Abstract

BRENNEMAN, KATHRYN ALETA. Lifting k-Involutions from the Root System to the Lie Algebra. (Under the direction of A. G. Helminck.)

We use the idea of admissibility as described in [Helminck(1988)]. Since every admissible (Γ, θ)-index is a combination of admissible absolutely irreducible (Γ, θ)-indices, it suffices to classify the admissible absolutely irreducible (Γ, θ)-indices and the admissible $\Bbbk$-involutions related to these indices. In this thesis, we begin the classification of $\Bbbk$-involutions by building the classification for the "first" non-algebraically-closed $\Bbbk$, the $\mathbb{R}$-involutions. It is a result of [Helminck(1988)] that the $\mathbb{R}$-involutions can be classified by commuting pairs of $\mathbb{C}$-involutions. Before we can classify the $\mathbb{R}$-involutions by way of commuting pairs of $\mathbb{C}$-involutions, we must first know how to lift $\mathbb{C}$-involutions. Since the existing results in the literature are incorrect [Watson(2010)], we begin with our own construction and analysis of $\mathbb{C}$-involutions.

Lifting k-Involutions from the Root System to the Lie Algebra

by
Kathryn Aleta Brenneman

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APPROVED BY:

Kailash Misra	Amassa Faunteleroy
Ernest Stitzinger	A. G. Helminck Chair of Advisory Committee

Dedication

To my parents and all my friends and family who have supported me along the way.
I would not have done it without you.

Biography

The author was born in Fort Wayne, Indiana, and attended Canterbury School for her primary and secondary education graduating high school in the class of 2002. She discovered her passion for mathematics and puzzle solving during her undergraduate career at Smith College of Northampton, MA, whence she earned her Bachelors degree in Mathematics and Computer Science in 2006. While at Smith, she was also épée squad captain of the Hell's Belles Fencing Club and took up karate while at NC State.

Kate loves languages, their histories, and linguistic theory, and has studied to varying degrees French, Old English, Old Norse, Icelandic, and, of course, Mathematics. In her travels, she picked up smatterings of Italian, Spanish, and Mandarin Chinese. She is often to be found enjoying sunshine, atmosphere, and life in general, in a local coffee shop.

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I would like to thank my advisor for his longstanding support and for seeing my potential before I could. Before him I had a long string of teachers and advisors who were my friends and mentors, and who still inspire me today through their lessons and advice that I carry with me always. [1]

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Part I

Background and Preliminaries

Chapter 1: Introduction

1.1 Background and Motivation

Symmetric spaces have been studied for over 100 years. Initially they were only studied over the real numbers, but in the last 25 to 30 years generalization of symmetric spaces over other fields have become important in other areas of mathematics as well. In the following we will give a brief introduction.

Definition 1.1.1. Let G be a group, $\theta \in \mathrm{Aut}(G)$ an involution, i.e. $\theta^2 = \mathrm{id}$ and

$$H = G^\theta = \{x \in G \mid \theta(x) \ = x\}$$

the *fixed point group* of θ. Let $\tau : G \to G$ be the map defined by $\tau(x) = x\theta(x)^{-1}$ and

$$Q = \tau(G) = \{x\theta(x)^{-1} \mid x \in G\}.$$

Then $Q \simeq G/H$ is called a *generalized symmetric space*. If G is a reductive linear algebraic group defined over a field $\Bbbk$ of characteristic not 2, then Q is also called a *symmetric* $\Bbbk$*-variety*. When $\Bbbk \neq \mathbb{R}$, then it is usually called a *reductive symmetric space*.

If $\Bbbk = \mathbb{R}$, then Q is also called an *affine symmetric space*. Moreover if H is compact, then X is also called a *Riemannian symmetric space*. These symmetric spaces play an essential role in many areas of mathematics including mathematical physics, Lie theory, representation theory and differential geometry. We note that with this definition every linear algebraic group is a symmetric $\Bbbk$-variety.

Example 1.1.2. *Group case:* Consider $G_1 = G \times G$ and $\theta(x, y) = (y, x)$, then $H = \{(x, x) \mid x \in G\} \simeq G$ embedded diagonally and $Q = \{(x, x^{-1}) \mid x \in G\} \simeq G$ embedded anti-diagonally.

1.2 Fine structure of involutions and $\Bbbk$-groups

There is a natural fine structure related to reductive algebraic groups, which comes from the root system and associated Weyl group of a maximal torus. For a reductive algebraic group G defined over a field $\Bbbk$ the $\Bbbk$-structure of the group gives rise to a natural fine structure as well, which is related to a maximal $\Bbbk$-split torus. For an involution θ of a reductive group G it was shown in Helminck [Helminck(1988)] that there is a similar fine structure related to maximal θ-split tori. Here a torus A is called *θ-split* if $\theta(a) = a^{-1}$ for all $a \in A$. To study $\Bbbk$-involutions one needs to combine the fine structure related to the involution with the fine structure related to the $\Bbbk$-structure of the group. The natural fine structure related to symmetric $\Bbbk$-varieties and $\Bbbk$-involutions comes from the maximal $(\theta, \Bbbk)$-split tori. In this case, a torus A is called *$(\theta, \Bbbk)$-split* if it is both θ-split and $\Bbbk$-split. In this section we will introduce all this fine structure.

1.2.1 Root Data

To deal with the notion of root system in reductive groups it is quite useful to work with the notion of root datum.

Notation 1.2.2. If T is a torus of G, then we denote by

$$\begin{aligned} X &= X^*(T), && \text{the set of characters,} \\ \Phi &= \Phi(T), && \text{the set of roots,} \\ X^\vee &= X_*(T), && \text{the set of one parameter subgroups,} \\ \Phi^\vee &= \Phi^\vee(T), && \text{the set of co-roots, and} \\ W &= W(T), && \text{the Weyl group of } T \text{ with respect to } G. \end{aligned}$$

Definition 1.2.3. If T is a torus in a reductive group G, such that $\Phi(T)$ is a root system with Weyl group $W(T)$, then the *root datum* associated to the pair (G, T) is the quadruple $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$.

Remark 1.2.4. If T_1 and T_2 are tori and ϕ is a homomorphism of T_1 into T_2, then the mapping ${}^t\phi$ of $X^*(T_2)$ into $X^*(T_1)$, defined by

$$ {}^t\phi(\chi_2) := \chi_2 \circ \phi, \quad \chi_2 \in X^*(T_2) \tag{1.2.4 (a)}$$

is a module homomorphism. If ϕ is an isomorphism, then ${}^t\phi^{-1}$ is a module isomorphism from $(X^*(T_1), \Phi(T_1))$ onto $(X^*(T_2), \Phi(T_2))$.

1.2.5 Actions on root data

In the study of $\mathbb{k}$-involutions one has to combine the $\mathbb{k}$-structure of the group with the structure of the involution. For this one has to combine the actions on the related root data. This can be seen as follows. Let G be a reductive $\mathbb{k}$-group, T a maximal $\mathbb{k}$-torus of G, $X = X^*(T)$, $\Phi = \Phi(T)$, $\mathbb{K}$ a finite Galois extension of $\mathbb{k}$ which splits T and $\Gamma = \mathrm{Gal}(\mathbb{K}/\mathbb{k})$ the Galois group of $\mathbb{K}/\mathbb{k}$. If $\phi \in \mathrm{Aut}(G,T)$ is defined over $\mathbb{k}$, then $\phi^\star := {}^t(\phi|T)^{-1}$ satisfies

$$\sigma\phi^\star = \phi^\star\sigma \text{ for all } \sigma \in \Gamma. \tag{1.2.5 (a)}$$

If $\theta \in \mathrm{Aut}(G,T)$ is a $\mathbb{k}$-involution, then we will also write θ for $\theta^\star := {}^t(\theta|T)^{-1} \in \mathrm{Aut}(X,\Phi)$. Both Γ and θ act on (X,Φ). Let $\mathcal{E}_\theta = \{\mathrm{id}, -\theta\} \subset \mathrm{Aut}(X,\Phi)$ be the subgroup spanned by $-\theta|T$. Let $\mathcal{E}_\Gamma \subset \mathrm{Aut}(X,\Phi)$ be the subgroup corresponding to the action of Γ on (X,Φ) and let $\Gamma_\theta = \mathcal{E}_\Gamma.\mathcal{E}_\theta$ be the subgroup of $\mathrm{Aut}(X,\Phi)$ generated by $\mathcal{E}_\Gamma$ and $\mathcal{E}_\theta$. It was shown in [Helminck(2000)] that Γ_θ is a finite subgroup of $\mathrm{Aut}(X,\Phi)$. The actions of Γ, θ, resp. Γ_θ on (X,Φ) all lead to natural restricted root systems and as it turns out these are precisely the restricted root systems related to a maximal $\mathbb{k}$-split, θ-split resp. $(\theta,\mathbb{k})$-split torus. Since all three of these actions on the root datum can be described in a similar manner we will consider in the remainder of this section the action of an arbitrary finite group $\mathcal{E}$ on (X,Φ).

1.2.6. Let $\Psi = (X,\Phi,X^\vee,\Phi^\vee)$ be a root datum with $\Phi \neq \emptyset$ and let $\mathcal{E}$ be a finite group acting on Ψ. For $\sigma \in \mathcal{E}$ and $\chi \in X$ we will also write χ^σ or $\sigma(\chi)$ for the element $\sigma.\chi \in X$. Write $W = W(\Phi)$ for the Weyl group of Φ. Now define the following:

$$X_0 = X_0(\mathcal{E}) := \{\chi \in X \mid \sum_{\sigma\in\mathcal{E}} \chi^\sigma = 0\} \text{ and } \Phi_0 = \Phi_0(\mathcal{E}) := \Phi \cap X_0. \tag{1.2.6 (a)}$$

Then X_0 is a co-torsion free submodule of X and Φ_0 is a closed subsystem of Φ, both invariant under the action of $\mathcal{E}$. Denote the Weyl group of Φ_0 by W_0 and identify it with the subgroup of $W(\Phi)$ generated by the reflections $s_\alpha, \alpha \in \Phi_0$. Put $W^{\mathcal{E}} := \{w \in W \mid w(X_0) = X_0\}$, $\overline{X}_{\mathcal{E}} := X/X_0(\mathcal{E})$ and let π be the natural projection from X to $\overline{X}_{\mathcal{E}}$. If we take $A = \{t \in T \mid \chi(t) = e \text{ for all } \chi \in X_0\}$ to be the annihilator of X_0 and $Y = X^*(A)$, then Y may be identified with $\overline{X}_{\mathcal{E}} = X/X_0$. Let $\overline{\Phi}_{\mathcal{E}} := \pi(\Phi - \Phi_0(\mathcal{E}))$ denote the set of *restricted roots of* Φ *relative to* $\mathcal{E}$.

Remark 1.2.7. In the case that $\mathcal{E} = \Gamma$, then X_0 is the annihilator of a maximal $\mathbb{k}$-split torus A of T. Similarly in the case that $\mathcal{E} = \mathcal{E}_\theta$, then X_0 is the annihilator of a maximal θ-split torus A of G. In both these cases, if A is maximal $\mathbb{k}$-split resp. θ-split in G then $\overline{\Phi}_{\mathcal{E}}$ is the root system of A with Weyl group $\overline{W}_{\mathcal{E}}$.

We define now an order on (X,Φ) related to the action of $\mathcal{E}$ as follows.

Definition 1.2.8. A linear order on X which satisfies

$$\text{if } \chi \succ 0 \ \text{ and } \ \chi \notin X_0, \text{ then } \ \chi^{\sigma} \succ 0 \ \text{ for all } \ \sigma \in \mathcal{E} \tag{1.2.8 (a)}$$

is called a *$\mathcal{E}$-linear order*. A fundamental system of Φ with respect to a $\mathcal{E}$-linear order is called a *$\mathcal{E}$-fundamental system of* Φ or a *$\mathcal{E}$-basis of* Φ.

A $\mathcal{E}$-linear order on X induces linear orders on $Y = X/X_0$ and X_0, and conversely, given linear orders on X_0 and on Y, these uniquely determine a $\mathcal{E}$-linear order on X, which induces the given linear orders (i.e. if $\chi \notin X_0$, then define $\chi \succ 0$ if and only if $\pi(\chi) \succ 0$). Instead of the above $\mathcal{E}$-linear order one could give a more general definition of a linear order on X using only the fact that X_0 is a co-torsion free submodule of X. In the following we give a number of properties of an $\mathcal{E}$-linear order on X.

1.2.9 Restricted fundamental system

Fix a $\mathcal{E}$-linear order $\succ$ on X, let Δ be a $\mathcal{E}$-fundamental system of Φ and let Δ_0 be a fundamental system of Φ_0 with respect to the induced order on X_0. Let $A = \{t \in T \mid \chi(t) = e \text{ for all } \chi \in X_0\}$ be the annihilator of X_0 and define $\overline{\Delta}_{\mathcal{E}} = \pi(\Delta - \Delta_0)$. This is called a *restricted fundamental system* of Φ relative to A or also a *restricted fundamental system* of $\overline{\Phi}_{\mathcal{E}}$. The following proposition lists some properties of these fundamental systems.

Proposition 1.2.10. *Let X, X_0, Φ, Φ_0, $\overline{\Phi}_{\mathcal{E}}$, etc. be defined as above and let Δ, Δ' be $\mathcal{E}$-fundamental systems of Φ. Then we have the following*

(1) $\Delta_0 = \Delta \cap \Phi_0$.

(2) $\Delta = \Delta'$ *if and only if* $\Delta_0 = \Delta'_0$ *and* $\overline{\Delta}_{\mathcal{E}} = \overline{\Delta}'_{\mathcal{E}}$.

(3) If $\overline{\Delta}_{\mathcal{E}} = \overline{\Delta}'_{\mathcal{E}}$, then there exists a unique $w' \in W_0$ such that $\Delta' = w'\Delta$.

1.2.11 Restricted Weyl group

There is a natural (Weyl) group associated with the set of restricted roots, which is related to $W^{\mathcal{E}}/W_0$. Since W_0 is a normal subgroup of $W^{\mathcal{E}}$, every $w \in W^{\mathcal{E}}$ induces an automorphism of $\overline{X}_{\mathcal{E}} = X/X_0 = Y$. Denote the induced automorphism by $\pi(w)$. Then $\pi(w\chi) = \pi(w)\pi(\chi)$ $(\chi \in X)$. Define $\overline{W}_{\mathcal{E}} := \{\pi(w) \mid w \in W^{\mathcal{E}}\}$. We call this the *restricted Weyl group* with respect to the action of $\mathcal{E}$ on X. It is not necessarily a Weyl group in the sense of Bourbaki [Bourbaki(1981), Ch.VI,no.1], however, we do have the following.

Proposition 1.2.12. *Let* X, X_0, Φ, Φ_0, $\overline{\Phi}_{\mathcal{E}}$, Δ, Δ_0, $\overline{\Delta}_{\mathcal{E}}$, W_0, $W^{\mathcal{E}}$, $\overline{W}_{\mathcal{E}}$ *be defined as above and let A be the annihilator of X_0. Then we have:*

(1) If $w \in W^{\mathcal{E}}$, then $w(\Delta)$ is an $\mathcal{E}$-fundamental system.

(2) Given $w \in W^{\mathcal{E}}$, $w \in W_0$ iff $\pi(w) = 1$ iff $\pi(w)\overline{\Delta}_{\mathcal{E}} = \overline{\Delta}_{\mathcal{E}}$.

(3) $\overline{W}_{\mathcal{E}} \approx W^{\mathcal{E}}/W_0$.

(4) $W^{\mathcal{E}}/W_0 \approx N_G(A)/Z_G(A)$, where $N_G(A)$ and $Z_G(A)$ are respectively the normalizer and centralizer of A in G.

Remarks 1.2.13. (1) In the case that A is a maximal $\Bbbk$-split, θ-split or $(\theta, \Bbbk)$-split torus, then $\overline{\Phi}_{\mathcal{E}}$ is actually a root system with Weyl group $\overline{W}_{\mathcal{E}}$. (2) In the remainder of this section we will also write $\overline{\Phi}$, $\overline{\Delta}$, $\overline{W}$ instead of $\overline{\Phi}_{\mathcal{E}}$, $\overline{\Delta}_{\mathcal{E}}$, $\overline{W}_{\mathcal{E}}$ whenever it causes no confusion.

1.2.14 Action of $\mathcal{E}$ on Δ

From Proposition 1.2.12 it follows that $W^{\mathcal{E}}$ acts on the set of $\mathcal{E}$-fundamental systems of Φ. There is also a natural action of $\mathcal{E}$ on this set. If Δ is a $\mathcal{E}$-fundamental system of Φ, and $\sigma \in \mathcal{E}$, then the $\mathcal{E}$-fundamental system $\Delta^\sigma := \{\alpha^\sigma \mid \alpha \in \Delta\}$ gives the same restricted basis as Δ, i.e. $\overline{\Delta}^\sigma = \overline{\Delta}$. This follows from the fact that $\alpha_i \equiv \alpha_i^\sigma \mod X_0$ for all $\alpha_i \in \Delta,\ \sigma \in \mathcal{E}$. From Proposition 1.2.10 it follows that there is a unique element $w_\sigma \in W_0$ such that $\Delta^\sigma = w_\sigma \Delta$. This means we can define a new operation of $\mathcal{E}$ on X as follows:

$$\chi^{[\sigma]} := w_\sigma^{-1}\chi^\sigma, \quad \chi \in X, \quad \sigma \in \mathcal{E}. \tag{1.2.14 (a)}$$

It is easily verified that $\chi \to \chi^{[\sigma]}$ is an automorphism of the triple (X, Φ, Δ) and that $\chi^{[\sigma][\gamma]} = \chi^{[\sigma\gamma]}$ for all $\sigma,\ \gamma \in \mathcal{E},\ \chi \in X$.

1.3 (Γ, θ)-indices

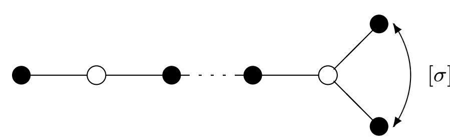
The actions of Γ and θ on the root datum can be described by an index. These indices not only determine the fine structure of restricted root systems with multiplicities etc. of the corresponding $\Bbbk$-group and symmetric variety, but also play an important role in the classifications of $\Bbbk$-groups and symmetric varieties (or equivalently involutions of reductive groups). In this section we extend these indices to get an index that describes the action of a $\Bbbk$-involution. Similar as for $\Bbbk$-groups and symmetric varieties this index describes the fine structure of restricted root systems with multiplicities etc. of the corresponding symmetric $\Bbbk$-variety, and again plays an important role in the classification of $\Bbbk$-involutions.

1.3.1 The index of $\mathcal{E}$

Throughout this section let Ψ be a semisimple root datum with $\Phi \neq \emptyset$, $\mathcal{E}$ a (finite) group acting on Ψ, as in 1.2.6, Δ a $\mathcal{E}$-basis of Φ and $\Delta_0 = \Delta_0(\mathcal{E}) = \Delta \cap X_0(\mathcal{E})$. In equation (1.2.14 (a)) we defined an action of $\mathcal{E}$ on Δ, which we denote by $[\sigma]$. The action of $\mathcal{E}$ on Ψ is essentially determined by Δ, Δ_0 and $[\sigma]$. Following Tits [Tits(1966)] we will call the quadruple (X, Δ, Δ_0, $[\sigma]$) an *index of* $\mathcal{E}$ or an $\mathcal{E}$*-index*. We will also use the name $\mathcal{E}$*-diagram*, following the notation in Satake [Satake(1971), 2.4].

1.3.2

As in [Tits(1966)] we make a diagrammatic representation of the index of $\mathcal{E}$ by coloring black those vertices of the ordinary Dynkin diagram of Φ, which represent roots in $\Delta_0(\mathcal{E})$ and indicating the action of $[\sigma]$ on Δ by arrows. An example in type D_ℓ is:



To use these $\mathcal{E}$-indices in the characterization of isomorphy classes of reductive $\Bbbk$-groups or involutions, we need a notion of isomorphism between them.

Definition 1.3.3. Let Ψ and Ψ' be semisimple root data and $\mathcal{E}$ a group acting on them. A *congruence* φ of the $\mathcal{E}$-index (X, Δ, Δ_0, $[\sigma]$) of Ψ onto the $\mathcal{E}$-index (X', Δ', Δ'_0, $[\sigma]'$) of Ψ' is an isomorphism which maps (X, Δ, Δ_0) $\to$ (X', Δ', Δ'_0), and satisfies $[\sigma]' = \varphi[\sigma]\varphi^{-1}$.

For $\Bbbk$-involutions it suffices to consider two actions of $\mathcal{E}$ on the same root datum. In that case we will also use the term *isomorphic* $\mathcal{E}$*-indices* instead of congruent $\mathcal{E}$-indices. In this case one can differentiate between inner and outer automorphisms.

Definition 1.3.4. Let Ψ be a root datum and $\mathcal{E}_1$, $\mathcal{E}_2 \subset \mathrm{Aut}(\Psi)$ the subgroups of $\mathrm{Aut}(\Psi)$ corresponding to actions of $\mathcal{E}$ on Ψ. Two indices (X, Δ, $\Delta_0(\mathcal{E}_1)$, $[\sigma]_1$) and $(X, \Delta', \Delta'_0(\mathcal{E}_2), [\sigma]_2)$ are said to be $W(\Phi)$*-isomorphic* (resp. $\mathrm{Aut}(\Phi)$*-isomorphic*) if there is a $w \in W(\Phi)$ (resp. $w \in \mathrm{Aut}(\Phi)$), which maps $(\Delta, \Delta_0(\mathcal{E}_1))$ onto $(\Delta', \Delta'_0(\mathcal{E}_2))$ and satisfies $w[\sigma]_1 w^{-1} = [\sigma]_2$. Instead of $W(\Phi)$-isomorphic we will also simply use the term *isomorphic*.

Remark 1.3.5. An index of $\mathcal{E}$ may depend on the choice of the $\mathcal{E}$-basis of Φ, i.e. for two $\mathcal{E}$-bases Δ, Δ', the corresponding indices $(X, \Delta, \Delta_0(\mathcal{E}), [\sigma])$ and $(X, \Delta', \Delta'_0(\mathcal{E}), [\sigma]')$ need not be isomorphic. However this cannot happen if $\overline{\Phi}_{\mathcal{E}}$ is a root system with Weyl group $\overline{W}_{\mathcal{E}}$:

Proposition 1.3.6. *Let Ψ be a semisimple root datum and $\mathcal{E} \subset \mathrm{Aut}(\Psi)$ a group acting on Ψ such that $\overline{\Phi}_{\mathcal{E}}$ is a root system with Weyl group $\overline{W}_{\mathcal{E}}$. If Δ, Δ' are $\mathcal{E}$-bases of Φ, then $(X, \Delta, \Delta_0(\mathcal{E}), [\sigma])$ and $(X, \Delta', \Delta'_0(\mathcal{E}), [\sigma]')$ are isomorphic.*

Theorem 1.3.7. *Let G_1, G_2 be connected semisimple groups defined over $\Bbbk$. For $i = 1, 2$ let T_i be a maximal $\Bbbk$-torus of G_i, $\Psi_i = (X^*(T_i), \Phi(T_i), X_*(T_i), \Phi^\vee(T_i))$ the root datum corresponding to (G_i, T_i), $\mathcal{E}$ a (finite) group acting on Ψ_i, $X_0(\mathcal{E}, T_i) = \{\chi \in X^*(T_i) \mid \sum_{\sigma\in\mathcal{E}} \chi^\sigma = 0\}$, $A_i = \{t \in T_i \mid \chi(t) = e \text{ for all } \chi \in X_0(\mathcal{E}, T_i)\}$ the annihilator of $X_0(\mathcal{E}, T_i)$, $\Delta(T_i)$ a $\mathcal{E}$-basis of $\Phi(T_i)$, $\Delta_0(T_i) = \Delta(T_i) \cap X_0(\mathcal{E})$ and $[\sigma]_i$ the action of $\mathcal{E}$ on $\Delta(T_i)$. If $\varphi : (G_1, T_1, A_1) \to (G_2, T_2, A_2)$ is a $\Bbbk$-isomorphism and $\varphi^\star = {}^t(\varphi|T_1)^{-1}$ is as in* (1.2.4 (a))*, then there exists a unique $w \in W^{\mathcal{E}}(T_2)$ such that $w(\varphi^\star(\Delta(T_1))) = \Delta(T_2)$ and $\varphi^{[\star]} := w\varphi^\star$ is a congruence from $(X^*(T_1), \Delta(T_1), \Delta_0(T_1), [\sigma]_1)$ to $(X^*(T_2), \Delta(T_2), \Delta_0(T_2), [\sigma]_2)$.*

Definition 1.3.8. If $\phi : (G_1, T_1, A_1) \to (G_2, T_2, A_2)$ is a $\Bbbk$-isomorphism as in Theorem 1.3.7, then we will call the congruence $\varphi^{[\star]} := w\varphi^\star$ of the $\mathcal{E}$-indices $(X^*(T_1), \Delta(T_1), \Delta_0(T_1), [\sigma]_1)$ and $(X^*(T_2), \Delta(T_2), \Delta_0(T_2), [\sigma]_2)$ the *congruence associated with φ*.

In the cases of $\mathcal{E} = \mathcal{E}_\theta$ and $\mathcal{E} = \Gamma$ we get the well known θ-*index* and Γ-*index*, which are essential in the respective classifications. Since the classification of $\Bbbk$-involutions depends on a classification of these, we will briefly review these in the next subsections. First, though, we need a notion of irreducibility for $\mathcal{E}$-indices.

Definition 1.3.9. Let $\mathcal{E} \subset \mathrm{Aut}(X, \Phi)$ be a subgroup and Δ a $\mathcal{E}$-basis of Φ. An index $\mathcal{D} = (X, \Delta, \Delta_0, [\sigma])$ is $\mathcal{E}$-*irreducible* if Δ is not the union of two mutually orthogonal $[\sigma]$-invariant (non-empty) subsystems Δ', Δ''. The system $\mathcal{D}$ is *absolutely irreducible* if Δ is connected. In the case $\mathcal{E} = \mathcal{E}_\Gamma$ (resp. $\mathcal{E}_\theta$) we will also call an $\mathcal{E}$-irreducible index a $\Bbbk$-*irreducible index* (resp. θ-*irreducible index*).

By abuse of notation, we will say $\theta \in \mathrm{Aut}(X, \Phi)$ is an *irreducible* (resp. *absolutely irreducible*) involution if it has θ-irreducible (resp. absolutely irreducible) index.

1.3.10 θ-INDICES

In this subsection we discuss the index associated with an involutorial automorphism of a reductive algebraic group. Let G be a reductive algebraic group, $\theta \in \mathrm{Aut}(G)$ an involution and T a θ-stable maximal torus of G. Again, write $X = X^*(T)$, $\Phi = \Phi(T)$ and let $\mathcal{E}_\theta = \{\mathrm{id}, -\theta\} \subset \mathrm{Aut}(X, \Phi)$ be the subgroup spanned by $-\theta|T$. In this case we will also write $X_0(\theta)$, $\overline{X}_\theta$, $\Phi_0(\theta)$,

$\overline{\Phi}_\theta$, $W_1(\theta)$, $\overline{W}_\theta$, $\Delta_0(\theta)$, $\overline{\Delta}_\theta$ instead of, respectively, $X_0(\mathcal{E}_\theta)$, $\overline{X}_{\mathcal{E}_\theta}$, $\Phi_0(\mathcal{E}_\theta)$, $\overline{\Phi}_{\mathcal{E}_\theta}$, $W_0(\mathcal{E}_\theta)$, $W_1(\mathcal{E}_\theta)$, $\overline{W}_{\mathcal{E}_\theta}$, $\Delta_0(\mathcal{E}_\theta)$, $\overline{\Delta}_{\mathcal{E}_\theta}$.

Definition 1.3.11. A $\mathcal{E}_\theta$-order on X will also be called a θ-*order* on X, a $\mathcal{E}_\theta$-basis of Φ a θ-*basis* of Φ, and a $\mathcal{E}_\theta$-index a θ-*index*.

Let Δ be a θ-basis of Φ. To find the θ-index we need to find the action of $[-\theta]$ on (X, Φ, Δ). Since $\theta(-\Delta)$ is also a θ-basis of Φ with the same restricted basis, it follows from Proposition 1.2.10 that there is $w_0(\theta) \in W_0(\theta)$ such that $w_0(\theta)\theta(\Delta) = -\Delta$. Put $\theta^* = \theta^*(\Delta) = -w_0(\theta)\theta$. Then $\theta^* = [-\theta]$. Note that $\theta^*(\Delta) \in \mathrm{Aut}(X, \Phi, \Delta) := \{\phi \in \mathrm{Aut}(X, \Phi) \mid \phi(\Delta) = \Delta\}$, $\theta^*(\Delta)^2 = \mathrm{id}$ and $\theta^*(\Delta_0(\theta)) = \Delta_0(\theta)$.

Proposition 1.3.12. *Let A be a maximal θ-split torus of G, $T \supset A$ a maximal torus and Δ a θ-basis of $\Phi(T)$. The θ-index $(X,\ \Delta,\ \Delta_0,\ \theta^*)$ is uniquely determined (up to congruence) by the isomorphy class of θ.*

1.3.13 Γ-indices

In this subsection we introduce the index related to the isomorphy classes of semisimple $\Bbbk$-groups. For the remainder of this section let G be a reductive $\Bbbk$-group, A a $\Bbbk$-split torus of G, $T \supset A$ a maximal $\Bbbk$-torus, $\mathbb{K}$ the smallest Galois extension of $\Bbbk$ which splits T. Let $\Gamma = \mathrm{Gal}(\mathbb{K}/\Bbbk)$ be the Galois group of $\mathbb{K}/\Bbbk$, $X = X^*(T)$, $\Phi = \Phi(T)$, $X_0 = X_0(\Gamma)$, $\Phi_0 = \Phi_0(\Gamma)$, etc. Let $G_0 = G(\Phi_0)$ denote the connected semisimple subgroup of G generated by $\{U_\alpha \mid \alpha \in \Phi_0\}$. Note that the group G_0 is the semisimple part of $Z_G(A)$. If A is a maximal $\Bbbk$-split torus, then G_0 is anisotropic over $\Bbbk$ and is uniquely determined (up to $\Bbbk$-isomorphy) by the $\Bbbk$-isomorphism class of G. In that case G_0 is also called the $\Bbbk$-*anisotropic kernel* of G.

1.3.14

Let Δ be a Γ-basis of Φ, and let $\Delta_0 = \Delta \cap X_0$. As in (1.2.14 (a)) we have an action of Γ on Δ, which we denote by $[\sigma]$. The 4-tuple $(X,\ \Delta,\ \Delta_0,\ [\sigma])$ is called the Γ-*index of* (G, T, A). If A is a maximal $\Bbbk$-split torus of G, then we will also call this the Γ-index of G. It was shown by Tits [Tits(1966)] that the $\Bbbk$-isomorphism class of G uniquely determines, up to congruence, the Γ-index of G. Using Proposition 1.3.6 this can also be seen easily as follows. Let G_1, G_2 be connected semisimple groups defined over $\Bbbk$ and $\phi : G_1 \to G_2$ a $\Bbbk$-isomorphism. For $i = 1, 2$ let $A_i \subset G_i$ be a maximal $\Bbbk$-split torus, $T_i \supset A_i$ a maximal $\Bbbk$-torus of G_i and $\Delta(T_i)$ a Γ-basis of $\Phi(T_i)$. Now $\phi(A_1)$ is a maximal $\Bbbk$-split torus of G_2, hence there exists a $g \in G_\Bbbk$ such that $\mathrm{Int}(g)\phi(A_1) = A_2$. Then $\mathrm{Int}(g)\phi(T_1) \supset A_2$ is a maximal $\Bbbk$-torus. Let $\mathbb{K}$ be the smallest Galois extension of $\Bbbk$ which splits T_1 and T_2. Then there exists $x \in G_\mathbb{K}$ such that $\mathrm{Int}(x)\,\mathrm{Int}(g)\phi(T_1) = T_2$. Let $\phi_1 = \mathrm{Int}(x)\,\mathrm{Int}(g)\phi$. Then $\phi_1 : (G_1, T_1, A_1) \to (G_2, T_2, A_2)$ is a $\mathbb{K}$-isomorphism and by Theorem 1.3.7 $\varphi_1^\star = {}^t(\varphi_1|T_1)^{-1}$ as in (1.2.4 (a)) (modulo a Weyl group element of $W(T_2)$) is a congruence from the Γ-index of (G_1, T_1, A_1) onto the Γ-index of (G_2, T_2, A_2). Summarized we have now the following result:

Proposition 1.3.15 ([Tits(1966)])**.** *The* $\Bbbk$*-isomorphism class of* G *uniquely determines (up to congruence) the* Γ*-index* $(X,\ \Delta,\ \Delta_0(\Gamma),\ [\sigma])$ *of* G.

1.3.16 (Γ, θ)-indices

In this subsection we discuss indices related to the isomorphy classes of $\Bbbk$-involutions. Let G be a connected semisimple $\Bbbk$-group, $\theta \in \mathrm{Aut}(G)$ an $\Bbbk$-involution, A a $(\theta, \Bbbk)$-split torus of G, $T \supset A$ a θ-stable maximal $\Bbbk$-torus of G and $X = X^*(T)$, $\Phi = \Phi(T)$. Let $\mathbb{K}$ be a finite Galois extension of $\Bbbk$ which splits T, $\Gamma = \mathrm{Gal}(\mathbb{K}/\Bbbk)$ the Galois group of $\mathbb{K}/\Bbbk$ and $\mathcal{E}_\theta = \{1, -\theta\} \subset \mathrm{Aut}(X, \Phi)$ be the subgroup spanned by $-\theta|T$ as in 1.3.10. Let $\mathcal{E}_\Gamma \subset \mathrm{Aut}(X, \Phi)$ be the subgroup corresponding to the action of Γ on (X, Φ) and let $\Gamma_\theta = \mathcal{E}_\Gamma.\mathcal{E}_\theta$ the subgroup of $\mathrm{Aut}(X, \Phi)$ generated by $\mathcal{E}_\Gamma$ and $\mathcal{E}_\theta$. As in 1.2.6 (a) let $X_0 = X_0(\Gamma_\theta)$, $\Phi_0 = \Phi_0(\Gamma_\theta)$, etc. We will also use the notation $\Phi_0(\Gamma, \theta)$ (resp. $\Delta_0(\Gamma, \theta)$) for $\Phi_0(\Gamma_\theta)$ (resp. $\Delta_0(\Gamma_\theta)$). In addition, let $G_0 = G(\Phi_0)$ denote the connected semisimple subgroup of G generated by $\{U_\alpha \mid \alpha \in \Phi_0\}$. The group G_0 is the semisimple part of $Z_G(A)$. Moreover $\overline{\Phi}_{\Gamma_\theta} = \Phi(A)$ is the set of restricted roots of A, which, by [Helminck and Wang(1993), 5.9] is a root system if A is a maximal $(\theta, \Bbbk)$-split torus of G. Let Δ be a Γ_θ-bases of Φ, and let $\Delta_0 = \Delta \cap X_0$. Similar as in (1.2.14 (a)) we have an action of Γ_θ on Δ, which we denote by $[\sigma]$. The 4-tuple $(X,\ \Delta,\ \Delta_0,\ [\sigma])$ is called *the* Γ_θ*-index of* (G, T, A, θ). If A is a maximal $(\theta, \Bbbk)$-split torus of G, then we will also call this *the* Γ_θ*-index of* (G, T, θ). In the case of θ-indices or Γ-indices the indices did not depend on the choice of the maximal torus, when one choose the torus A involved to be maximal. The above Γ_θ-index of (G, T, θ) depends on the choice of $T \supset A$. For example one can choose T such that T_θ^- is maximal θ-split or one can choose T such that T_θ^+ is a maximal torus of $Z_G(A) \cap H$. In most cases this leads to non congruent Γ_θ-indices. We can obtain a Γ_θ-index uniquely determined by the isomorphy class of the $\Bbbk$-involution by taking A maximal $(\theta, \Bbbk)$-split and $T \supset A$ a θ-standard maximal $\Bbbk$-torus of $Z_G(A)$, i.e. T contains a maximal $\Bbbk$-split torus and T_θ^- is a maximal θ-split $\Bbbk$-torus of G. We will call a Γ_θ-index of (G, T, A, θ) a Γ_θ*-index of* (G, θ) if A is a maximal $(\theta, \Bbbk)$-split and $T \supset A$ a θ-standard maximal $\Bbbk$-torus of G. This index is uniquely determined by the isomorphy class of the $\Bbbk$-involution θ:

Proposition 1.3.17. *Let* θ_1 *be a* $\Bbbk$*-involution of* G*. The* $\Bbbk$*-isomorphism class of* θ_1 *uniquely determines (up to congruence) the* Γ_θ*-index* $(X,\ \Delta,\ \Delta_0(\Gamma_\theta),\ [\sigma])$ *of* (G, θ_1).

Remark 1.3.18. Similar as for the θ-index and Γ-index one easily determines the restricted root system of a maximal $(\theta, \Bbbk)$-split torus of G from the Γ_θ-index $(X,\ \Delta,\ \Delta_0,\ [\sigma])$ of (G, θ).

1.3.19 (Γ, θ)-ORDER

The Γ_θ-index of (G, θ), as defined above, corresponds to a Γ_θ-order on (X, Φ). However there is a lot of additional structure present, which is not represented in the Γ_θ-index. We also have a θ-index and a Γ-index. This can be seen as follows. Assume A is a maximal $(\theta, \Bbbk)$-split torus of G, $\tilde{A} \supset A$ a maximal $\Bbbk$-split torus of G and $T \supset \tilde{A}$ a θ-standard maximal $\Bbbk$-torus. Let $X = X^*(T)$ and $\Phi = \Phi(T)$. Then we have the usual Γ-order on (X, Φ). On the other hand since T_θ^- is a maximal θ-split torus of G, we also have a θ-order on (X, Φ). Finally since A is maximal $(\theta, \Bbbk)$-split we also have a Γ_θ-order. All these can be defined simultaneously on (X, Φ) as follows.

Definition 1.3.20. Let Ψ be a semisimple root datum and let Γ, θ act on (X, Φ). A linear order on X which is simultaneously a Γ-, θ- and Γ_θ-order is called a (Γ, θ)-order. A fundamental system of Φ with respect to a (Γ, θ)-order is called a (Γ, θ)-*fundamental system of* Φ.

From the above remarks it follows that if A, A_1, S, T are as above, then a (Γ, θ)-order on (X, Φ) exists. However not every Γ_θ-order is a (Γ, θ)-order. Another characterization of a (Γ, θ)-order is given in the following result.

