should continue to study teacher learning of LTs and the ways in which teachers use LTs in instruction. Currently, there are eighteen different trajectories (Daro et al., 2011) and few researchers have addressed how teachers negotiate multiple trajectories in their teaching practices over time. This is especially vital in the elementary grades where teaching and learning of concepts, for example counting and addition and subtraction, are often intertwined. Those who work to empirically develop students' learning trajectories must consider the grain size appropriate for teachers and continue to develop resources that can be used by professional developers such as student work samples and videos of student interviews and whole class instruction. Research on professional development should persist to understand how teachers come to know about LTs and to design professional development experiences that are effective in promoting teacher learning. Similar case studies should be conducted using other learning trajectories as the context to confirm, refine, or challenge the findings presented here.

Finally, research must consider the impact of teachers' uses of LTs on student learning. What is the impact of teachers' participation in professional development on LTs for student learning? Is there a relationship between teachers' uses of LTs and student learning? If so, what is the nature of the relationship? If trajectories support teachers to consider the range of ideas present in their classrooms, then a consequence would be to examine issues of equity in relation to teachers' uses of LTs. Do LTs support teachers in providing meaningful learning opportunities for *all* students?

Closing Remarks

This study contributes to the research on the utility of learning trajectories to support attention to students' mathematical thinking. The EPLT proved useful for considering instructional tasks that spanned multiple proficiency levels and provided teachers with information about short term learning goals in relation to the broader mathematical ideas outlined in the trajectory. Specific information about common approaches and misconceptions supported the teachers in anticipating how students would approach intended tasks. During instruction, teachers used the LT to attend to the processes students used to arrive at their answers, moving beyond seeing student work as merely right or wrong, and to consider the levels of sophistication among students' conceptions. Attention to the process also provided the teachers with evidence of their students' knowledge, informing future instruction.

The findings suggest variation in the ways that teachers use LTs to plan lessons, implement instruction, and assess students' understandings. Through an in depth study of individual teachers' uses of one particular LT, differences emerged across the cases. Teachers ranged from not using the trajectory to focus on student thinking, to using it purposefully to calibrate tasks, attend to levels of sophistication among students' approaches, and to structure lessons that facilitated students' movement to more sophisticated ideas. The differences among emergent, initial, and proficient users highlight key aspects to attend to as we continue to study the ways in which teachers use LTs to focus on students' mathematical thinking.

REFERENCES

Ainley, J., Pratt, D., & Hansen, A. (2006). Connecting engagement and focus in pedagogical task design. *British Educational Research Journal*, *32*(1), 23-38.

Atlas.ti. (version 6.2) [Computer software]. Berlin, Germany: Scientific Software Development GmbH.

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *The Elementary School Journal*, *93*(4), 373-397.

Ball, D. L, & Cohen, C. K. (1999). Developing practice, developing practitioners: Toward a practice-based theory of professional education. In G. Sykes and L. Darling-Hammond (Eds.), *Teaching as the learning profession: Handbook of policy and practice* (pp. 3-32). San Francisco: Jossey Bass.

Bardsley, M. E. (2006). *Pre-kindergarten teachers' use and understanding of hypothetical learning trajectories in mathematics education* (unpublished doctoral dissertation). State University of New York, Buffalo, NY.

Barrett, J. E., Clements, D. H., Klanderman, D., Pennisi, S., & Polaki, M. V. (2006). Students' coordination of geometric reasoning and measuring strategies on a fixed perimeter task: Developing mathematical understanding of linear measurement. *Journal for Research in Mathematics Education, 37*(3), 187-221.

Battista, M. T. (2004). Applying cognition-based assessment to elementary school students' development of understanding area and volume measurement. *Mathematical Thinking and Learning*, *6*(2), 185-204.

Black, P., & Wiliam, D. (1998). Assessment and classroom learning. *Assessment in Education: Principles, Policy, & Practice, 5*(1), 7-75.

Black, P., Harrison, C., Lee, C., Marshall, B., & Wiliam, D. (2004). *Assessment for learning: Putting it into practice.* Buckingham, UK: Open University Press.

Bransford, J. D., Brown, A. L., & Cocking, R. R. (Eds.). (2000). *How people learn: Brain, mind, experience, and school*. Washington, DC: National Academy Press.

Brown, A. (1988). Twelve middle school teachers' planning. *The Elementary School Journal, 89*(1), 69-87.

Campbell, P. F, Nishio, M., Smith, T. M., Clark, L. M., Conant, D., Rust, A.,. . . Griffin, M. (2011). *The relationship between teachers' mathematical content and pedagogical knowledge, teachers' perceptions, and student achievement*. Paper presented at the Annual meeting of the Research Presession of the Special Interest Group for Research in Mathematics Education and the National Council of Teachers of Mathematics, Indianapolis, IN.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). *Children's mathematics: Cognitively guided instruction.* Portsmouth, NH: Heinemann

Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C., & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal, 26*(4), 499-531.

Clark, L. M., DePiper, J. N., Jones, T., Nishio, M., & Campbell, P. F. (2011). *Teacher variables associated with teachers' beliefs and awareness of their students'* *mathematical dispositions.* Paper presented at the Annual meeting of the Research Presession of the Special Interest Group for Research in Mathematics Education and the National Council of Teachers of Mathematics, Indianapolis, IN.

Clark, C., & Lampert M. (1986). The study of teacher thinking: Implications for teacher education. *Journal of Teacher Education , 37*(5), 27-31.

Clements, D. H., & Sarama, J. (2004). Learning trajectories in mathematics education. *Mathematical Thinking and Learning, 6*(2), 81-89.

Clements, D. H., & Sarama, J. (2007). *Building Blocks—SRA real math, grade pre-K.* Columbus, OH: SRA-McGraw Hill.

Clements, D. H., & Sarama, J. (2008). Experimental evaluation of the effects of a research-based preschool mathematics curriculum. *American Educational Research Journal, 45*, 443-494.

Clements, D. H., & Sarama, J. (2009). *Learning and teaching early math: The learning trajectories approach*. New York: Routledge.

Clements, D. H., Sarama, J., Spitler, M. E., Lange, A. A., and Wolfe, C. B., (2011). Mathematics learned by young children in an intervention based on learning trajectories: A large-scale cluster randomized trial. *Journal for Research in Mathematics Education, 42*(2), 127-166.

Clements, D., Wilson, D., & Sarama, J. (2004). Young children's composition of geometric figures: A learning trajectory. *Mathematical Thinking and Learning, 6(2),* 163-184.

Cobb, P. (1995). Continuing the Conversation: A Response to Smith. *Educational Researcher, 24(7)*, 25-27.

Cobb, P. (2000). Conducting classroom teaching experiments in collaboration with teachers. In R. Lesh & E. Kelly (Eds.), *New methodologies in mathematics and science education (pp. 307-334).* Mahwah, NJ: Lawrence Erlbaum Associates.

Cobb, P. (2007). Putting philosophy to work: Coping with multiple theoretical perspectives. In F. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (p. 3-38). Charlotte, NC: Information Age Publishing.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher, 32*, 9-13.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist*, *31*(3/4), 175-190.

Cobb, P., Yackel, E., & Wood, T. (1992). A constructivist alternative to the representational view of mind in mathematics education. *Journal for Research in Mathematics Education*, *23*(1), 2-33.

Cohen, D. K., Raudenbush, S. W., & Ball, D. L. (2003). Resources, instruction, and research. *Education Evaluation and Policy Analysis*, *25*, 119-142.

Common Core State Standards. (2010). *The common core state standards in mathematics.* Retrieved from http://www.corestandards.org/

Confrey, J. (1988). Multiplication and Splitting: Their Role in Understanding Exponential Functions. In M. Behr, C. LaCompagne, & M. Wheeler (Eds.), *Proceedings of the*

Tenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 250-259), DeKalb, IL: Northern Illinois University Press.

Confrey, J. (2006). The evolution of design studies as methodology. In K. Sawyer (Ed.), *The Cambridge Handbook of the Learning Sciences* (pp.131-151). Cambridge, NY: Cambridge University Press.

Confrey, J. (2012). Better measurement of higher-cognitive processes through learning trajectories and diagnostic assessments in mathematics: The challenge in adolescence. In V. Reyna, M. Dougherty, S. B. Chapman, & J. Confrey (Eds.), *The adolescent brain: Learning, reasoning, and decision making* (pp. 155-182). Washington, DC: American Psychological Association.

Confrey, J., & Lachance, A. (2000). Transformative teaching experiments through conjecture driven research design. In A. E. Kelly & R. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 231-265). Mahwah, NJ: Lawrence Erlbaum Associates.

Confrey, J., Maloney, A., Nguyen, K., Mojica, G., & Myers, M. (2009). Equipartitioning/splitting as a foundation of rational number reasoning using learning trajectories. *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education,* (pp. 345 – 353), Thessaloniki, Greece.

Confrey, J., Maloney, A. P., Wilson, P. H., & Nguyen, K. H. (2010). *Understanding over time: The cognitive underpinnings of learning trajectories.* Paper presented at the annual meeting of the American Education Research Association, Denver, CO.

Confrey, J., Rupp, A., Maloney, A., & Nguyen, K. H. (in review). Toward developing and representing a novel learning trajectory for equipartitioning: Literature review, diagnostic assessment design, and explanatory item response. *Journal for Research in Mathematics Education*.

Confrey, J., & Scarano, G. H. (1995). *Splitting Reexamined: Results from a Three-Year Longitudinal Study of Children in Grades Three to Five*. Paper presented at the 17th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Columbus, OH.

Corcoran, T., Mosher, F. A., & Rogat, A. (2009). Learning progressions in science: An evidence based approach to reform. Retrieved July 10, 2011, from http://www.cpre.org/images/stories/cpre_pdfs/lp_science_rr63.pdf

Corey, D. L., Peterson, B. E., Lewis, B. M., & Bukarau, J. (2010). Are there any place that students use their heads? Principles of high-quality Japanese mathematics instruction. *Journal for Research in Mathematics Education, 41*(5). 438-478.

Creswell, J. W. (2007). *Qualitative inquiry & research design: Choosing among five approaches* (2nd ed.). Thousand Oaks, CA: Sage.

Daro, P., Mosher, F. A., & Corcoran, T. (2011). Learning trajectories in mathematics: A foundation for standards, curriculum, assessment, and instruction. Retrieved May 3, 2011, from http://www.cpre.org/images/stories/cpre_pdfs/learning%20trajectories%20in%20math_ccii%20report.pdf

Decker, K. A., & Ware, H. W. (2001). Elementary teacher planning time: Teacher use, parent perception. Paper presented at the Annual Meeting of the American Educational Research Association, Seattle, WA.

Dewey, J. (1929). *The sources of a science of education.* New York: Horace Liveright.

Edgington, C., Sztajn, P., & Wilson, P. H., & Confrey, J. (2011). *Teachers' use of a learning trajectory for formative assessment.* Paper presented at the Thirty-third Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Reno, Nevada.

Ernst, P. (1991). *The philosophy of mathematics education*. London: The Falmer Press.

Ernst, P. (2005). What is social constructivism in the psychology of mathematics education? In P. Ernst (Ed.), *Constructing mathematical knowledge: Epistemology and mathematics education* (pp. 62-72). Bristol, PA: The Falmer Press.

Ernest, P. (2006). Reflections on theories of learning, *Zentralblatt für Didaktik der Mathematik*, Vol. 38 No. 1: 3-8.