Proposition 1.3.21. *Let Ψ be a semisimple root datum and assume Γ, θ act on (X, Φ). The following are equivalent:*

(1) (X, Φ) has a (Γ, θ)-order.

(2) $\Phi_0(\Gamma, \theta) = \Phi_0(\Gamma) \cup \Phi_0(\theta)$.

(3) If $\Phi_1 \subset \Phi_0(\Gamma, \theta)$ irreducible component then $\Phi_1 \subset \Phi_0(\theta)$ or $\Phi_1 \subset \Phi_0(\Gamma)$.

Remarks 1.3.22. (1) A (Γ, θ)-order, as above, is completely determined by the sextuple

$$(X,\, \Delta,\, \Delta_0(\Gamma),\, \Delta_0(\theta),\, [\sigma],\, \theta^*). \tag{1.3.22 (a)}$$

We will call this sextuple an *index of* (Γ, θ) or an (Γ, θ)-index. This terminology follows again Tits [Tits(1966)]. We will also use the name (Γ, θ)-diagram, following the notation in Satake [Satake(1971), 2.4]. (2) The above index of (Γ, θ) determines the indices of both Γ and θ and vice versa.

(3) We can make a diagrammatic representation of the (Γ, θ)-index by coloring black those vertices of the ordinary Dynkin diagram of Φ, which represent roots in $\Delta_0(\Gamma, \theta)$ and giving the vertices of $\Delta_0(\Gamma) \cup \Delta_0(\theta)$ which are not in $\Delta_0(\Gamma) \cap \Delta_0(\theta)$ a label Γ or θ if $\alpha \in \Delta_0(\Gamma) - \Delta_0(\Gamma) \cap \Delta_0(\theta)$ or $\alpha \in \Delta_0(\theta) - \Delta_0(\Gamma) \cap \Delta_0(\theta)$ respectively. The actions of $[\sigma]$ and θ^* are indicated by arrows. Here is an example with Φ of type D_{10}:

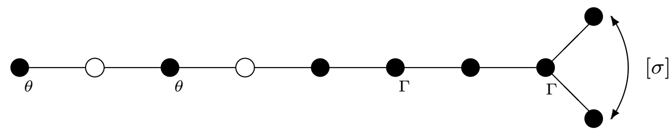
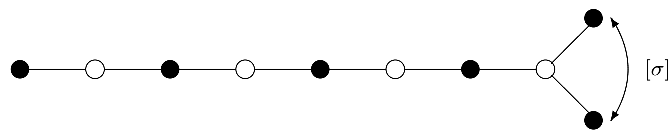
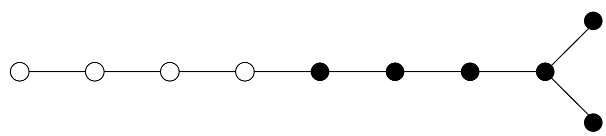


Figure 1.1: (Γ, θ)-index with Φ of type D_{10}

This (Γ, θ)-index is obtained by gluing together the indices





of Γ resp. θ with the above recipe. (4) A (Γ, θ)-index of Γ_θ may depend again on the choice of the (Γ, θ)-basis of Φ. However if Φ_{Γ_θ} is a root system, then it follows similar as in Proposition 1.3.6 that the (Γ, θ)-index is independent of the choice of the (Γ, θ)-basis.

1.4 Isomorphy classes of involutions and $\Bbbk$-groups

In each of the cases of symmetric varieties, symmetric $\Bbbk$-varieties and semisimple $\Bbbk$-groups there is a natural fine structure associated with these spaces. For a study of these spaces and their representation theory it is important to have a classification of these spaces together with this fine structure of restricted root systems with multiplicities and Weyl groups. This fine structure easily follows from the index as defined in section 1.3. On the other hand this index can also be used as an invariant to characterize the isomorphy classes. In the case of isomorphy classes of involutions these indices completely characterize the isomorphy classes. In the case of isomorphy classes of semisimple $\Bbbk$-groups one needs a second invariant to characterize the isomorphy classes and in the case of isomorphy classes of $\Bbbk$-involutions three invariants are needed. Since the classification of $\Bbbk$-involutions depends on the classifications of semisimple $\Bbbk$-groups (see [Tits(1966)]) and the classification of involutions over algebraically closed fields (see [Helminck(1988)]), we will first briefly review some facts about both these classifications, which will be needed later in the classification of the $\Bbbk$-involutions.

1.4.1 Characterization of the isomorphy classes of involutions

The classification of isomorphy classes of involutions can be reduced to a classification of $W(T)$-conjugacy classes of involutions normally related to a maximal torus T (see [Helminck(1988)]). In this subsection we briefly review these results. We use the same notation as in 1.3.10. In particular let G be a reductive algebraic group, $\theta \in \mathrm{Aut}(G)$ an involution and T a maximal torus of G. Write $X = X^*(T)$ and $\Phi = \Phi(T)$. To relate the isomorphy classes of involutions to the indices as in 1.3.10, we define the following:

Definition 1.4.2. Let T be a maximal torus of G. An automorphism θ of G of order ≤ 2 is said to be *normally related to* T if $\theta(T) = T$ and T_θ^- is a maximal θ-split torus of G.

Note that, since all maximal tori of G are conjugate under $\mathrm{Int}(G)$, every involutorial automorphism of G is conjugate to one which is normally related to T. The involutions normally related to T can be characterized now as follows (see [Helminck(1988), 3.7]).

Theorem 1.4.3. *Let $\theta_1, \theta_2 \in \mathrm{Aut}(G)$ be such that $\theta_1^2 = \theta_2^2 = \mathrm{id}$ and assume θ_1, θ_2 are normally related to T. Then we have the following:*

(1) θ_1 and θ_2 are conjugate under $\mathrm{Int}(G)$ if and only if $\theta_1|T$ and $\theta_2|T$ are conjugate under $W(T)$.

(2) θ_1 and θ_2 are conjugate under $\mathrm{Aut}(G)$ if and only if $\theta_1|T$ and $\theta_2|T$ are conjugate under $\mathrm{Aut}(T)$.

We showed in Proposition 1.3.12 that the G-isomorphy class determines the θ-index up to congruence. From Theorem 1.4.3 it follows now that these indices actually completely characterize the isomorphy classes. To formulate this result we need to define first a notion of admissibility.

Definition 1.4.1 (admissible involution)**.** Let $\theta \in \mathrm{Aut}(X, \Phi)$ be an involution. Then θ is called *admissible* if there exists an involution $\tilde{\theta} \in \mathrm{Aut}(G, T)$ such that $\tilde{\theta}|T = \theta$ and $T_{\tilde{\theta}}^-$ is a maximal $\tilde{\theta}$-split torus of G. If X is semisimple, then the indices of admissible involutions of (X, Φ) are called *admissible θ-indices.*

We have the following characterization of the isomorphy classes of involutions in terms of θ-indices. Note that these θ-indices yield most of the fine structure of the corresponding symmetric variety G/G_θ.

Theorem 1.4.4. *Let G, T be as above and assume G is semisimple. Then there is a bijection of the set of* $\mathrm{Int}(G)$ *(resp.* $\mathrm{Aut}(G)$*) conjugacy classes of involutorial automorphisms of G and the W-congruence (resp.* $\mathrm{Aut}(\Phi)$*-congruence) classes of indices of admissible involutions of* $(X^*(T), \Phi(T))$.

1.4.5 Characterization of the isomorphy classes of semisimple $\Bbbk$-groups

In the remainder of this section we give a characterization of the isomorphy classes of semisimple $\Bbbk$-groups. Most of these results can be found in [Tits(1966)] and [Satake(1971)].

1.4.6

We use the same notation as in the previous section. In particular let G be a connected semisimple groups defined over $\Bbbk$ and let $A \subset G$ be a maximal $\Bbbk$-split torus, $T \supset A$ a maximal $\Bbbk$-torus of G, $X = X^*(T)$, $\Phi = \Phi(T)$, $\mathbb{K}$ the smallest Galois extension of $\Bbbk$ which splits T and $\Delta = \Delta(T)$ a Γ-basis of $\Phi(T)$. In Proposition 1.3.15 we demonstrated that the Γ-index is an invariant for the isomorphy classes of semisimple $\Bbbk$-groups. Another invariant is the following. Let $G_0 = G(\Phi_0)$ denote the connected semisimple subgroup of G generated by $\{U_\alpha \mid \alpha \in \Phi_0\}$. The group G_0 is the semisimple part of $Z_G(A)$ and is $\Bbbk$-anisotropic if A is maximal $\Bbbk$-split. Let $T_0 = T \cap G_0$. This is a maximal $\Bbbk$-torus of G_0. Since all maximal $\Bbbk$-split tori of G are conjugate under $G_{\Bbbk}$, it follows that G_0 is uniquely determined (up to $\Bbbk$-isomorphism) by the $\Bbbk$-isomorphism class of G. We will call G_0 the $\Bbbk$-*anisotropic kernel* of G. We have shown now that the $\Bbbk$-isomorphism class of G uniquely determines the Γ-index $(X, \Phi, \Delta_0(\Gamma), [\sigma])$ of G and the $\Bbbk$-anisotropic kernel G_0 of G. The following result shows that these two actually suffice to characterize the isomorphy classes (see [Tits(1966)] or [Satake(1971)]).

Theorem 1.4.7. *Let G, G' be connected semi-simple algebraic groups defined over $\Bbbk$. Let T, A, X, G_0, T_0, etc., T', A', X', G_0', T_0' etc. be as defined above, and corresponding to G and G', respectively. There exists a $\Bbbk$-isomorphism $\varphi : (G, T, A) \to (G', T', A')$ if and only if the following conditions are satisfied:*

(i) There exists a congruence $\phi : (X, \Delta, \Delta_0(\Gamma), [\sigma]) \to (X', \Delta', \Delta_0'(\Gamma), [\sigma]')$ of the Γ-index of G onto the Γ-index of G'.

(ii) There exists a $\Bbbk$-isomorphism $\varphi_0 : (G_0, T_0) \to (G_0', T_0')$ such that the restriction ϕ_0 of ϕ to $(X_0, \Delta_0(\Gamma), [\sigma]|X_0)$ is associated to φ_0 as in 1.3.8 (i.e., $\varphi_0^{[\star]} = \phi_0$).

The Γ-indices, which belong to connected semi-simple groups will be called admissible. They are defined as follows:

Definition 1.4.8. If X is a free module of rank n, Δ a fundamental system of a root system Φ in X, $\Delta_0(\Gamma)$ a subset of Δ, and $[\,\cdot\,]$ a homomorphism of the Galois group Γ into $\mathrm{Aut}(X, \Delta, \Delta_0(\Gamma))$, we will say that the system $\mathcal{D} = (X,\ \Delta,\ \Delta_0(\Gamma),\ [\sigma])$ is *admissible* if there exists a connected semi-simple group G defined over $\Bbbk$ having $\mathcal{D}$ as Γ-index.

Remark 1.4.9. The above result reduces the problem of classifying connected semisimple algebraic groups defined over $\Bbbk$ to the following two problems:

(1) Classification of all admissible Γ-indices.

(2) Classification of all $\Bbbk$-anisotropic semisimple algebraic groups.

For arbitrary base fields not much is known about the $\Bbbk$-anisotropic semisimple algebraic groups. The first problem is discussed in Tits [Tits(1966)].

1.4.10

For $\Bbbk = \mathbb{R}$ every complex semisimple group contains a compact real form, which is unique up to isomorphism (see [Helgason(1978)]). So in this case there is a one-to-one correspondence between isomorphy classes of $\Bbbk$-anisotropic semisimple groups and isomorphy classes of complex semisimple groups. Since the latter (modulo the center) are completely characterized by the corresponding Dynkin diagram, the classification of real semisimple groups reduces to a classification of the admissible Γ-indices. For a $\mathfrak{p}$-adic field $\Bbbk = \mathbb{Q}_p$ the only $\Bbbk$-anisotropic semisimple groups are $\mathrm{SL}(1, \mathbb{K})$, where $\mathbb{K}/\Bbbk$ is a normal division algebra . So in particular the Γ-index of a $\Bbbk$-anisotropic semisimple group over $\mathbb{Q}_p$ can only consist of copies of the Dynkin diagrams of type A_n.

1.4.11 Isomorphy classes of $\Bbbk$-involutions

The characterization and classification of $\Bbbk$-involutions is much more complicated than simply combining the invariants characterizing involutions in the algebraically closed case and invariants characterizing the $\Bbbk$-structure of a reductive algebraic group defined over a field $\Bbbk$. In most cases a third invariant is needed and the interplay of these 3 invariants adds additional complications. In this subsection we briefly review the characterization of the $\Bbbk$-involutions. Let G be a reductive $\Bbbk$-group and θ a $\Bbbk$-involution of G. We will consider isomorphy classes of $\Bbbk$-involutions under the action of $\mathrm{Int}(G_\Bbbk)$, $\mathrm{Int}_\Bbbk(G)$ and $\mathrm{Aut}_\Bbbk(G)$. We will say that two $\Bbbk$-involutions are isomorphic under $G_\Bbbk$ (or $G_\Bbbk$-isomorphic) if they are isomorphic under $\mathrm{Int}(G_\Bbbk)$. We want to characterize the isomorphism classes in a such a way that we also get a classification of the natural root systems of the symmetric $\Bbbk$-varieties. This means we need to characterize the isomorphism classes of the $\Bbbk$-involutions on a fixed maximal $\Bbbk$-split torus. For this we define the following notion:

Definition 1.4.12. Let A be a maximal $\Bbbk$-split torus of G. A $\Bbbk$-involution θ of G is *normally related* to A if $\theta(A) = A$ and A_θ^- is a maximal $(\theta, \Bbbk)$-split torus of G.

Lemma 1.4.13. *Let A be a maximal $\Bbbk$-split torus of G and $T \supset A$ a maximal $\Bbbk$-torus. Every $\Bbbk$-involution is $G_\Bbbk$-isomorphic to one normally related to A and T.*

The admissible $\Bbbk$-involutions are now defined as follows.

Definition 1.4.2 (admissible $\Bbbk$-involution)**.** Let G be a reductive $\Bbbk$-group, A a maximal $\Bbbk$-split torus of G and $T \supset A$ a maximal $\Bbbk$-torus of G, $\mathbb{K} \supset \Bbbk$ a splitting extension for T. An involution $\theta \in \mathrm{Aut}(X^*(T), \Phi(T))$ is said to be an *admissible $\Bbbk$-involution* (with respect to $(G,\ T,\ A)$) if there exists a $\Bbbk$-involution $\tilde{\theta}$ of G, normally related to A and T such that $\tilde{\theta}|T = \theta$. We will call the (Γ, θ)-index corresponding to an admissible $\Bbbk$-involution an *admissible (Γ, θ)-index*.

Involutions normally related to A and T determine the (Γ, θ)-index up to congruence:

Theorem 1.4.14. *Let A be a maximal $\Bbbk$-split torus of G, $T \supset A$ a maximal $\Bbbk$-torus, $\mathbb{K}$ a finite Galois extension of $\Bbbk$ which splits T and $\Gamma = \mathrm{Gal}(\mathbb{K}/\Bbbk)$ the Galois group of $\mathbb{K}/\Bbbk$. Let θ_1, θ_2 be $\Bbbk$-involutions of G normally related to A and T. If θ_1 and θ_2 are isomorphic under $G_\Bbbk$ then the corresponding (Γ, θ)-indicies of θ_1 and θ_2 are congruent.*

The second invariant characterizing the $\Bbbk$-structure of a reductive algebraic group defined over $\Bbbk$ was the $\Bbbk$-anisotropic kernel. Combining this with the characterization of involutions over algebraically closed fields, we get that the induced involutions of the $\Bbbk$-anisotropic kernel need to be isomorphic. These involutions can be characterized as follows.

Theorem 1.4.15. *Let G be a connected semi-simple algebraic group defined over $\Bbbk$, A a maximal $\Bbbk$-split torus of G and θ_1, θ_2 $\Bbbk$-involutions of G, normally related to A. Then $\theta_1|Z_G(A)$ and $\theta_2|Z_G(A)$ are isomorphic under $G_\Bbbk$ if and only if θ_1 is $G_\Bbbk$-isomorphic to $\theta_2 \,\mathrm{Int}(a)$ for some $a \in A_{\theta_2}^-$.*

$\Bbbk$-inner elements

Denote the set of $a \in A_\theta^-$ such that $\theta \,\mathrm{Int}(a)$ is a $\Bbbk$-involution of G by $I_\Bbbk(A_\theta^-)$. This will be called the *set of $\Bbbk$-inner elements of* A_θ^-. Note that for any $a \in A_\theta^-$ the automorphism $\theta \,\mathrm{Int}(a)$ is an involution of G. So the question is for which $a \in A_\theta^-$ this involution is in fact a $\Bbbk$-involution of G. Since θ is a $\Bbbk$-automorphism this is equivalent to the condition that $\mathrm{Int}(a)$ is a $\Bbbk$-automorphism of G. What remains to determine is the different $G_\Bbbk$-isomorphic classes of involutions $\theta \,\mathrm{Int}(a)$ for $a \in I_\Bbbk(A_\theta^-)$. Unfortunately this isomerphy can not always be restricted to the normalizer in $G_\Bbbk$ of the maximal $\Bbbk$-split torus A. The main reason for this is that there is not always a unique $H_\Bbbk$-conjugacy class of maximal $(\theta, \Bbbk)$-split tori.

Reduction of the classification of $\mathbb{k}$-involutions

It was shown in [Helminck(2000)] that the classification of $\mathbb{k}$-involutions of G reduces to the following 3 problems:

(1) Classification of admissible (Γ, θ)-indices.

(2) Classification of the $G_{\mathbb{k}}$-isomorphy classes of $\mathbb{k}$-involutions of the $\mathbb{k}$-anisotropic kernel of G.

(3) Classification of the $G_{\mathbb{k}}$-isomorphy classes of $\mathbb{k}$-inner elements of G.

For more details, see [Helminck(2000)]. The admissible (Γ, θ)-indices determine most of the fine structure of the symmetric $\mathbb{k}$-varieties and a classification of these was included in [Helminck(2000)] as well. For $\mathbb{k}$ algebraically closed the full classification can be found in [Helminck(1988)]. For other fields a classification of the remaining two invariants is still lacking. In particular the case of symmetric $\mathbb{k}$-varieties over the $\mathfrak{p}$-adic numbers is of interest.

1.4.16 Involutions of compact real groups

For $\mathbb{k} = \mathbb{R}$ there is a one-to-one correspondence between isomorphy classes of $\mathbb{k}$-anisotropic semisimple groups and isomorphy classes of complex semisimple groups (see 1.4.10). For involutions of compact groups we have a similar correspondence. This can be seen as follows. If θ is an involution of a complex group G, then there exists a conjugation σ of a compact real form U of G such that $\theta\sigma = \sigma\theta$ (see [Helminck(1988), 10.3]). Then $\theta|U$ is an involution of U. Conversely any involution of U can be lifted to an involution of G by extending the base field. It is easy to show then that there exists a one to one correspondence between isomorphy classes of involutions of $\mathbb{k}$-anisotropic semisimple groups and isomorphy classes of involutions of complex semisimple groups (see [Helgason(1978), Chap. X, 1.4]). By Theorem 1.4.3 the latter are characterized by isomorphy classes of admissible involutions. This means that the classification of the $\mathbb{k}$-involutions reduces to the first and third problem in 1.4.11. A classification of the isomorphy classes of $\mathbb{k}$-involutions, for $\mathbb{k} = \mathbb{R}$, together with all the fine structure, can be found in [Helminck(1988)].

1.4.17

In this thesis we are concerned with the problem of explicitly constructing the $\mathbb{k}$-involution of G from a given (Γ, θ)-index using the a suitable Chevalley basis. By finding this $\mathbb{k}$-involution explicitly, we will be able to do computations symbolically in symmetric $\mathbb{k}$-varieties, which is impossible without an explicit realization of the involution. The first step in this process is to lift $\mathbb{C}$-involutions from the root system to involutions of the Lie algebra and to establish the algorithmic infrastructure for constructing the Lie algebra involutions from the root system data. The next step toward lifting $\Bbbk$-involution for arbitrary $\Bbbk$ is to lift $\mathbb{R}$-involutions. We can do this by lifting pairs of commuting $\mathbb{C}$-involutions as described below.

1.5 $\mathbb{R}$-involutions

1.5.1 Reduction to ordered pairs of commuting involutions of complex reductive groups

The isomorphic classes of involutions of real reductive groups correspond bijectively with the isomorphic classes of ordered pairs of involutions of a complex group. We review these results in this section.

1.5.2 Real forms and conjugations.

We reference Knapp [Knapp(2005), p. 34] for the following. Let V be a vector space over $\Bbbk$ and consider the tensor product space $V \otimes_{\Bbbk} \mathbb{K}$. If c is a member of $\mathbb{K}$, then multiplication by c, which we denote temporarily by $m(c)$, is $\Bbbk$ linear from $\mathbb{K}$ to $\mathbb{K}$. Thus $1 \otimes m(c)$ defines a $\Bbbk$ linear map of $V \otimes_{\Bbbk} \mathbb{K}$ to itself, and we define this to be scalar multiplication by c in $V \otimes_{\Bbbk} \mathbb{K}$. With the definition $V \otimes_{\Bbbk} \mathbb{K}$ becomes a vector space over $\mathbb{K}$. We write $V^{\mathbb{K}}$ for this vector space. The map $c \mapsto v \otimes 1$ allows us to identify V canonically with a subset of $V^{\mathbb{K}}$. If $\{v_i\}$ is a basis of V over $\Bbbk$, then $\{v_i \otimes 1\}$ is a basis of $V^{\mathbb{K}}$ over $\mathbb{K}$. If W is a vector space over the extension field $\mathbb{K}$ of $\Bbbk$, we can restrict the definition of scalar multiplication to scalars in $\Bbbk$, thereby obtaining a vector space over $\Bbbk$. This vector space we denote by $W^{\Bbbk}$. In the special case that $\Bbbk = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$ and V is a real vector space, the complex vector space $V^{\mathbb{C}}$ is called the *complexification* of V. By these definition we have the identification

$$(V^{\mathbb{C}})^{\mathbb{R}} = V \oplus iV \tag{1.5.2 (a)}$$

as real vector spaces, where V means $V \otimes 1$ in $V \otimes_{\mathbb{R}} \mathbb{C}$ and the i refers to the real linear transformation of multiplication by i.

Definition 1.5.3 (real form, [Knapp(2005), p. 35]). When a complex vector space W and a real vector space V are related by

$$W^{\mathbb{R}} = V \oplus iV,$$

we say that V is a *real form* of the complex vector space W.

Note that any real vector space is a real form of its complexification.

Definition 1.5.4 (conjugation)**.** In eq. (1.5.2 (a)) the $\mathbb{R}$ linear map that is 1 on V and -1 on iV, i.e. conjugate linear, is called the *conjugation* of the complex vector space $V^{\mathbb{C}}$ with respect to the real form V.

Similarly, if $\mathfrak{g}_0$ is a real Lie algebra, the complex Lie algebra $(\mathfrak{g}_0)^{\mathbb{C}}$ is called the *complexification of* $\mathfrak{g}_0$, and when a complex Lie algebra $\mathfrak{g}$ and a real Lie algebra $\mathfrak{g}_0$ are related as vector spaces over $\mathbb{R}$ by

$$\mathfrak{g}^{\mathbb{R}} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0 \qquad (1.5.4\ (a))$$

we say that $\mathfrak{g}_0$ is the *real form of* $\mathfrak{g}$. Any real Lie algebra is a real form of its complexification. The *conjugation of a complex Lie algebra* with respect to a real form is a Lie algebra isomorphism of $\mathfrak{g}^{\mathbb{R}}$ with itself.

1.5.5 Building $\mathbb{R}$-involutions from conjugations.

Let G be a connected reductive group defined over a field $\mathbb{k}$, let $G_{\mathbb{k}}$ denote the set of $\mathbb{k}$-rational points of G, and let $\mathfrak{g}$ denote the Lie algebra of G. If $\mathbb{k} = \mathbb{R}$, then the Galois group $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{k})$ has order 2. If $\delta \neq id \in \Gamma$, then δ acts on the complex Lie algebra $\mathfrak{g}$ of G as a *conjugation* , i.e.

- δ has order 2,
- δ is linear with respect to addition, and
- δ is conjugate linear with respect to scalar multiplication.

For the rest of this section, the symbol δ denotes the conjugation defined by the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

Definition 1.5.6 ($\mathfrak{g}^{\delta}$)**.** Let $\mathfrak{g}^{\delta}$ denote the set of fixed points in $\mathfrak{g}$ of the conjugation δ:

$$\mathfrak{g}^{\delta} = \{X \in \mathfrak{g} \mid \delta(X) = X\}$$

Lemma 1.5.7. $\mathfrak{g}^{\delta}$ *is exactly the (real) Lie algebra of the group* $G_{\mathbb{R}}$.

Lemma 1.5.8. *The Lie algebra* $\mathfrak{g}^{\delta}$ *is a* real form *of* $\mathfrak{g}$ *and there is a one-to-one correspondence between real forms of* $\mathfrak{g}$ *and conjugations of* $\mathfrak{g}$.

1.5.9 σ and δ commute

Let $\sigma \in \mathrm{Aut}(G_{\mathbb{R}})$ be a $\mathbb{R}$-involution. Denote the involution of $\mathfrak{g}^{\delta}$ induced by σ also by σ. By extending the base field we can lift our involution σ from $G_{\mathbb{R}}$ to G and similarly from $\mathfrak{g}^{\delta}$ to $\mathfrak{g}$. Since σ is an involution of $\mathfrak{g}^{\delta}$ we can write $\mathfrak{g}^{\delta} = \mathfrak{h} + \mathfrak{q}$ as a sum of eigenspaces, i.e.

$$\mathfrak{h} = \{X \in \mathfrak{g}^\delta \mid \sigma(X) = X\} \qquad (\sigma\text{-stable})$$
$$\mathfrak{q} = \{X \in \mathfrak{g}^\delta \mid \sigma(X) = -X\} \qquad (\sigma\text{-split})$$

The space $\mathfrak{q}$ is called a *local affine symmetric space.*

Note. From $\sigma(\mathfrak{g}^\delta) = \mathfrak{g}^\delta$ it follows that σ and δ commute.

1.5.10 Replace σ on $\mathfrak{g}^\delta$ with (σ, δ) on $\mathfrak{g}$

So, instead of considering the action of σ on $\mathfrak{g}^\delta$ we can consider the action of the pair (σ, δ) on the complex Lie algebra $\mathfrak{g}$. Simplifying further: Instead of the pair (σ, δ) of an involution and a conjugation commuting with it, consider a pair of commuting involutions by replacing δ with the Cartan involution, θ, corresponding to $\mathfrak{g}^\delta$.

1.5.11 Constructing the Cartan involution

In order to construct the Cartan involution we have to consider a compact real form. Helminck showed that there is a compact real form $\mathfrak{u}$ that is δ- and σ-stable.

Lemma 1.5.12. *Let $\mathfrak{u}$ be a compact real form of $\mathfrak{g}$ that is δ- and σ-stable and τ a conjugation of $\mathfrak{u}$. Then*

$$\sigma\tau = \tau\sigma \quad \text{and} \quad \delta\tau = \tau\delta$$

Let $\theta = \tau\delta = \delta\tau$. Since both δ and τ are conjugations (therefore of order 2), it follows that θ is an involution of $\mathfrak{g}^\delta$ and $\mathfrak{g}$ and $\theta\big|\mathfrak{g}^\delta$ is a *Cartan involution of* $\mathfrak{g}^\delta$. Furthermore,

$$\sigma\theta = \sigma\tau\delta = \tau\sigma\delta = \tau\delta\sigma = \theta\sigma \qquad (1.5.12\text{ (a)})$$

Remark 1.5.13. Up to isomorphy there exists a unique compact real form $\mathfrak{u}$ defined by the fact that the Killing form is negative definite on $\mathfrak{u}$ [Helminck(1988), 10.3].

1.5.14 An involution in $\mathbb{R} \longleftrightarrow$ commuting involutions in $\mathbb{C}$

Theorem 1.5.15. *$\exists$ bijection between the isomorphy classes of real forms of complex semisimple Lie algebras and the isomorphy classes of involutions of complex semisimple Lie algebras.*

$$[\mathfrak{g}_\mathbb{R} \subset \mathfrak{g}] \quad \longleftrightarrow \quad \left[\theta \in \mathrm{Aut}(\mathfrak{g}), \theta^2 = \mathrm{id}\right] \qquad (1.5.15\text{ (a)})$$

This gives us a bijection between involutions in $\mathrm{Aut}(\mathfrak{g}_{\mathbb{R}})$ and ordered pairs of involutions $(\sigma, \theta) \in \mathrm{Aut}(\mathfrak{g})$:

$$\sigma \in \mathrm{Aut}(\mathfrak{g}_{\mathbb{R}}), \sigma^2 = \mathrm{id} \xrightarrow[\text{bijection (1.5.15 (a))}]{\text{complexification}} (\sigma, \theta) \in \mathrm{Aut}(\mathfrak{g}) \text{ such that } \sigma\theta = \theta\sigma$$

$$\sigma \in \mathrm{Aut}(\mathfrak{g}_{\mathbb{R}}), \sigma^2 = \mathrm{id} \xleftarrow[\text{ensures 1-to-1}]{\text{ordered pair}} (\sigma, \theta) \in \mathrm{Aut}(\mathfrak{g}) \text{ such that } \sigma\theta = \theta\sigma$$

Remark 1.5.16. This means we can identify real algebras with Cartan involutions.

1.6 Summary of results in this thesis

Since every admissible (Γ, θ)-index is a combination of admissible absolutely irreducible (Γ, θ)-indices, it suffices to classify the admissible absolutely irreducible (Γ, θ)-indices and the admissible $\Bbbk$-involutions related to these indices. Hence, from now on we will restrict our attention to these. In the remainder of this thesis, we begin the classification of $\Bbbk$-involutions by building the classification for the "first" non-algebraically-closed $\Bbbk$, the $\mathbb{R}$-involutions. Before we can classify the $\mathbb{R}$-involutions by way of commuting pairs of $\mathbb{C}$-involutions, however, we must first know how to lift single $\mathbb{C}$-involutions. Since the existing results in the literature are incorrect [Watson(2010)], we begin with our own construction and analysis of $\mathbb{C}$-involutions.

In the following we use this notation:

Notation 1.6.1. Let $X, \Phi, \Delta, \overline{\Delta}_\theta, \mathfrak{g}, \mathfrak{t}, \theta$ be as in Section 1.2.9, and $\overline{\Delta} = \overline{\Delta}_\theta$. Let the set of X_α where $\alpha \in \Phi$ be a Chevalley basis of $\mathfrak{g}$. For an automorphism of order two τ of a vector space V we denote by $V_\tau^\pm$ the (± 1)-eigenspaces of τ in V.

The main result of Part II is the following theorem:

Theorem 1 (θ_Δ). *Let $\theta_\Delta \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{t})$ be the unique automorphism lifted from $\theta \in \mathrm{Aut}(X, \Phi)$ such that $\theta_\Delta|\mathfrak{t} = \theta$ and*

$$\theta_\Delta\left(X_{\alpha_i}\right) = X_{\alpha_i} \textit{ for all } \alpha_i \in \Delta.$$

Then θ_Δ is an involution for any admissible absolutely irreducible involution θ except when θ is of type $D_\ell^{(\ell-1)/2}$ IIIb.

In Part III we give a classification of admissible ordered pairs of commuting involutions, $(\sigma, \theta) \in \mathrm{Aut}(X, \Phi)$, for which we construct a standard lifted pair from $(\sigma_\Delta, \theta_\Delta) \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{t})$.

Definition 1.6.2 ([Helminck(1988), 6.11])**.** Let $\overline{\Delta}$ be a basis of $\Phi(T^{-}_{\sigma,\theta})$. A pair of commuting involutorial automorphisms $(\tilde{\sigma}, \tilde{\theta})$ of G is called a *standard pair* if, for all $\lambda \in \overline{\Delta}$ we have

$$\dim(\mathfrak{g}_\lambda)^{+}_{\tilde{\sigma}\tilde{\theta}} \geq \dim(\mathfrak{g}_\lambda)^{-}_{\tilde{\sigma}\tilde{\theta}}.$$

Helminck showed that there is always a standard pair and it is unique [Helminck(1988)], and in particular that it is useful because of the following result: For a subgroup $M \in G$, denote by $\mathcal{W}_M(A) = N_M(A)/Z_M(A)$ the Weyl group of A in M.

Proposition 1.6.3. *If (σ, θ) is standard, $H = G^\sigma$ and $K = G^\theta$ the fixed point subgroups of σ and θ in G respectively, and A a maximal (σ, θ)-split torus, then*

$$\mathcal{W}(A) = \mathcal{W}_H(A) = \mathcal{W}_K(A) = \mathcal{W}_{H\cap K}(A).$$

Hence we can compute representatives of the $\mathcal{W}(A)$ orbits of the $\Bbbk$-inner elements and therefore representatives of non-isomorphic involutions on the Lie algebra.

We find that in most cases, the corresponding pair $(\sigma_\Delta, \theta_\Delta) \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{t})$ is a pair of commuting involutions and is standard. We call these "Nice" pairs. The next largest class of (σ, θ) is the set of "Decent" pairs for which the pair $(\sigma_\Delta, \theta_\Delta)$ commutes but is not standard. We also call these pairs "Corrected-for-Standard." The few remaining cases fall into two classes: "Better-than-Okay," which is the closest that any pair containing θ of type $D_\ell^{(\ell-1)/2}$IIIb can come to being "Nice," and "Feisty," which describes those pairs for which the lifted involutions found in Part II do not commute and hence are also not standard.

Chapter 2: Results from S. Klein

We will take advantage of the following results from S. Klein [Klein(2009)], the interested reader may refer to the proofs there. By abuse of notation, we will refer to a Lie algebra with compact real form as a *compact Lie algebra*.

Notation 2.1.4 ([Klein(2009), 3]). Let $\mathfrak{g}$ be a compact semisimple Lie algebra. We consider the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$; via the complexification of the Lie bracket of $\mathfrak{g}$, $\mathfrak{g}^{\mathbb{C}}$ becomes a complex semisimple Lie algebra. It should be noted that the Killing form of $\mathfrak{g}^{\mathbb{C}}$ equals the complexification κ of the Killing form of $\mathfrak{g}$. The complexification $\mathfrak{t}^{\mathbb{C}}$ of the Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$, and we put for any $\alpha \in (\mathfrak{t}^{\mathbb{C}})^{\vee}$

$$\begin{aligned} \mathfrak{g}^{\mathbb{C}}_{\alpha} &:= \{X \in \mathfrak{g}^{\mathbb{C}} \mid \forall H \in \mathfrak{t}^{\mathbb{C}}, \mathrm{ad}(H)X = \alpha(H)X\} \\ &= \{X \in \mathfrak{g}^{\mathbb{C}} \mid \forall H \in \mathfrak{t}, \mathrm{ad}(H)X = \alpha(H)X\} \end{aligned}$$

The the root system $\{\alpha \in (\mathfrak{t}^{\mathbb{C}})^{\vee} \backslash \{0\} \mid \mathfrak{g}^{\mathbb{C}}_{\alpha} \neq \{0\}\}$ of $\mathfrak{g}^{\mathbb{C}}$ equals the roots system Φ of $\mathfrak{g}$, and we have the root space decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \sum_{\alpha \in \Phi} g^{\mathbb{C}}_{\alpha}.$$

Definition 2.1.5 (Chevalley basis of $\mathfrak{g}^{\mathbb{C}}$). A family of vectors $(X_\alpha)_{\alpha \in \Phi}$ is called a *Chevalley basis of* $\mathfrak{g}^{\mathbb{C}}$, if we have $X_\alpha \in \mathfrak{g}^{\mathbb{C}}_{\alpha}$ for every $\alpha \in \Phi$ and if there exists a family of *real* numbers $(N_{\alpha,\beta})_{\alpha,\beta \in \Phi}$, called the *Chevalley constants* corresponding to (X_α), so the for all $\alpha, \beta \in \Phi$ we have

$$[X_\alpha\ ,\ X_\beta] = \begin{cases} N_{\alpha,\beta} X_{\alpha+\beta} & \text{if } \alpha+\beta \in \Phi \\ \alpha^\vee & \text{if } \alpha+\beta = 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.1.5 (a)}$$

and

$$N_{-\alpha,-\beta} = -N_{\alpha,\beta} \tag{2.1.5 (b)}$$

For formal reasons we put $N_{\alpha,\beta} := 0$ whenever $\alpha, \beta \in \Phi$ such that $\alpha+\beta \notin \Phi$ and $\beta \neq -\alpha$.

Note. Though the Chevalley constants do depend on the choice of the Chevalley basis, their squares are uniquely determined by the structure of the Lie algebra (as shown below in proposition 2.1.8), and therefore the transition from one Chevalley basis to another can change the corresponding Chevalley constants only in sign.

Proposition 2.1.6 ([Klein(2009), 3.2])**.** *$\mathfrak{g}^{\mathbb{C}}$ has a Chevalley basis.*

Definition 2.1.7 (root string coefficients)**.** We will refer to the coefficients (p, q) defining the *α-string through β*, $\{\beta + k\alpha \mid -p \leq k \leq q\}$, as the *root string coefficients* corresponding to the ordered pair of roots (α, β).

Proposition 2.1.8 ([Klein(2009), 3.3])**.** *Let (X_α) be a Chevalley basis of $\mathfrak{g}^{\mathbb{C}}$ and $(N_{\alpha,\beta})$ be the corresponding Chevalley constants. Suppose $\alpha, \beta, \gamma, \delta \in \Phi$. Then we have*

(a) $N_{\beta,\alpha} = -N_{\alpha,\beta}$.

(b) $\kappa(X_\alpha\ ,\ X_{-\alpha}) = 1$, *where κ is the Killing form of $\mathfrak{g}^{\mathbb{C}}$.*

(c) $\overline{X_\alpha} = a \cdot X_{-\alpha}$, *with* $a := \kappa(X_\alpha\ ,\ \overline{X_\alpha}) < 0$.

(d) Suppose $\alpha + \beta + \gamma = 0$. Then we have $N_{\alpha,\beta} = N_{\beta,\gamma} = N_{\gamma,\alpha}$.