Fennema, E., Carpenter, T. P., Franke, M. L., Levi, L., Jacobs, V. R., & Empson, S. B. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education, 27*(4), 403-434.

Fernandez, C., & Cannon, J. (2005). What Japanese and U.S. teachers think about when constructing mathematics lessons: A preliminary investigation. *The Elementary School Journal, 105*(5), 481-498.

Franke, M. L., Carpenter, T. P., Levi, L., & Fennema, E. (2001). Capturing teachers' generative change: A follow-up study of professional development in mathematics. *American Educational Research Journal, 38*(3), 653-689.

Garet, M.S., Porter, A. C., Desimone, L., Birman, B. F., & Yoon, K. S. (2001). What makes professional development effective? Results from a national sample of teachers. *American Educational Research Journal, 38*(4), 915-945.

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom teaching on students' learning. In F. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (p. 371-404). Charlotte, NC: Information Age Publishing.

Hiebert, J., Morris, A. K., Berk, D., & Jansen, A. (2007). Preparing teachers to learn from teaching. *Journal of Teacher Education, 58*(1), 47-61.

Heritage, M. (2007). Formative assessment: What do teachers need to know and do? *The Phi Delta Kappan, 89*(2), 140-145.

Heritage, M. (2008). *Learning progressions: Supporting instruction and formative assessment.* Retrieved from Council of Chief State School Officers website: http://www.ccsso.org/Resources/Programs/Formative_Assessment_for_Students_and_Teachers_%28FAST%29.html

Heritage, M., Kim, J, Vendlinski, T., & Herman, J. (2009). From evidence to action: A seamless process in formative assessment? *Educational Measurement: Issues and Practice, 28*(3), 24-31.

Hill, H. C., & Ball, D. L. (2004). Learning mathematics for teaching: Results from California's mathematics professional development institutes. *Journal for Research in Mathematics Education, 35*, 330-351.

Jacobs, V. R., Lamb, L. C., Phillips, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education, 41*(2), 169-202.

John, P. D. (2006). Lesson planning and the student teacher: Re-thinking the dominant model. *Journal of Curriculum Studies, 38*(4), 483-498.

Kazemi, E., & Franke, M. L. (2004). Teacher learning in mathematics: Using student work to promote collective inquiry. *Journal of Mathematics Teacher Education, 7,* 203-235.

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. Lester, Jr. (Ed.), *Second handbook of research in mathematics teaching and learning* (pp. 629-667). Charlotte, NC: Information Age.

Lampert, M. (2001). *Teaching problems and the problems of teaching*. New Haven, CT: Yale University.

Lincoln, Y. S., & Guba, E. G. (1985). *Naturalistic inquiry*. Beverly Hills, CA: Sage.

Lubienski, S. T. (2000). Problem solving as a means toward mathematics for all: An exploratory look through a class lens. *Journal for Research in Mathematics Education, 31*(4), 454-482.

Maloney, A. P., & Confrey, J. (2010). The construction, refinement, and early validation of the equipartitioning learning trajectory. *9th international conference of the learning sciences*. Chicago, IL.

McCutcheon, G. (1980). How do elementary school teachers plan? The nature of planning and influences on it. *The Elementary School Journal, 81*(1), 4-23.

McKool, J. K. (2009). *Measurement learning trajectories: A tool for professional development* (unpublished doctoral dissertation). Illinois State University, Normal, Il.

McManus, S. (2008). *Attributes of effective formative assessment*. Retrieved from Council of Chief State School Officers website: http://www.ccsso.org/Resources/Programs/Formative_Assessment_for_Students_and_Teachers_%28FAST%29.html

Merriam, S. B. (1998). *Qualitative research and case study applications in education*. San Francisco: Jossey-Bass.

Miles, M. & Huberman, M. (1994). *Qualitative Data Analysis: An Expanded Sourcebook*. Thousand Oaks, CA: Sage.

Mojica, G. F. (2010). *Preparing pre-service elementary teachers to teach mathematics with learning trajectories* (Unpublished doctoral dissertation). North Carolina State University, Raleigh NC.

National Assessment Governing Board (2008). *Science framework for the 2009 National Assessment of Educational Progress.* Washington, DC: National Assessment Governing Board.

National Research Council (2001). *Knowing what students know.* Washington, DC: National Academy Press.

National Research Council (2007). *Taking science to school*. Washington, DC: National Academy Press.

Nguyen, K. (2010). *Investigating the role of equipartitioning and creating internal units in the construction of a learning trajectory for length and area* (unpublished doctoral dissertation). North Carolina State University, Raleigh, NC.

Oser, F. K., & Baeriswyl, F. J. (2001). Choreographies of teaching: Bridging instruction to learning. In V. Richardson (Ed.), *AERA's handbook of research on teaching (4th ed.,* pp. 1031–1065). Washington, DC: American Educational Research Association.

Patton, M. Q. (1990). *Qualitative evaluation and research methods* (2nd ed.). Newbury Park, CA: Sage.

Piaget, J. (1975). *The Equilibration of Cognitive Structures: The Central Problem of Cognitive Structures*. Chicago, IL: University of Chicago Press.

Piaget, J. (1970). *Genetic Epistemology*. New York, NY: Norton & Company.

Post, T., Cramer, K. A., Behr, M., Lesh, R., & Harel, G. (1993). Curriculum implications of research on the learning, teaching, and assessing of rational number concepts. In T. P. Carpenter, E. Fennema & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 327-362). Hillsdale, NJ: Erlbaum.

Powell, A.B., Francisco, J.M., & Maher, C.A. (2003). An analytic model for studying the development of learners' mathematical ideas and reasoning using videotape data. *Journal of Mathematical Behavior, 22*, 405-435.