(e) Suppose $\alpha + \beta + \gamma + \delta = 0$ such that none of the roots $\alpha, \beta, \gamma\delta$ is the negative of one of the others. Then we have $N_{\alpha,\beta}N_{\gamma,\delta} + N_{\beta,\gamma}N_{\alpha,\delta} + N_{\gamma,\alpha}N_{\beta,\delta} = 0$

(f) We have

$$N_{\alpha,\beta}^2 = \frac{q \cdot (1+p)}{2} \cdot \|\,\alpha\,\|^2,$$

*where $\{\beta + k\alpha \mid -p \leq k \leq q\}$ is the α-*string through *β; note that this implies that we have $N_{\alpha,\beta} \neq 0$ if $\alpha + \beta \in \Phi$.*

Proposition 2.1.9 ([Klein(2009), 3.4])**.** *Let (X_α) be a Chevalley basis of $\mathfrak{g}^{\mathbb{C}}$ with the corresponding constants $(N_{\alpha,\beta})$.*

(a) Let constants $z_\alpha \in \mathbb{C}^\times$ for every $\alpha \in \Phi$ be given, so that the following properties are satisfied:

(i) For every $\alpha \in \Phi$ we have $z_\alpha \cdot z_{-\alpha} = 1$.

(ii) For every $\alpha, \beta \in \Phi$ with $\alpha + \beta \in \Phi$ we have $\varepsilon_{\alpha,\beta} := \frac{z_\alpha \cdot z_\beta}{z_{\alpha+\beta}} \in \{\pm 1\}$.

Then $(z_\alpha \cdot X_\alpha)_{\alpha\in\Phi}$ is another Chevalley basis of $\mathfrak{g}^{\mathbb{C}}$, the corresponding Chevalley constants are $(\varepsilon_{\alpha,\beta} \cdot N_{\alpha,\beta})_{\alpha,\beta\in\Phi}$.

(b) Every Chevalley basis of $\mathfrak{g}^{\mathbb{C}}$ is obtained by the construction of (a).

Proposition 2.1.10 ([Klein(2009), 3.5])**.** *For every non-simple positive root $\alpha \in \Phi^+\backslash\Delta$, fix a decomposition $\alpha = \zeta_\alpha + \eta_\alpha$ with $\zeta_\alpha, \eta_\alpha \in \Phi^+$. Then there exists a Chevalley basis (X_α) (with corresponding Chevalley constants $(N_{\alpha,\beta})$ with the following properties:*

(i) For every $\alpha \in \Phi^+$ we have $X_{-\alpha} = -\overline{X_\alpha}$. (Compare Proposition 2.1.8 (c).)

(ii) For every $\alpha \in \Phi^+\backslash\Delta$ we have $N_{\zeta_\alpha,\eta_\alpha} > 0$.

Any two such Chevalley bases have the same Chevalley constants $(N_{\alpha,\beta})$.

To completely describe the Lie bracket of $\mathfrak{g}^{\mathbb{C}}$, it suffices to fix a Chevalley basis (X_α) of $\mathfrak{g}^{\mathbb{C}}$, and calculate the corresponding Chevalley constants $(N_{\alpha,\beta})$. Up to sign, they are determined by proposition 2.1.8(f). We now show how to calculate the sign correctly if (X_α) is a Chevalley basis of the kind of proposition 2.1.10, corresponding to a family of decompositions $(\alpha = \zeta_\alpha + \eta_\alpha)_{\alpha\in\Phi^+\backslash\Delta}$. By Proposition 2.1.8 (a) and (d) and eq. (2.1.5 (b)) we have for any $\alpha, \beta \in \Phi^+$ with $\beta \neq \alpha$

$$N_{-\alpha,-\beta} = -N_{\alpha,\beta} \quad \text{and} \quad N_{\alpha,-\beta} = -N_{-\alpha,\beta} = \begin{cases} N_{(\beta-\alpha),\alpha} & \text{if } \beta-\alpha \in \Phi^+ \\ N_{(\alpha-\beta),\beta} & \text{if } \beta-\alpha \in (-\Phi^+) \\ 0 & \text{otherwise} \end{cases} \qquad \text{(2.1.10 (a))}$$

The above identities show us that it suffices to calculate $N_{\alpha,\beta}$ for $\alpha, \beta \in \Phi^+$. This is achieved by the following algorithm, but first a definition.

Definition 2.1.11 ($\mathrm{ht}(\alpha) = \sum r_i$)**.** Write any root $\alpha \in \Phi$ as $\alpha = \sum_{\alpha_i\in\Delta} r_i\alpha_i$, then define the height of the root to be

$$\mathrm{ht}(\alpha) = \sum r_i \quad \text{where} \quad r_i \in \mathbb{Z} \text{ such that } r_i \geq 0 \text{ or } r_i \leq 0 \forall i$$

Note that $\mathrm{ht}(\alpha)$ may be positive or negative but never zero, and more often than not we will restrict ourselves to the positive roots so that $r_i \geq 0$.

Algorithm 2.2 ([Klein(2009), p. 34])**.**

(C1) [$N_{\alpha,\beta}$ with $\alpha+\beta \notin \Phi$.] Iterate the following for all $\alpha, \beta \in \Phi^+$:
If $\alpha+\beta \notin \Phi^+$, put $N_{\alpha,\beta} = 0$.

(C2) [Iterate on height.] Iterate steps **(C3)** to **(C8)** for $\mathrm{ht} = 2, \ldots, L$ where L denotes the maximal height of roots occurring in Φ.

(C3) [Iterate on roots of height ht.] Iterate steps **(C4)** to **(C8)** with α running through all positive roots in Φ of height ht.

(C4) Set $\zeta := \zeta_\alpha$ and $\eta = \eta_\alpha$.

(C5) [Calculate $N_{\zeta,\eta}$ and $N_{\eta,\zeta}$.] Let p be the smallest integer so that $\eta-(p+1)\zeta \notin \Phi$ holds and put $q := p - 2\frac{(\eta,\zeta)}{\|\zeta\|^2}$,

$$N_{\zeta,\eta} := \sqrt{\frac{q\cdot(1+p)}{2}} \cdot \|\zeta\|$$

and $N_{\eta,\zeta} := -N_{\zeta,\eta}$.

(C6) [Iterate on decompositions of α.] Iterate step **(C7)** for all pairs (γ,δ) of positive roots with $\gamma+\delta = \alpha$ and $\gamma, \delta \notin \{\zeta, \eta\}$.

(C7) [Calculate $N_{\gamma,\delta}$.] Put

$$n_{11} := \begin{cases} N_{\eta,(\gamma-\eta)} & \text{if } \gamma-\eta \in \Phi^+ \\ N_{\gamma,(\eta-\gamma)} & \text{if } \gamma-\eta \in -\Phi^+ \\ 0 & \text{otherwise} \end{cases} \qquad n_{12} := \begin{cases} N_{\delta,(\zeta-\delta)} & \text{if } \zeta-\delta \in \Phi^+ \\ N_{\zeta,(\delta-\zeta)} & \text{if } \zeta-\delta \in -\Phi^+ \\ 0 & \text{otherwise} \end{cases}$$

$$n_{21} := \begin{cases} N_{(\zeta-\gamma),\gamma} & \text{if } \zeta-\gamma \in \Phi^+ \\ N_{(\gamma-\zeta),\zeta} & \text{if } \zeta-\gamma \in -\Phi^+ \\ 0 & \text{otherwise} \end{cases} \qquad n_{22} := \begin{cases} N_{\eta,(\delta-\eta)} & \text{if } \delta-\eta \in \Phi^+ \\ N_{\delta,(\eta-\delta)} & \text{if } \delta-\eta \in -\Phi^+ \\ 0 & \text{otherwise} \end{cases}$$

Then put $N_{\gamma,\delta} := \frac{1}{N_{\zeta,\eta}} \cdot (n_{11}n_{12} + n_{21}n_{22})$

(C8) [End of loops.]

Remark 2.2.1. It is useful to note that since $\zeta + \eta = \alpha = \gamma + \delta$ it will always be the case that

$$\eta - \gamma = \delta - \zeta \iff \gamma - \eta = \zeta - \delta \tag{2.2.1 (a)}$$

and likewise

$$\delta - \eta = \zeta - \gamma \iff \eta - \delta = \gamma - \zeta \tag{2.2.1 (b)}$$

which implies that $n_{11} = 0 \iff n_{12} = 0$ and likewise for n_{21} and n_{22}. We will make use of this identity repeatedly in later proofs.

Lemma 2.2.2 (lifting constants on $\widetilde{X}_\alpha$)**.** *Let $(X_\alpha)_{\alpha\in\Phi}$ as in proposition 2.1.10 and $(\widetilde{X}_\alpha) = (z_\alpha X_\alpha)$ as in proposition 2.1.9. It follows immediately that if $\varphi \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{t})$ such that $\varphi^\vee|_{\mathfrak{t}^\vee} = \phi$ where $\phi \in \mathrm{Aut}(X, \Phi)$ defined by*

$$\varphi(X_\alpha) = c_\alpha^\varphi X_{\phi(\alpha)},$$

then

$$\varphi(\widetilde{X}_\alpha) = c_\alpha^\varphi \frac{z_\alpha}{z_{\phi(\alpha)}} \widetilde{X}_{\phi(\alpha)}.$$

Proof. This follows directly from the linearity of the Lie algebra automorphism:
Define $\varphi(X_\alpha)$ and $\widetilde{X}_\alpha$ as above, then

$$\varphi(\widetilde{X}_\alpha) = \varphi(z_\alpha X_\alpha) = z_\alpha \varphi(X_\alpha) = z_\alpha c_\alpha^\varphi X_{\phi(\alpha)} = z_\alpha c_\alpha^\varphi \left(\frac{1}{z_{\phi(\alpha)}} \widetilde{X}_{\phi(\alpha)} \right) = c_\alpha^\varphi \frac{z_\alpha}{z_{\phi(\alpha)}} \widetilde{X}_{\phi(\alpha)}.$$

□

Remark 2.2.3. It is useful to note that if φ is an involution on the chosen Chevalley basis (X_α), then it is also an involution on any other Chevalley basis. That is, if $\varphi^2(X_\alpha) = X_\alpha$ for all $\alpha \in \Phi$, then $\varphi^2(\widetilde{X}_\alpha) = z_\alpha \varphi^2(X_\alpha) = z_\alpha X_\alpha = \widetilde{X}_\alpha$ for all $\alpha \in \Phi$. This means that in lifting an involution from the root system to the Lie algebra it is sufficient to construct an involution on the chosen basis (X_α).

Chapter 3:

Root Decomposition and the Chevalley Constants

In order to optimize our computation, we use the root decompositions $\{\zeta_\alpha, \eta_\alpha\}$ derived during the computation of the positive roots.

Algorithm 3.1 (Construction of the Positive Roots, Φ^+)**.** Let ℓ denote the rank of the root system. **SetupLPrts[]:=Module[**

Local Variables: *i,j,ht,ialpha,ibeta,alpha,beta,diffroot,zerovector,ind,q,newroot;*

(start by constructing the simple roots *)*

LPrts:=$\{\alpha_1, \alpha_2, \ldots \alpha_\ell\}$

LPrtsHeight:=$\{\{1, \ldots \ell\}\}$; *(*LPrtsHeight is a list of lists each containing the indices of the roots of a particular height *)*

LPrtZetaEtaPQ:=ConstantArray[Undefined,ℓ];

zerovector:=ConstantArray[0,ℓ];

ht=1;

While [Length[LPrtsHeight] $\geq$ ht,

For Each *ialpha* $\in$ **LPrtsHeight[1] and** *ibeta* $\in$ **LPrtsHeight[ht]**

alpha=LPrts[*ialpha*]; *(* a simple root *)*

beta=LPrts[*ibeta*]; *(* a root of height ht *)*

diffroot=beta-alpha;

(if beta $\neq$ alpha and we're working with simple roots or beta-alpha isn't a root, then calculate q for the root string construction with p=0, otherwise do nothing *)*

If diffroot≠zerovector *AND*
(ht=1 *OR* **diffroot $\notin$ LPrts[LPrtsHeight[ht-1]]),**
then
$q = -((2$ RtInnerProd[beta,alpha]$)/$RtInnerProd[alpha,alpha]$)$;
For [$j = 1$, $j \leq q$, $j++$,

newroot=beta+j*alpha;
(if the newroot has not already been added to LPrts, then append it to the list,otherwise do nothing *)*
If newroot $\notin$ LPrts[*ibeta***...-1]**
then
AppendTo[LPrts,newroot];
(if the ht to which newroot belongs has not been created, then append an empty list to LPrtsHeight *)*
If Length[LPrtsHeight]<ht+j
then
AppendTo[LPrtsHeight,{}];
(put the index of new root in its ht *)*
AppendTo[LPrtsHeight[ht+j],Length[LPrts];

(record the initial decomposition of newroot along with its root string data p,q *)*

AppendTo[LPrtZetaEtaPQ,$\{\{ialpha,$ Position[LPrts,beta+(j-1) alpha]$\}, \{j-1, q-j+1\}\}$];];];

];

ht=ht+1];

LMaxHeight=ht-1;
LNumPrts=Length[LPrts];

];

As we proceed through the rest of this chapter (and beyond), we will find the following Chevalley identities in Table 3.1 useful.

Table 3.1: Chevalley Constant Identities

Identity =	Condition	Ref
$N_{-\alpha,-\beta} = -N_{\alpha,\beta}$		
$N_{\beta,\alpha} = -N_{\alpha,\beta}$		
${N_{\alpha,\beta}}^2 = \frac{q(1+p)\Vert \alpha \Vert^2}{2}$	where $\beta + n\alpha \in \Phi$, with $-p \leq n \leq q$.	
$N_{\alpha,\beta} = N_{\beta,\gamma} = N_{\gamma,\alpha}$	if $\alpha, \beta, \gamma \in \Phi$ and $\alpha + \beta + \gamma = 0$	
$N_{-\alpha,\alpha+\beta} = N_{-\beta,-\alpha} = -N_{-\alpha,-\beta}$	if $\alpha, \beta, \alpha + \beta \in \Phi$	
$N_{\alpha,\beta} = -N_{-\alpha,-\beta} = N_{-\alpha,\alpha+\beta}$	Combining the above relations.	
$= -N_{\alpha,-\alpha-\beta} = N_{-\alpha-\beta,\alpha}$		
$N_{\gamma,\delta} = \frac{1}{N_{\zeta,\eta}} \cdot (N_{-\eta,\gamma}N_{-\zeta,\delta} + N_{\gamma,-\zeta}N_{-\eta,\delta})$	$\zeta + \eta = \alpha = \gamma + \delta$	2.1.8(e) and 2.1.5 (b)
$N_{-\eta,\gamma} = N_{\gamma,(\eta-\gamma)}$		2.1.8(d)
$= -N_{-\gamma,(\gamma-\eta)}$		2.1.5 (b)
$= N_{(\gamma-\eta),-\gamma}$		2.1.8(a)
$= N_{\eta,(\gamma-\eta)}$		2.1.8 and 2.1.5 (b)

3.2 Chevalley orders

Definition 3.2.1 (lexicographic ordering)**.** We will say a decomposition of a root $\alpha = \gamma + \delta$ is *lexicographically ordered* if $\gamma = \sum_i^k r_j \alpha_j$ and $\delta = \sum_{j>i} s_j \alpha_j$. We will also denote this by

$$\gamma \prec_{lex} \delta.$$

Remark 3.2.2. Notice that for any non-simple positive root we take an initial *minimal decomposition* in the following sense. For $\alpha \in \Phi^+ \backslash \Delta$, we write $\alpha = \zeta_\alpha + \eta_\alpha$, where ζ_α is a simple root α_i with the smallest possible index i (i.e. the one encountered first by our algorithm). However, there may be some α_j in the support of α such that $j < i$, and hence the minimal decomposition may not be lexicographically ordered. By choosing $N_{\zeta_\alpha, \eta_\alpha} > 0$, we dictate the signs of the Chevalley constants for all other decompositions of α.

Definition 3.2.3 ($N_{\zeta_\alpha, \eta_\alpha}$ in minimal form)**.** We will say $N_{\zeta_\alpha, \eta_\alpha}$ is in *minimal form* if it corresponds to a minimal decomposition $\alpha = \zeta_\alpha + \eta_\alpha$ as described in remark 3.2.2.

3.2.4 Chevalley order for Type A

Lemma 3.2.5. *For any connected root (sub)system of type A,*

$$N_{\gamma,\delta} > 0 \iff \gamma \prec_{lex} \delta.$$

Proof. Given any non-simple positive root of height m, we choose an initial minimal decomposition $\alpha = \alpha_i + \beta_{i+1}$ such that $\beta_{i+1} = \sum_{i+1 \leq j \leq m} \alpha_j$ where $\alpha_i, \alpha_j \in \Delta$ and put

$$N_{\alpha_i, \beta_{i+1}} = \sqrt{\frac{q(1+p) \| \alpha_i \|^2}{2}} > 0.$$

Suppose for induction that $N_{\alpha_i, \sum_{j=i+1}^n \alpha_j} > 0$ for any $n < m$.

Note. This implies that we also assume that $N_{\sum_{j=i+1}^n \alpha_j, \alpha_i} < 0$.

We then determine the Chevalley constants for all other decompositions of α relative to the initial one. In type A, these are of the form $\alpha = (\alpha - \beta_k) + \beta_k$ where $\beta_k = \sum_{k \leq j \leq m} \alpha_j$ such that $k \geq i+2$. That is, if $k = i+2$, we find $N_{\alpha_i + \alpha_{i+1}, \beta_{i+2}}$ by step **(C7)**. **Case** $k = i+2$**:** In order to apply our rules from **(C7)** with

$$\zeta = \alpha_i, \quad \eta = \beta_{i+1}, \quad \gamma = \alpha_i + \alpha_{i+1}, \quad \delta = \beta_{i+2}, \tag{3.2.5 (a)}$$

we determine that

$$\gamma - \eta = \zeta - \delta = \alpha_i - \beta_{i+2} = \alpha_i - \sum_{i+2}^{m} \alpha_j \qquad \notin \Phi \qquad (3.2.5\ (b))$$

$$\delta - \eta = \zeta - \gamma = \alpha_i - (\alpha_i + \alpha_{i+1}) = -\alpha_{i+1} \qquad \in -\Phi^+ \qquad (3.2.5\ (c))$$

and, so, we find by Algorithm 2.2 and the definition of the minimal form:

$$\begin{aligned} n_{11} &= 0 & n_{12} &= 0 \\ n_{21} &= N_{\alpha_{i+1},\alpha_i} < 0 & n_{22} &= N_{\alpha_{i+2}+\cdots+\alpha_m,(\alpha_{i+1})} < 0, \end{aligned}$$

and

$$N_{\alpha_i+\alpha_{i+1},\beta_{i+2}} = \frac{1}{N_{\alpha_i,\beta_{i+1}}} \cdot (n_{11}n_{12} + n_{21}n_{22}) > 0.$$

Case $k = i + 3$:

Let

$$\zeta = \alpha_i, \quad \eta = \beta_{i+1}, \quad \gamma = (\alpha_i + \alpha_{i+1} + \alpha_{i+2}), \quad \delta = \beta_{i+3}, \qquad (3.2.5\ (d))$$

then

$$\gamma - \eta = (\alpha_i + \alpha_{i+1} + \alpha_{i+2}) - \beta_{i+1} = \alpha_i - \beta_{i+3} = \zeta - \delta \qquad \notin \Phi \qquad (3.2.5\ (e))$$

$$\zeta - \gamma = \alpha_i - (\alpha_i + \alpha_{i+1} + \alpha_{i+2}) = -\alpha_{i+1} - \alpha_{i+2} = \delta - \eta \qquad \in -\Phi^+. \qquad (3.2.5\ (f))$$

Hence

$$n_{11} = 0 \qquad n_{12} = 0 \qquad (3.2.5\ (g))$$

$$n_{21} = N_{((\alpha_i+\alpha_{i+1}+\alpha_{i+2})-\alpha_i),\alpha_i} < 0 \qquad n_{22} = N_{\beta_{i+3},(\beta_{i+1}-\beta_{i+3})} < 0 \qquad (3.2.5\ (h))$$

and so, again $N_{\gamma,\delta} > 0$. If we continue in this manner, we see

$$\gamma - \eta = \sum_{j=i}^{k-1} \alpha_j - \beta_{i+1} = \alpha_i - \beta_k = \zeta - \delta \qquad \notin \Phi \qquad (3.2.5\ (i))$$

$$\zeta - \gamma = \alpha_i - \sum_{j=i}^{k-1} \alpha_j = -\sum_{j=i+1}^{k-1} \alpha_j = \delta - \eta \qquad \in -\Phi^+. \qquad (3.2.5\ (j))$$

It follows from induction on the index k, that if the root decomposition (γ, δ) is in lexicographic order, the corresponding Chevalley constant will be positive. □

Definition 3.2.6 (diagrammatically ordered)**.** Assuming we draw and label the simple Dynkin diagrams as in Helgason [Helgason(1978)], we will say a root decomposition $\alpha = \zeta + \eta$ used to index a Chevalley constant $N_{\zeta,\eta}$ is *diagrammatically ordered* if ζ is supported on roots which come before the support of η in the Dynkin diagram when reading left to right.

Lemma 3.2.7. *Let Φ be a simple root system of type A, B, C, D, E, F, or G. Let the Chevalley constants $(N_{\alpha,\beta})$ be chosen as in Algorithm 2.2. Let $\alpha \in \Phi^+ \backslash \Delta$ be a non-simple positive root that is only supported on simple roots which form a type A subsystem. Let $\alpha = \alpha_i + \beta$ be the minimal decomposition of α, then any diagrammatically ordered decomposition $\alpha = \gamma + \delta$ will have positive Chevalley constant:*

$$N_{\gamma,\delta} = N_{\alpha_i,\beta} > 0.$$

Proof. Let $\alpha = \sum_{j=i}^{m} \alpha_j \in \Phi^+$ where $1 \leq i < m \leq \ell - 1$. Then the minimal decomposition of α is

$$\alpha = \alpha_i + \beta \tag{3.2.7 (a)}$$

where $\beta = \sum_{j=i+1}^{m} \alpha_j$ is of height $\operatorname{ht}\alpha - 1$. By construction, $\| \alpha_i \| = \| \alpha_j \|$ for all $i, j < \ell$ and

$$N_{\alpha_i,\beta} = +\sqrt{\frac{q(1+p)\| \alpha_i \|^2}{2}} = +\sqrt{\frac{\| \alpha_i \|^2}{2}} > 0.$$

We claim $N_{\alpha_i,\beta} = N_{\alpha_i+\alpha_{i+1},\beta-\alpha_{i+1}} = \cdots = N_{\sum_{j=i}^{m-1} \alpha_j,\alpha_m} > 0$. We will prove this using the following relation:

$$N_{\gamma,\delta} = \frac{1}{N_{\zeta,\eta}} \cdot (N_{-\eta,\gamma} N_{-\zeta,\delta} + N_{\gamma,-\zeta} N_{-\eta,\delta}) \quad \text{such that} \quad \zeta + \eta = \alpha = \gamma + \delta \tag{3.2.7 (b)}$$

where we take $\zeta = \alpha_i$ and $\eta = \beta = \sum_{j=i+1}^{m} \alpha_j$:

$$N_{\gamma,\delta} = \frac{1}{N_{\alpha_i,\beta}} \cdot (N_{-\beta,\gamma} N_{-\alpha_i,\delta} + N_{\gamma,-\alpha_i} N_{-\beta,\delta}) \quad \text{such that} \quad \alpha_i + \beta = \alpha = \gamma + \delta \tag{3.2.7 (c)}$$

Base Case: Let $\gamma = \alpha_i + \alpha_{i+1}$ and $\delta = \beta - \alpha_{i+1}$, then noting that we have

$$\begin{aligned} \gamma - \eta &= (\alpha_i + \alpha_{i+1}) - (\beta) = \alpha_i + \alpha_{i+1} - (\alpha_{i+1} + \cdots + \alpha_m) = \alpha_i - \alpha_{i+2} - \cdots && \notin \Phi \\ \delta - \eta &= (\beta - \alpha_{i+1}) - (\beta) = -\alpha_{i+1} = (\alpha_i) - (\alpha_i + \alpha_{i+1}) = \zeta - \gamma && \in -\Phi^+, \end{aligned}$$

we see that

$$\begin{aligned}
N_{\alpha_i+\alpha_{i+1},\beta-\alpha_{i+1}} &= \frac{1}{N_{\alpha_i,\beta}} \cdot (N_{-\beta,\alpha_i+\alpha_{i+1}} N_{-\alpha_i,\beta-\alpha_{i+1}} + N_{\alpha_i+\alpha_{i+1},-\alpha_i} N_{-\beta,\beta-\alpha_{i+1}}) \\
&= \frac{1}{N_{\alpha_i,\beta}} \cdot (0 + N_{\alpha_i+\alpha_{i+1},-\alpha_i} N_{-\beta,\beta-\alpha_{i+1}}) \\
&= \frac{1}{N_{\alpha_i,\beta}} \cdot (0 + (N_{\alpha_{i+1},\alpha_i})(-N_{\beta-\alpha_{i+1},-\beta})) && \textit{by eq. } (2.1.10(a)) \\
&= \frac{1}{N_{\alpha_i,\beta}} \cdot (0 + (-N_{\alpha_i,\alpha_{i+1}})(-N_{\alpha_{i+1},\beta-\alpha_{i+1}})) && \textit{by eq. } (2.1.10(a)).
\end{aligned}$$

Now, since both $(N_{\alpha_i,\alpha_{i+1}})$ and $(N_{\alpha_{i+1},\beta-\alpha_{i+1}})$ are in minimal form, we have

$$N_{\alpha_i,\alpha_{i+1}} = \sqrt{\frac{\| \alpha_i \|^2}{2}} = N_{\alpha_i,\beta} \text{ and } N_{\alpha_{i+1},\beta-\alpha_{i+1}} = \sqrt{\frac{\| \alpha_{i+1} \|^2}{2}} = \sqrt{\frac{\| \alpha_i \|^2}{2}} = N_{\alpha_i,\beta}.$$

Hence $N_{\alpha_i+\alpha_{i+1},\beta-\alpha_{i+1}} = N_{\alpha_i,\beta}$. **Inductive step:** Suppose now for roots $\rho = \sum_{j<\ell}^{p} \alpha_j \in \Phi^+$, such that $\operatorname{ht}\rho < \operatorname{ht}\alpha$ that all diagrammatic decompositions correspond to positive Chevalley constants:

$$N_{(\alpha_i+\cdots+\alpha_k),(\alpha_{k+1}+\cdots+\alpha_p)} > 0$$

Let $\gamma = \alpha_i + \alpha_{i+1} + \cdots + \alpha_k$ and $\delta = \beta - (\alpha_{i+1} + \cdots + \alpha_k)$, then

$$\begin{aligned}
\zeta - \delta = \gamma - \eta &= (\alpha_i + \alpha_{i+1} + \cdots + \alpha_k) - (\beta) \\
&= (\alpha_i + \alpha_{i+1} + \cdots + \alpha_k) - (\alpha_{k+1} + \cdots + \alpha_m) && \notin \Phi
\end{aligned}$$

$$\begin{aligned}
\zeta - \gamma = \delta - \eta &= (\beta - (\alpha_{i+1} + \cdots + \alpha_k)) - (\beta) \\
&= -(\alpha_{i+1} + \cdots + \alpha_k) \\
&= (\alpha_i) - (\alpha_i + \alpha_{i+1} + \cdots + \alpha_k) && \in -\Phi^+.
\end{aligned}$$

$$\begin{aligned}
\Rightarrow N_{\gamma,\delta} &= \frac{1}{N_{\zeta,\eta}} \cdot (n_{11}n_{12} + n_{21}n_{22}) \\
&= \frac{1}{N_{\alpha_i,\beta}} \left(0 + N_{(\alpha_i+\alpha_{i+1}+\cdots+\alpha_k-\alpha_i),\alpha_i} \cdot N_{\beta-(\alpha_{i+1}+\cdots+\alpha_k),(\beta-\beta-(\alpha_{i+1}+\cdots+\alpha_k))}\right) \\
&= \frac{1}{N_{\alpha_i,\beta}} \left(-N_{\alpha_i,\alpha_{i+1}+\cdots+\alpha_k}\right) \cdot \left(N_{(\alpha_{i+1}+\cdots+\alpha_k)+(\alpha_{k+1}+\cdots+\alpha_m),-(\alpha_{i+1}+\cdots+\alpha_k)}\right) \\
&= \frac{1}{N_{\alpha_i,\beta}} \left(-N_{\alpha_i,\alpha_{i+1}+\cdots+\alpha_k}\right) \cdot \left(-N_{-(\alpha_{i+1}+\cdots+\alpha_k),(\alpha_{i+1}+\cdots+\alpha_k)+(\alpha_{k+1}+\cdots+\alpha_m)}\right) \\
&= \frac{1}{N_{\alpha_i,\beta}} \left(-N_{\alpha_i,\alpha_{i+1}+\cdots+\alpha_k}\right) \cdot \left(-N_{(\alpha_{i+1}+\cdots+\alpha_k),(\alpha_{k+1}+\cdots+\alpha_m)}\right).
\end{aligned}$$

Since all roots of a type A subsystem have the same length, the Chevalley constants in the previous equation all have the same magnitude because the ordered pairs of roots all have root string coefficients $(p, q) = (0, 1)$:

$$N_{\alpha_i,\beta} = \frac{\| \alpha_i \|}{\sqrt{2}} = N_{\alpha_i,\alpha_{i+1}+\cdots+\alpha_k} = N_{(\alpha_{i+1}+\cdots+\alpha_k),(\alpha_{k+1}+\cdots+\alpha_m)}$$

Hence $N_{\gamma,\delta} = N_{\alpha_i,\beta} > 0$ for any diagrammatically ordered decomposition of α. $\square$

3.2.8 Chevalley order for Type B

Lemma 3.2.9 (Type B Chevalley ordering)**.** *Let Φ be a root system of type B_ℓ. Let α_ℓ be the short simple root and $(N_{\alpha,\beta})$ be chosen as in Algorithm 2.2. If $\alpha = \sum_{j=i}^{\ell} r_j\alpha_j = \gamma + \delta$ such that (γ, δ) are in lexicographic order with $\gamma = \sum_{j=i}^{k} s_j\alpha_j$, with $s_j > 0$, and $\delta = \sum_{j=m+1}^{\ell} (r_j - s_j)\alpha_j$ and*

(1) (γ, δ) have disjoint support in Δ, then $N_{\gamma,\delta} > 0$.

(2) (γ, δ) do not have disjoint support in Δ (i.e. $m + 1 \leq k \leq \ell$), then $N_{\gamma,\delta} < 0$.

Remark 3.2.10. Note that for $\alpha = \sum_{j=i}^{\ell} r_j\alpha_j \in \Phi^+$ we have that $s_i \leq r_i \leq 1$ and so we can simply choose the smallest index i so that $r_i = 1 = s_i$ by our ordering.

Proof. By our construction, we assume that non-simple positive roots in type B have the form

$$\alpha_i + \cdots + \alpha_m + 2\alpha_{m+1} + \cdots + 2\alpha_\ell \quad \text{such that} \quad i \leq m \leq \ell, \tag{3.2.10 (a)}$$

and we have the following identities to work with:

$$\eta - \delta = (\alpha - \zeta) - (\alpha - \gamma) = -\zeta + \gamma = \gamma - \zeta \tag{3.2.10 (b)}$$

$$\eta - \gamma = \eta - \delta + \delta - \gamma = \gamma - \zeta + \delta - \gamma = \delta - \zeta. \tag{3.2.10 (c)}$$

This follows directly from lemma 3.2.7.

Suppose $\alpha = \sum_{j=i}^{\ell} r_j\alpha_j = \zeta + \eta = \gamma + \delta \in \Phi^+\backslash\Delta$ such that (ζ, η) and (γ, δ) are both ordered decompositions of α. Let (ζ, η) be the minimal decomposition of α such that $\zeta \in \Delta$ and $\eta = \alpha - \zeta$ be , then $N_{\zeta,\eta} = N_{\zeta,\alpha-\zeta} > 0$. Without loss of generality, let $(\gamma, \delta) \neq (\zeta, \eta)$ be any other ordered decomposition of α.

By the skew-symmetry of the Lie bracket, it is sufficient to consider the diagrammatically-ordered pairs (γ, δ).

Case 1: (γ, δ) have disjoint support in Δ.

Case (a): $\alpha = \sum_i^n \alpha_j$ with $i < n \leq \ell$. In this case any decomposition (γ, δ) of α must have disjoint supports in Δ. We take $\zeta = \alpha_i$, $\gamma = \sum_i^m \alpha_j$, and $\delta = \sum_{m+1}^n \alpha_j$. By following an argument similar to that of the proof for lemma 3.2.7, we find that $N_{\gamma,\delta} > 0$.

Case (b): $\alpha = \sum_{j=i}^{\ell} r_j \alpha_j = \sum_{j=i}^{m} \alpha_j + 2\sum_{j=m+1}^{\ell} \alpha_j$ such that $i \leq m < \ell$. In this case, our construction gives us the following minimal decomposition,

$$\zeta = \alpha_{m+1} \text{ and } \eta = \sum_{j=i}^{m+1} \alpha_j + 2 \sum_{j=m+2}^{\ell} \alpha_j.$$

Note that in assuming that $N_{\zeta,\eta} > 0$ we determine that the Chevalley constant corresponding to the lexicographically-ordered pair $N_{\eta,\zeta} < 0$, which is in accordance with the statement of (1).

(γ, δ) have disjoint supports and are in lexicographic order:

$$\gamma = \sum_{j=i}^{k} \alpha_j, \text{ such that } i \leq k < m, \text{ and } \delta = \sum_{j=k+1}^{m} \alpha_j + 2 \sum_{j=m+1}^{\ell} \alpha_j$$

We see that

$$\eta - \delta = \gamma - \zeta = \sum_{j=i}^{k} \alpha_j - \alpha_{m+1} \tag{3.2.10 (d)}$$

$$\notin \Phi \text{ since } k < m \tag{3.2.10 (e)}$$

and

$$\eta - \gamma = \delta - \zeta = \left(\sum_{j=k+1}^{m} \alpha_j + 2 \sum_{j=m+1}^{\ell} \alpha_j \right) - \alpha_{m+1} \tag{3.2.10 (f)}$$

$$= \left(\sum_{j=k+1}^{m+1} \alpha_j + 2 \sum_{j=m+2}^{\ell} \alpha_j \right) \tag{3.2.10 (g)}$$

$$\in \Phi^+ \subset \Phi \tag{3.2.10 (h)}$$

Since

$$N_{\gamma,\delta} = \frac{1}{N_{\zeta,\eta}} \left(n_{11} n_{12} + n_{21} n_{22} \right)$$

such that

$$n_{11} := \begin{cases} N_{\eta,(\gamma-\eta)} & \text{if } \gamma-\eta \in \Phi^+ \\ N_{\gamma,(\eta-\gamma)} & \text{if } \gamma-\eta \in -\Phi^+ \\ 0 & \text{otherwise} \end{cases} \qquad n_{12} := \begin{cases} N_{\delta,(\zeta-\delta)} & \text{if } \zeta-\delta \in \Phi^+ \\ N_{\zeta,(\delta-\zeta)} & \text{if } \zeta-\delta \in -\Phi^+ \\ 0 & \text{otherwise} \end{cases}$$

$$n_{21} := \begin{cases} N_{(\zeta-\gamma),\gamma} & \text{if } \zeta-\gamma \in \Phi^+ \\ N_{(\gamma-\zeta),\zeta} & \text{if } \zeta-\gamma \in -\Phi^+ \\ 0 & \text{otherwise} \end{cases} \qquad n_{22} := \begin{cases} N_{\eta,(\delta-\eta)} & \text{if } \delta-\eta \in \Phi^+ \\ N_{\delta,(\eta-\delta)} & \text{if } \delta-\eta \in -\Phi^+ \\ 0 & \text{otherwise} \end{cases},$$

in this case we have

$$\begin{aligned} N_{\gamma,\delta} &= \frac{1}{N_{\zeta,\eta}}\left(n_{11}n_{12}\right) = \frac{N_{\gamma,(\eta-\gamma)}N_{\zeta,(\delta-\zeta)}}{N_{\zeta,\eta}} \\ &= \frac{N_{\sum_{j=i}^{k}\alpha_j,\left(\sum_{j=k+1}^{m+1}\alpha_j+2\sum_{j=m+2}^{\ell}\alpha_j\right)}N_{\alpha_{m+1},\left(\sum_{j=k+1}^{m+1}\alpha_j+2\sum_{j=m+2}^{\ell}\alpha_j\right)}}{N_{\alpha_{m+1},\sum_{j=i}^{m+1}\alpha_j+2\sum_{j=m+2}^{\ell}\alpha_j}}. \end{aligned}$$

Then by straightforward (but somewhat tedious) induction on k and m and the definition of the minimal form we have $N_{\gamma,\delta} > 0$ using the base case where $\gamma = \alpha_i$ and $\delta = \alpha_{i+1} + 2\alpha_{i+2}$ such that $k = i, m = k+1 = i+1, \ell = m+1 = i+2$:

$$\begin{aligned} N_{\gamma,\delta} &= \frac{1}{N_{\zeta,\eta}}\left(n_{11}n_{12}\right) = \frac{N_{\gamma,(\eta-\gamma)}N_{\zeta,(\delta-\zeta)}}{N_{\zeta,\eta}} \\ &= \frac{N_{\alpha_i,\alpha_{i+1}+\alpha_{i+2}}N_{\alpha_{i+2},\alpha_{i+1}+\alpha_{i+2}}}{N_{\alpha_{i+2},\alpha_i+\alpha_{i+1}+\alpha_{i+2}}} \\ &= \frac{(+)(+)}{(+)} \\ &> 0 \text{ by Case 1(a) and the minimal form.} \end{aligned}$$

Case 2: (γ, δ) do not have disjoint support in Δ.

As in Case 1(b) we write $\alpha = \sum_{j=i}^{\ell} r_j\alpha_j = \sum_{j=i}^{m}\alpha_j + 2\sum_{j=m+1}^{\ell}\alpha_j$ such that $i \le m < \ell$, and have the minimal decomposition $\zeta = \alpha_{m+1}$ and $\eta = \sum_{j=i}^{m+1}\alpha_j + 2\sum_{j=m+2}^{\ell}\alpha_j$. In this case, $\gamma = \sum_{j=i}^{k} s_j\alpha_j$ and $\delta = \sum_{j=m+1}^{\ell}(r_j - s_j)\alpha_j$ such that $m+1 \le k \le \ell$, and we see that

$$\begin{aligned}
\eta - \delta = \gamma - \zeta &= \sum_{j=i}^{k} s_j \alpha_j - \alpha_{m+1} \\
&= \left(\sum_{i}^{m+1} \alpha_j + 2 \sum_{m+2}^{k} \alpha_j \right) - \alpha_{m+1} \\
&= \sum_{i}^{m} \alpha_j \\
&\in \Phi^+ \text{ only if } k = m+1
\end{aligned}$$

and

$$\begin{aligned}
\eta - \gamma = \delta - \zeta &= \sum_{j=m+1}^{\ell} (r_j - s_j)\alpha_j - \alpha_{m+1} \\
&= \left(\alpha_{m+1} + \sum_{m+2}^{k} \alpha_j + 2 \sum_{k+1}^{\ell} \alpha_j \right) - \alpha_{m+1} \\
&\in \Phi^+ \text{ only if } k > m+1.
\end{aligned}$$

Hence $\eta - \delta = \gamma - \zeta \in \Phi \Rightarrow \eta - \gamma = \delta - \zeta \notin \Phi$ and vice versa, and $N_{\gamma,\delta} < 0$:

$$N_{\gamma,\delta} = \begin{cases} \frac{1}{N_{\zeta,\eta}}(N_{(\gamma-\zeta),\zeta} \cdot N_{\delta,(\eta-\delta)}) = (+)((+) \cdot (-)) & \text{if } k = m+1 \\ \frac{1}{N_{\zeta,\eta}}(N_{\gamma,(\eta-\gamma)} \cdot N_{\zeta,(\delta-\zeta)}) = (+)((-) \cdot (+)) & \text{if } k > m+1. \end{cases}$$

This again follows by a straightforward induction using the base case where $m = i, m+1 = i+1, k = i+2$, and $\ell = i+2$.

□

3.2.11 Chevalley orders for Types C and D

We find the following two results by a similar proofs process:

Note. In our construction, we assume that non-simple positive roots in type C have the form:

$$\alpha_i + \cdots + \alpha_{k-1} + 2\alpha_k + \cdots + 2\alpha_{\ell-1} + \alpha_\ell$$

such that $i \le k \le \ell - 1$ or $i + 1 \le k - 1 \le \ell$.

Lemma 3.2.12 (Type C Chevalley ordering)**.** *Let Φ be a root system of type C_ℓ. Let α_ℓ be the long simple root and $(N_{\alpha,\beta})$ be chosen as in Algorithm 2.2. Let $\alpha = \sum_i^n r_j \alpha_j = \gamma + \delta \in \Phi^+$ with*

$r_j \geq 0$ such that (γ, δ) is any ordered decomposition such that

$$\gamma = \sum_{x}^{m} s_j\alpha_j, \ \textit{with } 0 < s_j \leq r_j$$

and

$$\delta = \sum_{y}^{n} (r_j - s_j)\alpha_j, \ \textit{with } m < n$$

then $N_{\gamma,\delta} > 0$.

Remark 3.2.13. Note that in this ordering the important thing is that the simple root of greatest index α_n with $r_n \neq 0$ is only in the support of δ.

If Φ is of type D, we assume that non-simple positive roots have the form:

$$\sum_{i}^{\ell} r_j\alpha_j \ \text{ such that } \ r_i = 1, \ r_{\ell-1}, r_\ell \leq 1$$

and if $r_k = 2$ then $r_j = 2 \ \forall \ j$ such that $k \leq j \leq \ell - 2$ and $r_{\ell-1} = r_\ell \leq 1 = 1$.

Lemma 3.2.14 (Type D Chevalley ordering)**.** *Let Φ be a root system of type D_ℓ. Let $\alpha_{\ell-1}$ and α_ℓ be the spin nodes and $(N_{\alpha,\beta})$ be chosen as in Algorithm 2.2. If $\alpha = \sum_i^n r_j\alpha_j = \gamma + \delta$ such that (γ, δ) are in lexicographic order with*

$$\gamma = \alpha_i + \sum_{i+1}^{k} s_j\alpha_j \ \textit{and } \delta = \sum_{m}^{n} (r_j - s_j)\alpha_j \ \textit{such that } i \leq k < n$$

such that

(1) (γ, δ) have disjoint support in Δ and

(a) $k > \ell - 2$ such that δ is *one of the spin nodes, i.e. $\delta = \alpha_{\ell-1}$ or $\delta = \alpha_\ell \Rightarrow N_{\gamma,\delta} < 0$.*

(b) $k \leq \ell - 2$ and $k < m$ such that α is in a type A subsystem or δ is supported on both spin nodes $\Rightarrow N_{\gamma,\delta} > 0$.

(2) (γ, δ) do not have disjoint support in Δ (i.e. $m \leq k \leq \ell - 2 \Rightarrow r_k = 2$), then $N_{\gamma,\delta} < 0$.

Remark 3.2.15. Note that by our ordering if $\alpha = \sum_{j=i}^n r_j\alpha_j \in \Phi^+$, we have $r_i = 1 = s_i$ and α_n is only in the support of δ except in the case where $\delta = \alpha_{\ell-1}$ and $n = \ell$.

Proof Outline. Let $\alpha = \sum_i^n r_j\alpha_j = \gamma + \delta$ such that $\gamma = \alpha_i + \sum_{i+1}^k s_j\alpha_j$ and $\delta = \sum_m^n (r_j - s_j)\alpha_j$ where $0 \leq s_j \leq 1, i \leq k < n$.

(1) (γ, δ) have disjoint support in Δ and

(a) $k > \ell - 2$ ($\Rightarrow \alpha = \sum_i^n \alpha_j$) such that δ *is* one of the spin nodes, i.e. $\delta = \alpha_{\ell-1}$ or $\delta = \alpha_\ell$

$$\gamma = \sum_i^{\ell-2} \alpha_j + \alpha_{\ell-1} \text{ or } \gamma = \sum_i^{\ell-2} \alpha_j + \alpha_\ell.$$

$\Rightarrow N_{\gamma,\delta} < 0$.

(b) $k \leq \ell - 2$ and $k < m$ such that α is in a type A subsystem or δ is supported on both spin nodes.

Case 1: If α is in a type A subsystem such that

$$\gamma = \sum_i^{k} \alpha_j \text{ and } \delta = \sum_{k+1}^{\ell-2} \alpha_j + \alpha_{\ell-1} \text{ or } \delta = \sum_{k+1}^{\ell-2} \alpha_j + \alpha_\ell.$$

Case 2: If α is supported on both spin nodes such that

$$\gamma = \sum_i^{k} \alpha_j \text{ and } \delta = \sum_{k+1}^{\ell-3} r_j \alpha_j + (\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell).$$

$\Rightarrow N_{\gamma,\delta} > 0$.

(2) $\alpha = \alpha_i + \sum_{i+1}^{\ell} r_j \alpha_j$ such that (γ, δ) do not have disjoint support in Δ. That is, in following decomposition $m \leq k \leq \ell - 2$ such that $r_k = 2$:

$$\gamma = \sum_i^{k} s_j \alpha_j \text{ and } \delta = \sum_m^{\ell-3} (r_j - s_j)\alpha_j + (\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell)$$

such that $s_j = 0\ \forall j > k$ and $s_j = 1\ \forall i \leq j \leq k$. $\Rightarrow N_{\gamma,\delta} < 0$.

$\square$

3.3 Chevalley constants for diagramatically ordered root decompositions.

Let Φ be a connected root system and Δ the set of simple roots. Let the Chevalley constants be chosen by Algorithm 2.2 with the ordering of the positive roots as determined by Algorithm 3.1. Let $\alpha_m, \alpha_{m+1} \in \Delta$ be simple roots such that $\langle \alpha_m, \alpha_{m+1} \rangle \neq 0$ so that $\alpha_m + \alpha_{m+1} \in \Phi^+$. For simplicity, assume without loss of generality that $\alpha_m \prec \alpha_{m+1}$ as indicated by the indices so that $N_{\alpha_m, \alpha_{m+1}} > 0$ is in minimal form.

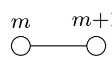
3.3.1 Disjoint diagrammatic decompositions.

Notation 3.3.2. Write $\alpha = \gamma + \delta$ in diagrammatic order such that their supports on the simple roots are disjoint. We will use the following notation

- $\Phi_\gamma =< \mathrm{supp}_\Delta(\gamma) >$, the root system generated by the support of γ on the simple roots.
- $\Phi_\delta =< \mathrm{supp}_\Delta(\delta) >$, the root system generated by the support of δ on the simple roots. Note that by our diagrammatic ordering, for any $\alpha_i \in \mathrm{supp}_\Delta(\gamma)$ and $\alpha_j \in \mathrm{supp}_\Delta(\delta)$ we assume that $i < j$.
- $\Phi_\alpha =< \mathrm{supp}_\Delta(\alpha) >$, the root system generated by the support of α on the simple roots. Note that $\Phi_\alpha = \Phi_\gamma \oplus \Phi_\delta$.
- $\alpha_i, \alpha_m \in \mathrm{supp}_\Delta(\gamma)$ such that i is the least index and m the maximum in the support of γ.
- $\alpha_{m+1} \in \mathrm{supp}_\Delta(\delta)$ such that $m+1$ is the least index in the support of δ and adjacent to α_m.

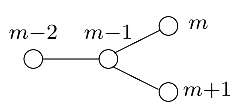
By the above notation we assume that $\mathrm{ht}(\gamma) = m - i$ and $\mathrm{ht}(\delta) \geq 1$.

Case 1: Φ_γ is type A. In this case γ is connected to δ according to the Cartan matrix of Φ in one of the following ways.

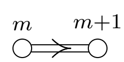
Case (a): $\langle \alpha_m, \alpha_{m+1} \rangle = -1$, such that 

$\Rightarrow \Phi_\delta$ is type A, B, C, D, E, or F. Note that in types D and E, the $m+1$ index may, in fact, correspond to a different index in the diagram, though this changes nothing about the relation between the simple roots.

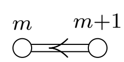
Case (b): $\langle \alpha_m, \alpha_{m+1} \rangle = 0$ but $\langle \alpha_{m-1}, \alpha_m \rangle = -1$ and $\langle \alpha_{m-1}, \alpha_{m+1} \rangle = -1$, such that



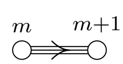
$\Rightarrow \Phi$ is type D or E and Φ_δ is type A.

Case (c): $\langle \alpha_m, \alpha_{m+1} \rangle = -2$, such that 

$\Rightarrow \Phi_\delta$ is type B or F.

Case (d): $\langle \alpha_{m+1}, \alpha_m \rangle = -2$, such that 

$\Rightarrow \Phi_\delta$ is type C.

Case (e): $\langle \alpha_m, \alpha_{m+1} \rangle = -3$, such that 

$\Rightarrow \Phi$ is type G and the only root with disjoint decomposition is $\alpha_m + \alpha_{m+1}$ and hence only the minimal decomposition has $N_{\gamma,\delta} = N_{\alpha_m, \alpha_{m+1}} > 0$.

Case 2: Φ_γ is not of type A. In this case, we have two options for Φ:

Case (a): Φ is of type F_4. In this case, Φ_γ is of type B_3 and Φ_δ is of type A_1.

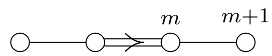


Figure 3.1

Case (b): Φ is of type E_6, E_7 or E_8. In this case, Φ_γ is of type D_5 or type E. In either of these cases, Φ_δ is of type A.

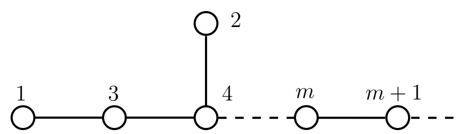


Figure 3.2

3.3.3 General observations about decompositions when $(\gamma, \delta) \neq (\zeta, \eta)$.

Let $\alpha = \sum_i^n r_j\alpha_j = \gamma + \delta = \zeta + \eta$ such that the pair (γ, δ) is lexicographically ordered and the pair (ζ, η) is the minimal decomposition of α. Recall that the minimal decomposition is not always lexicographically ordered. We then have the following cases.

Case 1: γ and δ have disjoint support: $\mathrm{supp}_\Delta(\gamma) \cap \mathrm{supp}_\Delta(\delta) = \emptyset$.

$$\gamma = \sum_i^m r_j\alpha_j \text{ such that } i \leq m \qquad \delta = \sum_{m+1}^n r_j\alpha_j \text{ such that } m+1 \leq n$$

$\Rightarrow i < m + 1 \leq n.$

Case (a): $\mathrm{supp}_\Delta(\zeta) \cap \mathrm{supp}_\Delta(\eta) = \emptyset$.

Case i: $\zeta = \alpha_i \Rightarrow \eta = \sum_{i+1}^n r_j\alpha_j$, and since $i < m + 1$ and $r_i = 1$:

$$\gamma - \eta = \zeta - \delta = \alpha_i - \sum_{m+1}^n r_j\alpha_j \qquad \notin \Phi$$
$$\delta - \eta = \zeta - \gamma = \alpha_i - \sum_i^m r_j\alpha_j = -\sum_{i+1}^m r_j\alpha_j \qquad \in -\Phi^+$$

Case ii: $\zeta = \alpha_n \Rightarrow \eta = \sum_i^{n-1} r_j\alpha_j$, and since $n \geq m+1$ and $r_n = 1$:

$$\begin{aligned} \gamma - \eta = \zeta - \delta &= \alpha_n - \sum_{m+1}^{n} r_j\alpha_j = -\sum_{m+1}^{n-1} r_j\alpha_j & \in -\Phi^+ \\ \delta - \eta = \zeta - \gamma &= \alpha_n - \sum_{i}^{m} r_j\alpha_j & \notin \Phi \end{aligned}$$

Case (b): $\mathrm{supp}_\Delta(\zeta) \cap \mathrm{supp}_\Delta(\eta) = \zeta$.

Case i: $\zeta = \alpha_i \Rightarrow \eta = (r_i - 1)\alpha_i + \sum_{i+1}^{n} r_j\alpha_j$ such that $i < m+1$

$$\gamma - \eta = \zeta - \delta \notin \Phi \quad \text{and} \quad \delta - \eta = \zeta - \gamma \in -\Phi^+$$

follows from the same argument as above.

Case ii: $\zeta = \alpha_k$ such that $i < k < n$ and $r_k \geq 2$

$\Rightarrow \eta = \sum_i^{k-1} r_j\alpha_j + (r_k - 1)\alpha_k + \sum_{k+1}^{n} r_j\alpha_j$

$$\gamma - \eta = \zeta - \delta = \alpha_k - \sum_{m+1}^{n} r_j\alpha_j \qquad \begin{cases} \in -\Phi^+ & \text{only if } n > k \geq m+1 \\ \notin \Phi & \text{if } i < k \leq m \end{cases}$$

$$\delta - \eta = \zeta - \gamma = \alpha_k - \sum_{i}^{m} r_j\alpha_j \qquad \begin{cases} \in -\Phi^+ & \text{only if } i < k \leq m \\ \notin \Phi & \text{if } n > k \geq m+1 \end{cases}$$

Case iii: $\zeta = \alpha_n \Rightarrow \eta = \sum_i^{n-1} r_j\alpha_j + (r_n - 1)\alpha_n$

$$\gamma - \eta = \zeta - \delta \in -\Phi^+ \quad \text{and} \quad \delta - \eta = \zeta - \gamma \notin \Phi$$

follows from the same argument as above.

Case 2: γ and δ do not have disjoint support: $\mathrm{supp}_\Delta(\gamma) \cap \mathrm{supp}_\Delta(\delta) \neq \emptyset$. If there is such a decomposition of α then it must be the case that $\exists$ a minimum k such that $r_k \geq 2$ in the coefficients of the simple roots composing α.

$$\gamma = \sum_{i}^{m} r_j\alpha_j \text{ such that } i \leq m \qquad \delta = \sum_{k}^{n} r_j\alpha_j \text{ such that } i < k \leq m \leq n$$

Case (a): $\mathrm{supp}_\Delta(\zeta) \cap \mathrm{supp}_\Delta(\eta) = \emptyset$.

Case i: $\zeta = \alpha_i \Rightarrow \eta = \sum_{i+1}^{n} r_j\alpha_j$, $i < k \leq m$ (otherwise $\zeta = \gamma$) and $r_i = 1$:

$$\begin{aligned} \gamma - \eta = \zeta - \delta &= \alpha_i - \sum_{k}^{n} r_j\alpha_j & \notin \Phi \\ \delta - \eta = \zeta - \gamma &= \alpha_i - \sum_{i}^{m} r_j\alpha_j = -\sum_{i+1}^{m} r_j\alpha_j & \in -\Phi^+ \end{aligned}$$

Case ii: $\zeta = \alpha_n \Rightarrow \eta = \sum_i^{n-1} r_j\alpha_j$, and since $r_n = 1$ we know that $n > m \geq k$:

$$\gamma - \eta = \zeta - \delta = \alpha_n - \sum_k^n r_j\alpha_j = -\sum_k^{n-1} r_j\alpha_j \qquad \in -\Phi^+$$
$$\delta - \eta = \zeta - \gamma = \alpha_n - \sum_i^m r_j\alpha_j \qquad \notin \Phi$$

Case (b): $\mathrm{supp}_\Delta(\zeta) \cap \mathrm{supp}_\Delta(\eta) = \zeta$.

Case i: $\zeta = \alpha_i \Rightarrow \eta = (r_i - 1)\alpha_i + \sum_{i+1}^n r_j\alpha_j$ such that $i < m + 1$

$$\gamma - \eta = \zeta - \delta \notin \Phi \quad \text{and} \quad \delta - \eta = \zeta - \gamma \in -\Phi^+$$

follows from the same argument as above.

Case ii: $\zeta = \alpha_k$ such that $i < k < n$ and $r_k \geq 2$
$\Rightarrow \eta = \sum_i^{k-1} r_j\alpha_j + (r_k - 1)\alpha_k + \sum_{k+1}^n r_j\alpha_j$

$$\gamma - \eta = \zeta - \delta = \alpha_k - \sum_k^n r_j\alpha_j \qquad \begin{cases} \in -\Phi^+ & \text{only if } n > k \geq m+1 \\ \notin \Phi & \text{if } i < k \leq m \end{cases}$$
$$\delta - \eta = \zeta - \gamma = \alpha_k - \sum_i^m r_j\alpha_j \qquad \begin{cases} \in -\Phi^+ & \text{only if } i < k \leq m \\ \notin \Phi & \text{if } n > k \geq m+1 \end{cases}$$

Case iii: $\zeta = \alpha_n \Rightarrow \eta = \sum_i^{n-1} r_j\alpha_j + (r_n - 1)\alpha_n$

$$\gamma - \eta = \zeta - \delta \in -\Phi^+ \quad \text{and} \quad \delta - \eta = \zeta - \gamma \notin \Phi$$

follows from the same argument as above.

In the next Part, we will explore lifting single involutions from the root system to the Lie algebra over $\mathbb{C}$ using the results of Part I.

Part II

Single Involutions

Chapter 4: $\mathbb{C}$-Involution Preliminaries

Since every admissible (Γ, θ)-index is a combination of admissible *absolutely irreducible* (Γ, θ)-indices corresponding to connected (Γ, θ)-diagrams, it suffices to classify the admissible absolutely irreducible (Γ, θ)-indices and the admissible $\Bbbk$-involutions related to these indices. Hence, from now on we will restrict our attention to these. In the remainder of this thesis, we begin the classification of $\Bbbk$-involutions by building the classification for the "first" non-algebraically-closed $\Bbbk$, the $\mathbb{R}$-involutions. Before we can classify the $\mathbb{R}$-involutions by way of commuting pairs of $\mathbb{C}$-involutions, however, we must first know how to lift single $\mathbb{C}$-involutions. Since the existing results in the literature are incorrect, we begin with our own construction and analysis of $\mathbb{C}$-involutions. The main result of this Part is the following theorem which follows from the results of Chapter 6.

Theorem 1 (θ_Δ)**.** *Let $\theta_\Delta \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{t})$ be the unique automorphism lifted from $\theta \in \mathrm{Aut}(X, \Phi)$ such that $\theta_\Delta|\mathfrak{t} = \theta$ and*

$$\theta_\Delta\left(X_{\alpha_i}\right) = X_{\alpha_i} \text{ for all } \alpha_i \in \Delta.$$

Then θ_Δ is an involution for any admissible absolutely irreducible involution θ except when θ is of type $D_\ell^{(\ell-1)/2}$ IIIb.

We begin with some definitions and notation:

4.1 Definitions and Notation

4.1.1 Admissible involutions

We will use the same notation as in Part I. Recall,

Definition 1.4.1 (admissible involution)**.** Let $\theta \in \mathrm{Aut}(X, \Phi)$ be an involution. Then θ is called *admissible* if there exists an involution $\tilde{\theta} \in \mathrm{Aut}(G, T)$ such that $\tilde{\theta}|T = \theta$ and $T^-_{\tilde{\theta}}$ is a maximal $\tilde{\theta}$-split torus of G. If X is semisimple, then the indices of admissible involutions of (X, Φ) are called *admissible θ-indices.*

Equivalently, in terms of the Lie algebra,

Definition 4.1.2 (admissible involution, [Daniel and Helminck(2008)]:3.3.2)**.** An involution θ of $(X(\mathfrak{t}), \Phi(\mathfrak{t}))$ will be called *admissible* if there is an involution $\tilde{\theta}$ of $(\mathfrak{g}, \mathfrak{t})$ such that $\tilde{\theta}$ induces θ on $(X(\mathfrak{t}), \Phi(\mathfrak{t}))$ and such that $\mathfrak{t}^-_{\theta} = \{X \in \mathfrak{t} \mid \theta(X) = -X\}$ is a maximal toral subalgebra contained in $\mathfrak{p}$.

Definition ($\mathrm{ht}(\alpha) = \sum r_i$)**.** Write any root $\alpha \in \Phi(T)$ as $\alpha = \sum_{\alpha_i \in \Delta} r_i \alpha_i$, then define the height of the root to be

$$\mathrm{ht}(\alpha) = \sum r_i \tag{4.1.2 (a)}$$

Note that this may be positive or negative but never zero.

Definition 4.1.3 ($\tilde{\theta}$ lifting constants)**.** The action of any automorphism $\tilde{\theta} \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{t})$ lifted from an involution $\theta \in \mathrm{Aut}(X, \Phi)$ is determined by its action on the root spaces

$$\tilde{\theta} : X_{\alpha_i} \mapsto c^{\tilde{\theta}}_{\alpha} X_{\theta(\alpha_i)} \quad \forall \alpha \in \Phi. \tag{4.1.3 (a)}$$

We will call these constants $c^{\tilde{\theta}}_{\alpha}$ the *lifting constants* for $\tilde{\theta}$.

Lemma 4.1.4 ($c^{\tilde{\theta}}_{\alpha+\beta}$)**.** *For any $\tilde{\theta} \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{t})$ such that $\tilde{\theta}|_{\mathfrak{t}^\vee} = \theta$ and for any $\alpha, \beta \in \Phi$ such that $\alpha + \beta \in \Phi$,*

$$c^{\tilde{\theta}}_{\alpha+\beta} = c^{\tilde{\theta}}_{\alpha}\, c^{\tilde{\theta}}_{\beta} \frac{N_{\theta(\alpha),\theta(\beta)}}{N_{\alpha,\beta}} \tag{4.1.4 (a)}$$

Corollary 4.1.5 ($c^{\tilde{\theta}}_{\alpha}$)**.** *For any $\alpha \in \Phi$, $c^{\tilde{\theta}}_{\alpha}$ is determined by the $c^{\tilde{\theta}}_{\alpha_i}$ for $\alpha_i \in \Delta$.*

The following unique automorphism was defined by Steinberg (see [Steinberg(1968), Theorem29]):

Definition. Let Δ be a basis of Φ. For an involution $\theta \in \mathrm{Aut}(X, \Phi)$ let $\theta_\Delta \in \mathrm{Aut}(G, T)$ denote the unique automorphism of G such that

$$\theta_\Delta(x_\alpha(\xi)) = x_{\theta(\alpha)}(\xi) \text{ for all } \alpha \in \Delta, \xi \in \overline{\mathbb{k}}. \tag{4.1.5 (a)}$$

From [Steinberg(1968), Theorem 29] it follows that $c^{\theta_\Delta}_{\alpha} = \pm 1$ for all $\alpha \in \Phi$.

Due to the relation between a maximal torus $T \subset G$ and its Lie algebra $\mathfrak{t} \in \mathfrak{g}$, it is sufficient to work with the corresponding involution on the Lie algebra whose structure constants are likewise defined. In fact, the lifting constants are the same in both definitions.

Definition 4.1.6 (θ_Δ)**.** Let Δ be a basis of Φ. For an involution $\theta \in \mathrm{Aut}(X, \Phi)$ let $\theta_\Delta \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{t})$ denote the unique automorphism of $\mathfrak{g}$ such that

$$\theta_\Delta(X_{\alpha_i}) = X_{\theta(\alpha_i)} \text{ for all } \alpha_i \in \Delta \tag{4.1.6 (a)}$$

and

$$\theta_\Delta(X_\alpha) = c_\alpha^{\theta_\Delta} X_{\theta(\alpha)} \text{ for all } \alpha \in \Phi \tag{4.1.6 (b)}$$

in general.

In order to determine the lifting constants $c_{\alpha+\beta}^{\theta_\Delta}$ for a non-simple root, $\alpha + \beta$, we must establish an order on the Chevalley basis since the Jacobi identity on $\mathfrak{g}$ enforces that $N_{\alpha,\beta} = -N_{\beta,\alpha}$. We will choose a natural ordering given by the Dynkin diagram and our construction of the positive roots by Algorithm 2.2. In Chapter 3, we determined the partial orders for the simple root systems given by our construction which we will use to prove results about θ_Δ.

Example 4.1.7. In type A, recall that the Chevalley order is both diagrammatic and lexicographic on the simple roots such that not only $N_{\alpha_1,\alpha_2+\alpha_3} > 0$ but also $N_{\alpha_1+\alpha_2,\alpha_3} > 0$.

In the non-simply laced cases, we have differing results:

Example 4.1.8. In type C, our Chevalley ordering is again diagrammatic, i.e. if i is the greatest index such that α_i is in the support of γ and j the least index such that α_j is in the support of δ, then $N_{\gamma,\delta} > 0$ only if $i \leq j$. For example, if $\alpha = \alpha_1 + \alpha_2$ and $\beta = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5$, then $N_{\alpha,\beta} > 0$ and if $\alpha = \alpha_4 + \alpha_5$ and $\beta = \alpha_3 + \alpha_4$, then $N_{\alpha,\beta} < 0$.

In type B, however, the Chevalley ordering is only lexicographic if the supports of the decomposing roots on the simple roots are disjoint. For example in type B_3,

$$N_{\alpha_1+\alpha_2,\alpha_3} > 0 \quad \text{but} \quad N_{\alpha_1+\alpha_2,\alpha_2+2\alpha_3} < 0.$$

Lemma 4.1.9. *Any automorphism of a root system $\phi \in \mathrm{Aut}(X, \Phi)$ preserves the magnitude of the Chevalley constants*

$$|N_{\alpha,\beta}| = |N_{\phi(\alpha),\phi(\beta)}|. \tag{4.1.9 (a)}$$

Proof. To be an automorphism of a root system, ϕ must preserve the geometry of the system and therefore also the root strings: If $\{\beta + n\alpha \in \Phi | -p \leq n \leq q\}$ is a root string in Φ then $\{\phi(\beta) + n\phi(\alpha) \in \Phi | -p \leq n \leq q\}$ must also be. In preserving the geometry, ϕ must preserve the inner product $(\alpha,\beta)=(\phi(\alpha),\phi(\beta))$ and so $(\alpha,\alpha)=\|\,\alpha\,\|^2=\|\,\phi(\alpha)\,\|^2=(\phi(\alpha),\phi(\alpha))$. Hence

$$N^2_{\alpha,\beta}=\frac{q(1+p)\|\,\alpha\,\|^2}{2}=\frac{q(1+p)\|\,\phi(\alpha)\,\|^2}{2}=N^2_{\phi(\alpha),\phi(\beta)}$$

and so

$$|N_{\alpha,\beta}|=\left|\sqrt{\frac{q(1+p)\|\,\alpha\,\|^2}{2}}\right|=|N_{\phi(\alpha),\phi(\beta)}|.$$

□

In addition to the results above, the following proposition will allow us to determine when θ_Δ is an involution. It will be pivotal in the classification of the absolutely irreducible $\theta\in\mathrm{Aut}(\Phi)$.

Proposition 4.1.10. *$\theta_\Delta\in\mathrm{Aut}(\mathfrak{g},\mathfrak{t})$ if and only if $c^{\theta_\Delta}_{\theta(\alpha)}=1$ for all $\alpha\in\Delta$.*

Proof. By definition,

$$\begin{aligned}
\theta_\Delta(H)&=H\ \ \forall H\in\mathfrak{t} && \Rightarrow \theta^2_\Delta(H)=H\\
\theta_\Delta(X_\alpha)&=X_{\theta(\alpha)}\ \ \forall\alpha\in\Delta && \Rightarrow \theta^2_\Delta(X_\alpha)=c^{\theta_\Delta}_{\theta(\alpha)}X_\alpha\ \ \forall\alpha\in\Delta \qquad (4.1.10\ (a))\\
\theta_\Delta(X_{\alpha+\beta})&=c^{\theta_\Delta}_{\alpha+\beta}X_{\theta(\alpha+\beta)}\ \ \forall\alpha+\beta\in\Phi && \Rightarrow \theta^2_\Delta(X_{\alpha+\beta})=c^{\theta_\Delta}_{\alpha+\beta}c^{\theta_\Delta}_{\theta(\alpha+\beta)}X_{\alpha+\beta}
\end{aligned}$$

(level 1) ($\Rightarrow$) If θ_Δ is an involution then $c^{\theta_\Delta}_{\theta(\alpha)}=1\ \ \forall\alpha\in\Delta$.
($\Leftarrow$) If $c^{\theta_\Delta}_{\theta(\alpha)}=1$ for all $\alpha\in\Delta$ then $c^{\theta_\Delta}_{\alpha+\beta}c^{\theta_\Delta}_{\theta(\alpha+\beta)}=1$, and this implies $\theta^2_\Delta(X_\gamma)=X_\gamma$ for all $\gamma\in\Phi$:
By the skew symmetry of the Lie bracket, we have the following identities to work with for any $\alpha,\beta,\alpha+\beta\in\Phi$:

$$c^{\theta_\Delta}_{\alpha+\beta}=\frac{N_{\theta(\alpha),\theta(\beta)}}{N_{\alpha,\beta}}\ c^{\theta_\Delta}_{\alpha}c^{\theta_\Delta}_{\beta} \qquad (4.1.10\ (b))$$

and

$$\frac{1}{c^{\theta_\Delta}_{\alpha}}=c^{\theta_\Delta}_{-\alpha} \qquad (4.1.10\ (c))$$

So,

$$c^{\theta_\Delta}_{\alpha+\beta}=\frac{N_{\theta(\alpha),\theta(\beta)}}{N_{\alpha,\beta}}\ c^{\theta_\Delta}_{\alpha}c^{\theta_\Delta}_{\beta} \qquad (4.1.10\ (d))$$

$$c^{\theta_\Delta}_{\theta(\alpha+\beta)}=c^{\theta_\Delta}_{\theta(\alpha)+\theta(\beta)}=\frac{N_{\alpha,\beta}}{N_{\theta(\alpha),\theta(\beta)}}\ c^{\theta_\Delta}_{\theta(\alpha)}c^{\theta_\Delta}_{\theta(\beta)} \qquad (4.1.10\ (e))$$

and

$$\begin{aligned}
c^{\theta_\Delta}_{\alpha+\beta}c^{\theta_\Delta}_{\theta(\alpha+\beta)}&=\frac{N_{\theta(\alpha),\theta(\beta)}}{N_{\alpha,\beta}}\frac{N_{\alpha,\beta}}{N_{\theta(\alpha),\theta(\beta)}}\ c^{\theta_\Delta}_{\alpha}c^{\theta_\Delta}_{\beta}\ c^{\theta_\Delta}_{\theta(\alpha)}c^{\theta_\Delta}_{\theta(\beta)}\\
&=c^{\theta_\Delta}_{\alpha}c^{\theta_\Delta}_{\beta}\ c^{\theta_\Delta}_{\theta(\alpha)}c^{\theta_\Delta}_{\theta(\beta)} \qquad (4.1.10\ (f))
\end{aligned}$$

(level 2) Let $\alpha, \beta \in \Delta, \alpha+\beta \in \Phi$, then

$$\begin{aligned} c^{\theta^2_\Delta}_{\alpha+\beta} = c^{\theta_\Delta}_{\alpha+\beta} c^{\theta_\Delta}_{\theta(\alpha+\beta)} &= c^{\theta_\Delta}_{\alpha} c^{\theta_\Delta}_{\beta}\ c^{\theta_\Delta}_{\theta(\alpha)} c^{\theta_\Delta}_{\theta(\beta)} \\ &= 1 \cdot 1 \cdot c^{\theta_\Delta}_{\theta(\alpha)} c^{\theta_\Delta}_{\theta(\beta)} \quad \text{(by defintion)} \\ &= 1 \quad \forall \alpha, \beta \in \Delta \end{aligned}$$

(level 3) Let $\alpha, \beta, \gamma \in \Delta, \alpha+\beta, \alpha+\beta+\gamma \in \Phi$, then

$$\begin{aligned} c^{\theta^2_\Delta}_{\alpha+\beta+\gamma} = c^{\theta_\Delta}_{\alpha+\beta+\gamma} c^{\theta_\Delta}_{\theta(\alpha+\beta+\gamma)} &= c^{\theta_\Delta}_{\alpha+\beta} c^{\theta_\Delta}_{\gamma}\ c^{\theta_\Delta}_{\theta(\alpha+\beta)} c^{\theta_\Delta}_{\theta(\gamma)} \\ &= c^{\theta_\Delta}_{\alpha+\beta} c^{\theta_\Delta}_{\theta(\alpha+\beta)}\ c^{\theta_\Delta}_{\gamma} c^{\theta_\Delta}_{\theta(\gamma)} \quad \text{(const's commute)} \\ &= 1 \cdot c^{\theta_\Delta}_{\gamma} c^{\theta_\Delta}_{\theta(\gamma)} \quad \text{(by level 2)} \\ &= 1 \quad \forall \alpha, \beta, \gamma \in \Delta \quad \text{(by level 1)} \end{aligned}$$

Inductive assumption: Assume $\forall \alpha = \sum_{j=1}^m \alpha_j \in \Phi \quad \alpha_i \in \Delta, m \leq k-1$, $c^{\theta^2_\Delta}_{\alpha} = c^{\theta_\Delta}_{\alpha} c^{\theta_\Delta}_{\theta(\alpha)} = 1$.
(level k) Let $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_i$, $\beta = \alpha_{i+1} + \cdots + \alpha_k$, $\alpha + \beta \in \Phi$ then

$$\begin{aligned} c^{\theta^2_\Delta}_{\alpha+\beta} = c^{\theta_\Delta}_{\alpha+\beta} c^{\theta_\Delta}_{\theta(\alpha+\beta)} &= c^{\theta_\Delta}_{\alpha} c^{\theta_\Delta}_{\beta}\ c^{\theta_\Delta}_{\theta(\alpha)} c^{\theta_\Delta}_{\theta(\beta)} \\ &= c^{\theta_\Delta}_{\alpha} c^{\theta_\Delta}_{\theta(\alpha)}\ c^{\theta_\Delta}_{\beta} c^{\theta_\Delta}_{\theta(\beta)} \\ &= 1 \cdot 1 \\ &= 1 \end{aligned}$$

Thus by induction, if $c^{\theta_\Delta}_{\theta(\alpha)} = 1 \quad \forall \alpha \in \Delta$, then $c^{\theta^2_\Delta}_{\alpha} = c^{\theta_\Delta}_{\alpha} c^{\theta_\Delta}_{\theta(\alpha)} = 1 \quad \forall \alpha \in \Phi^+$ and therefore $\forall \alpha \in \Phi$ by equation 4.1.10 (c). $\square$

Chapter 5: Correction Vectors

5.1 Calculating Correction Vectors for Admissible Involutions

By definition, any admissible involution θ can be lifted and therefore by proposition 5.2.2, there exists an $h \in T$ such that $\theta_\Delta \mathrm{Int}(h)$ is an involution. If θ_Δ is already an involution then $h = id \in T$ so that $\theta_\Delta = \theta_\Delta \mathrm{Int}(h)$.
Suppose we have θ_Δ such that for some $\alpha_i \in \Delta$,

$$\theta_\Delta^2(X_{\alpha_i}) = c_{\alpha_i}^{\theta_\Delta} c_{\theta(\alpha_i)}^{\theta_\Delta} X_{\alpha_i} \neq X_{\alpha_i}. \tag{5.1.0 (a)}$$

For admissible involutions θ, we find the only occurrence of this is when

$$\theta_\Delta^2(X_{\alpha_i}) = -1 \cdot X_{\alpha_i}. \tag{5.1.0 (b)}$$

5.1.1 $\mathrm{Int}(h)(X_\alpha)$

Let $h \in G$ and $H \in \mathfrak{g}$ such that $H = S\Lambda S^{-1}$, $H = \sum y_j H_j$ where $H_j \in \mathfrak{t}$ such that $H_j = \alpha_j^\vee = \frac{2H_{\alpha_j}}{(\alpha_j,\alpha_j)}$, so that $H = y_1H_1 + y_2H_2 + \cdots + y_\ell H_\ell$ where $y_i \in \mathbb{C}$ for any $H \in \mathfrak{t}$ and

$$\alpha(H_j) = \alpha\left(\frac{2H_{\alpha_j}}{(\alpha_j,\alpha_j)}\right) = \frac{2(\alpha,\alpha_j)}{(\alpha_j,\alpha_j)} = \langle \alpha, \alpha_j \rangle.$$

Then

$$
\begin{aligned}
\operatorname{Int}(h)(X_\alpha) = hX_\alpha h^{-1} = \operatorname{Ad}(h)(X_\alpha) &= \operatorname{Ad}(\exp(H))(X_\alpha) \\
&= \exp(\operatorname{ad}(H)(X_\alpha)) \\
&= X_\alpha + [H\ ,\ X_\alpha] + \frac{[H\ ,\ [H\ ,\ X_\alpha]]}{2!} + \cdots \\
&= X_\alpha + \alpha(H)X_\alpha + \frac{\alpha(H)^2 X_\alpha}{2!} + \cdots \\
&= \sum_{k=0}^{\infty} \frac{(\alpha(H))^k (X_\alpha)}{k!} \\
&= \exp\left(\alpha\left(H\right)\right) X_\alpha \\
&= \exp\left(\alpha\left(\sum y_j H_j\right)\right) X_\alpha \\
&= \exp\left(\sum y_j \alpha\left(H_j\right)\right) X_\alpha \\
&= \exp\left(\sum y_j \langle \alpha, \alpha_j \rangle\right) X_\alpha \\
&= e^{\left(\sum y_j \langle \alpha, \alpha_j \rangle\right)} X_\alpha .
\end{aligned}
$$

5.1.2 $\theta_\Delta \operatorname{Int}(t)(X_\alpha)$

In order to determine an involution from θ_Δ, we can find a vector $H \in \mathfrak{t}$ corresponding to an $h \in T$ such that

$$
\begin{aligned}
\theta_\Delta \operatorname{Ad}(h)(X_\alpha) &= \theta_\Delta \left(e^{\left(\sum y_j \langle \alpha, \alpha_j \rangle\right)} X_\alpha \right) \\
&= c_\alpha^{\theta_\Delta} \cdot e^{\left(\sum y_j \langle \alpha, \alpha_j \rangle\right)} X_{\theta(\alpha)}
\end{aligned}
$$

and

$$
\begin{aligned}
\left(\theta_\Delta \operatorname{Ad}(h)\right)^2 (X_\alpha) &= c_\alpha^{\theta_\Delta} e^{\left(\sum y_j \langle \alpha, \alpha_j \rangle\right)} \cdot c_{\theta(\alpha)}^{\theta_\Delta} e^{\left(\sum y_j \langle \theta(\alpha), \alpha_j \rangle\right)} X_\alpha \\
&= c_\alpha^{\theta_\Delta} c_{\theta(\alpha)}^{\theta_\Delta} \cdot e^{\left(\sum y_j \langle \alpha, \alpha_j \rangle\right)} e^{\left(\sum y_j \langle \theta(\alpha), \alpha_j \rangle\right)} X_\alpha \\
&= c_\alpha^{\theta_\Delta} c_{\theta(\alpha)}^{\theta_\Delta} \cdot e^{\left(\sum y_j \langle \alpha, \alpha_j \rangle + \sum y_j \langle \theta(\alpha), \alpha_j \rangle\right)} X_\alpha \\
&= X_\alpha .
\end{aligned}
$$

In fact, by a result of Helminck (see proposition 5.2.2), we know that we can limit our search to $h \in T_\theta^+$. By the above we need to solve the equation

$$
c_\alpha^{\theta_\Delta} c_{\theta(\alpha)}^{\theta_\Delta} \cdot e^{\left(\sum y_j \langle \alpha, \alpha_j \rangle\ + \sum y_j \langle \theta(\alpha), \alpha_j \rangle\right)} = 1
$$

or, since $c_\alpha^{\theta_\Delta} \neq 0$,

$$e^{\left(\sum y_j \langle \alpha,\alpha_j\rangle\ + \sum y_j \langle \theta(\alpha),\alpha_j\rangle\right)} = \frac{1}{c_\alpha^{\theta_\Delta} c_{\theta(\alpha)}^{\theta_\Delta}}.$$

We find that $\frac{1}{c_\alpha^{\theta_\Delta} c_{\theta(\alpha)}^{\theta_\Delta}} = \pm 1$. Let $k \in \mathbb{Z}$, then

<u>Case 1:</u> $\frac{1}{c_\alpha^{\theta_\Delta} c_{\theta(\alpha)}^{\theta_\Delta}} = +1$:

$$e^{\left(\sum y_j \langle \alpha,\alpha_j\rangle + \sum y_j \langle \theta(\alpha),\alpha_j\rangle\right)} = 1 = e^{2ki\pi}$$

giving us the equation

$$\begin{aligned} \left(\sum y_j \langle \alpha, \alpha_j\rangle + \sum y_j \langle \theta(\alpha), \alpha_j\rangle\right) &= 2ki\pi \\ \sum y_j \left(\langle \alpha, \alpha_j\rangle + \langle \theta(\alpha), \alpha_j\rangle\right) &= 2ki\pi. \end{aligned}$$

<u>Case 2:</u> $\frac{1}{c_\alpha^{\theta_\Delta} c_{\theta(\alpha)}^{\theta_\Delta}} = -1$:

$$e^{\left(\sum y_j \langle \alpha,\alpha_j\rangle + \sum y_j \langle \theta(\alpha),\alpha_j\rangle\right)} = -1 = e^{(2k+1)i\pi}$$

corresponding to

$$\sum y_j \left(\langle \alpha, \alpha_j\rangle + \langle \theta(\alpha), \alpha_j\rangle\right) = (2k+1)i\pi.$$

In order to ensure that $\theta_\Delta \operatorname{Ad}(h)$ is indeed an involution, we must satisfy a system of such equations. Given that we have previously seen that all the lifting constants $c_\alpha^{\theta_\Delta}$ are determined by the set $\{c_{\alpha_i}^{\theta_\Delta} | \alpha_i \in \Delta\}$, and therefore also all the $c_{\theta(\alpha)}^{\theta_\Delta}$, we need only consider the system of equations corresponding to the set of $\alpha_i \in \Delta$.