Putnam, R. T., & Borko, H. (2000). What do new views of knowledge and thinking have to say about research on teacher learning? *Educational Researcher, 29*(4), 4-15.

Rasmussen, C., & Marrongelle, K. (2006). Pedagogical content tools: Integrating student reasoning and mathematics in instruction. *Journal for Research in Mathematics Education, 37*(5), 388-420.

Sato, M., Wei, R. C., & Darling-Hammond, L. (2008). Improving teachers' assessment practices through professional development: The case of National Board Certification. *American Educational Research Journal*, *45*(3), 669-700.

Schoenfeld, A. (2011). *How we think: A theory of goal-oriented decision making and its educational applications*. New York: Routledge.

Schwandt, T.A. (1998) Constructivist, interpretivist approaches to human inquiry. In: N.K. Denzin and Y.S. Lincoln (Eds.), *The landscape of qualitative research: Theories and issues* (pp. 221-259). London: Sage.

Sherin, B., & Fuson, K. (2005). Multiplication strategies and the appropriate of computational resources. *Journal for Research in Mathematics Education, 36*(4), 347-395.

Sherin, M. G., & van Es, E. A. (2009). Effects of video club participation on teachers' professional vision. *Journal of Teacher Education, 60*, 20-37, doi: 10.1177/0022487108328155.

Sikes, K. A. (2008). *Investigating teachers' perceptions of the data-driven decision making process at a Georgia elementary school* (unpublished doctoral dissertation). Cambridge College, Augusta, GA.

Silver, E. A., Clark, L. M., Ghousseini, H. N., Charalambous, C. Y., & Sealy, J. T. (2007). Where is the mathematics? Examining teachers' mathematics learning opportunities in practice-based professional learning tasks. *Journal of Mathematics Teacher Education, 10*, 261-277. doi 10.1007/210857-007-9039-7.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114-145.

Simon, M. A. (2000). Research on the development of mathematics teachers: The teacher development experiment. In R. Lesh & E. Kelly (Eds.), *New methodologies in mathematics and science education* (pp. 335-359). Mahwah, NJ: Lawrence Erlbaum Associates.

Smith, M. S., & Stein, M. K. (2011). *5 practices for orchestrating productive mathematics discussions.* Restin, VA: National Council of Teachers of Mathematics.

Stake, R. (1995). *The art of case study research*. Thousand Oaks, CA: Sage.

Stein, M., Engle, R., Smith, M., & Hughes, E. (2008). Orchestrating productive mathematical discussions: five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning, 10(4),* 313-340.

Strauss, A., & Corbin, J. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (2nd ed.). Thousand Oaks, CA: Sage.

Superfine, A. C. (2008). Planning for mathematics instruction: A model of experienced teachers' planning processes in the context of a reform mathematics curriculum. *The Mathematics Educator*, *18*(2), 11-22.

Von Glasersfeld, E. (1984). An introduction to radical constructivism. In P. Watzlawick (Ed.), *The invented reality: How do we know what we believe we know? Contributions to constructivism* (pp. 17 – 40). New York: W.W. Norton & Co.

Vygotsky, L. (1986). *Thought and Language*. Cambridge: MIT Press.

Webster-Wright, A. (2009). Framing professional development through understanding authentic professional learning. *Review of Educational Research, 79,* 702- 739.

Wickstrom, M., Baek, J., Barrett, J. E., Cullen, C. J., & Tobias, J. M. (2012). *Teacher's noticing of children's understanding of linear measurement.* Paper presented at the Thirty-fourth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Kalamazoo, Michigan.

Wiliam, D., Lee, C., Harrison, C, & Black, P. (2004). Teachers developing assessment for learning: Impact on student achievement. *Assessment in Education: Principles, Policy and Practice, 11*(1), 49-65.

Wilson, P. H. (2009). *Teacher's uses of a learning trajectory for equipartitioning* (Unpublished doctoral dissertation). North Carolina State University, Raleigh NC.

Wilson, P. H., & Confrey, J. (2011). *Making sense of student work with learning trajectories.* Paper accepted at the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Reno, Nevada.

Wilson, P. H., Edgington, C., & Confrey, J. (2010). *Equipartitioning materials for professional development*, Unpublished manuscript, Raleigh, NC.

Wilson, P. H., Sztajn, P., Edgington, C, & DeCuir-Gunby, J. (2012). *Learning trajectories based instruction codebook for pedagogical practices*, Unpublished manuscript, Raleigh, NC.

Yin, R. K. (2003). *Case study research: Design and methods* (3rd ed.). Thousand Oaks, CA: Sage.

Yinger, R. J. (1980). A study of teacher planning. *The Elementary School Journal*, *80*(3), 107 127.

Zahorik, J.A. (1975). Teachers' planning models. *Educational Leadership*, *33*(2), 134-139.

APPENDICES

Appendix A: Teachers' Beliefs about Mathematics and Mathematics Teaching

This scale presents a listing of sentences. You are to indicate the degree to which you agree or disagree with the opinion or belief expressed in each of the sentences.

> If you strongly disagree with the opinion or belief expressed in a sentence, circle the letters SD to the right of that sentence.
>
> If you disagree with the opinion or belief expressed in a sentence, but not so strongly, circle the letter D to the right of that sentence.
>
> If you are not sure how you feel about the opinion or belief expressed in a sentence, that is you cannot decide or you do not really have an opinion, circle the letter N to the right of that sentence.
>
> If you agree with the opinion or belief expressed in a sentence, circle the letter A to the right of that sentence.
>
> If you strongly agree with the opinion or belief expressed in a sentence, circle the letters SA to the right of that sentence.

There are no "right" or "wrong" answers. The only correct responses are those that reflect what you believe to be true. Be sure to respond to each item in a way that reflects your personal beliefs.

Do not spend too much time pondering each sentence. Read a sentence carefully and then indicate your opinion.

Be sure to respond to every statement.

Mid-Atlantic Center for Mathematics Teaching and Learning
2226 Benjamin Building
University of Maryland
College Park, MD 20742-1175

Developed for use in the MAC-MTL project "Relationship between Teachers' Mathematical Knowledge and Beliefs and Student Achievement" with the support of the National Science Foundation, Grants No. ESI-0353360 and DRL-0426253. Item development and inclusion reflect the opinions of the authors and not necessarily those of the Foundation.

QuickTime™ and a
TIFF (LZW) decompressor
are needed to see this picture.

		Strongly Disagree	Disagree	Not Sure No Opinion	Agree	Strongly Agree
1.	The best way to teach students to solve mathematics problems is to model how to solve one kind of problem at a time until the students achieve mastery and then to foster frequent practice.	SD	D	N	A	SA
2.	I learn about my students' perceptions of their mathematical ability through explicitly asking them (e.g., students write about it, one-on-one discussions, group discussions).	SD	D	N	A	SA
3.	When teaching multi-step word problems, I allow students who are struggling with basic computation to use a calculator.	SD	D	N	A	SA
4.	I like to use mathematics problems that can be solved in many different ways.	SD	D	N	A	SA
5.	When planning mathematics lessons, teachers need to focus explicitly on rules and procedures.	SD	D	N	A	SA
6.	For the majority of my students, I have a good sense of whether or not they see how the mathematics we do in class connects to their everyday lives.	SD	D	N	A	SA
7.	My primary role as a mathematics teacher is to identify and teach the mathematics content that is on the state's mathematics assessment.	SD	D	N	A	SA
8.	I often learn from my students during mathematics class because my students come up with ingenious ways of solving problems that I never thought of.	SD	D	N	A	SA
9.	In my class it is just as important for students to learn data analysis and probability as it is to learn a long division algorithm.	SD	D	N	A	SA

10.	I have a good sense of what my unsuccessful students perceive as challenges to their mathematical performance.	SD	D	N	A	SA
11.	During mathematics class, a teacher should limit the questions he or she asks of a student to those questions that the teacher knows the student can answer correctly.	SD	D	N	A	SA
12.	Mathematics skills are mastered incrementally, so instruction should only focus on one skill at a time, ordered by difficulty, and not move on until most students have mastered that skill.	SD	D	N	A	SA
13.	Students in a mathematics class should be expected to question other students, to question the teacher, and to answer other students' questions.	SD	D	N	A	SA
14.	In order to prepare students for assessments, when students are working on a problem in mathematics, I highlight more than one approach to solving that problem.	SD	D	N	A	SA
15.	A lot of things in mathematics must simply be accepted as true and remembered.	SD	D	N	A	SA
16.	Prior achievement in mathematics determines a student's potential for learning mathematics in the future.	SD	D	N	A	SA
17.	Students who perceive themselves as 'bad' at mathematics need different mathematical experiences than students who see themselves as 'good' at mathematics.	SD	D	N	A	SA

18.	If students use calculators, they will not master the basic skills they need to know.	SD	D	N	A	SA
19.	Students learn mathematics best by paying attention when their teacher demonstrates what to do, by asking questions if they do not understand, and then by practicing.	SD	D	N	A	SA
20.	If students can use manipulatives correctly, then they understand the underlying mathematics.	SD	D	N	A	SA
21.	I learn about my students' perceptions of connections between mathematics and their everyday lives through explicitly asking them (e.g., students write about it, one-on-one discussions, group discussions).	SD	D	N	A	SA
22.	Learning mathematics requires a good memory because you must remember how to carry out procedures and, when solving an application problem, you have to remember which procedure to use.	SD	D	N	A	SA
23.	Grouping students for cooperative work during instruction is not efficient because teachers do not know what students have learned or who did the work.	SD	D	N	A	SA
24.	Students learn mathematics best by working to solve accessible problems that entail a solution process that has not been demonstrated to them.	SD	D	N	A	SA
25.	When two students solve the same mathematics problem correctly using two different strategies, they should share the steps they went through	SD	D	N	A	SA

26.	During mathematics class, I do not necessarily answer students' questions immediately but rather let them struggle and puzzle things out for themselves.	SD	D	N	A	SA
27.	For the majority of my students, I have a good sense of their motivations for wanting to succeed in mathematics.	SD	D	N	A	SA
28.	I like my students to master basic mathematical operations before they tackle complex problems.	SD	D	N	A	SA
29.	During mathematics class, students should be asked to solve problems and complete activities by relying on their own thinking without teachers modeling an approach.	SD	D	N	A	SA
30.	When working in groups, if students are not progressing, the teacher should help students who are stuck by demonstrating the intended solution.	SD	D	N	A	SA
31.	When teaching an objective in the curriculum, a teacher should use knowledge of students' prior mathematical performance to vary problem difficulty across groups of students.	SD	D	N	A	SA
32.	During mathematics class, discussion should focus on students' ideas and approaches, no matter whether their answers are correct or incorrect.	SD	D	N	A	SA
33.	Students can figure out how to solve many mathematics problems without being told what to do.	SD	D	N	A	SA
34.	For the majority of my students, I have a good sense of whether they see themselves as 'good' or 'bad' at mathematics.	SD	D	N	A	SA

35.	The majority of my students think ‘doing mathematics’ is mainly about figuring out what rule or procedure to apply and then doing so.	SD	D	N	A	SA
36.	To teach mathematics, first model the activity, then provide some practice and immediate feedback, and, finally, clarify what the assignment is and how it is to be completed.	SD	D	N	A	SA
37.	I learn about my students’ perceptions of what ‘doing mathematics’ means through explicitly asking them (e.g., students write about it, one-on-one discussions, group discussions).	SD	D	N	A	SA
38.	Students should be homogeneously grouped for instruction and assigned to a curriculum on the basis of their prior mathematical performance.	SD	D	N	A	SA
39.	Students who hold the perception that mathematics is important and connected to their future need to engage in mathematical activities that are different than those needed by students who do not think mathematics is important and connected to their future.	SD	D	N	A	SA
40.	A teacher’s role is to ask students questions and to question their answers so that students will make sense of the mathematics.	SD	D	N	A	SA

Appendix B: Pre-lesson Questionnaire

Name:

Date:

1. What are the learning goals for this particular lesson?

2. Describe any changes or adaptations to the lesson you have made since the planning meeting. Why did you choose to make these changes?