5.2 Correction Vector Algorithm for θ_Δ

In the following example, θ_Δ is not the standard involution and we show the relation between the two.

Example 5.2.0.1 (Involutions in $\mathrm{sl}(2,\mathbb{C})$)**.** In $\mathrm{sl}(2,\mathbb{C})$ there is only one conjugacy class of involutions, which may be represented by abuse of notation by $\theta(X) = -X^T$, corresponding to the opposition involution on the root system A_1, $\theta(\alpha) = -\alpha = -\theta^*(\alpha) = -\operatorname{id}^*(\alpha)$. If we take a natural basis of $\mathrm{sl}(2,\mathbb{C})$, it is clear by simple calculation that the lifting constants

$c^\theta_\alpha = -1 = c^\theta_{-\alpha}$:

$$H_\alpha = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \quad X_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_{-\alpha} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

then

$$\begin{aligned} [H_\alpha\ ,\ X_\alpha] &= aX_\alpha - (-a)X_\alpha = 2aX_\alpha \\ [H_\alpha\ ,\ X_{-\alpha}] &= -aX_{-\alpha} - aX_{-\alpha} = -2aX_{-\alpha} \end{aligned}$$

and

$$\begin{aligned} \theta: \quad H_\alpha &\mapsto \begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix} = -H_\alpha = H_{-\alpha} = H_{\theta(\alpha)} \\ X_\alpha &\mapsto \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = -X_{-\alpha} = -X_{\theta(\alpha)} = c^\theta_\alpha X_{\theta(\alpha)} \\ X_{-\alpha} &\mapsto \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = -X_\alpha = -X_{\theta(-\alpha)} = c^\theta_{-\alpha} X_{\theta(-\alpha)} \end{aligned}$$

Composition with

$$\operatorname{Int}(h_\alpha) = \operatorname{Int}\left(\exp\left(\frac{i\pi}{2a}H_\alpha\right)\right) = \operatorname{Int}\begin{pmatrix} e^{\frac{i\pi}{2}} & 0 \\ 0 & e^{-\frac{i\pi}{2}} \end{pmatrix}$$

gives us another involution that has the same action on the root space, but different lifting constants:

$$\begin{aligned} \theta\operatorname{Int}(h_\alpha): \quad & H_\alpha \xmapsto{\operatorname{Int}(h_\alpha)} \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \overset{\theta}{\mapsto} \begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix} = -H_\alpha = H_{-\alpha} = H_{\theta(\alpha)} \\ & X_\alpha \xmapsto{\operatorname{Int}(h_\alpha)} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \overset{\theta}{\mapsto} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = X_{-\alpha} = X_{\theta(\alpha)} = c^{\theta\operatorname{Int}(h_\alpha)}_\alpha X_{\theta(\alpha)} \\ & X_{-\alpha} \xmapsto{\operatorname{Int}(h_\alpha)} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \overset{\theta}{\mapsto} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = X_\alpha = X_{\theta(-\alpha)} = c^{\theta\operatorname{Int}(h_\alpha)}_{-\alpha} X_{\theta(-\alpha)} \end{aligned}$$

So in this example we have shown that $\theta_\Delta = \theta\operatorname{Int}(h_\alpha)$.
This process allows us to choose the $\tilde{\theta} \in \operatorname{Aut}(G,T)$ to which we lift $\theta \in \operatorname{Aut}(\Phi)$.

For the admissible involutions, θ , if θ_Δ is not an involution on $\mathfrak{g}$, and hence not on G, then by the following results we can find a correction vector $H \in \mathfrak{t}^+_\theta$ such that $\theta_\Delta\operatorname{Int}(\exp(H))$ is an involution in $\mathfrak{g}$. First, the following lemma is well-known:

Lemma 5.2.1. *The exponential map from the Lie algebra of a torus to the torus is surjective:*

$$\exp : \mathfrak{t} \to T$$

Proposition 5.2.2 ([Helminck(2000), 10.8])**.** *Let $\theta \in \mathrm{Aut}(X, \Phi)$ be an involution and Δ a basis of Φ (T, G). Then the following are equivalent :*

(1) θ can be lifted.

(2) There is a $h \in T$ such that θ_Δ Int(h) is an involution.

(3) There is a $h \in T^+_\theta$ such that θ_Δ Int(h) is an involution.

(4) There is a $h \in T$ such that $c^{\theta_\Delta}_{\theta(\alpha)} = \alpha(\theta(h)h)$ for all $\alpha \in \Delta$.

(5) There is a $h \in T^+_\theta$ such that $c^{\theta_\Delta}_{\theta(\alpha)} = \alpha(h)$ for all $\alpha \in \Delta$.

Corollary 5.2.3. *Let $\theta \in$ Aut (X, Φ) be an involution and Δ a basis of Φ (T, G). Then the following are equivalent :*

(1) θ can be lifted.

(2) There is a $H \in \mathfrak{t}$ so that $\exp(H) = h \in T$ such that $\theta_\Delta \, \mathrm{Int}(h) = \theta_\Delta \, \mathrm{Ad}(\exp(H)) = \theta_\Delta \exp(\mathrm{ad}(H))$ is an involution.

(3) There is a $H \in \mathfrak{t}^+_\theta$ so that $\exp(H) = h \in T^+_\theta$ such that $\theta_\Delta \exp(\mathrm{ad}(H))$ is an involution.

The following algorithm is implemented in our *k-Involutions* Mathematica package as

`getCorrectionVectors[myInvol_admissibleInvolution]`.

It is the method by which we correct θ_Δ to an involution and follows from the computations above.

Algorithm 5.3 (Correction Vector for θ_Δ)**.**

Input: θ_Δ

Output: all solutions $H \in \mathfrak{t}^+_\theta$ such that $c^{\theta_\Delta \exp(\mathrm{ad}(H))}_\alpha c^{\theta_\Delta \exp(\mathrm{ad}(H))}_{\theta(\alpha)} = 1$

As usual, let ℓ be the rank of Φ, $\theta \in \mathrm{Aut}(\Phi)$ the root action corresponding to θ_Δ, $H_j = \alpha_j^\vee \in \mathfrak{t}$, and denote $\vec{H} = (H_1, H_2, \ldots, H_\ell)$.

(CorrVec 1) If θ_Δ is an involution, then put $H = 0$ and $c^{\theta_\Delta \exp(\mathrm{ad}(H))}_\alpha = c^{\theta_\Delta}_\alpha$.
Go to **(CorrVec 3)**.

(CorrVec 2) If θ_Δ is an not involution, then

(a) Let $\vec{y} = (y_i)_1^\ell$ be the coefficient vector of H as in Section 5.1.1,

(b) Put $M = (\langle \alpha_i, \alpha_j \rangle) + (\langle \theta(\alpha_i), \alpha_j \rangle)$ for $\alpha_i, \alpha_j \in \Delta$ so that $c_{\alpha_i}^{\theta_\Delta \exp(\mathrm{ad}(H))} c_{\theta(\alpha_i)}^{\theta_\Delta \exp(\mathrm{ad}(H))} = c_{\alpha_i}^{\theta_\Delta} c_{\theta(\alpha_i)}^{\theta_\Delta} e^{(M.\vec{y})_i}$,

(c) Let $\vec{y}_{\mathrm{form}} = \sum b_j \beta_j$ for $\beta_j \in \Delta_0$, $b_j \in \mathbb{C} \Rightarrow H \in \mathfrak{t}_\theta^+$, and

(d) Solve the system of equations $c_{\alpha_i}^{\theta_\Delta \exp(\mathrm{ad}(H))} c_{\theta(\alpha_i)}^{\theta_\Delta \exp(\mathrm{ad}(H))} = 1$ for $\vec{y}$ with the restriction that $\vec{y}$ is of the form $\vec{y}_{\mathrm{form}}$.

(CorrVec 3) Return all solutions $H = \vec{y}.\vec{H}$ along with the general form of $c_\alpha^{\theta_\Delta \exp(\mathrm{ad}(H))}$and assumptions for each. End.

5.3.1 D_5^2IIIb Example

Let $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ be a basis for a root system Φ of type D_5 and θ an involution on Φ of Cartan Type D_5^2IIIb. Then the θ-diagram is shown below in Figure 5.1 and the θ-action on Δ and the lifting constants for θ_Δ are shown in Table 5.2.

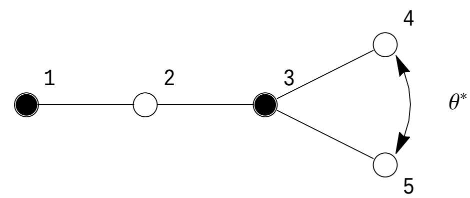


Figure 5.1: D_5^2IIIb θ-diagram

Table 5.2: D_5^2IIIb: θ_Δ Lifting Constants on the α-basis

$\alpha_i \mapsto$	$\theta(\alpha_i)$	$c_{\alpha_i}^{\theta_\Delta}$	$c_{\theta(\alpha_i)}^{\theta_\Delta}$
$\alpha_1 \mapsto$	α_1	1	1
$\alpha_2 \mapsto$	$-\alpha_1 - \alpha_2 - \alpha_3$	1	1
$\alpha_3 \mapsto$	α_3	1	1
$\alpha_4 \mapsto$	$-\alpha_3 - \alpha_5$	1	-1
$\alpha_5 \mapsto$	$-\alpha_3 - \alpha_4$	1	-1

We choose a Chevalley basis with respect to Δ and a maximal toral subalgebra, $\mathfrak{t}$, of the corresponding complex Lie algebra, $\mathfrak{g}$, such that H_i corresponds to α_i in the normal way and therefore the H_i's form a basis of $\mathfrak{t}$ as a vector space. Thus any $H \in \mathfrak{t}$ may be expressed as

$$H = y_1 H_1 + y_2 H_2 + y_3 H_3 + y_4 H_4 + y_5 H_5$$

such that $y_i \in \mathbb{C}$.
In order to find the correction vector $H \in \mathfrak{t}$ such that $\theta_\Delta \exp(\mathrm{ad}(H))$ is an involution, we need to solve the following system:

$$\begin{aligned}
e^{\sum_{j=1}^{5} y_j(\langle \alpha_1, \alpha_j\rangle + \langle \theta(\alpha_1), \alpha_j\rangle)} &= 1 \\
e^{\sum_{j=1}^{5} y_j(\langle \alpha_2, \alpha_j\rangle + \langle \theta(\alpha_2), \alpha_j\rangle)} &= 1 \\
e^{\sum_{j=1}^{5} y_j(\langle \alpha_3, \alpha_j\rangle + \langle \theta(\alpha_3), \alpha_j\rangle)} &= 1 \qquad (5.3.1\text{ (a)}) \\
e^{\sum_{j=1}^{5} y_j(\langle \alpha_4, \alpha_j\rangle + \langle \theta(\alpha_4), \alpha_j\rangle)} &= -1 \\
e^{\sum_{j=1}^{5} y_j(\langle \alpha_5, \alpha_j\rangle + \langle \theta(\alpha_5), \alpha_j\rangle)} &= -1
\end{aligned}$$

which can be solved as the following linear system where $k_i \in \mathbb{Z}$:

$$\begin{aligned}
\sum_{j=1}^{5} y_j \left(\langle \alpha_1, \alpha_j\rangle + \langle (\alpha_1), \alpha_j\rangle\right) &= 2k_1 i\pi \\
\sum_{j=1}^{5} y_j \left(\langle \alpha_2, \alpha_j\rangle + \langle (-\alpha_1 - \alpha_2 - \alpha_3), \alpha_j\rangle\right) &= 2k_2 i\pi \\
\sum_{j=1}^{5} y_j \left(\langle \alpha_3, \alpha_j\rangle + \langle (\alpha_3), \alpha_j\rangle\right) &= 2k_3 i\pi \qquad (5.3.1\text{ (b)}) \\
\sum_{j=1}^{5} y_j \left(\langle \alpha_4, \alpha_j\rangle + \langle (-\alpha_3 - \alpha_5), \alpha_j\rangle\right) &= (2k_4 + 1) i\pi \\
\sum_{j=1}^{5} y_j \left(\langle \alpha_5, \alpha_j\rangle + \langle (-\alpha_3 - \alpha_4), \alpha_j\rangle\right) &= (2k_5 + 1) i\pi.
\end{aligned}$$

Cartan Matrix for Φ:

$$(\langle \alpha_i, \alpha_j \rangle) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

θ-Permuted Cartan Matrix:

$$(\langle \theta(\alpha_i), \alpha_j \rangle) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & 1 \\ 0 & -1 & 2 & -1 & -1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & -1 & 1 \end{pmatrix}$$

y_i-Coefficient Matrix:

$$M = (\langle \alpha_i, \alpha_j \rangle) + (\langle \theta(\alpha_i), \alpha_j \rangle) = \begin{pmatrix} 4 & -2 & 0 & 0 & 0 \\ -2 & 2 & -2 & 1 & 1 \\ 0 & -2 & 4 & -2 & -2 \\ 0 & 1 & -2 & 3 & -1 \\ 0 & 1 & -2 & -1 & 3 \end{pmatrix}$$

So, we solve the equation

$$(c_{\alpha_i}^{\theta_\Delta} c_{\theta(\alpha_i)}^{\theta_\Delta} e^{(M.\vec{y})_i})_\Delta = \begin{pmatrix} e^{4y_1-2y_2} \\ e^{-2y_1+2y_2-2y_3+y_4+y_5} \\ e^{-2y_2+4y_3-2y_4-2y_5} \\ e^{y_2-2y_3+3y_4-y_5} \\ e^{y_2-2y_3-y_4+3y_5} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

and by restricting to our $\vec{y}_{\text{form}}$, this becomes

$$(c_{\alpha_i}^{\theta_\Delta} c_{\theta(\alpha_i)}^{\theta_\Delta} e^{(M.\vec{y}_{\text{form}})_i})_\Delta = \begin{pmatrix} e^{4y_1} \\ e^{-2y_1-2y_3} \\ e^{4y_3} \\ e^{-2y_3} \\ e^{-2y_3} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

giving us the solution

$$\{y_1, y_2, y_3, y_4, y_5\} = \left\{ \frac{1}{2} i\pi \left(4x_1 - 1\right), 0, \frac{1}{2} i\pi \left(4x_3 - 1\right), 0, 0 \right\}$$

or

$$H = \frac{1}{2} i\pi \left(\left(4x_1 - 1\right) H_{\alpha_1} + \left(4x_3 - 1\right) H_{\alpha_3} \right)$$

where $x_i \in \mathbb{Z}$. For computation purposes we choose the simplest representation where $x_i = 0$:

$$H = -\frac{1}{2} i\pi \left(H_{\alpha_1} + H_{\alpha_3}\right).$$

Hence we find the lifting constants for the corrected automorphism

$$\theta_\Delta \exp\left(\mathrm{ad}\left(-\frac{1}{2} i\pi \left(H_{\alpha_1} + H_{\alpha_3}\right)\right)\right)$$

and see that it is clearly an involution:

$\alpha_i \mapsto$	$\theta(\alpha_i)$	$c_{\alpha_i}^{\theta_\Delta \operatorname{Int}(h)}$	$c_{\theta(\alpha_i)}^{\theta_\Delta \operatorname{Int}(h)}$
$\alpha_1 \mapsto$	α_1	-1	-1
$\alpha_2 \mapsto$	$-\alpha_1 - \alpha_2 - \alpha_3$	-1	-1
$\alpha_3 \mapsto$	α_3	-1	-1
$\alpha_4 \mapsto$	$-\alpha_3 - \alpha_5$	i	$-i$
$\alpha_5 \mapsto$	$-\alpha_3 - \alpha_4$	i	$-i$

Chapter 6: Lifting to θ_Δ over $\mathbb{C}$

This results in this chapter can be generalized to any algebraically closed field of characteristic not 2. We restrict our attention to the complex numbers.

6.1 General results

Recall, we define the height of a root to be the sum of the coefficients in the basis:

$$\mathrm{ht}(\alpha) = \sum_{i=1}^{n} r_i$$

when $\alpha = \sum_{i=1}^{n} r_i \alpha_i$ for $\alpha_i \in \Delta$.

Lemma 6.1.1 (Black roots)**.** *Any fixed root $\alpha \in \Phi_o(\theta)$ has $c_\alpha^{\theta_\Delta} = 1$:*

$$\theta_\Delta : X_\alpha \mapsto X_\alpha \quad \forall \alpha \in \Phi_0(\theta) \tag{6.1.1 (a)}$$

Proof. For any $\alpha = \sum_{\Delta_0(\theta)} s_i \alpha_i \in \Phi_0(\theta)$, either $\alpha = \alpha_i \in \pm\Delta_0$ in which case $c_\alpha^{\theta_\Delta} \overset{\text{def}}{=} 1$ or $\alpha = \beta + \gamma$ such that $\beta, \gamma \in \Phi_0$. Which gives us

$$\frac{N_{\theta(\beta),\theta(\gamma)}}{N_{\beta,\gamma}} = \frac{N_{\beta,\gamma}}{N_{\beta,\gamma}} = 1$$

and

$$c_\alpha^{\theta_\Delta} \overset{\text{def}}{=} c_\beta^{\theta_\Delta} c_\gamma^{\theta_\Delta} \frac{N_{\theta(\beta),\theta(\gamma)}}{N_{\beta,\gamma}} = c_\beta^{\theta_\Delta} c_\gamma^{\theta_\Delta},$$

so, by simple induction $c_\alpha^{\theta_\Delta} = 1$ for any $\alpha \in \Phi_0(\theta)$. □

Alternatively, recall that by definition we take the black dots of a θ-diagram to be the simple roots fixed by θ, see for example Figures 6.1 and 6.2. Since θ acts linearly on Φ, it is also true for any $\alpha = \sum_{\Delta_0(\theta)} s_i\alpha_i \in \Phi_0(\theta)$ that $\theta(\alpha) = \theta\left(\sum_{\Delta_0(\theta)} s_i\alpha_i\right) = \sum_{\Delta_0(\theta)} s_i\theta(\alpha_i) = \sum_{\Delta_0(\theta)} s_i\alpha_i = \alpha$.

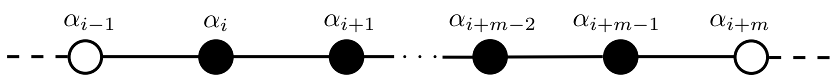


Figure 6.1: Black middle with $\theta^* = \text{id}$: $\alpha = \sum s_i\alpha_i \in \Phi_0$.

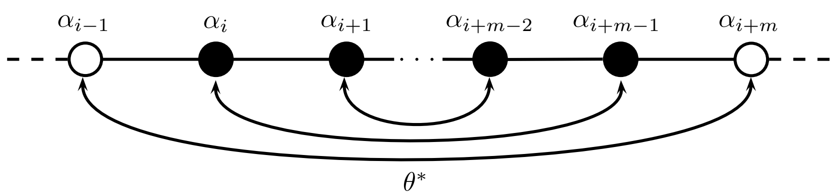


Figure 6.2: Black middle with $\theta^* \neq \text{id}$: $\alpha = \sum s_i\alpha_i \in \Phi_0$.

Remark 6.1.2. We also have the relation that if $\theta(\alpha_i) = -\alpha_i$ then $c^{\theta_\Delta}_{\theta(\alpha_i)} = \dfrac{1}{c^{\theta_\Delta}_{\alpha_i}} = 1$ if $\alpha_i \in \Delta$.

Lemma 6.1.3 (All white dots and $\theta^* = \text{id}$)**.** *If $-\text{id} = \theta \in \text{Aut}(X, \Phi)$, corresponding to the involution diagram of all white dots and no arches, then any automorphism $\tilde{\theta} \in \text{Aut}(\mathfrak{g}, \mathfrak{t})$ lifted from θ must be an involution. In particular, θ_Δ is an involution with structure constants*

$$c^{\theta_\Delta}_\alpha = (-1)^{\text{ht}(\alpha)-1}.$$

Proof. First we will show that any automorphism $\tilde{\theta} \in \text{Aut}(\mathfrak{g}, \mathfrak{t})$ lifted from $-\text{id} = \theta \in \text{Aut}(X, \Phi)$ is an involution and then we will prove the claim about the lifting constants, $c^{\theta_\Delta}_\alpha$.

(Part 1) In this case, $\theta(\alpha) = -\alpha$ for all $\alpha \in \Phi$. We know that for any set of structure constants for a lifted involution, $c^{\tilde{\theta}}_{-\alpha} = 1/c^{\tilde{\theta}}_\alpha$. Hence

$$\tilde{\theta}^2(X_\alpha) = c^{\tilde{\theta}}_\alpha c^{\tilde{\theta}}_{\theta(\alpha)} X_\alpha = c^{\tilde{\theta}}_\alpha \frac{1}{c^{\tilde{\theta}}_\alpha} X_\alpha = X_\alpha$$

and $\tilde{\theta}$ is an involution of the Lie algebra. In the particular case of θ_Δ,

$$\theta_\Delta{}^2(X_\alpha) = c_\alpha^{\theta_\Delta} c_{\theta(\alpha)}^{\theta_\Delta} X_\alpha = (\pm 1)\frac{1}{(\pm 1)} X_\alpha = (\pm 1)^2 X_\alpha = X_\alpha.$$

(Part 2) Next we will prove $c_\alpha^{\theta_\Delta} = (-1)^{\mathrm{ht}(\alpha)-1}$ for all $\alpha \in \Phi$. By definition $c_{-\alpha}^{\theta_\Delta} = c_\alpha^{\theta_\Delta}$ and $(-1)^{n-1} = (-1)^{-n-1}$ for any $n \in \mathbb{Z}$, so, it is sufficient to consider only $\alpha \in \Phi^+$.

Case 1: $\mathrm{ht}(\alpha) = 1$. By definition, for any simple root $\alpha_i \in \Delta$, $c_{\alpha_i}^{\theta_\Delta} = 1 = (-1)^0$.

Case 2: $\mathrm{ht}(\alpha) = 2$. Assume $\alpha, \beta \in \Delta$ are simple roots. Recalling Table 3.1 and eq. (4.1.10 (b)):

$$N_{\alpha,\beta} = -N_{-\alpha,-\beta} = -N_{\theta(\alpha),\theta(\beta)}$$

giving us

$$c_{\alpha+\beta}^{\theta_\Delta} = c_\alpha^{\theta_\Delta} c_\beta^{\theta_\Delta} \frac{N_{\theta(\alpha),\theta(\beta)}}{N_{\alpha,\beta}} = c_\alpha^{\theta_\Delta} c_\beta^{\theta_\Delta} \frac{N_{-\alpha,-\beta}}{N_{\alpha,\beta}} = c_\alpha^{\theta_\Delta} c_\beta^{\theta_\Delta} \frac{-N_{\alpha,\beta}}{N_{\alpha,\beta}} = -1 \cdot c_\alpha^{\theta_\Delta} c_\beta^{\theta_\Delta}$$

hence

$$c_{\alpha+\beta}^{\theta_\Delta} = -1 \cdot c_\alpha^{\theta_\Delta} c_\beta^{\theta_\Delta} = -1 \cdot 1 \cdot 1 = (-1)^1.$$

Case k: $\mathrm{ht}(\alpha) = k$.

$$c_\alpha^{\theta_\Delta} = (-1)^{k-1}.$$

Case k+1: $\mathrm{ht}(\alpha + \alpha_i) = k + 1$. Assume $\mathrm{ht}(\alpha) = k$ and $\mathrm{ht}(\alpha_i) = 1$, then $\mathrm{ht}(\alpha + \alpha_i) = k + 1$ by definition. Then

$$\begin{aligned} c_{\alpha+\alpha_i}^{\theta_\Delta} &= c_\alpha^{\theta_\Delta}\ c_\alpha^{\theta_\Delta}\ \frac{N_{-\alpha,-\alpha_i}}{N_{\alpha,\alpha_i}} && \text{(6.1.3 (a))} \\ &= -1 \cdot c_\alpha^{\theta_\Delta}\ c_{\alpha_i}^{\theta_\Delta} && \text{(6.1.3 (b))} \\ &= -1 \cdot (-1)^{k-1} \cdot 1 && \text{(6.1.3 (c))} \\ &= (-1)^{(k+1)-1}. && \text{(6.1.3 (d))} \end{aligned}$$

Therefore by induction, $c_\alpha^{\theta_\Delta} = (-1)^{\mathrm{ht}(\alpha)-1}$ for all $\alpha \in \Phi$.

□

Remark 6.1.4. $w_0(\theta)$ only acts non-trivially on roots containing or adjacent to roots containing $\alpha_i \in \Delta_0(\theta)$. This is because all other roots are orthogonal to the roots in Φ_0 and therefore also to the reflections that compose $w_0(\theta)$. Hence for any root $\alpha \in \Phi \backslash \Phi_0$ that is acted on trivially by $w_0(\theta)$:

$$\theta(\alpha) = -\operatorname{id}\ \theta^*\ w_0(\theta)(\alpha) = -\theta^*(\alpha)$$

Lemma 6.1.5. *Remark 6.1.4, in turn, implies that θ_Δ acts as involution on the root spaces corresponding to $\alpha \in \Phi\backslash\Phi_0$ upon which $w_0(\theta)$ acts trivially.*

Proof. Let $\alpha \in \Phi\backslash\Phi_0$ upon which $w_0(\theta)$ acts trivially as in remark 6.1.4. Since we've already proved this to be true in the case when $\theta^* = \operatorname{id}$, it is sufficient to prove the result when $\theta^* \neq \operatorname{id}$.

Base Case 1: Consider first $\alpha = \alpha_i \in \Delta\backslash\Delta_0$:

$$\begin{aligned} \theta_\Delta^2(X_{\alpha_i}) &= c_{\alpha_i}^{\theta_\Delta} c_{\theta(\alpha_i)}^{\theta_\Delta} X_{\alpha_i} \\ &= 1 \cdot c_{-\alpha_j}^{\theta_\Delta} X_{\alpha_i} && \text{such that } \alpha_j \in \Delta \\ &= X_{\alpha_i} && \text{by remark 6.1.2.} \end{aligned}$$

Base Case 2: Let $\alpha = \alpha_i + \alpha_{i+1}$ such that $\alpha_{i-1} \dots \alpha_{i+1} \in \Delta\backslash\Delta_0$. Then

$$c_{\theta(\alpha)}^{\theta_\Delta} = c_{\theta(\alpha_i)+\theta(\alpha_{i+1})}^{\theta_\Delta} = c_{\theta(\alpha_i)}^{\theta_\Delta} \cdot c_{\theta(\alpha_{i+1})}^{\theta_\Delta} \cdot \frac{N_{\theta(\theta(\alpha_i)),\theta(\theta(\alpha_{i+1}))}}{N_{\theta(\alpha_i),\theta(\alpha_{i+1})}} = \frac{N_{\alpha_i,\alpha_{i+1}}}{N_{\theta(\alpha_i),\theta(\alpha_{i+1})}}.$$

In types A and E,

$$c_{\theta(\alpha)}^{\theta_\Delta} = \frac{N_{\alpha_i,\alpha_{i+1}}}{N_{\theta(\alpha_i),\theta(\alpha_{i+1})}} = \frac{N_{\alpha_i,\alpha_{i+1}}}{N_{-\alpha_j,-\alpha_{j-1}}} = \frac{N_{\alpha_i,\alpha_{i+1}}}{N_{\alpha_{j-1},\alpha_j}} = 1 = c_\alpha^{\theta_\Delta} \Rightarrow \theta_\Delta^2(X_\alpha) = X_\alpha.$$

In type D, this can only occur when $\alpha_i = \alpha_{\ell-2}$ and α_{i+1} is a spin node. Since $\theta^* : \alpha_{\ell-1} \leftrightarrow \alpha_\ell$, it is sufficient to consider one of these cases:

$$c_{\theta(\alpha)}^{\theta_\Delta} = \frac{N_{\alpha_i,\alpha_{i+1}}}{N_{\theta(\alpha_i),\theta(\alpha_{i+1})}} = \frac{N_{\alpha_{\ell-2},\alpha_{\ell-1}}}{N_{\theta(\alpha_{\ell-2}),\theta(\alpha_{\ell-1})}} = \frac{N_{\alpha_{\ell-2},\alpha_{\ell-1}}}{N_{-\alpha_{\ell-2},-\alpha_\ell}} = \frac{N_{\alpha_{\ell-2},\alpha_{\ell-1}}}{-N_{\alpha_{\ell-2},\alpha_\ell}},$$

and

$$c_\alpha^{\theta_\Delta} = \frac{N_{\theta(\alpha_{\ell-2}),\theta(\alpha_{\ell-1})}}{N_{\alpha_{\ell-2},\alpha_{\ell-1}}} = \frac{N_{-\alpha_{\ell-2},-\alpha_\ell}}{N_{\alpha_{\ell-2},\alpha_{\ell-1}}} = \frac{-N_{\alpha_{\ell-2},\alpha_\ell}}{N_{\alpha_{\ell-2},\alpha_{\ell-1}}} = \frac{1}{c_{\theta(\alpha)}^{\theta_\Delta}},$$

hence again $\theta_\Delta^2(X_\alpha) = X_\alpha$.

The result follows by induction on $\operatorname{ht}(\alpha)$. ∎

6.2 $\boldsymbol{\theta}$-DIAGRAM COMPONENTS.

In this section we make a few remarks pertaining to the diagrammatic representation of irreducible θ-indices.

Example 6.2.1. Consider the following θ-diagram component in Figure 6.3 and suppose the white dots continue on to the left and the black dots in the middle. Then $w_0(\theta)(\alpha_{i-1}) = \alpha_{i-1}$ whereas $w_0(\theta)(\alpha_i) \neq \alpha_i$ because α_i is adjacent to a fixed root and α_{i-1} is not.

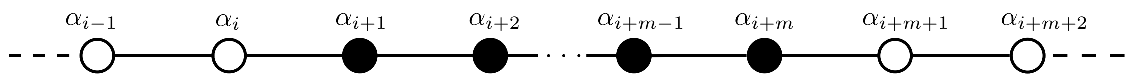


Figure 6.3: White adjacent to black middle with $\theta^* = \mathrm{id}$.

Remark 6.2.2 (θ action on white segments)**.** Let $\alpha_i \in \Delta\backslash\Delta_0$ such that its immediate neighboring α_j are also white ($\in \Delta\backslash\Delta_0$), then θ acts on α_i only by $-\mathrm{id}\,.\theta^*$. If $\theta^* = \mathrm{id}$ then $\theta : \alpha_i \mapsto -\alpha_i$, if $\theta^* \neq \mathrm{id}$ then $\theta : \alpha_i \mapsto -\alpha_j$ for some $\alpha_j \in \Delta\backslash\Delta_0$. In either case $c^{\theta_\Delta}_{\theta(\alpha_i)} = 1$ and hence θ_Δ acts as an involution on the corresponding root spaces X_{α_i}.

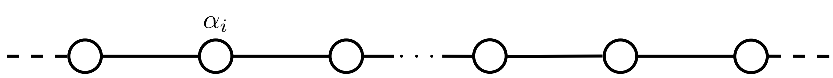


Figure 6.4: Type A segment with only white dots with $\theta^* = \mathrm{id}$: $\alpha_i \in \Delta\backslash\Delta_0$.

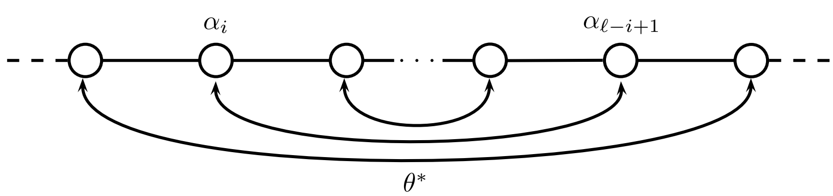


Figure 6.5: In type A, the diagram automorphism maps $\alpha_i \leftrightarrow \alpha_{\ell-i+1}$ for all $\alpha_i \in \Delta$.

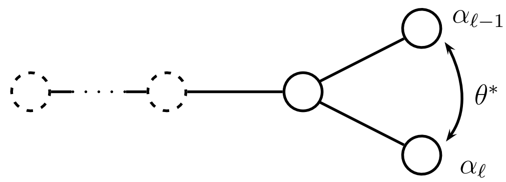


Figure 6.6: In type D, the diagram automorphism maps $\alpha_\ell \leftrightarrow \alpha_{\ell-1}$ and fixes all other $\alpha_i \in \Delta$.

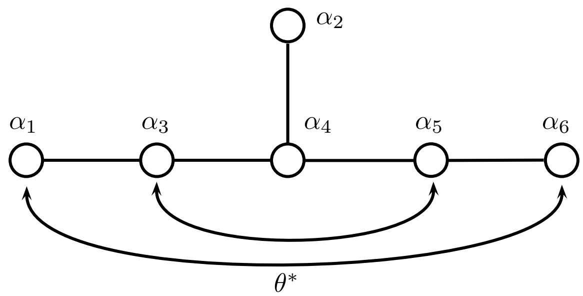


Figure 6.7: In type E_6, the diagram automorphism maps $\alpha_1 \leftrightarrow \alpha_6$ and $\alpha_3 \leftrightarrow \alpha_5$ and fixes α_2 and α_4.

Proof. This follows directly from remark 6.1.4 above and the definition of a diagram automorphism. $\square$

Lemma 6.2.3 (White next to black middle)**.** *θ_Δ acts as an involution on X_{α_i} for any root supported on an simple root like α_i in Figure 6.8 with white dots to the left and black dots to the right all simply laced.*

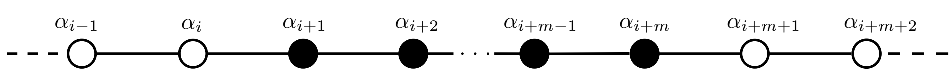


Figure 6.8: White adjacent to black middle with $\theta^* = \mathrm{id}$.