3. Anticipate what you think will happen as you implement this lesson. What particular conceptions do you expect your students to hold? What understandings do you think your students will gain? What misconceptions or difficulties do you think will arise? How do you know?

4. What will you look for as you monitor your students' work? Why?

5. What is your plan for selecting and/or sequencing student work for discussion? Why?

Appendix C: Observation Protocol

Day: Lesson :
Time: Duration of lesson:
Teacher:
Math topic addressed:

1. General description of lesson:

2. What aspects of the EPLT were present in the proposed task?
   a. Levels:
   b. Task parameters:

3. What aspects of the EPLT were addressed in the lesson:
   a. Levels:
   b. Task parameters:

4. In what ways was the task open or closed? How were students guided?

5. In what ways does the task address the intended learning goals?

6. How did the teacher demonstrate listening to students?

7. How did the teacher use students' ideas and solutions in the lesson?

IQA Ratings for lesson observation (CSE Report 681 p. 46-52)
Accountable talk:
1. Participation ________
2. Teacher Linking ________
3. Student Linking ________
4. Asking ________
5. Providing evidence ________

Academic Rigor
1. Potential of task ________
2. Implementation of task ________
3. Discussion following task ________

Clear Expectation
1. Clarity of expectation ________

7. Lesson highlights:
8. Notes:

Appendix D: Post-Lesson Interview Protocol

1. What would you like to tell me about the lesson I observed?

2. What was the most important moment in the lesson for you? Why?
    a. In what ways did the lesson go as you intended or envisioned?

3. What did you do to better understand your students' thinking?

4. What do you think your students learned? How do you know?

5. What aspects of your instruction helped your students' learning? What aspects hindered your students' learning?

6. What were you looking for during the lesson in terms of student work?
    If needed, prompt:
    a. To what degree did students work as you had anticipated?
    b. Did something surprise you?

7. How did you select and sequence students' ideas to share in class?
    If needed, prompt:
    a. What were you thinking about as you had to make these decisions?

8. What connections were you hoping students would make across different approaches or strategies? Were they able to make those connections?

9. Did you make any changes to the lesson during instruction? If so, why?

10. Did you draw upon ideas from the EPLT as you implemented your lesson? If so, what were they?

11. Based on this lesson, what will you teach next? Why?

Appendix E: Lesson #1 Task

Fair Shares

1. You and your partner have some counters. Spill your counters. Partner 1 gets all the red counters and Partner 2 gets all the yellow counters.
   How many red? ________How many yellow? _______ Do you have a fair share?

   Circle one: yes no

   How do you know?
   ____

2. Flip your counters to be all red. Now share them fairly between yourselves. Show how you shared them with words, numbers, or pictures. Make sure to explain your thinking!

3. What do you call each partner's share? ___________________________________

4. Suppose you and your partner join another 2 people and share the counters fairly. Would each person's share be greater or fewer than when you just shared it between the two of you?

   Circle one: greater fewer
   How do you know? ____

5. Show how you would share the counters among all 4 of you. How many counters would each person get? _________

   Explain your thinking with words, numbers, or pictures.

6. Imagine that one person left and you wanted to share the same counters among just 3 people. Would each person's share be greater or fewer that it was in question 5?

   Circle one: greater fewer

   How do you know? ____

7. Show how you would share the counters among 3 of you. How many counters would each person get? ________

   Explain your thinking with words, numbers, or pictures.

Appendix F: Lesson #2 Task

Fair Shares Lesson #2

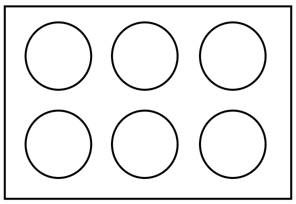
Launch: Begin by asking students to share a rectangle (or circle or square) fairly for two people. Give half the class a larger rectangle and half the class a smaller rectangle. Ask the class what they would name each person's share. Discuss the importance of indicating the whole: Each part can be called "one-half" but ½ of the smaller rectangle is not the same as ½ of the larger rectangle.
Explore: Connect naming half of a whole to naming half of a collection. Work through #1 together, emphasizing that each person gets 3 counters (candies, marbles, stickers, etc), or 3 out of 6 counters, or ½ of the 6 counters. Have students complete the worksheet.
Discuss: Have students share their strategies for sharing the collections and how they named each person's share. Discuss what students' noticed was the same or different from sharing the rectangles to sharing the collections. Emphasize that we can name each part as ½ of the whole.

Name:

Fair Shares

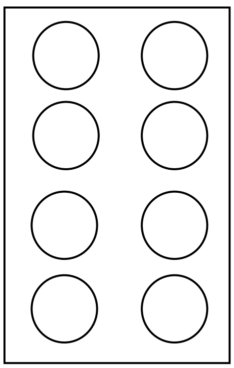
Share the counters fairly between two friends. Color one friend's share red and color the other friend's share blue. Describe how much each friend gets.

1.