Proof. First note that $\theta : \alpha_i \mapsto -\mathrm{id}\, w_o(\theta)(\alpha_i) = -\alpha_i - \alpha_{i+1} - \cdots - \alpha_{i+m}$. For cleaner notation, let $\beta = \alpha_{i+1} + \cdots + \alpha_{i+m}$ be the sum of the black roots, so, $\theta(\alpha_i) = -\alpha_i - \beta$. Then

$$c^{\theta_\Delta}_{\theta(\alpha_i)} = c^{\theta_\Delta}_{-\alpha_i-\beta} = c^{\theta_\Delta}_{-\alpha_i} \cdot c^{\theta_\Delta}_{-\beta} \cdot \frac{N_{\theta(-\alpha_i),\theta(-\beta)}}{N_{-\alpha_i,-\beta}}$$

By definition 4.1.6 and lemma 6.1.1, we have

$$c^{\theta_\Delta}_{-\alpha_i} = \frac{1}{c^{\theta_\Delta}_{\alpha_i}} = 1 = c^{\theta_\Delta}_{-\beta}$$

and, so,

$$c^{\theta_\Delta}_{\theta(\alpha_i)} = \frac{N_{\alpha_i+\beta,-\beta}}{-N_{\alpha_i,\beta}} = \frac{-N_{-\beta,\alpha_i+\beta}\beta}{-N_{\alpha_i,\beta}} = \frac{-N_{\alpha_i,\beta}}{-N_{\alpha_i,\beta}} = 1$$

□

6.3 Lifting Absolutely Irreducible Involutions over $\mathbb{C}$

Theorem 1 (θ_Δ). *Let $\theta_\Delta \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{t})$ be the unique automorphism lifted from $\theta \in \mathrm{Aut}(X, \Phi)$ such that $\theta_\Delta|\mathfrak{t} = \theta$ and*

$$\theta_\Delta\left(X_{\alpha_i}\right) = X_{\alpha_i} \text{ for all } \alpha_i \in \Delta.$$

Then θ_Δ is an involution for any admissible absolutely irreducible involution θ except when θ is of type $D_\ell^{(\ell-1)/2}$ IIIb.

Proof. This follows directly from the results in Table 6.1 and the explicit computations for the exceptional root systems E, F, and G found below. □

6.4 The lifting constants: $c^{\theta_\Delta}_{\theta(\alpha_i)}$

Lemma 6.4.1. *Let $\alpha \in \Phi^+\backslash\Delta$ and (ζ, η) be the minimal decomposition of α, then*

$$c^{\theta_\Delta}_\alpha = c^{\theta_\Delta}_\eta \iff N_{\theta(\zeta),\theta(\eta)} > 0.$$

Proof. We know $|N_{\theta(\zeta),\theta(\eta)}| = |N_{\zeta,\eta}|$ and hence $\dfrac{N_{\theta(\zeta),\theta(\eta)}}{N_{\zeta,\eta}} = \pm 1$. By the definition of the minimal decomposition, we also know that $N_{\zeta,\eta} > 0$. Therefore

$$c^{\theta_\Delta}_\alpha = c^{\theta_\Delta}_\zeta \cdot c^{\theta_\Delta}_\eta \cdot \frac{N_{\theta(\zeta),\theta(\eta)}}{N_{\zeta,\eta}} = 1 \cdot c^{\theta_\Delta}_\eta \cdot \frac{N_{\theta(\zeta),\theta(\eta)}}{N_{\zeta,\eta}} = c^{\theta_\Delta}_\eta$$

if and only if $N_{\theta(\zeta),\theta(\eta)} > 0$. □

6.4.2 $c^{\theta_\Delta}_{\theta(\alpha_i)}$ for the Classical Root Systems

For ease of reference, we present the θ-permuted lifting constants $c^{\theta_\Delta}_{\alpha_i}$ for the simple roots of the classical Lie algebras in the following Table 6.1. Proofs which are immediate from previously results are omitted.

Table 6.1: $c^{\theta_\Delta}_{\theta(\alpha_i)}$ for Admissible Involutions on the Simple Roots of Classical Lie Algebras.

$\theta(\alpha_i) = \alpha$	$c^{\theta_\Delta}_{\theta(\alpha_i)} = c^{\theta_\Delta}_{\alpha}$	θ-type(s)	
$\theta(\alpha_i) = \alpha_i$	1	AII, AIIIa, BI,CIIa, DIa	(1)
$\theta(\alpha_i) = -\alpha_i$	1	AI, BI, CI, DIa, DIb	(2)
$\theta(\alpha_i) = -\alpha_j$	1	DIb ($\ell - 1 \leftrightarrow \ell$ odd)	(3)
$\theta(\alpha_i) = -\alpha_{\ell-i+1}$	1	AIIIa	(4)

Table 6.1: Continued

$\theta(\alpha_i) = \alpha$	$c^{\theta_\Delta}_{\theta(\alpha_i)} = c^{\theta_\Delta}_{\alpha}$	θ-type(s)
$\theta(\alpha_i) = -\alpha_{i-1} - \alpha_i - \alpha_{i+1}$	$1:$ *Proof.* Let $\gamma = \alpha_{i-1}$ and $\delta = \alpha_i + \alpha_{i+1}$, then $\theta(\gamma) = \gamma$ and $\theta(\delta) = -\alpha_i - \gamma$, so $$\begin{aligned} c^{\theta_\Delta}_{\theta(\alpha_i)} &= 1 \cdot c^{\theta_\Delta}_{-\delta} \cdot \frac{N_{-\gamma,\gamma+\alpha_i}}{-N_{\gamma,\delta}} \\ &= c^{\theta_\Delta}_{-\alpha_i} \cdot c^{\theta_\Delta}_{-\alpha_{i+1}} \cdot \frac{N_{\theta(-\alpha_i),\theta(-\alpha_{i+1})}}{N_{-\alpha_i,-\alpha_{i+1}}} \frac{N_{\gamma,\alpha_i}}{-N_{\gamma,\alpha_i+\alpha_{i+1}}} \\ &= 1 \cdot 1 \cdot \frac{N_{\gamma+\delta,-\alpha_{i+1}}}{-N_{\alpha_i,\alpha_{i+1}}} \frac{N_{\gamma,\alpha_i}}{-N_{\gamma,\alpha_i+\alpha_{i+1}}} \\ &= \frac{N_{\alpha_{i-1}+\alpha_i,\alpha_{i+1}}}{-N_{\alpha_i,\alpha_{i+1}}} \frac{N_{\alpha_{i-1},\alpha_i}}{-N_{\alpha_{i-1},\alpha_i+\alpha_{i+1}}} = 1 \end{aligned}$$ by disjoint decomposition and lexicographic ordering (see lemma 3.2.7). □	AII, CIIa, DIIIa, DIIIb (5)
$\theta(\alpha_i) = -\alpha_{i-1} - \alpha_i - \alpha_{i+2}$	$1 : c^{\theta_\Delta}_{\theta(\alpha)} = c^{\theta_\Delta}_{-\alpha_{\ell-3}} \cdot c^{\theta_\Delta}_{-\alpha_{\ell-2}-\alpha_\ell} \cdot \frac{N_{\theta(-\alpha_{\ell-3}),\theta(-\alpha_{\ell-2}-\alpha_\ell)}}{N_{-\alpha_{\ell-3},-\alpha_{\ell-2}-\alpha_\ell}}$ This minimal disjoint decomposition behaves like the lexicographically-ordered one in row (5).	DIIIa ($i = \ell - 2$) (6)
$\theta(\alpha_i) = -\alpha_i - \alpha_{i+1} - \alpha_{i+2}$	$1 : c^{\theta_\Delta}_{-\alpha_{\ell-1}} \cdot c^{\theta_\Delta}_{-\alpha_{\ell-2}-\alpha_\ell} \cdot \frac{N_{\theta(-\alpha_{\ell-1}),\theta(-\alpha_{\ell-2}-\alpha_\ell)}}{N_{-\alpha_{\ell-1},-\alpha_{\ell-2}-\alpha_\ell}}$ This minimal disjoint decomposition behaves like the lexicographically-ordered one in row (5).	DIa ($i = \ell - 2$) (7)

Table 6.1: Continued

<table>
<tr><th>$\theta(\alpha_i) = \alpha$</th><th>$c^{\theta_\Delta}_{\theta(\alpha_i)} = c^{\theta_\Delta}_{\alpha}$</th><th>$\theta$-type(s)</th></tr>
<tr><td>$\theta(\alpha_i) = -\alpha_{i+1} - \alpha_{i+2} \cdots - \alpha_{i+m+1}$</td><td>1 :

Proof. Let $\gamma = \alpha_{i+1} + \alpha_{i+2} \cdots + \alpha_{i+m} \in \Phi_0(\theta)$ and $\delta = \alpha_{i+m+1} \in \Delta$, then $\theta(\gamma) = \gamma$ and $\theta(\delta) = -\alpha_i - \gamma$, and $c^{\theta_\Delta}_{\delta} = 1 = c^{\theta_\Delta}_{\gamma}$, so
$$\begin{aligned} c^{\theta_\Delta}_{\theta(\alpha_i)} &= c^{\theta_\Delta}_{-\gamma} \cdot c^{\theta_\Delta}_{-\delta} \cdot \frac{N_{\theta(-\gamma),\theta(-\delta)}}{N_{-\gamma,-\delta}} \\ &= 1 \cdot \frac{N_{\theta(-\gamma),\theta(-\delta)}}{N_{-\gamma,-\delta}} = \frac{-1}{-1} \cdot \frac{N_{\theta(\gamma),\theta(\delta)}}{N_{\gamma,\delta}} \\ &= \frac{N_{\gamma,-\alpha_i-\gamma}}{N_{\gamma,\delta}} = \frac{-N_{-\gamma,\gamma+\alpha_i}}{N_{\gamma,\delta}} = \frac{-N_{\gamma,\alpha_i}}{N_{\gamma,\delta}} = \frac{N_{\alpha_i,\gamma}}{N_{\gamma,\delta}} \end{aligned}$$
Using lemma 3.2.7, we know that both the numerator and the denominator are positive, and since the magnitude of the Chevalley constants are all the same in a type A subsystem, we have:
$$c^{\theta_\Delta}_{\theta(\alpha_i)} = \frac{N_{\alpha_i,\alpha_{i+1}+\alpha_{i+2}\cdots+\alpha_{i+m}}}{N_{\alpha_{i+1}+\alpha_{i+2}\cdots+\alpha_{i+m},\alpha_{i+m+1}}} = +1.$$
□</td><td>AIIIa (8)</td></tr>
<tr><td>$\theta(\alpha_i) = -\alpha_{i-m-1} - \alpha_{i-m} \cdots - \alpha_{i-1}$</td><td>1 : Follows by similar argument as in row (8).</td><td>AIIIa (9)</td></tr>
</table>

Table 6.1: Continued

$\theta(\alpha_i) = \alpha$	$c^{\theta_\Delta}_{\theta(\alpha_i)} = c^{\theta_\Delta}_{\alpha}$	θ-type(s)
$\theta(\alpha_i) = -\alpha_i - 2\sum_{i+1}^{\ell} \alpha_j$	$1:$ *Proof.* Let $\gamma = \alpha_{i+1}$ and $\delta = \alpha_i + \alpha_{i+1} + 2\sum_{i+2}^{\ell}$ such that $\theta(\alpha_i) = -\gamma - \delta$, $\zeta_\delta = \alpha_i$ and $\eta_\delta = \alpha_{i+1} + 2\sum_{i+2}^{\ell}$. Then we know that $\zeta_\delta \in \Delta$ and $\gamma, \eta_\delta \in \Phi_0(\theta)$ such that $c^{\theta_\Delta}_{\gamma} = 1 = c^{\theta_\Delta}_{\zeta_\delta} = c^{\theta_\Delta}_{\eta_\delta}$. We also know $\theta(\zeta_\delta) = \theta(\alpha_i) = -\gamma - \zeta_\delta - \eta_\delta$ and $\theta(\delta) = -\zeta_\delta - \gamma$. Hence, $\begin{aligned} c^{\theta_\Delta}_{\theta(\alpha_i)} &= c^{\theta_\Delta}_{-\gamma} \cdot c^{\theta_\Delta}_{-\delta} \cdot \frac{N_{\theta(-\gamma),\theta(-\delta)}}{N_{-\gamma,-\delta}} \\ &= 1 \cdot \left(c^{\theta_\Delta}_{-\zeta_\delta} \cdot c^{\theta_\Delta}_{-\eta_\delta} \cdot \frac{N_{\theta(-\zeta_\delta),\theta(-\eta_\delta)}}{N_{-\zeta_\delta,-\eta_\delta}} \right) \frac{N_{\theta(-\gamma),\theta(-\delta)}}{N_{-\gamma,-\delta}} \\ &= \left(\frac{N_{\theta(\zeta_\delta),\theta(\eta_\delta)}}{N_{\zeta_\delta,\eta_\delta}} \right) \frac{N_{\theta(\gamma),\theta(\delta)}}{N_{\gamma,\delta}} \\ &= \left(\frac{N_{-\zeta_\delta-\eta_\delta-\gamma,\eta_\delta}}{N_{\zeta_\delta,\eta_\delta}} \right) \frac{N_{\gamma,-\zeta_\delta-\gamma}}{N_{\gamma,\delta}} \end{aligned}$ By lemma 3.2.9, given a lexicographic decomposition (γ, δ), $N_{\gamma,\delta} > 0$ if γ and δ have disjoint support and $N_{\delta,\gamma} > 0$ if they do not. Hence $c^{\theta_\Delta}_{\theta(\alpha_i)} = \left(\frac{N_{\eta_\delta,\zeta_\delta+\gamma}}{N_{\zeta_\delta,\eta_\delta}} \right) \frac{N_{\zeta_\delta,\gamma}}{N_{\gamma,\delta}} = 1 \cdot 1 = 1.$ □	BI (10)

Table 6.1: Continued

$\theta(\alpha_i) = \alpha$	$c^{\theta_\Delta}_{\theta(\alpha_i)} = c^{\theta_\Delta}_{\alpha}$	θ-type(s)
$\theta(\alpha_i) = -\alpha_{i-1} - \alpha_i - 2\sum_{i+1}^{\ell-1} \alpha_j - \alpha_\ell$	1 : *Proof.* Let $\gamma = \alpha_{i-1}$ and $\delta = \alpha_i + 2\sum_{i+1}^{\ell-1}\alpha_j + \alpha_\ell$, then $\theta(\gamma) = \gamma$ and $\theta(\delta) = -\alpha_i - \gamma$, so by the work in row (8) we have $c^{\theta_\Delta}_{\theta(\alpha_i)} = c^{\theta_\Delta}_{\delta} \cdot \frac{N_{\alpha_i,\alpha_{i-1}}}{N_{\gamma,\delta}}.$ Now letting $\zeta_\delta = \alpha_i$ and $\eta_\delta = \delta - \alpha_i \in \Phi_0(\theta)$, and noting that $\theta(\zeta_\delta) = \theta(\alpha_i) = -\gamma - \delta = -\gamma - \eta_\delta - \zeta_\delta$, $c^{\theta_\Delta}_{\theta(\alpha_i)} = 1 \cdot \frac{N_{\theta(\zeta_\delta),\theta(\eta_\delta)}}{N_{\zeta_\delta,\eta_\delta}} \cdot (-1)$ $= -1 \cdot \frac{N_{-\gamma-\eta_\delta-\zeta_\delta,\eta_\delta}}{N_{\alpha_i,\delta-\alpha_i}} = -1 \cdot \frac{N_{\eta_\delta,\gamma+\zeta_\delta}}{N_{\alpha_i,\delta-\alpha_i}}$ $= -1 \cdot \frac{N_{2\alpha_{i+1}+2\sum_{i+2}^{\ell-1}\alpha_j+\alpha_\ell,\alpha_{i-1}+\alpha_i}}{N_{\alpha_i,\delta-\alpha_i}}$ $= -1 \cdot -1 = 1$ by lemma 3.2.12. $\square$	CIIa (11)
$\theta(\alpha_i) = -2\alpha_{i-1} - \alpha_i$	1 : Follows by similar argument.	CIIb ($i = \ell$) (12)

Table 6.1: Continued

$\theta(\alpha_i) = \alpha$	$c^{\theta_\Delta}_{\theta(\alpha_i)} = c^{\theta_\Delta}_{\alpha}$	θ-type(s)
$\theta(\alpha_i) = -\alpha_i - 2\sum_{i+1}^{\ell-2}\alpha_j - \alpha_{\ell-1} - \alpha_\ell$	1 : *Proof.* Let $\gamma = \alpha_{i+1}$ and $\delta = \alpha_i + \alpha_{i+1} + 2\sum_{i+2}^{\ell-2}\alpha_j + \alpha_{\ell-1} + \alpha_\ell$ $c^{\theta_\Delta}_{\theta(\alpha_i)} = c^{\theta_\Delta}_{-\gamma} \cdot c^{\theta_\Delta}_{-\delta} \cdot \frac{N_{\theta(-\gamma),\theta(-\delta)}}{N_{-\gamma,-\delta}}$ $= 1 \cdot c^{\theta_\Delta}_{-\delta} \cdot \frac{N_{-\gamma,\alpha_i+\gamma}}{-N_{\gamma,\delta}} = 1 \cdot c^{\theta_\Delta}_{\delta} \cdot \frac{-N_{\alpha_i,\gamma}}{-N_{\gamma,\delta}} = c^{\theta_\Delta}_{\delta}$ Now let $\zeta_\delta = \alpha_i \in \Delta$ and $\eta_\delta = \delta - \alpha_i \in \Phi_0(\theta)$: $= c^{\theta_\Delta}_{\zeta_\delta} \cdot c^{\theta_\Delta}_{\eta_\delta} \cdot \frac{N_{\theta(\zeta_\delta),\theta(\eta_\delta)}}{N_{\zeta_\delta,\eta_\delta}}$ $= 1 \cdot 1 \cdot \frac{N_{-\zeta_\delta-\gamma-\eta_\delta,\eta_\delta}}{N_{\zeta_\delta,\eta_\delta}} = \frac{N_{\eta_\delta,\zeta_\delta+\gamma}}{N_{\zeta_\delta,\eta_\delta}}$ Then we see that we have a reverse lexicographic non-disjoint form in the numerator and a lexicographic disjoint form in the denominator, and hence by lemma 3.2.14: $c^{\theta_\Delta}_{\theta(\alpha_i)} = \frac{N_{\alpha_{i+1}+2\sum_{i+2}^{\ell-2}\alpha_j+\alpha_{\ell-1}+\alpha_\ell,(\alpha_i+\alpha_{i+1})}}{N_{\alpha_i,\delta-\alpha_i}} = 1.$ □	DIa ($i < \ell - 2$) (13)

Table 6.1: Continued

$\theta(\alpha_i) = \alpha$	$c^{\theta_\Delta}_{\theta(\alpha_i)} = c^{\theta_\Delta}_{\alpha}$	θ-type(s)
$\theta(\alpha_i) = -\alpha_{\ell-2} - \alpha_{i\pm 1}$	-1 : *Proof.* It is sufficient to prove the case when $i = \ell$. $$\begin{aligned} c^{\theta_\Delta}_{\theta(\alpha_i)} &= c^{\theta_\Delta}_{-\alpha_{\ell-2}} \cdot c^{\theta_\Delta}_{-\alpha_\ell} \cdot \frac{N_{\theta(-\alpha_{\ell-2}),\theta(-\alpha_\ell)}}{N_{-\alpha_{\ell-2},-\alpha_\ell}} \\ &= 1 \cdot 1 \cdot \frac{-N_{\theta(\alpha_{\ell-2}),\theta(\alpha_\ell)}}{-N_{\alpha_{\ell-2},\alpha_\ell}} \\ &= \frac{-N_{\alpha_{\ell-2},-\alpha_{\ell-2}-\alpha_{\ell-1}}}{-N_{\alpha_{\ell-2},\alpha_\ell}} \\ &= \frac{N_{-\alpha_{\ell-2},\alpha_{\ell-2}+\alpha_{\ell-1}}}{-N_{\alpha_{\ell-2},\alpha_\ell}} = \frac{N_{\alpha_{\ell-2},\alpha_{\ell-1}}}{-N_{\alpha_{\ell-2},\alpha_\ell}} = -1 \end{aligned}$$ by definition of the minimal form. $\square$	DIIIb ($i = \ell - 1$ or ℓ) (14) Note: This appears to be the only instance when a simple rootis mapped to a root of height 2.

6.5 Type A

θ_Δ is always an involution.

Theorem 6.5.1 (Type A: $\theta_\Delta \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{t})$ is an involution)**.** *For any admissible absolutely irreducible involution $\theta \in \mathrm{Aut}(X, \Phi)$ where Φ is a type A root system, $\theta_\Delta \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{t})$ is an involution.*

Proof. Follows directly from the results in Table 6.1. □

$c_\alpha^{\theta_\Delta}$ Formulas

Remark 6.5.2. Let $\theta \in \mathrm{Aut}(X, \Phi)$ have absolutely irreducible θ-index, then there are two cases in Type A:

Case 1: θ^* is the identity (i.e. $\theta = -id \cdot w_0(\theta)$) $\Rightarrow$ θ is an outer automorphism (i.e. acts by transpose inverse on the group level and by $-\mathrm{id} \cdot X^T$ on the Lie algebra) and $c_\alpha^{\theta_\Delta} = (-1)^{ht(\alpha)-1}$.

Case 2: θ^* is not the identity (i.e. $\theta^* = -\theta\ w_0(\theta) = -w_0$) $\Rightarrow$ θ is an inner automorphism (i.e. acts by conjugation on the group level and by ad on the Lie algebra) and $c_\alpha^{\theta_\Delta} = 1$.

6.6 Type B

θ_Δ is always an involution.

Theorem 6.6.1 (Type B: $\theta_\Delta \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{t})$ is an involution)**.** *For any admissible absolutely irreducible involution $\theta \in \mathrm{Aut}(X, \Phi)$ where Φ is a type B root system, $\theta_\Delta \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{t})$ is an involution.*

Proof. Follows directly from the results in Table 6.1. □

$c_\alpha^{\theta_\Delta}$ Formulas

Lemma 6.6.2 (Type $B_\ell^p I$: θ_Δ Constants)**.** *For any involution θ of type $B_\ell^p I$, write $\alpha \in \Phi^+(T)$ as $\alpha = \sum r_i \alpha_i + \sum s_j \beta_j$ where $\alpha_i \in \Delta \backslash \Delta_0$ and $\beta_j \in \Delta_0$, then the structure constants for θ_Δ are determined by $\sum r_i$:*

$$c_\alpha^{\theta_\Delta} = (-1)^{\sum r_i - 1} \tag{6.6.2 (a)}$$

Note. Roots that include an odd number of white roots have $+1$ coefficient and roots with even number of white roots have coefficient -1.

Proof. Follows from relations of the Chevalley constants and the action of the involution. □

6.7 Type C

θ_Δ is always an involution.

Theorem 6.7.1 (θ_Δ in Type C)**.** *For any admissible absolutely irreducible involution $\theta \in \mathrm{Aut}(X, \Phi)$ where Φ is a type C root system, $\theta_\Delta \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{t})$ is an involution.*

Proof. Follows directly from the results in Table 6.1. □

6.8 Type D

6.8.1 θ_Δ is almost always an involution.

DIa & DIb Write $\alpha = \sum_{\alpha_i \in \Delta \setminus \Delta_0} r_i \alpha_i + \sum_{\beta_j \in \Delta_0} s_j \beta_j$, then

$$c_\alpha^{\theta_\Delta} = (-1)^{\sum r_i - 1} \tag{6.8.1 (a)}$$

Note. DIb falls into the general case for $-id = \theta$, i.e. $c_\alpha^{\theta_\Delta} = (-1)^{\sum r_i - 1} = (-1)^{\mathrm{ht}(\alpha) - 1}$.

DIIIa For type DIIIa, defined only for even rank, θ_Δ is and involution and we can define the structure constants piecewise: Let $\ell = \mathrm{rank}(\Phi)$, and write $\alpha = \sum_{i=1}^{\ell} r_i \alpha_i$.

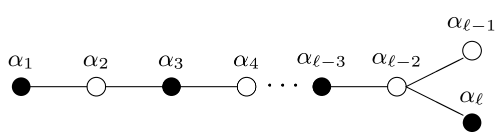


Figure 6.9: D_ℓ^pIIIa, ℓ even and $p = \ell/2$

DIIIb This is the only case when θ_Δ is not an involution. As we observed in Table 6.1, this is precisely because there are $\alpha_i \in \Delta$ such that $c_{\theta(\alpha_i)}^{\theta_\Delta} = -1 \neq 1$. Rather unsurprisingly, it is the spin nodes that require the correction of θ_Δ. To correct θ_Δ, we find a (non-unique) correction vector $H \in \mathfrak{t}_\theta^+$ corresponding to $h \in T_\theta^+$ of the following form:

$$H = \frac{i\pi}{2} \sum_{\alpha_i \in \Delta_0(\theta)} (4x_i - 1) H_{\alpha_i}, \text{ where } x_i \in \mathbb{Z}.$$

Using this correction vector we find our lifted involution

$$\theta_\Delta \,\mathrm{Int}(h) = \theta_\Delta \exp\left(\mathrm{ad}\left(H\right)\right).$$

For computation, we may choose a simplest form of H such that $x_i = 0$:

$$H = -\frac{i\pi}{2} \sum_{\alpha_i \in \Delta_0(\theta)} H_{\alpha_i}.$$

Example 6.8.2 (Lifting θ of type D_5^2IIIb.)**.** Recall the example from Chapter 5:

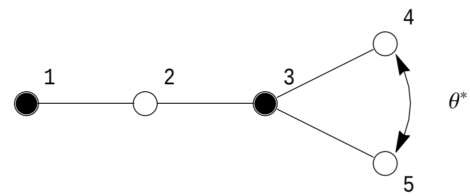


Figure 6.10: D_5^2IIIb θ-diagram

Table 6.2: D_5^2IIIb: θ_Δ Lifting Constants on the α-basis

$\alpha \mapsto$	$\theta(\alpha)$	$c_\alpha^{\theta_\Delta}$	$c_{\theta(\alpha)}^{\theta_\Delta}$
$\alpha_1 \mapsto$	α_1	1	1
$\alpha_2 \mapsto$	$-\alpha_1 - \alpha_2 - \alpha_3$	1	1
$\alpha_3 \mapsto$	α_3	1	1
$\alpha_4 \mapsto$	$-\alpha_3 - \alpha_5$	1	-1
$\alpha_5 \mapsto$	$-\alpha_3 - \alpha_4$	1	-1

Table 6.3: D_5^2IIIb: $\theta_\Delta \exp\left(\mathrm{ad}\left(-\frac{1}{2} i\pi \left(H_{\alpha_1} + H_{\alpha_3}\right)\right)\right)$ Lifting Constants on the α-basis

$\alpha \mapsto$	$\theta(\alpha)$	$c_\alpha^{\theta_\Delta \,\mathrm{Int}(h)}$	$c_{\theta(\alpha)}^{\theta_\Delta \,\mathrm{Int}(h)}$
$\alpha_1 \mapsto$	α_1	-1	-1
$\alpha_2 \mapsto$	$-\alpha_1 - \alpha_2 - \alpha_3$	-1	-1
$\alpha_3 \mapsto$	α_3	-1	-1
$\alpha_4 \mapsto$	$-\alpha_3 - \alpha_5$	i	$-i$
$\alpha_5 \mapsto$	$-\alpha_3 - \alpha_4$	i	$-i$

6.9 Type E

θ_Δ is always an involution.

Lemma 6.9.1 (Type E: $c^{\theta_\Delta}_{\theta(\alpha_2)} = 1$)**.** *For any absolutely irreducible involution in type E,* $c^{\theta_\Delta}_{\alpha_2} = 1$*.*

Proof. For any absolutely irreducible θ on a type E root system $\theta|_{\alpha_2} = \mathrm{id}$ or $\theta|_{\alpha_2} = -\mathrm{id}$. Hence by previous results $c^{\theta_\Delta}_{\alpha_2} = 1$. □

Henceforth we will only address the lifting constants of the remaining simple roots in Δ in order to show $c^{\theta_\Delta}_{\alpha_i} = 1$ and hence that θ_Δ is an involution.

Type E rank 6

EI All white dots with $\theta^* = id$: The lifting constants are $c^{\theta_\Delta}_{\alpha} = (-1)^{\mathrm{ht}(\alpha)-1}$ (see lemma 6.1.3).

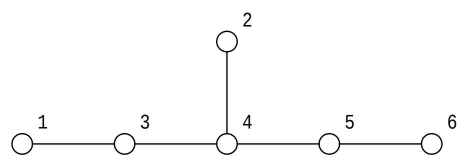


Figure 6.11: E_6^6I θ-diagram

Table 6.4: E_6^6I: θ_Δ Lifting Constants on the α-basis

$\alpha \mapsto$	$\theta(\alpha)$	$c^{\theta_\Delta}_{\alpha}$	$c^{\theta_\Delta}_{\theta(\alpha)}$
$\alpha_1 \mapsto$	$-\alpha_1$	1	1
$\alpha_2 \mapsto$	$-\alpha_2$	1	1
$\alpha_3 \mapsto$	$-\alpha_3$	1	1
$\alpha_4 \mapsto$	$-\alpha_4$	1	1
$\alpha_5 \mapsto$	$-\alpha_5$	1	1
$\alpha_6 \mapsto$	$-\alpha_6$	1	1

EII θ_Δ is an involution: θ acts as $-\mathrm{id}$ on $\{\alpha_2, \alpha_4\}$, therefore by lemma 6.1.3, $c^{\theta_\Delta}_{\theta(\alpha_2)} = 1 = c^{\theta_\Delta}_{\theta(\alpha_4)}$. Similarly, θ acts as $-\theta^*$ on the remaining simple roots and so for $\alpha \in \Delta$ such that $\theta^*(\alpha) \neq \alpha$, $\theta(\alpha_i) = -\alpha_j$ and so $c^{\theta_\Delta}_{\theta(\alpha_i)} = c^{\theta_\Delta}_{-\alpha_j} = 1$. Hence, $c^{\theta_\Delta}_{\alpha_i} c^{\theta_\Delta}_{\theta(\alpha_i)} = 1$ for all $\alpha_i \in \Delta$ and θ_Δ is an involution.

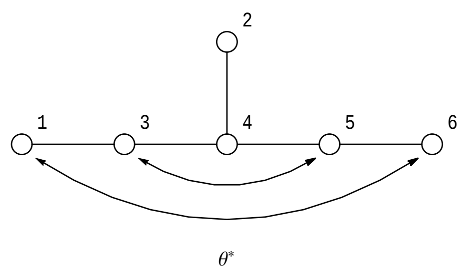


Figure 6.12: E_6^4II θ-diagram

Table 6.5: E_6^4II: θ_Δ Lifting Constants on the α-basis

$\alpha \mapsto$	$\theta(\alpha)$	$c^{\theta_\Delta}_{\alpha}$	$c^{\theta_\Delta}_{\theta(\alpha)}$
$\alpha_1 \mapsto$	$-\alpha_6$	1	1
$\alpha_2 \mapsto$	$-\alpha_2$	1	1
$\alpha_3 \mapsto$	$-\alpha_5$	1	1
$\alpha_4 \mapsto$	$-\alpha_4$	1	1
$\alpha_5 \mapsto$	$-\alpha_3$	1	1
$\alpha_6 \mapsto$	$-\alpha_1$	1	1

EIII For θ of type EIII, θ_Δ is an involution. It is sufficient to show that $\alpha_1, \alpha_6 \in \Delta \backslash \Delta_0$ have $c^{\theta_\Delta}_{\theta(\alpha)} = 1$ which in fact follows from the arguments presented for type A.

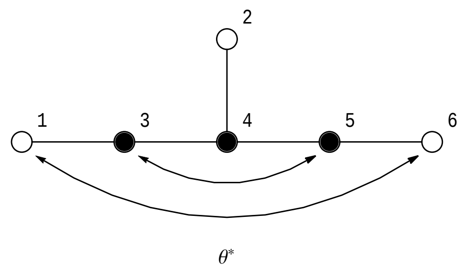


Figure 6.13: E_6^2III θ-diagram

Table 6.6: E_6^2III: θ_Δ Lifting Constants on the α-basis

$\alpha \mapsto$	$\theta(\alpha)$	$c^{\theta_\Delta}_{\alpha}$	$c^{\theta_\Delta}_{\theta(\alpha)}$
$\alpha_1 \mapsto$	$-\alpha_3 - \alpha_4 - \alpha_5 - \alpha_6$	1	1
$\alpha_2 \mapsto$	$-\alpha_2 - \alpha_3 - 2\alpha_4 - \alpha_5$	1	1
$\alpha_3 \mapsto$	α_3	1	1
$\alpha_4 \mapsto$	α_4	1	1
$\alpha_5 \mapsto$	α_5	1	1
$\alpha_6 \mapsto$	$-\alpha_1 - \alpha_3 - \alpha_4 - \alpha_5$	1	1

EIV For θ of type EIV, θ_Δ is an involution. It is sufficient to show that $\alpha_1, \alpha_6 \in \Delta \backslash \Delta_0$ have $c^{\theta_\Delta}_{\theta(\alpha)} = 1$.

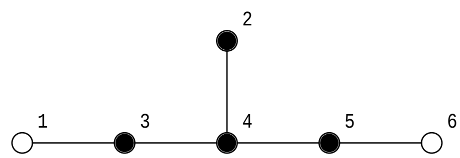


Figure 6.14: E_6^2IV θ-diagram

Table 6.7: E_6^2IV: θ_Δ Lifting Constants on the α-basis

$\alpha \mapsto$	$\theta(\alpha)$	$c^{\theta_\Delta}_{\alpha}$	$c^{\theta_\Delta}_{\theta(\alpha)}$
$\alpha_1 \mapsto$	$-\alpha_1 - \alpha_2 - 2\alpha_3 - 2\alpha_4 - \alpha_5$	1	1
$\alpha_2 \mapsto$	α_2	1	1
$\alpha_3 \mapsto$	α_3	1	1
$\alpha_4 \mapsto$	α_4	1	1
$\alpha_5 \mapsto$	α_5	1	1
$\alpha_6 \mapsto$	$-\alpha_2 - \alpha_3 - 2\alpha_4 - 2\alpha_5 - \alpha_6$	1	1

Type E rank 7

EV All white dots with $\theta^* = id$: The lifting constants are $c_\alpha^{\theta_\Delta} = (-1)^{\mathrm{ht}(\alpha)-1}$ (see lemma 6.1.3).

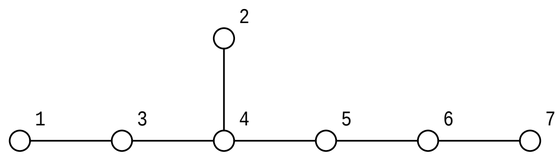


Figure 6.15: E_7^7V θ-diagram

Table 6.8: E_7^7V: θ_Δ Lifting Constants on the α-basis

$\alpha \mapsto$	$\theta(\alpha)$	$c_\alpha^{\theta_\Delta}$	$c_{\theta(\alpha)}^{\theta_\Delta}$
$\alpha_1 \mapsto$	$-\alpha_1$	1	1
$\alpha_2 \mapsto$	$-\alpha_2$	1	1
$\alpha_3 \mapsto$	$-\alpha_3$	1	1
$\alpha_4 \mapsto$	$-\alpha_4$	1	1
$\alpha_5 \mapsto$	$-\alpha_5$	1	1
$\alpha_6 \mapsto$	$-\alpha_6$	1	1
$\alpha_7 \mapsto$	$-\alpha_7$	1	1

EVI For θ of type EVI, θ_Δ is an involution.

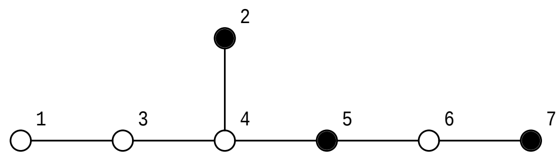


Figure 6.16: E_7^4VI θ-diagram

Table 6.9: E_7^4VI: θ_Δ Lifting Constants on the α-basis

$\alpha \mapsto$	$\theta(\alpha)$	$c_\alpha^{\theta_\Delta}$	$c_{\theta(\alpha)}^{\theta_\Delta}$
$\alpha_1 \mapsto$	$-\alpha_1$	1	1
$\alpha_2 \mapsto$	α_2	1	1
$\alpha_3 \mapsto$	$-\alpha_3$	1	1
$\alpha_4 \mapsto$	$-\alpha_2 - \alpha_4 - \alpha_5$	1	1
$\alpha_5 \mapsto$	α_5	1	1
$\alpha_6 \mapsto$	$-\alpha_5 - \alpha_6 - \alpha_7$	1	1
$\alpha_7 \mapsto$	α_7	1	1

EVII

Lemma 6.9.2. *For θ of type EVII, θ_Δ is an involution.*

Proof. It is sufficient to show that $\alpha_i \in \Delta\backslash\Delta_0$ have $c^{\theta_\Delta}_{\theta(\alpha_i)} = 1$. This follows from the proof of Section 6.9 and $\square$

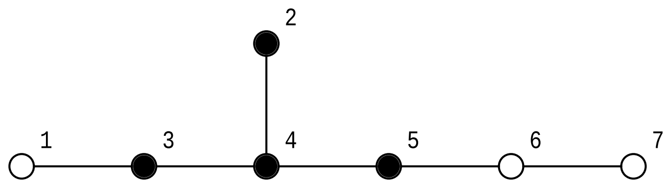


Figure 6.17: E_7^3VII θ-diagram

Table 6.10: E_7^3VII: θ_Δ Lifting Constants on the α-basis

$\alpha \mapsto$	$\theta(\alpha)$	$c^{\theta_\Delta}_{\alpha}$	$c^{\theta_\Delta}_{\theta(\alpha)}$
$\alpha_1 \mapsto$	$-\alpha_1 - \alpha_2 - 2\alpha_3 - 2\alpha_4 - \alpha_5$	1	1
$\alpha_2 \mapsto$	α_2	1	1
$\alpha_3 \mapsto$	α_3	1	1
$\alpha_4 \mapsto$	α_4	1	1
$\alpha_5 \mapsto$	α_5	1	1
$\alpha_6 \mapsto$	$-\alpha_2 - \alpha_3 - 2\alpha_4 - 2\alpha_5 - \alpha_6$	1	1
$\alpha_7 \mapsto$	$-\alpha_7$	1	1

Type E rank 8

EVIII All white dots with $\theta^* = id$: The lifting constants are $c_\alpha^{\theta_\Delta} = (-1)^{\mathrm{ht}(\alpha)-1}$ (see lemma 6.1.3).

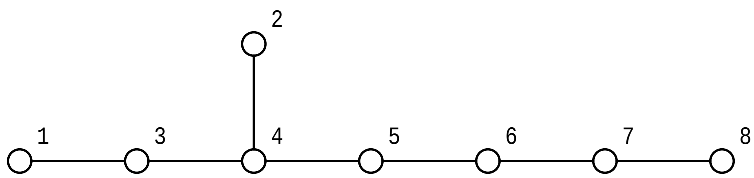


Figure 6.18: E_8^8VIII θ-diagram

Table 6.11: E_8^8VIII: θ_Δ Lifting Constants on the α-basis

$\alpha \mapsto$	$\theta(\alpha)$	$c_\alpha^{\theta_\Delta}$	$c_{\theta(\alpha)}^{\theta_\Delta}$
$\alpha_1 \mapsto$	$-\alpha_1$	1	1
$\alpha_2 \mapsto$	$-\alpha_2$	1	1
$\alpha_3 \mapsto$	$-\alpha_3$	1	1
$\alpha_4 \mapsto$	$-\alpha_4$	1	1
$\alpha_5 \mapsto$	$-\alpha_5$	1	1
$\alpha_6 \mapsto$	$-\alpha_6$	1	1
$\alpha_7 \mapsto$	$-\alpha_7$	1	1
$\alpha_8 \mapsto$	$-\alpha_8$	1	1

EIX It from the proofs of Section 6.9 and lemma 6.1.3 that For θ of type EIX, θ_Δ is an involution.