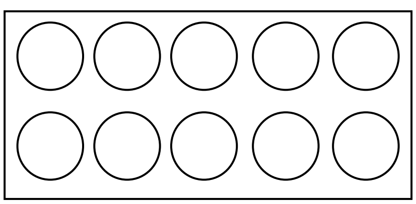
How much does each friend get?

2.



How much does each friend get?

3.



How much does each friend get?

4. What is the same or different about each problem? Explain your thinking.

Appendix G: The Wrapping Paper Task (as presented in the LTBI professional development)

Task

Juana is wrapping holiday gifts. She has [*insert number*] gifts to wrap and needs to create equal-sized pieces from the large piece of wrapping paper drawn below. Help Juana make equal-sized pieces of paper to wrap each gift by drawing a line where she should cut the paper.

Note: In order to see a range of abilities from students, vary the number of gifts used in the task. Suggested ordering for the number of gifts: 2, 4, 6, 9, 5. K-2 students should begin with 2 and try 4 and 6. 3-5 students should begin with 6, drop back to 4 if too difficult or progress through 9 and 5.

Discussion Questions

Justification

- How do you know that each piece of wrapping paper is the same size?
- How could you convince someone that each piece of wrapping paper is the same size?

Naming

- What would you call each piece of wrapping paper? What about if you used a mathematical name?
- If you were a mathematician, how would you name a piece of wrapping paper?
- Is there another way to name each of the equal-sized pieces of wrapping paper?

Emergent Relationships – Compensation

- If there are more [*fewer*] presents to wrap, is the size of each equal-sized piece of paper larger, smaller, or the same size as before?
- How much larger or smaller is a piece of paper now that we have [#] pieces than it was when we had [#] of presents?
- If we had [*twice as many*, *half as many*, *3 times as many*, *one third as many*, *etc*] gifts to wrap, how could you describe the size of each piece of wrapping paper?

Emergent Relationships – Composition of Splits & Multiple Methods

- Is there another way to cut the wrapping paper to make equal-sized pieces?
- How many different ways are there to cut the wrapping paper to make equal-sized pieces?
- Grades 3-5: In which ways does cutting for 6 help you think, or not, about how to cut for 9? What are similarities and what are differences between cutting 6 and cutting 9 pieces?

Emergent Relationships – Transitivity

- Suppose Mark cut 6 [*or another composite #*] strips of wrapping paper but Shauna cut out three rectangles by making two vertical cuts and one horizontal cut [*or some factor equivalent*]. Are Shauna's pieces the same amount of wrapping paper as Mark's? How do you know?

Appendix H: Lara's Wrapping Paper Task

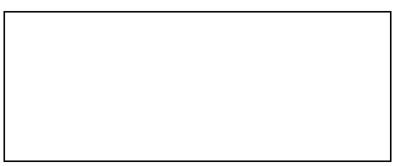
Juana's Wrapping Paper

Juana is wrapping holiday gifts. She has gifts to wrap and needs to create equal-sized pieces from the large piece of wrapping paper drawn below. Help Juana make equal-sized pieces of paper to wrap each gift by drawing a line where she should cut the paper.

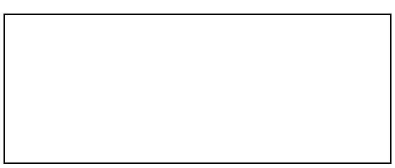
1. Draw how the paper would be cut if you needed to wrap **4** equal sized gifts:

What do you call each piece of wrapping paper?__________________________

How could you show another way to cut the wrapping paper to make **4** equal sized gifts?



2. Draw how the paper would be cut if you needed to wrap **8** equal sized gifts:



What do you call each piece of wrapping paper?________________________

How could you show another way to cut the wrapping paper to make **8** equal sized gifts?

3. Draw how the paper would be cut if you needed to wrap **3** equal sized gifts:

What do you call each piece of wrapping paper?___________________________

How could you show another way to cut the wrapping paper to make **3** equal sized gifts?

4. Draw how the paper would be cut if you needed to wrap **6** equal sized gifts:

What do you call each piece of wrapping paper?____________________________

How could you show another way to cut the wrapping paper to make **6** equal sized gifts?