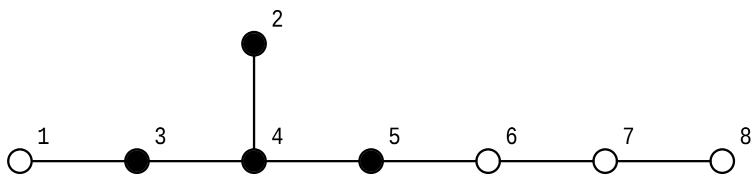


Figure 6.19: E_8^4IX θ-diagram

Table 6.12: E_8^4IX: θ_Δ Lifting Constants on the α-basis

$\alpha \mapsto$	$\theta(\alpha)$	$c_\alpha^{\theta_\Delta}$	$c_{\theta(\alpha)}^{\theta_\Delta}$
$\alpha_1 \mapsto$	$-\alpha_1 - \alpha_2 - 2\alpha_3 - 2\alpha_4 - \alpha_5$	1	1
$\alpha_2 \mapsto$	α_2	1	1
$\alpha_3 \mapsto$	α_3	1	1
$\alpha_4 \mapsto$	α_4	1	1
$\alpha_5 \mapsto$	α_5	1	1
$\alpha_6 \mapsto$	$-\alpha_2 - \alpha_3 - 2\alpha_4 - 2\alpha_5 - \alpha_6$	1	1
$\alpha_7 \mapsto$	$-\alpha_7$	1	1
$\alpha_8 \mapsto$	$-\alpha_8$	1	1

6.10 Type F

θ_Δ is always an involution.

FI All white dots with $\theta^* = id$: The lifting constants are $c_\alpha^{\theta_\Delta} = (-1)^{\mathrm{ht}(\alpha)-1}$ (see lemma 6.1.3).

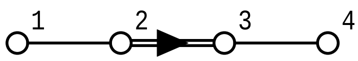


Figure 6.20: F_4^4I θ-diagram

Table 6.13: F_4^4I: θ_Δ Lifting Constants on the α-basis

$\alpha \mapsto$	$\theta(\alpha)$	$c_\alpha^{\theta_\Delta}$	$c_{\theta(\alpha)}^{\theta_\Delta}$
$\alpha_1 \mapsto$	$-\alpha_1$	1	1
$\alpha_2 \mapsto$	$-\alpha_2$	1	1
$\alpha_3 \mapsto$	$-\alpha_3$	1	1
$\alpha_4 \mapsto$	$-\alpha_4$	1	1

FII Write $\alpha = \sum r_i\alpha_i + \sum s_j\beta_j$ where $\alpha_i \in \Delta\backslash\Delta_0$ and $\beta_j \in \Delta_0$. Then the lifting constants are

$$c_{\theta_\Delta} = \begin{cases} 1 & \text{if } r_4 = 0 \text{ (all black dots)} \\ (-1)^{\sum s_j} & \text{if } r_4 \neq 0 \end{cases}$$

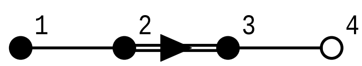


Figure 6.21: F_4^1II θ-diagram

Table 6.14: F_4^1II: θ_Δ Lifting Constants on the α-basis

$\alpha \mapsto$	$\theta(\alpha)$	$c_\alpha^{\theta_\Delta}$	$c_{\theta(\alpha)}^{\theta_\Delta}$
$\alpha_1 \mapsto$	α_1	1	1
$\alpha_2 \mapsto$	α_2	1	1
$\alpha_3 \mapsto$	α_3	1	1
$\alpha_4 \mapsto$	$-\alpha_1 - 2\alpha_2 - 3\alpha_3 - \alpha_4$	1	1

6.11 Type G

θ_Δ is an involution.

All white dots with $\theta^* = id$: The lifting constants are $c_\alpha^{\theta_\Delta} = (-1)^{\mathrm{ht}(\alpha)-1}$ (see lemma 6.1.3).

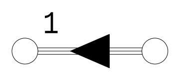


Figure 6.22: G_2^2 θ-diagram

Table 6.15: G_2^2: θ_Δ Lifting Constants on the α-basis

$\alpha \mapsto$	$\theta(\alpha)$	$c_\alpha^{\theta_\Delta}$	$c_{\theta(\alpha)}^{\theta_\Delta}$
$\alpha_1 \mapsto$	$-\alpha_1$	1	1
$\alpha_2 \mapsto$	$-\alpha_2$	1	1

Part III

Commuting Pairs of Involutions

Chapter 7: Introduction

As discussed in Part I, we will now classify commuting pairs of involutions over the algebraically closed field $\mathbb{C}$ of the simple Lie algebras. From now on we will let $\Phi(\mathfrak{g}, \mathfrak{t})$ be an irreducible root system and hence $\mathfrak{g}$ a simple Lie algebra. Let $(\sigma, \theta) \in \mathrm{Aut}(X, \Phi)$ be an ordered pair of commuting admissible involutions of the root system, $(\tilde{\sigma}, \tilde{\theta}) \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{t})$ be the corresponding arbitrary lifted involutions of the Lie algebra and $(\sigma_\Delta,\ \theta_\Delta) \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{t})$ be as in definition 4.1.6. For each ordered pair of admissible involutions in $\mathrm{Aut}(\Phi)$ we will find the standard pair (see definition 7.1.11), and for this we will need the notion of the signature of a lifted pair and some more notation.

Notation 7.0.1. The following notation will appear in our discussion of commuting pairs of involutions:

Table 7.1: Notation

Symbol	:	Definition
G	:	a semisimple (usually simple) Lie group
T	:	a fixed maximal torus (i.e. maximal connected diagonalizable subgroup) of G
$\mathfrak{g}$	:	the simple Lie algebra of G
$\mathfrak{t}$	:	a maximal toral subalgebra of $\mathfrak{g}$
$\mathfrak{a}$	:	a toral subalgebra $\subset \mathfrak{t}$
Φ	:	$\Phi(T)$ the root system with respect to T
X	:	$\mathbb{Z}\Phi$ - an integer lattice of Φ, may be a root or weight lattice, ($X^*(T)$ characters)
W	:	$W(T)$ Weyl group with respect to T

Table 7.1: Continued

Symbol	:	Definition
Ψ	:	$(X^*(T)\ ,\ \Phi(T)\ ,\ X_*(T)\ ,\ \Phi^\vee(T))$, the root datum corresponding to T
$N_{\alpha,\beta}$	:	Chevalley constants
$c_\alpha^{\sigma_\Delta}, c_\alpha^{\theta_\Delta}$	:	lifting constants of σ_Δ and θ_Δ respectively
$c_\alpha^{\sigma\theta}$	:	lifting constants of the involution lifted from $\sigma\theta = \theta\sigma$
$\langle \alpha_i, \alpha_j \rangle$	:	The product defined by the Killing form on $\mathfrak{g}$ that relates the root system and its dual.
T_σ^-, T_σ^+	:	the σ-split and σ-stable tori respectively, likewise for $T_\theta^\pm$
$T_{(\sigma,\theta)}^-, T_{(\sigma,\theta)}^+$	:	the (σ,θ)-split and (σ,θ)-stable tori respectively
$T_{\sigma\theta}^-, T_{\sigma\theta}^+$	:	the $\sigma\theta$-split and $\sigma\theta$-stable tori respectively
$\mathfrak{g}(A,\lambda)$	:	For $\lambda \in \Phi(A)$ let $\mathfrak{g}(A,\lambda) = \{X \in \mathfrak{g} \mid [H,X] = \lambda(H)X\ \forall H \in \mathfrak{a}\}$ be the corresponding root space. Since $\sigma\theta(\lambda) = \lambda$, we have $\sigma\theta(\mathfrak{g}(A,\lambda)) = \mathfrak{g}(A,\lambda)$.
$\mathfrak{g}(A,\lambda)_{\sigma\theta}^\pm$	:	$\{X \in \mathfrak{g}(A,\lambda) \mid \sigma\theta(X) = \pm X\}$
$m^\pm(\lambda,\sigma\theta)$	:	$\dim_{\Bbbk}\mathfrak{g}(A,\lambda)_{\sigma\theta}^\pm$
$\Phi(T,\lambda)$	:	$\{\alpha \in \Phi(T) \mid \alpha\|A = \lambda\}$
$m_{\text{split}}(\lambda,\sigma\theta)$	:	$\|\Phi_{\text{split}}(\lambda,\sigma\theta)\|$ such that $\Phi_{\text{split}}(\lambda,\sigma\theta) = \{\alpha \in \Phi(T,\lambda) \mid \sigma\theta(\alpha) \neq \alpha\}$
$m_{\text{check}}(\lambda,\sigma\theta)$	:	$\|\Phi_{\text{check}}(\lambda,\sigma\theta)\|$ such that $\Phi_{\text{check}}(\lambda,\sigma\theta) = \{\alpha \in \Phi(T,\lambda) \mid \sigma\theta(\alpha) = \alpha\}$
$m(\lambda)$	:	$\begin{aligned}\dim_{\Bbbk}\mathfrak{g}(A,\lambda) &= m^+(\lambda,\sigma\theta) + m^-(\lambda,\sigma\theta) = \lvert\Phi(T,\lambda)\rvert \\ &= m_{\text{split}}(\lambda,\sigma\theta) + m_{\text{check}}(\lambda,\sigma\theta)\end{aligned}$

In this chapter we will discuss the following lifting admissible involutions to simple Lie algebras. Let us first recall some definitions.

7.1 Preliminaries

7.1.1 Admissible pairs

The formal definition of an admissible pair of commuting involutions is as follows:

Definition 7.1.2 (admissible pair,[Helminck(1988), 5.18])**.** Let T be a maximal torus of G. A pair of commuting involutorial automorphisms (σ,θ) of $(X^*(T),\Phi(T))$ is said to be *admissible* (with respect to G) if there exists a pair of commuting involutorial automorphisms $(\widetilde{\sigma},\widetilde{\theta})$ of G, normally related to T and such that $\widetilde{\sigma}\big|_T = \sigma$, $\widetilde{\theta}\big|_T = \theta$.

We will however use the following equivalent definition which is more combinatorial.

Definition 7.1.1 (admissible pair)**.** Let (σ, θ) be a pair of commuting involutions of (X, Φ). Then (σ, θ) is admissible if and only if (σ, θ) is basic and both σ and θ are admissible [Helminck(1988), 7.11].

Such that "basic" is defined as follows:

Definition 7.1.3 (basic)**.** A pair of commuting involutions $(\sigma, \theta)$of (X, Φ) is called *basic* if Φ has a strong (σ, θ)-basis Δ for which $w_0(\theta), w_0(\sigma), \sigma^*, \theta^*$ mutually commute, where a (σ, θ)-order $\succ$ on Φ is called a *strong* (σ, θ)*-order* if it is simultaneously a σ- and a θ-order of Φ. A basis of Φ with respect to a strong (σ, θ)-order will be called a *strong* (σ, θ)*-basis*.

We again have nice combinatorial equivalent definition:

Theorem 7.1.4 (Combinatorial Conditions of basic (σ, θ))**.** *Let σ, θ be related involution of (X, Φ) and Δ a [Helminck(1988), relating basis, 7.12] of Φ with respect to (σ, θ). Then (σ, θ) is basic if and only if*

(1) σ^* and θ^* commute,

(2) $\Delta_0(\theta)$ is σ^*-stable and $\Delta_0(\sigma)$ is θ^*-stable

(3) for every connected component Δ_1 of $\Delta_0(\theta) \cup \Delta_0(\sigma)$ we have $\Delta_1 \subset \Delta_0(\theta)$ or $\Delta_1 \subset \Delta_0(\sigma)$. [Helminck(1988), 7.16]

Definition 7.1.5 ([Helminck(1988), 5.2])**.** A torus A of G is called (σ, θ)*-split* if A is σ- and θ-split. A torus T of G, which is σ- and θ-stable shall be called (σ, θ)*-stable*. We then put

$$T^-_{(\sigma,\theta)} = \{t \in T \mid \sigma(t) = \theta(t) = t^{-1}\}^0.$$

7.1.6 ([Helminck(1988), 5.6])**.** Let T be a (σ, θ)-stable maximal torus of G, denote by $\Psi = (X^*(T),\ \Phi(T),\ X_*(T),\ \Phi^\vee(T))$ the corresponding root datum and write $A = T^-_{(\sigma,\theta)}$. For the moment we do not yet assume that A is a maximal (σ, θ)-split torus of G. Using the notations of Part I we have the following identifications:

Lemma 7.1.7. *Let T, Ψ, σ, θ and A be as above. Then*

(i) $X_0(\sigma, \theta) = \{\chi \in X^*(T) \mid \chi(A) = 1\}$;

(ii) $\overline{\Phi}_{(\sigma,\theta)} = \Phi(A)$;

(iii) $W_1(\sigma, \theta) = \{w \in W(T) \mid w(A) = A\}$ *and* $W_0(\sigma, \theta) = \{w \in W(T) \mid w|A = \mathrm{id}\}$;

(iv) $W(A) \approx W_1(\sigma, \theta)/w_0(\sigma, \theta) \approx \overline{W}_{(\sigma,\theta)}$.

Lemma 7.1.8 ([Helminck(1988), 5.10])**.** *If $\Phi(G,T)$ is irreducible, i.e. G is simple, and $\sigma \neq \mathrm{id}, \theta \neq \mathrm{id}$, then non-trivial (σ,θ)-split tori exist.*

Proposition 7.1.9 ([Helminck(1988), 5.13])**.** *There exist (σ,θ)-stable maximal tori T of G such that $T^-_{(\sigma,\theta)}$ is a maximal (σ,θ)-split torus of G, T^-_σ is a maximal σ-split torus of G and T^-_θ is a maximal θ-split torus of G. Moreover all such maximal tori of G are conjugate under $(G_\sigma \cap G_\theta)^0$.*

[Helminck(1988), Section 6] Let T be a (σ,θ)-stable maximal torus of G and $A = T^-_{(\sigma,\theta)}$ a non-trivial maximally (σ,θ)-split torus of G with (not necessarily reduced) root system $\Phi(A) = \overline{\Phi}(\sigma,\theta)$, the restricted root system of the pair of involutions, (σ,θ).

Definition 7.1.10. For $\lambda \in \Phi(A)$ call $m(\lambda)$ the *multiplicity* of λ and $(m^+(\lambda,\sigma\theta), m^-(\lambda,\sigma\theta))$ the *signature* of λ.

For every $\lambda \in \overline{\Phi}(\sigma,\theta)$, the signature is determined by the dimension of the ±1 eigenspace of the lifted involution $\widetilde{\sigma}\widetilde{\theta}$ in the Lie algebra, $\mathfrak{g}$. For $\lambda \in \Phi(A)$ let

$$\mathfrak{g}(A,\lambda) = \{X \in \mathfrak{g} \mid [H,X] = \lambda(H)X \quad \forall H \in \mathfrak{a}\}$$

be the corresponding root space. Since $\sigma\theta(\lambda) = \lambda$, we have $\sigma\theta(\mathfrak{g}(A,\lambda)) = \mathfrak{g}(A,\lambda)$.

$\mathfrak{g}(A,\lambda)$	:	$\{X \in \mathfrak{g} \mid [H,X] = \lambda(H)X \quad \forall H \in \mathfrak{a}\}$
$\mathfrak{g}(A,\lambda)^\pm_{\sigma\theta}$	:	$\{X \in \mathfrak{g}(A,\lambda) \mid \sigma\theta(X) = \pm X\}$
$m^\pm(\lambda,\sigma\theta)$	:	$\dim_{\Bbbk}\mathfrak{g}(A,\lambda)^\pm_{\sigma\theta}$
$\Phi(T,\lambda)$	:	$\{\alpha \in \Phi(T) \mid \alpha\vert A = \lambda\}$
$m(\lambda)$	:	$\dim_{\Bbbk}\mathfrak{g}(A,\lambda) = m^+(\lambda,\sigma\theta) + m^-(\lambda,\sigma\theta) = \vert\Phi(T,\lambda)\vert$

Now, we can define the standard pair in terms of the signature.

Definition 7.1.11 ([Helminck(1988), 6.11])**.** A pair of commuting involutorial automorphisms $(\widetilde{\sigma},\widetilde{\theta})$ of G is called a *standard pair* if $m^+(\lambda,\sigma\theta) \geq m^-(\lambda,\sigma\theta)$ for any maximal (σ,θ)-split torus A of G and any $\lambda \in \Phi(A)$ such that $\frac{1}{2}\lambda \notin \Phi(A)$.

7.2 Finding the signature for $\alpha \in \Phi(T,\lambda)$

For $\alpha \in \Phi(T,\lambda)$, we get two cases. Let $\lambda = \pi_{\sigma\theta}(\alpha)$.

Case 1: $\sigma\theta(\alpha) \neq \alpha$. If $\sigma\theta(\alpha) \neq \alpha$, then it must be the case that $\sigma\theta(\alpha) = \beta \in \Phi(T,\lambda)$ and $\sigma\theta(\beta) = \alpha$. In this case, the corresponding subspace splits nicely regardless of the lifting constants, $c^{\sigma\theta}_\alpha$, into the two eigenspaces: one vector to the $+1$- and one to the -1-

eigenspace. i.e.

$$\begin{aligned}
\sigma\theta(X_\alpha + \sigma\theta(X_\alpha)) &= \sigma\theta(X_\alpha + c_\alpha^{\sigma\theta} X_{\sigma\theta(\alpha)}) \\
&= c_\alpha^{\sigma\theta} X_\alpha + c_\alpha^{\sigma\theta} c_{\sigma\theta(\alpha)}^{\sigma\theta} X_\alpha \\
&= \sigma\theta(X_\alpha) + X_\alpha \\
&= X_\alpha + \sigma\theta(X_\alpha) && \in \mathfrak{g}(A,\lambda)^+_{\sigma\theta}
\end{aligned}$$

$$\begin{aligned}
\sigma\theta(X_\alpha - \sigma\theta(X_\alpha)) &= \sigma\theta(X_\alpha) - X_\alpha \\
&= -(X_\alpha - \sigma\theta(X_\alpha)) && \in \mathfrak{g}(A,\lambda)^-_{\sigma\theta}
\end{aligned}$$

Case 2: $\sigma\theta(\alpha) = \alpha$. If $\alpha \in \Phi(T,\lambda)$ such that $\sigma\theta(\alpha) = \alpha$, then since σ and θ are both involutions:

$$\sigma\theta(\alpha) = \alpha \qquad \Longleftrightarrow \qquad \sigma(\alpha) = \theta(\alpha)$$

and

$$\begin{aligned}
\lambda &= \frac{1}{4}\left(\alpha - \sigma(\alpha) - \theta(\alpha) + \sigma\theta(\alpha)\right) \\
&= \frac{1}{4}\left(2\alpha - \sigma(\alpha) - \theta(\alpha)\right) \\
&= \frac{1}{4}\left((\alpha - \sigma(\alpha)) + (\alpha - \theta(\alpha))\right) \\
&= \frac{1}{2}\left(\alpha - \sigma(\alpha)\right) \quad \in \overline{\Phi}_\sigma \cap \overline{\Phi}_\sigma .
\end{aligned}$$

In this case, there are two possible forms for λ.

Case (a): If $\sigma(\alpha) = \theta(\alpha) = -\alpha$, then

$$\lambda = \frac{1}{4}\left(2\alpha + 2\alpha\right) = \alpha.$$

Notice that this means $\alpha \in \overline{\Phi}_\sigma \cap \overline{\Phi}_\theta$ and $H_\alpha \in T^-_{(\sigma,\theta)}$.

Case (b): If $\sigma(\alpha) = \theta(\alpha) \neq -\alpha$, then

$$\lambda = \frac{1}{2}\left(\alpha - \sigma(\alpha)\right).$$

7.2.1 What we know about the lifting constants.

In the first case when $\sigma\theta(\alpha) \neq \alpha$, as seen above, the lifting constants do not effect the the dimension of the eigenspaces. In the second case when $\sigma\theta(\alpha) = \alpha$, however, we can see easily that $c_\alpha^{\sigma\theta} = \pm 1$: Given that we assume $\tilde{\sigma}$ and $\tilde{\theta}$ commute on the Lie algebra, and hence $\tilde{\sigma}\tilde{\theta}$ an involution, we know

$$\tilde{\sigma}\tilde{\theta}(X_\alpha) = c_\alpha^{\sigma\theta} X_{\sigma\theta(\alpha)} = c_\alpha^{\sigma\theta} X_\alpha \tag{7.2.1 (a)}$$

and

$$c_\alpha^{\sigma\theta} \cdot c_{\sigma\theta(\alpha)}^{\sigma\theta} = (c_\alpha^{\sigma\theta})^2 = 1 \tag{7.2.1 (b)}$$

which implies that

$$c_\alpha^{\sigma\theta} = \pm 1. \tag{7.2.1 (c)}$$

In this case, we also know that $\sigma(\alpha) = \theta(\alpha)$ and, so,

$$c_\alpha^{\sigma\theta} = c_\alpha^{\theta} \cdot c_{\theta(\alpha)}^{\sigma} = c_\alpha^{\theta} \cdot c_{\sigma(\alpha)}^{\sigma} = c_\alpha^{\sigma} \cdot c_{\theta(\alpha)}^{\theta} \tag{7.2.1 (d)}$$

Again considering the two possible forms of λ:

Definition 7.2.2. Let $\theta \in \mathrm{Aut}(G)$ be an involution stabilizing T. Then a root $\alpha \in \Phi(T)$ is called

- *θ-singular*, if $\theta(\alpha) = \pm\alpha$ and $\theta | Z_G((\mathrm{Ker}\alpha)^0) \neq \mathrm{id}$;
- *real with respect to θ*, if $\theta(\alpha) = -\alpha$;
- *noncompact imaginary with respect to θ*, if $\theta(\alpha) = \alpha$ and α is θ-singular. In this case $c_{\alpha,\theta} = -1$, as follows also by simple computation in SL_2.
- *compact imaginary with respect to θ*, if $\theta(\alpha) = \alpha$ and α is not θ-singular, then $c_{\alpha,\theta} = 1$.

7.2.3 $c_\alpha^{\sigma\theta}$ WHEN $\sigma(\alpha) = \theta(\alpha) = -\alpha$

When α is real with respect to σ and θ, and imaginary with respect to $\sigma\theta$. This indicates that $c_\alpha^{\sigma\theta}$ will be decided by whether or not α is also $\sigma\theta$-singular. Equivalently, if $c_\alpha^{\sigma} = c_\alpha^{\theta}$, then $c_\alpha^{\sigma\theta} = 1$ and α is compact imaginary with respect to $\sigma\theta$. If $c_\alpha^{\sigma} = -c_\alpha^{\theta}$, then $c_\alpha^{\sigma\theta} = -1$ and α is noncompact imaginary with respect to $\sigma\theta$. To determine the lifting constant, we can consider the action of each involution restricted to $\mathfrak{g}_\alpha$. By our maximality condition on T, $\mathfrak{g}_\alpha$ is one dimensional and so the subalgebra $\mathfrak{t}_\alpha \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \approx \mathrm{sl}(2,\mathbb{C})$. Recall our example in $\mathrm{sl}(2,\mathbb{C})$,

Example 5.2.0.1 (Involutions in $\mathrm{sl}(2,\mathbb{C})$)**.** In $\mathrm{sl}(2,\mathbb{C})$ there is only one conjugacy class of involutions, which may be represented by abuse of notation by $\theta(X) = -X^T$, corresponding to the opposition involution on the root system A_1, $\theta(\alpha) = -\alpha = -\theta^*(\alpha) = -\mathrm{id}^*(\alpha)$. If we take a natural basis of $\mathrm{sl}(2,\mathbb{C})$, it is clear by simple calculation that the lifting constants $c_\alpha^\theta = -1 = c_{-\alpha}^\theta$:

$$H_\alpha = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \quad X_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_{-\alpha} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

then

$$[H_\alpha\ ,\ X_\alpha] = aX_\alpha - (-a)X_\alpha = 2aX_\alpha$$
$$[H_\alpha\ ,\ X_{-\alpha}] = -aX_{-\alpha} - aX_{-\alpha} = -2aX_{-\alpha}$$

and

$$\begin{aligned}
\theta: \quad H_\alpha &\mapsto \begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix} = -H_\alpha = H_{-\alpha} = H_{\theta(\alpha)} \\
X_\alpha &\mapsto \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = -X_{-\alpha} = -X_{\theta(\alpha)} = c^\theta_\alpha X_{\theta(\alpha)} \\
X_{-\alpha} &\mapsto \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = -X_\alpha = -X_{\theta(-\alpha)} = c^\theta_{-\alpha} X_{\theta(-\alpha)}
\end{aligned}$$

Composition with

$$\mathrm{Int}(h_\alpha) = \mathrm{Int}\left(\exp\left(\frac{i\pi}{2a}H_\alpha\right)\right) = \mathrm{Int}\begin{pmatrix} e^{\frac{i\pi}{2}} & 0 \\ 0 & e^{-\frac{i\pi}{2}} \end{pmatrix}$$

gives us another involution that has the same action on the root space, but different lifting constants:

$$\begin{aligned}
\theta\,\mathrm{Int}(h_\alpha): \qquad H_\alpha &\xmapsto{\mathrm{Int}(h_\alpha)} \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \overset{\theta}{\mapsto} \begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix} &&= -H_\alpha = H_{-\alpha} = H_{\theta(\alpha)} \\
X_\alpha &\xmapsto{\mathrm{Int}(h_\alpha)} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \overset{\theta}{\mapsto} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} &&= X_{-\alpha} = X_{\theta(\alpha)} = c^{\theta\,\mathrm{Int}(h_\alpha)}_\alpha X_{\theta(\alpha)} \\
X_{-\alpha} &\xmapsto{\mathrm{Int}(h_\alpha)} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \overset{\theta}{\mapsto} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} &&= X_\alpha = X_{\theta(-\alpha)} = c^{\theta\,\mathrm{Int}(h_\alpha)}_{-\alpha} X_{\theta(-\alpha)}
\end{aligned}$$

So in this example we have shown that $\theta_\Delta = \theta\,\mathrm{Int}(h_\alpha)$.
This process allows us to choose the $\tilde{\theta} \in \mathrm{Aut}(G, T)$ to which we lift $\theta \in \mathrm{Aut}(\Phi)$.

7.2.4 $c^{\sigma\theta}_\alpha$ WHEN $\sigma(\alpha) = \theta(\alpha) \neq -\alpha$

This case is in fact composed of several smaller cases which analyzed in a similar manner.

Chapter 8: Lifting Pairs of Commuting Involutions

8.1 Terminology

Let us first define some short-hand terminology to make discussing these pairs easier.

Definition 8.1.1. We will use the following to refer to different categories of involutions:

Nice Pair	We will call a pair of involutions (σ, θ) "nice" if both σ_Δ and θ_Δ are involutions, $\sigma_\Delta\theta_\Delta = \theta_\Delta\sigma_\Delta$ and $(\sigma_\Delta, \theta_\Delta)$ is standard.
Simply Lifted	Another name for "Nice." We will also say a single involution θ is "simply lifted" or "nice" if θ_Δ is an involution.
Better-than-Okay Pair	We will call a pair of involutions (σ, θ) "better-than-okay" if at least one of σ_Δ and θ_Δ is *not* an involution, but the (possibly both) corrected involutions commute $\sigma_\Delta \operatorname{Int}(h_\sigma)\theta_\Delta \operatorname{Int}(h_\theta) = \theta_\Delta \operatorname{Int}(h_\theta)\sigma_\Delta \operatorname{Int}(h_\sigma)$ and $(\sigma_\Delta \operatorname{Int}(h_\sigma), \theta_\Delta \operatorname{Int}(h_\theta))$ is standard. Note that w.l.o.g. $\operatorname{Int}(h_\sigma)$ may act as the identity.
Decent Pair	We will say a pair of involutions (σ, θ) is "decent" if the lifted pair $(\tilde{\sigma}, \tilde{\theta})$ commutes but is not standard. In this case, we correct to the standard pair $(\tilde{\sigma}, \tilde{\theta} \operatorname{Int}(h))$ where $h \in T^+_{\sigma\theta}$.
Corrected for Standard	Another name for "Decent".
Feisty Pair	We will say a pair of involutions (σ, θ) is "feisty" if our lifted single involutions $\tilde{\sigma}$ and $\tilde{\theta}$ do not commute. It happens to be the case that these only occur for a handful of involutions such that $\tilde{\sigma} = \sigma_\Delta$ and $\tilde{\theta} = \theta_\Delta$. Again, we correct to the standard pair $(\tilde{\sigma}, \tilde{\theta} \operatorname{Int}(h))$ where $h \in T^+_{\sigma\theta}$.

Definition 8.1.2. Given any lifted pair of involutions $(\tilde{\sigma}, \tilde{\theta})$, we will say the pair $(\tilde{\sigma}, \tilde{\theta}\,\mathrm{Int}(h))$ is a *corrected* pair of $(\tilde{\sigma}, \tilde{\theta})$. We will refer to $h \in T$ or its corresponding $H \in \mathfrak{t}$ such that $\mathrm{Int}(h) = \exp(\mathrm{ad}(H))$ as the *correction vector* for this pair. Since we "correct" to the standard pair (or to an involution), to avoid confusion we will distinguish these shifted pairs as having "*standard*" or "*non-standard*" correction vectors.

8.2 Constructing the standard pair

8.2.1 Lifting Algorithm for Commuting Pairs (σ, θ)

The following algorithm is implemented in our *k-involutions* Mathematica package as

`makeAdmissiblePair[{sigma_admissibleInvolution, theta_admissibleInvolution}]`.

It is the method by which we find the standard pair for each admissible pair of commuting involutions $(\sigma, \theta) \subset \mathrm{Aut}(\Phi)$.

Algorithm 8.3 (Lifting a Commuting Pair of Involutions)**.**

Input: $(\tilde{\sigma}, \tilde{\theta})$ **with all data including the root actions of each**

Output: Standard pair with all correction vectors. See Algorithm 8.4.

(LiftPair 1) Compute the root action of $\sigma\theta$.

(LiftPair 2) Compute the projected roots $\pi_{(\sigma,\theta)}(\Phi^+)$ and determine $\overline{\Phi}^+$ and $\overline{\Delta}$.

(LiftPair 3) For each restricted root $\lambda \in \overline{\Phi}^+$, record the preimage of the projection map $\pi^{-1}_{(\sigma,\theta)}(\lambda) \subset \Phi^+ = \{\alpha \in \Phi(T) \mid \alpha|A = \lambda\} = \Phi(T, \lambda)$.

(LiftPair 4) Record data for computing the signature of the correction vectors. For each $\lambda \in \overline{\Phi}^+$:

$$m(\lambda) = m^+(\lambda, \sigma\theta) + m^-(\lambda, \sigma\theta) \tag{8.3.0 (a)}$$

$$= m_{\text{split}}(\lambda, \sigma\theta) + m_{\text{check}}(\lambda, \sigma\theta) \tag{8.3.0 (b)}$$

and

$$m^{\pm}(\lambda, \sigma\theta) = m^{\pm}_{\text{split}}(\lambda) + m^{\pm}_{\text{check}}(\lambda) \tag{8.3.0 (c)}$$

(a) Determine $\Phi_{\text{split}}(\lambda, \sigma\theta) = \{\alpha \in \Phi(T, \lambda) \mid \sigma\theta(\alpha) \neq \alpha\}$.

Note. $m^+_{\text{split}}(\lambda) = m^-_{\text{split}}(\lambda)$, and so does not indicate whether or not $(\tilde{\sigma}, \tilde{\theta})$ is standard.

(b) Determine $\Phi_{\text{check}}(\lambda, \sigma\theta) = \{\alpha \in \Phi(T, \lambda) \mid \sigma\theta(\alpha) = \alpha\}$.

Note. $m_{\text{check}}(\lambda, \sigma\theta) = m^+_{\text{check}}(\lambda) + m^-_{\text{check}}(\lambda)$ and $(\tilde{\sigma}, \tilde{\theta})$ is standard if $m^+_{\text{check}}(\lambda) \geq m^-_{\text{check}}(\lambda)$ which is true if and only if $m^+_{\text{check}}(\lambda) - m^-_{\text{check}}(\lambda) \geq 0$.

(c) `getReducedRoots` since our definition of standard only depends on $\lambda \in \Phi(A)$ such that $\frac{1}{2}\lambda \notin \Phi(A)$.

(LiftPair 5) Call Algorithm 8.4: `getCorrectionVectors[myPair_admissiblePair]`.

(LiftPair 6) `getSignatures` to differentiate Standard and Non-Standard Correction Vectors, noting that lifted pairs with the same signature are isomorphic.

(LiftPair 7) Choose a representative Standard Correction Vector and use this to determine the lifting constants $c_\alpha^{\sigma\theta \exp(\text{ad}(H))}$ of the standard pair. Whenever possible choose H such that $\exp(\text{ad}(H)) = \text{id}$.

(LiftPair 8) End.

8.3.1 Correction Vector Algorithm for $(\tilde{\sigma}, \tilde{\theta})$

Assuming $\tilde{\sigma}$ and $\tilde{\theta}$ are involutions on the Lie algebra such that $\sigma\theta = \theta\sigma$ on the root system, if $\tilde{\sigma}\tilde{\theta}(X_\alpha) \neq \tilde{\sigma}\tilde{\theta}(X_\alpha)$ for some $\alpha \in \Phi$ we need to solve the equations

$$\tilde{\sigma}\; \tilde{\theta}\; e^{\alpha(H)}(X_\alpha) = \tilde{\theta}\; e^{\alpha(H)}\; \tilde{\sigma}(X_\alpha) \tag{8.3.1 (a)}$$

and

$$\left(\tilde{\theta}\; e^{\alpha(H)}\right)^2 (X_\alpha) = X_\alpha. \tag{8.3.1 (b)}$$

This gives us the condition that $\tilde{\theta}e^{\text{ad}(H)}$ is an involution that commutes with $\tilde{\sigma}$ and hence that $\tilde{\sigma}\tilde{\theta}e^{\text{ad}(H)}$ is an involution on the Lie algebra. We can limit our search still further by applying Proposition 5.2.2 to the involution $\sigma\theta$:

8.3.2. Suppose $(\sigma,\ \theta)$ is admissible then $\sigma\theta = \theta\sigma \iff (\sigma\theta)^2 = \text{id}$. By abuse of notation denote $\tilde{\sigma}|T = \sigma$ and likewise for $\tilde{\theta}$. Let Δ be a relating basis so that $\sigma(T) = \theta(T) = T$. Recall

$$T^-_{\sigma,\,\theta} = \{t \in T \mid \sigma(t) = \theta(t) = t^{-1}\}^0$$
$$T^+_{\sigma,\,\theta} = \{t \in T \mid \sigma(t) = \theta(t) = t\}^0.$$

Similarly,

$$T^-_\sigma = \{t \in T \mid \sigma(t) = t^{-1}\}^0$$
$$T^+_\sigma = \{t \in T \mid \sigma(t) = t\}^0 = (T \cap G_\sigma)^0$$

and likewise for T_θ^- and T_θ^+. So we have the identities,

$$T_{\sigma,\,\theta}^+ = \left(T_\sigma^+ \cap T_\theta^+\right)$$
$$T_{\sigma,\,\theta}^- = \left(T_\sigma^- \cap T_\theta^-\right).$$

By definition, $(\sigma,\ \theta)$ is admissible if and only if $\exists \tilde\sigma, \tilde\theta \in \mathrm{Aut}(G,T)$ such that

$$\tilde\sigma|T = \sigma \text{ and } \tilde\theta|T = \theta$$
$$\tilde\sigma^2 = \mathrm{id} = \tilde\theta^2$$
$$\tilde\sigma\tilde\theta = \tilde\theta\tilde\sigma$$

which is true if and only if

$$(\tilde\sigma\tilde\theta)^2 = \tilde\sigma\theta\tilde\theta\tilde\sigma = \mathrm{id}$$

$\Rightarrow \tilde\sigma\tilde\theta \in \mathrm{Aut}(G,T)$ is an involution.

Suppose we lift (σ,θ) to $(\widehat\sigma,\widehat\theta)$ such that $\widehat\sigma^2 = \mathrm{id} = \widehat\theta^2$ and $\widehat\sigma|T = \sigma$ and $\widehat\theta|T = \theta$ but $\widehat\sigma\widehat\theta \neq \widehat\theta\widehat\sigma$ and so $\tilde\sigma\tilde\theta \neq \mathrm{id}$. Since (σ,θ) is admissible, $\exists(\tilde\sigma,\tilde\theta)$ as above such that

$$\tilde\sigma\tilde\theta = \widehat\sigma\widehat\theta\,\mathrm{Int}(t) \text{ for some } t \in T_{\sigma\theta}^+$$

such that by Proposition 5.2.2 $(\tilde\sigma\tilde\theta)^2 = \mathrm{id} = (\widehat\sigma\widehat\theta\,\mathrm{Int}(t))^2$ and $\widehat\sigma\widehat\theta\mathrm{Int}(t) = \mathrm{Int}(\sigma\theta(t))\widehat\sigma\widehat\theta = \mathrm{Int}(t)\widehat\sigma\widehat\theta$.

Note. $T_{\sigma\theta}^+ = \{t \in T \mid \sigma\theta(t) = t\}^0 = (T \cap G_{\sigma\theta})^0 = T_{\sigma\theta}^+ \times T_{\sigma,\,\theta}^-$, so $t = t_+t_-$ where $t_+ \in T_{\sigma,\,\theta}^+$ and $t_- \in T_{\sigma,\,\theta}^-$.