Are the 6 pieces in each of your two drawings the same size? ________________

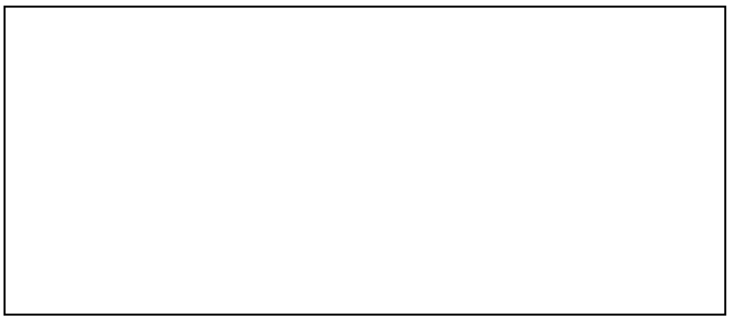
____

How do you know? ____

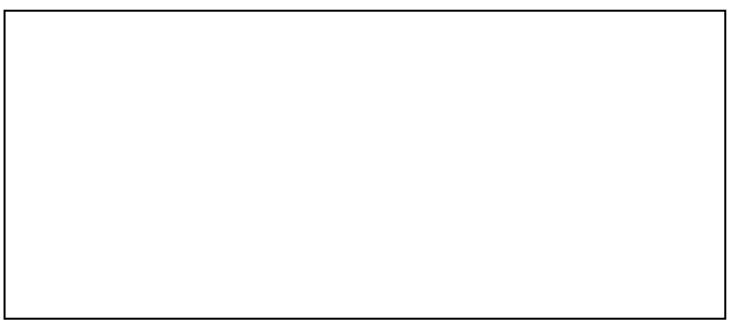
____

Appendix I: Ellen, Bianca, and Tracy's Wrapping Paper Task

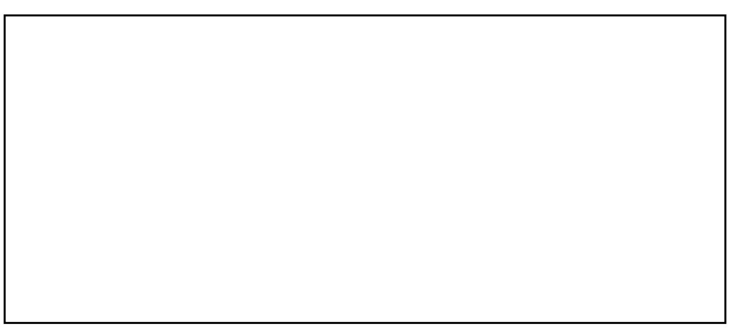
Juana's Wrapping Paper



How much does each person get?



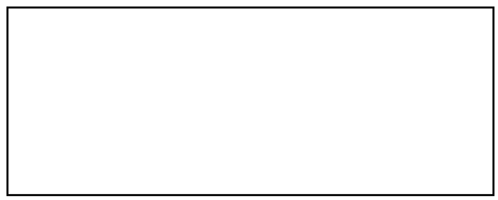
How much does each person get?



How much does each person get?

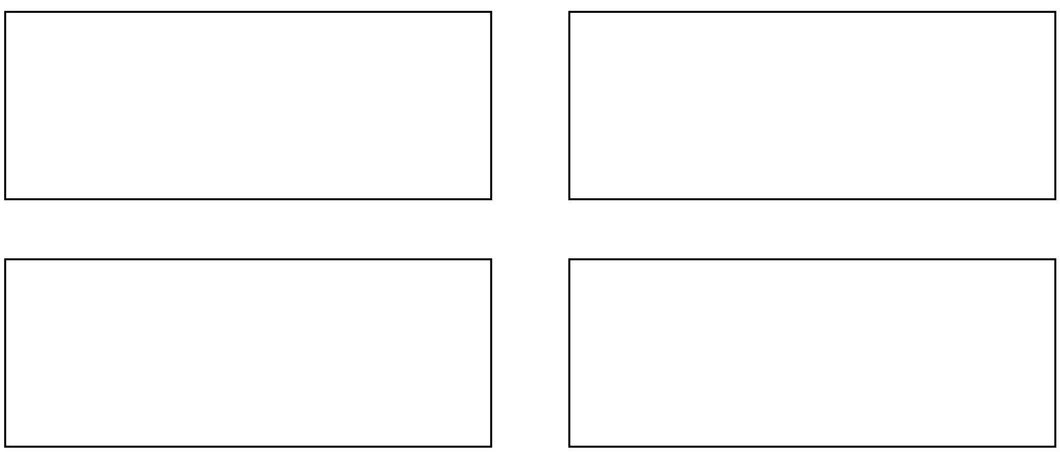
Appendix J: Emma's Wrapping Paper Task

Over the holidays, Juana was wrapping gifts. She has **2** gifts to wrap and needs to create equal sized pieces from the large piece of wrapping paper drawn below. Help Juana make equal-sized pieces of paper to wrap each gift by drawing a line where she should cut the paper.



What would you call each piece of wrapping paper? ______________________________

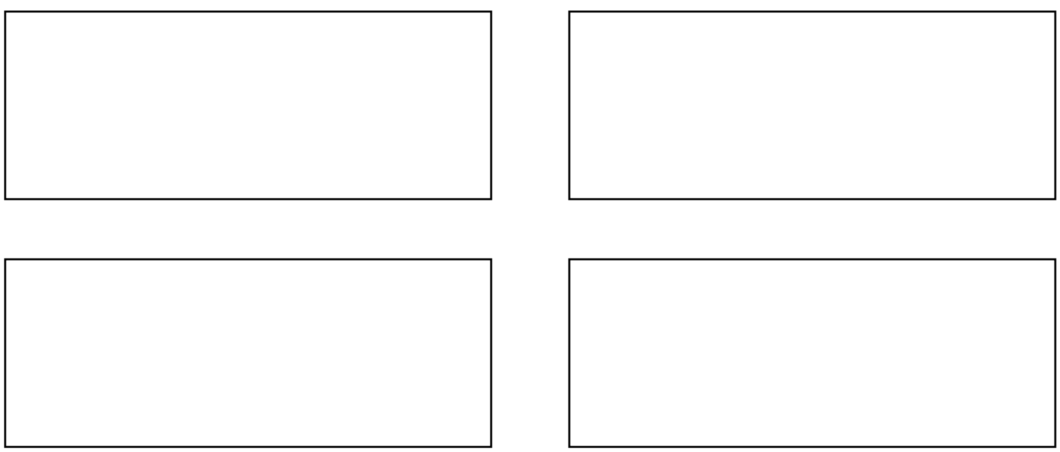
Can you cut the wrapping paper into 2 equal parts any other ways?



Over the holidays, Juana was wrapping gifts. Now she has **<u>4</u>** gifts to wrap and needs to create equal sized pieces from the large piece of wrapping paper drawn below. Help Juana make equal-sized pieces of paper to wrap each gift by drawing a line where she should cut the paper.

What would you call each piece of wrapping paper? ________________________

Can you cut the wrapping paper into 4 equal parts any other ways?



Over the holidays, Juana was wrapping gifts. Now she has <u>**8**</u> gifts to wrap and needs to create equal sized pieces from the large piece of wrapping paper drawn below. Help Juana make equal-sized pieces of paper to wrap each gift by drawing a line where she should cut the paper.

What would you call each piece of wrapping paper? ________________________

Can you cut the wrapping paper into 8 equal parts any other ways?