So in order to determine conditions on t, we observe the following.

$$\begin{aligned}\mathrm{id} &= (\widehat\sigma\widehat\theta\,\mathrm{Int}(t)) \cdot (\widehat\sigma\widehat\theta\,\mathrm{Int}(t))\\ &= (\widehat\sigma\widehat\theta)^2\,\mathrm{Int}(\sigma\theta(t)t),\end{aligned}$$

for $t \in T_\theta^+\sigma\theta$,

$$\begin{aligned}&= (\widehat\sigma\widehat\theta)^2\,\mathrm{Int}(t^2)\\ \Rightarrow (\widehat\sigma\widehat\theta)^2 &= \mathrm{Int}(t^2)^{-1} = \mathrm{Int}(t^{-1})^2.\end{aligned}$$

Now, given $t \in T_{\sigma\theta}^{+}$, we also have

$$\begin{aligned}\left[\widehat{\theta}\mathrm{Int}(t)\right]^2 &= \widehat{\theta}^2\mathrm{Int}(\theta(t)t)\\ &= \mathrm{id}\,\mathrm{Int}(\theta(t)t)\\ &= \mathrm{Int}(t_+t_-^{-1}t_+t_-)\\ &= \mathrm{Int}(t_+^2) = \mathrm{id}\end{aligned}$$

if $t_+^2 \in Z(G)$.

It follows that:

Proposition 8.3.3. *Let $(\widehat{\sigma}, \widehat{\theta})$ be as above, then $\widehat{\sigma}\widehat{\theta}\,\mathrm{Int}(t) = \widehat{\theta}\,\mathrm{Int}(t)\widehat{\sigma}$ if and only if $t = t_+t_- \in T_{\sigma\theta}^{+}$ such that $t_+^2 \in Z(G)$ is a quadratic element. Hence $t \in T_\sigma^+ \cap T_\theta^+$ or $t \in T_\sigma^- \cap T_\theta^-$.*

The following algorithm is implemented in our *k-involutions* Mathematica package as

`getCorrectionVectors[myPair_admissiblePair]`.

It is the method by which we correct the pair $(\tilde{\sigma}, \tilde{\theta})$ to the standard pair and determine the quadratic elements corresponding to standard and non-standard lifted pairs of involutions.

Algorithm 8.4 (Correction Vectors for a Pair of Involutions)**.**

Let $\sigma, \theta \in \mathrm{Aut}(\Phi)$ be commuting admissible involutions on the root system. Let $\tilde{\sigma}, \tilde{\theta} \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{t})$ be their lifted involutions of the Lie algebra. Note that in most cases $(\tilde{\sigma}, \tilde{\theta}) = (\sigma_\Delta, \theta_\Delta)$, though we do not assume this to be true. It is again sufficient to do computations only on the positive roots Φ^+.

For the purposes of this algorithm we will use the following notation:

$\overline{\Phi} = \overline{\Phi}(\sigma, \theta) \quad \overline{\Delta} = \overline{\Delta}(\sigma, \theta)$

Input: $(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\theta}})$

Output: **The set of all correction vectors for which $\tilde{\boldsymbol{\sigma}}\tilde{\boldsymbol{\theta}} = \tilde{\boldsymbol{\theta}}\tilde{\boldsymbol{\sigma}}$, the set of Standard Correction Vectors, the set of Non-Standard Correction Vectors.**

(PairCV 1) Let $\vec{y} = (y_i)_1^\ell$ be the coefficient vector of H such that $H = \vec{y}.\vec{H}_{\alpha_i}$ for which we will solve.

(PairCV 2) For each $\alpha_i \in \Delta$, solve the system of equations given by

$$c_{\alpha_i}^{\tilde{\theta}} \cdot e^{\alpha_i(H)} \cdot c_{\theta(\alpha_i)}^{\tilde{\sigma}} = c_{\alpha_i}^{\tilde{\sigma}} \cdot c_{\sigma(\alpha_i)}^{\tilde{\theta}} \cdot e^{\sigma(\alpha_i)(H)} \tag{8.4.0 (a)}$$

$$c_{\alpha_i}^{\tilde{\theta}} c_{\theta(\alpha_i)}^{\tilde{\theta}} = 1. \tag{8.4.0 (b)}$$

(a) Find $H \in \mathfrak{t}^-_{\sigma,\,\theta}$ such that $(\tilde{\sigma},\ \tilde{\theta}\exp(\mathrm{ad}(H)))$ is standard if one exists.

(b) If $\nexists H \in \mathfrak{t}^-_{\sigma,\,\theta}$ such that $(\tilde{\sigma},\ \tilde{\theta}\exp(\mathrm{ad}(H)))$ is standard, then find $H \in \mathfrak{t}^+_{\sigma,\,\theta}$ such that it is.

(c) Find quadratic elements $H \in \mathfrak{t}^-_{\sigma,\,\theta}$ such that $(\tilde{\sigma},\ \tilde{\theta}\exp(\mathrm{ad}(H)))$ is non-standard to find other non-isomorphic pairs.

(PairCV 3) End.

Chapter 9: Results

9.1 Nice Pairs: Both σ_Δ and θ_Δ are involutions, $\sigma_\Delta\theta_\Delta = \theta_\Delta\sigma_\Delta$ and $(\sigma_\Delta, \theta_\Delta)$ is standard.

Remark 9.1.1 ($\theta_\Delta = \sigma_\Delta$)**.** If $\theta_\Delta(X_\alpha) = \sigma_\Delta(X_\alpha)$ for all $\alpha \in \Phi$, then σ_Δ and θ_Δ commute.

Proposition 9.1.2 ($\Delta_0(\sigma) = \Delta_0(\theta) = \emptyset$)**.** *If $\Delta_0(\sigma) = \Delta_0(\theta) = \emptyset$, then σ_Δ and θ_Δ commute.*

Proof. If $\Delta_0(\sigma) = \Delta_0(\theta) = \emptyset$, then σ_Δ and θ_Δ commute:
If $\Delta_0(\sigma) = \Delta_0(\theta) = \emptyset$ then

$$\sigma = -\,\mathrm{id}\ \sigma^* \qquad \text{and} \qquad \theta = -\,\mathrm{id}\ \theta^* \tag{9.1.2 (a)}$$

Case 1: $\sigma = \theta$. In this case, $\sigma_\Delta = \theta_\Delta$ by definition and and hence they commute. In fact, we know that $\sigma^* = \mathrm{id}$ or $\sigma^* \neq \mathrm{id}$ and we showed in part I that in either of these instances, σ_Δ is an involution and so $\sigma_\Delta\theta_\Delta = \mathrm{id}$.

Case 2: $\sigma \neq \theta$. If $\sigma \neq \theta$, then w.l.o.g. we can assume that $\sigma^* = \mathrm{id} \neq \theta^*$ and so

$$\sigma\theta = \theta^* = \theta\sigma. \tag{9.1.2 (b)}$$

As in Case 1, we know that σ_Δ and θ_Δ are both involutions defined so that $c_{\alpha_i}^{\theta_\Delta} = 1 = c_{\alpha_i}^{\sigma_\Delta}$

for any simple root $\alpha_i \in \Delta$. Therefore we can deduce the following:

$$\sigma_\Delta \theta_\Delta (X_{\alpha_i}) = c^{\theta_\Delta}_{\alpha_i} c^{\sigma_\Delta}_{-\theta^*(\alpha_i)} X_{-\theta^*(\alpha_i)} \tag{9.1.2 (c)}$$

and

$$\theta_\Delta \sigma_\Delta (X_{\alpha_i}) = c^{\sigma_\Delta}_{\alpha_i} c^{\theta_\Delta}_{-\alpha_i} X_{-\theta^*(\alpha_i)} \tag{9.1.2 (d)}$$

where for any $\alpha_i \in \Delta$

$$c^{\theta_\Delta}_{\alpha_i} c^{\sigma_\Delta}_{-\theta^*(\alpha_i)} = c^{\sigma_\Delta}_{-\theta^*(\alpha_i)} = c^{\sigma_\Delta}_{\theta^*(\alpha_i)} = c^{\sigma_\Delta}_{\alpha_j} = 1 \quad \text{such that } \alpha_j \in \Delta \tag{9.1.2 (e)}$$

and

$$c^{\sigma_\Delta}_{\alpha_i} c^{\theta_\Delta}_{-\alpha_i} = c^{\theta_\Delta}_{\alpha_i} = 1. \tag{9.1.2 (f)}$$

Therefore σ_Δ and θ_Δ commute.

□

Remark 9.1.3. For all simple root systems the pair $(\theta_\Delta, \theta_\Delta \operatorname{Int}(\varepsilon_i))$ is nice except in the one case in type D when θ_Δ is not an involution. In the following sections we describe the other nice pairs. But first, a little explanation of the tables in this Chapter.

9.1.4 Standard vs. Non-Standard Correction

We will present our results for the standard and non-standard correction vectors in the following type of table. In these tables, we have represented the restricted roots $\lambda_i \in \overline{\Phi}(\sigma,\theta)$ and their corresponding fundamental weights ω_{λ_i} in terms of the simple roots Δ of Φ. To reduce the size of the tables in the classification, however, we have often omitted these explicit representations. Here, $a_{\alpha_i}, b_{\alpha_i} \in \mathbb{Q}$ such that $\lambda_i = \sum_\Delta a_j\alpha_j$, $\omega_{\lambda_i} = \sum_\Delta b_j\alpha_j$, $x_j, y_j \in \mathbb{C}$, and ω_{λ_i} acts on $\overrightarrow{H}$ such that $x_j.\omega_{\lambda_j}.\overrightarrow{H} = x_j \cdot \sum_\Delta b_j H_{\alpha_j}$.

Table 9.1: Correction Vector Table Form

(σ,θ)-type	restricted basis		fundamental weights		Standard	Non-Standard
$\Phi^{p,q}(\text{Cartan Type }\sigma, \text{Cartan Type }\theta, \varepsilon_i)$	λ_i	$\{a_{\alpha_1},\ldots,a_{\alpha_\ell}\}$	ω_{λ_i}	$\{b_{\alpha_1},\ldots,b_{\alpha_\ell}\}$	$\{x_j.\omega_{\lambda_j}.\overrightarrow{H}\}$	$\{y_j.\omega_{\lambda_j}.\overrightarrow{H}\}$

For Example, let $z_i \in \mathbb{Z}$

Table 9.2: Decent Pairs in Type A$_7$

(σ,θ)-type	restricted basis		fundamental weights		Standard Correction	Non-Standard Correction
$A_7^{3,4}(II, IIIb, \varepsilon_i)$	λ_1	$\{\frac{1}{4},\frac{1}{2},\frac{1}{4},0,\frac{1}{4},\frac{1}{2},\frac{1}{4}\}$	ω_{λ_1}	$\{\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{4}\}$	$(2i\pi(2z_1+1)).\omega_{\lambda_2}.\overrightarrow{H}$	$(4i\pi z_1).\omega_{\lambda_1}.\overrightarrow{H}$
	λ_2	$\{0,0,\frac{1}{2},1,\frac{1}{2},0,0\}$	ω_{λ_2}	$\{\frac{1}{4},\frac{1}{2},\frac{3}{4},1,\frac{3}{4},\frac{1}{2},\frac{1}{4}\}$		$(4i\pi z_1).\omega_{\lambda_2}.\overrightarrow{H}$

9.1.5 Type A

Even Rank: All pairs are nice.
Odd Rank: (σ, θ) such that $\sigma^* = \theta^*$, note this automatically includes $(\theta_\Delta, \theta_\Delta \operatorname{Int}(\varepsilon_i))$ or $\Delta_0(\sigma,\theta) = \emptyset$.

Nice Pairs in Type $A_{\ell=2k}$.

Table 9.3: Nice Pairs in Type $A_{\ell=2k}$

(σ,θ)-type	restricted basis	fundamental weights	Standard Correction	Non-Standard Correction
$A_\ell^{\ell,\ell}(I,I,\varepsilon_i)$ $1 \le q \le p = \ell$	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(2i\pi z_1)\,.\omega_{\lambda_q}.\overrightarrow{H}$ $(2i\pi z_1)\,.\omega_{\lambda_\ell}.\overrightarrow{H}$	$(i\pi\,(2z_1+1))\,.\omega_{\lambda_q}.\overrightarrow{H}$ $(i\pi\,(2z_1+1))\,.\omega_{\lambda_\ell}.\overrightarrow{H}$
$A_\ell^{p,k}(IIIa,IIIa,\varepsilon_i)$ $1 \le p \le k \le \ell/2$ $1 \le q \le p-1$	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(4i\pi z_1)\,.\omega_{\lambda_q}.\overrightarrow{H}$ $(8i\pi z_1)\,.\omega_{\lambda_p}.\overrightarrow{H}$	$(2i\pi\,(2z_1+1))\,.\omega_{\lambda_q}.\overrightarrow{H}$ $(4i\pi\,(2z_1+1))\,.\omega_{\lambda_p}.\overrightarrow{H}$
$A_\ell^{\ell,p}(I,IIIa,\varepsilon_i)$ $1 \le p \le \ell/2$ $1 \le q \le p-1$	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(2i\pi z_1)\,.\omega_{\lambda_q}.\overrightarrow{H}$ $(4i\pi z_1)\,.\omega_{\lambda_p}.\overrightarrow{H}$	$\{\}$

Nice Pairs in Type $A_{\ell=2k-1}$.

Table 9.4: Nice Pairs in Type $A_{\ell=2k-1}$

(σ,θ)-type	restricted basis	fundamental weights	Standard Correction	Non-Standard Correction
$A_\ell^{\ell,\ell}(I,I,\varepsilon_i)$ $1\le q\le p=\ell$	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(2i\pi z_1).\omega_{\lambda_q}.\overrightarrow{H}$	$(i\pi(2z_1+1)).\omega_{\lambda_q}.\overrightarrow{H}$
$A_\ell^{\ell,(\ell-1)/2}(I,II,\varepsilon_i)$ $1\le q\le p=(\ell-1)/2$	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(2i\pi z_1).\omega_{\lambda_q}.\overrightarrow{H}$	$\{\}$
$A_\ell^{\ell,(\ell+1)/2}(I,IIIb,\varepsilon_i)$ $1\le q\le p=(\ell+1)/2$	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(2i\pi z_1).\omega_{\lambda_q}.\overrightarrow{H}$	$(i\pi(2z_1+1)).\omega_{\lambda_p}.\overrightarrow{H}$
$A_\ell^{(\ell-1)/2,(\ell-1)/2}(II,II,\varepsilon_i)$ $1\le q\le p=(\ell-1)/2$	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(4i\pi z_1).\omega_{\lambda_q}.\overrightarrow{H}$	$(2i\pi(2z_1+1)).\omega_{\lambda_q}.\overrightarrow{H}$
$A_\ell^{(\ell-1)/2,(\ell+1)/2}(II,IIIb,\varepsilon_i)$ $\ell\ne 3+4k$ $1\le q<p=(\ell-1)/2$	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(4i\pi z_1).\omega_{\lambda_p}.\overrightarrow{H}$ $(8i\pi z_1).\omega_{\lambda_p}.\overrightarrow{H}$	$\{\}$
$A_\ell^{p,k}(IIIa,IIIa,\varepsilon_i)$ $1\le p\le k\le(\ell-1)/2$ $1\le q<p$	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(4i\pi z_1).\omega_{\lambda_q}.\overrightarrow{H}$ $(8i\pi z_1).\omega_{\lambda_p}.\overrightarrow{H}$	$(2i\pi(2z_1+1)).\omega_{\lambda_q}.\overrightarrow{H}$ $(4i\pi(2z_1+1)).\omega_{\lambda_p}.\overrightarrow{H}$
$A_\ell^{p,(\ell+1)/2}(IIIa,IIIb,\varepsilon_i)$ $1\le q<p<(\ell+1)/2$	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(4i\pi z_1).\omega_{\lambda_q}.\overrightarrow{H}$ $(4i\pi z_1).\omega_{\lambda_p}.\overrightarrow{H}$	$(2i\pi(2z_1+1)).\omega_{\lambda_q}.\overrightarrow{H}$
$A_\ell^{(\ell+1)/2,(\ell+1)/2}(IIIb,IIIb,\varepsilon_i)$ $1\le q<p=(\ell+1)/2$	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(4i\pi z_1).\omega_{\lambda_q}.\overrightarrow{H}$ $(2i\pi z_1).\omega_{\lambda_p}.\overrightarrow{H}$	$(2i\pi(2z_1+1)).\omega_{\lambda_q}.\overrightarrow{H}$ $(i\pi(2z_1+1)).\omega_{\lambda_p}.\overrightarrow{H}$

9.1.6 Type B

Nice Pairs in Type $B_{\ell=2k}$

Table 9.5: Nice Pairs in Type $B_{\ell=2k}$

(σ,θ)-type	restricted basis	fundamental weights	Standard Correction	Non-Standard Correction
$B_\ell^{p,k}(I,I,\varepsilon_i)$ p and k both odd $1 \le p \le k \le \ell-3$ $1 \le q \le p-1$	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(2i\pi z_1)\,.\omega_{\lambda_q}.\overrightarrow{H}$ $(4i\pi z_1)\,.\omega_{\lambda_p}.\overrightarrow{H}$	$(i\pi\,(2z_1+1))\,.\omega_{\lambda_q}.\overrightarrow{H}$ $(2i\pi\,(2z_1+1))\,.\omega_{\lambda_p}.\overrightarrow{H}$
$B_\ell^{p,k}(I,I,\varepsilon_i)$ p and k both even $2 \le p \le k \le \ell$ $1 \le q \le p-1$	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(2i\pi z_1)\,.\omega_{\lambda_q}.\overrightarrow{H}$ $(4i\pi z_1)\,.\omega_{\lambda_p}.\overrightarrow{H}$	$(i\pi\,(2z_1+1))\,.\omega_{\lambda_q}.\overrightarrow{H}$ $(2i\pi\,(2z_1+1))\,.\omega_{\lambda_p}.\overrightarrow{H}$
$B_\ell^{p,\ell-1}(I,I,\varepsilon_i)$ $1 \le p \le \ell-1$ $1 \le q \le p-1$	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(2i\pi z_1)\,.\omega_{\lambda_q}.\overrightarrow{H}$ $(4i\pi z_1)\,.\omega_{\lambda_p}.\overrightarrow{H}$ $(2i\pi\,(2z_1+1))\,.\omega_{\lambda_p}.\overrightarrow{H}$	$(i\pi\,(2z_1+1))\,.\omega_{\lambda_q}.\overrightarrow{H}$ $(2i\pi z_1)\,.\omega_{\lambda_p}.\overrightarrow{H}$

NICE PAIRS IN TYPE $B_{\ell=2k-1}$

Table 9.6: Nice Pairs in Type $B_{\ell=2k-1}$

(σ,θ)-type	restricted basis	fundamental weights	Standard Correction	Non-Standard Correction
$B_\ell^{p,k}(I,I,\varepsilon_i)$ p and k both odd $1 \leq p \leq k \leq \ell$ $1 \leq q \leq p-1$	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(2i\pi z_1).\omega_{\lambda_q}.\overrightarrow{H}$ $(4i\pi z_1).\omega_{\lambda_p}.\overrightarrow{H}$	$(i\pi(2z_1+1)).\omega_{\lambda_q}.\overrightarrow{H}$ $(2i\pi(2z_1+1)).\omega_{\lambda_p}.\overrightarrow{H}$
$B_\ell^{p,k}(I,I,\varepsilon_i)$ p and k both even $1 \leq p \leq k \leq \ell-1$ $1 \leq q \leq p-1$	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(2i\pi z_1).\omega_{\lambda_q}.\overrightarrow{H}$ $(4i\pi z_1).\omega_{\lambda_p}.\overrightarrow{H}$	$(i\pi(2z_1+1)).\omega_{\lambda_q}.\overrightarrow{H}$ $(2i\pi(2z_1+1)).\omega_{\lambda_p}.\overrightarrow{H}$

9.1.7 Type C

All admissible pairs are nice.

Table 9.7: Nice Pairs in Type $C_{\ell=2k-1}$

(σ,θ)-type	restricted basis	fundamental weights	Standard Correction	Non-Standard Correction
$C_\ell^{\ell,\ell}(I,I,\varepsilon_i), \ell \neq 4k+3$ $2 \leq q \leq \ell-1$ is even $2 \leq p \leq \ell$ is odd	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(2i\pi z_1).\omega_{\lambda_q}.\overrightarrow{H}$ $(4i\pi z_1).\omega_{\lambda_p}.\overrightarrow{H}$	$(i\pi(2z_1+1)).\omega_{\lambda_q}.\overrightarrow{H}$ $(2i\pi(2z_1+1)).\omega_{\lambda_p}.\overrightarrow{H}$
$C_\ell^{\ell,\ell}(I,I,\varepsilon_i), \ell = 4k+3$ $2 \leq q \leq \ell-1$ is even $2 \leq p \leq \ell$ is odd	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(2i\pi z_1).\omega_{\lambda_q}.\overrightarrow{H}$ $(4i\pi z_1).\omega_{\lambda_p}.\overrightarrow{H}$ $(i\pi(2z_1+1)).\omega_{\lambda_{\ell-1}}.\overrightarrow{H}$	$(i\pi(2z_1+1)).\omega_{\lambda_q}.\overrightarrow{H}$ $(2i\pi(2z_1+1)).\omega_{\lambda_p}.\overrightarrow{H}$ $\left(\frac{1}{2}i\pi(2z_1+1)\right).\omega_{\lambda_{\ell-1}}.\overrightarrow{H}$
$C_\ell^{\ell,p}(I,IIa,\varepsilon_i)$ $1 \leq p \leq (\ell-1)/2$ $1 \leq q \leq p-1$	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(2i\pi z_1).\omega_{\lambda_q}.\overrightarrow{H}$ $(4i\pi z_1).\omega_{\lambda_p}.\overrightarrow{H}$	$\{\}$
$C_\ell^{p,k}(IIa,IIa,\varepsilon_i)$ $1 \leq p \leq k < (\ell-1)/2$ $1 \leq q \leq p-1$	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(4i\pi z_1).\omega_{\lambda_q}.\overrightarrow{H}$ $(8i\pi z_1).\omega_{\lambda_p}.\overrightarrow{H}$	$(2i\pi(2z_1+1)).\omega_{\lambda_q}.\overrightarrow{H}$ $(4i\pi(2z_1+1)).\omega_{\lambda_p}.\overrightarrow{H}$
$C_\ell^{p,(\ell-1)/2}(IIa,IIa,\varepsilon_i)$ p is even $1 \leq q < p \leq (\ell-1)/2$	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(4i\pi z_1).\omega_{\lambda_q}.\overrightarrow{H}$ $(8i\pi z_1).\omega_{\lambda_p}.\overrightarrow{H}$	$(2i\pi(2z_1+1)).\omega_{\lambda_q}.\overrightarrow{H}$ $(4i\pi(2z_1+1)).\omega_{\lambda_p}.\overrightarrow{H}$
$C_\ell^{p,(\ell-1)/2}(IIa,IIa,\varepsilon_i)$ p is odd $1 \leq q < p \leq (\ell-1)/2$	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(4i\pi z_1).\omega_{\lambda_q}.\overrightarrow{H}$ $(4i\pi z_1).\omega_{\lambda_p}.\overrightarrow{H}$	$(2i\pi(2z_1+1)).\omega_{\lambda_q}.\overrightarrow{H}$

Table 9.8: Nice Pairs in Type $C_{\ell=2k}$

(σ,θ)-type	restricted basis	fundamental weights	Standard Correction	Non-Standard Correction
$C_{\ell}^{\ell,\ell}(I,I,\varepsilon_i)$ $2\leq q\leq \ell$ is even $2\leq p\leq \ell-1$ is odd	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(2i\pi z_1).\omega_{\lambda_q}.\overrightarrow{H}$ $(4i\pi z_1).\omega_{\lambda_p}.\overrightarrow{H}$	$(i\pi(2z_1+1)).\omega_{\lambda_q}.\overrightarrow{H}$ $\left(\frac{1}{2}i\pi(2z_1+1)\right).\omega_{\lambda_{\ell=4n}}.\overrightarrow{H}$ $(2i\pi(2z_1+1)).\omega_{\lambda_p}.\overrightarrow{H}$
$C_{\ell}^{\ell,\ell/2}(I,IIb,\varepsilon_i)$ $2\leq q\leq \ell/2$ is even $1\leq p\leq \ell/2-1$ is odd	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(2i\pi z_1).\omega_{\lambda_q}.\overrightarrow{H}$ $(4i\pi z_1).\omega_{\lambda_p}.\overrightarrow{H}$	$(i\pi(2z_1+1)).\omega_{\lambda_{\ell/2=4n}}.\overrightarrow{H}$
$C_{\ell}^{\ell/2,\ell/2}(IIb,IIb,\varepsilon_i)$ $2\leq q\leq \ell/2$ is even $1\leq p\leq \ell/2-1$ is odd	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(4i\pi z_1).\omega_{\lambda_p}.\overrightarrow{H}$ $(4i\pi z_1).\omega_{\lambda_q}.\overrightarrow{H}$	$(2i\pi(2z_1+1)).\omega_{\lambda_q}.\overrightarrow{H}$ $(i\pi(2z_1+1)).\omega_{\lambda_{\ell/2=4n}}.\overrightarrow{H}$

9.1.8 Type D

Nice Pairs in Type D_ℓ where ℓ is Even or Odd.

Table 9.9: Nice Pairs in Type D_ℓ where ℓ is even or odd

(σ,θ)-type	restricted basis	fundamental weights	Standard Correction	Non-Standard Correction
$D_\ell^{p,k}(Ia, Ia, \varepsilon_i)$ $p \leq k,$ both even or both odd $1 \leq q < p$	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(2i\pi z_1)\,.\omega_{\lambda_q}.\overrightarrow{H}$ $(4i\pi z_1)\,.\omega_{\lambda_p}.\overrightarrow{H}$	$(i\pi\,(2z_1+1))\,.\omega_{\lambda_q}.\overrightarrow{H}$ $(2i\pi\,(2z_1+1))\,.\omega_{\lambda_p}.\overrightarrow{H}$
$D_\ell^{p,\ell}(Ia, Ib, \varepsilon_i)$ $1 \leq p < \ell$ $1 \leq q < p$	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(2i\pi z_1)\,.\omega_{\lambda_q}.\overrightarrow{H}$ $(2i\pi z_1)\,.\omega_{\lambda_p}.\overrightarrow{H}$	$(i\pi\,(2z_1+1))\,.\omega_{\lambda_q}.\overrightarrow{H}$
$D_\ell^{\ell,\ell}(Ib, Ib, \varepsilon_i)$ $1 \leq q \leq p = \ell$	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(2i\pi z_1)\,.\omega_{\lambda_q}.\overrightarrow{H}$	$(i\pi\,(2z_1+1))\,.\omega_{\lambda_q}.\overrightarrow{H}$

Nice Pairs in Type $D_{\ell=2k}$

Table 9.10: Nice Pairs in Type D_{2k}

(σ,θ)-type	restricted basis	fundamental weights	Standard Correction	Non-Standard Correction
$D_{\ell}^{2p,\ell/2}(Ia, IIIa, \varepsilon_i)$ $1 \leq q < p$	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(2i\pi z_1)\,.\omega_{\lambda_q}.\overrightarrow{H}$ $(4i\pi z_1)\,.\omega_{\lambda_p}.\overrightarrow{H}$	$\{\}$
$D_{\ell}^{\ell,\ell/2}(Ib, IIIa, \varepsilon_i)$ $1 \leq q < p = \ell/2$	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(2i\pi z_1)\,.\omega_{\lambda_q}.\overrightarrow{H}$ $(2i\pi z_1)\,.\omega_{\lambda_p}.\overrightarrow{H}$	$(i\pi\,(2z_1+1))\,.\omega_{\lambda_p}.\overrightarrow{H}$
$D_{\ell}^{\ell/2,\ell/2}(IIIa, IIIa, \varepsilon_i)$ $1 \leq q < p = \ell/2$	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(4i\pi z_1)\,.\omega_{\lambda_q}.\overrightarrow{H}$ $(2i\pi z_1)\,.\omega_{\lambda_p}.\overrightarrow{H}$	$(2i\pi\,(2z_1+1))\,.\omega_{\lambda_q}.\overrightarrow{H}$ $(i\pi\,(2z_1+1))\,.\omega_{\lambda_p}.\overrightarrow{H}$

9.1.9 Type E

Rank 6: We only get nice pairs and feisty pairs.

Ranks 7 & 8: All admissible pairs are nice.

Nice Pairs in Type E_6

Table 9.11: Nice Pairs in Type E_6

(σ,θ)-type	restricted basis	fundamental weights	Standard Correction	Non-Standard Correction
	λ_1 $\{1,0,0,0,0,0\}$	ω_{λ_1} $\left\{\frac{4}{3},1,\frac{5}{3},2,\frac{4}{3},\frac{2}{3}\right\}$		
	λ_2 $\{0,1,0,0,0,0\}$	ω_{λ_2} $\{1,2,2,3,2,1\}$		
$E_6^{6,6}(I,I,\varepsilon_i)$	λ_3 $\{0,0,1,0,0,0\}$	ω_{λ_3} $\left\{\frac{5}{3},2,\frac{10}{3},4,\frac{8}{3},\frac{4}{3}\right\}$	$(2i\pi z_1)\,.\omega_{\lambda_q}.\vec{H}$	$(i\pi\,(2z_1+1))\,.\omega_{\lambda_q}.\vec{H}$
$1\leq q\leq p=6$	λ_4 $\{0,0,0,1,0,0\}$	ω_{λ_4} $\{2,3,4,6,4,2\}$		
	λ_5 $\{0,0,0,0,1,0\}$	ω_{λ_5} $\left\{\frac{4}{3},2,\frac{8}{3},4,\frac{10}{3},\frac{5}{3}\right\}$		
	λ_6 $\{0,0,0,0,0,1\}$	ω_{λ_6} $\left\{\frac{2}{3},1,\frac{4}{3},2,\frac{5}{3},\frac{4}{3}\right\}$		
	λ_1 $\left\{\frac{1}{2},0,0,0,0,\frac{1}{2}\right\}$	ω_{λ_1} $\left\{1,1,\frac{3}{2},2,\frac{3}{2},1\right\}$		
$E_6^{6,4}(I,II,\varepsilon_i)$	λ_2 $\{0,1,0,0,0,0\}$	ω_{λ_2} $\{1,2,2,3,2,1\}$	$(2i\pi z_1)\,.\omega_{\lambda_q}.\vec{H}$	$(i\pi\,(2z_1+1))\,.\omega_{\lambda_{q=2k}}.\vec{H}$
$1\leq q\leq p=4$	λ_3 $\left\{0,0,\frac{1}{2},0,\frac{1}{2},0\right\}$	ω_{λ_3} $\left\{\frac{3}{2},2,3,4,3,\frac{3}{2}\right\}$		
	λ_4 $\{0,0,0,1,0,0\}$	ω_{λ_4} $\{2,3,4,6,4,2\}$		
$E_6^{6,2}(I,IV,\varepsilon_i)$	λ_1 $\left\{1,\frac{1}{2},1,1,\frac{1}{2},0\right\}$	ω_{λ_1} $\left\{\frac{2}{3},\frac{1}{2},\frac{5}{6},1,\frac{2}{3},\frac{1}{3}\right\}$	$(2i\pi z_1)\,.\omega_{\lambda_q}.\vec{H}$	$\{\}$
$1\leq q\leq p=2$	λ_2 $\left\{0,\frac{1}{2},\frac{1}{2},1,1,1\right\}$	ω_{λ_2} $\left\{\frac{1}{3},\frac{1}{2},\frac{2}{3},1,\frac{5}{6},\frac{2}{3}\right\}$		
	λ_1 $\left\{\frac{1}{2},0,0,0,0,\frac{1}{2}\right\}$	ω_{λ_1} $\left\{1,1,\frac{3}{2},2,\frac{3}{2},1\right\}$		
$E_6^{4,4}(II,II,\varepsilon_i)$	λ_2 $\{0,1,0,0,0,0\}$	ω_{λ_2} $\{1,2,2,3,2,1\}$	$(2i\pi z_1)\,.\omega_{\lambda_{q=2k}}.\vec{H}$	$(i\pi\,(2z_1+1))\,.\omega_{\lambda_{q=2k}}.\vec{H}$
$1\leq q\leq p=4$	λ_3 $\left\{0,0,\frac{1}{2},0,\frac{1}{2},0\right\}$	ω_{λ_3} $\left\{\frac{3}{2},2,3,4,3,\frac{3}{2}\right\}$	$(4i\pi z_1)\,.\omega_{\lambda_{q=2k-1}}.\vec{H}$	$(2i\pi\,(2z_1+1))\,.\omega_{\lambda_{q=2k-1}}.\vec{H}$
	λ_4 $\{0,0,0,1,0,0\}$	ω_{λ_4} $\{2,3,4,6,4,2\}$		

Table 9.11: Continued

(σ,θ)-type	restricted basis		fundamental weights		Standard Correction	Non-Standard Correction
$E_6^{4,2}(II,III,\varepsilon_i)$	λ_1	$\left\{\frac{1}{2},0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\}$	ω_{λ_1}	$\left\{\frac{1}{2},\frac{1}{2},\frac{3}{4},1,\frac{3}{4},\frac{1}{2}\right\}$	$(4i\pi z_1).\omega_{\lambda_1}.\overrightarrow{H}$	$(2i\pi(2z_1+1)).\omega_{\lambda_2}.\overrightarrow{H}$
	λ_2	$\left\{0,1,\frac{1}{2},1,\frac{1}{2},0\right\}$	ω_{λ_2}	$\left\{\frac{1}{2},1,1,\frac{3}{2},1,\frac{1}{2}\right\}$	$(4i\pi z_1).\omega_{\lambda_2}.\overrightarrow{H}$	
$E_6^{2,2}(III,III,\varepsilon_i)$	λ_1	$\left\{\frac{1}{2},0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\}$	ω_{λ_1}	$\left\{\frac{1}{2},\frac{1}{2},\frac{3}{4},1,\frac{3}{4},\frac{1}{2}\right\}$	$(8i\pi z_1).\omega_{\lambda_1}.\overrightarrow{H}$	$(4i\pi(2z_1+1)).\omega_{\lambda_1}.\overrightarrow{H}$
	λ_2	$\left\{0,1,\frac{1}{2},1,\frac{1}{2},0\right\}$	ω_{λ_2}	$\left\{\frac{1}{2},1,1,\frac{3}{2},1,\frac{1}{2}\right\}$	$(4i\pi z_1).\omega_{\lambda_2}.\overrightarrow{H}$	$(2i\pi(2z_1+1)).\omega_{\lambda_2}.\overrightarrow{H}$
$E_6^{2,2}(III,IV,\varepsilon_i)$	λ_1	$\left\{\frac{1}{2},\frac{1}{2},\frac{3}{4},1,\frac{3}{4},\frac{1}{2}\right\}$	ω_{λ_1}	$\left\{\frac{1}{4},\frac{1}{4},\frac{3}{8},\frac{1}{2},\frac{3}{8},\frac{1}{4}\right\}$	$(8i\pi z_1).\omega_{\lambda_1}.\overrightarrow{H}$	$\{\}$
$E_6^{2,2}(IV,IV,\varepsilon_i)$	λ_1	$\left\{1,\frac{1}{2},1,1,\frac{1}{2},0\right\}$	ω_{λ_1}	$\left\{\frac{2}{3},\frac{1}{2},\frac{5}{6},1,\frac{2}{3},\frac{1}{3}\right\}$	$(4i\pi z_1).\omega_{\lambda_1}.\overrightarrow{H}$	$(2i\pi(2z_1+1)).\omega_{\lambda_1}.\overrightarrow{H}$
	λ_2	$\left\{0,\frac{1}{2},\frac{1}{2},1,1,1\right\}$	ω_{λ_2}	$\left\{\frac{1}{3},\frac{1}{2},\frac{2}{3},1,\frac{5}{6},\frac{2}{3}\right\}$	$(4i\pi z_1).\omega_{\lambda_2}.\overrightarrow{H}$	$(2i\pi(2z_1+1)).\omega_{\lambda_2}.\overrightarrow{H}$

Nice Pairs in Type E_7

Table 9.12: Nice Pairs in Type E_7

(σ,θ)-type	restricted basis	fundamental weights	Standard Correction	Non-Standard Correction
$E_7^{7,7}(V,V,\varepsilon_i)$	λ_1 $\{1,0,0,0,0,0,0\}$	ω_{λ_1} $\{2,2,3,4,3,2,1\}$	$(2i\pi z_1)\,.\omega_{\lambda_q}.\overrightarrow{H}$	$(i\pi\,(2z_1+1))\,.\omega_{\lambda_p}.\overrightarrow{H}$
	λ_2 $\{0,1,0,0,0,0,0\}$	ω_{λ_2} $\left\{2,\frac{7}{2},4,6,\frac{9}{2},3,\frac{3}{2}\right\}$		$(i\pi\,(2z_1+1))\,.\omega_{\lambda_q}.\overrightarrow{H}$
	λ_3 $\{0,0,1,0,0,0,0\}$	ω_{λ_3} $\{3,4,6,8,6,4,2\}$		
	λ_4 $\{0,0,0,1,0,0,0\}$	ω_{λ_4} $\{4,6,8,12,9,6,3\}$		
	λ_5 $\{0,0,0,0,1,0,0\}$	ω_{λ_5} $\left\{3,\frac{9}{2},6,9,\frac{15}{2},5,\frac{5}{2}\right\}$		
	λ_6 $\{0,0,0,0,0,1,0\}$	ω_{λ_6} $\{2,3,4,6,5,4,2\}$		
	λ_7 $\{0,0,0,0,0,0,1\}$	ω_{λ_7} $\left\{1,\frac{3}{2},2,3,\frac{5}{2},2,\frac{3}{2}\right\}$		
$E_7^{7,4}(V,VI,\varepsilon_i)$	λ_1 $\{1,0,0,0,0,0,0\}$	ω_{λ_1} $\{2,2,3,4,3,2,1\}$	$(2i\pi z_1)\,.\omega_{\lambda_q}.\overrightarrow{H}$	$(i\pi\,(2z_1+1))\,.\omega_{\lambda_1}.\overrightarrow{H}$
$1\leq q\leq p=4$	λ_2 $\{0,0,1,0,0,0,0\}$	ω_{λ_2} $\{3,4,6,8,6,4,2\}$		$(i\pi\,(2z_1+1))\,.\omega_{\lambda_2}.\overrightarrow{H}$
	λ_3 $\left\{0,\frac{1}{2},0,1,\frac{1}{2},0,0\right\}$	ω_{λ_3} $\left\{2,3,4,6,\frac{9}{2},3,\frac{3}{2}\right\}$		
	λ_4 $\left\{0,0,0,0,\frac{1}{2},1,\frac{1}{2}\right\}$	ω_{λ_4} $\left\{1,\frac{3}{2},2,3,\frac{5}{2},2,1\right\}$		
$E_7^{7,3}(V,VII,\varepsilon_i)$	λ_1 $\left\{1,\frac{1}{2},1,1,\frac{1}{2},0,0\right\}$	ω_{λ_1} $\left\{1,1,\frac{3}{2},2,\frac{3}{2},1,\frac{1}{2}\right\}$	$(2i\pi z_1)\,.\omega_{\lambda_q}.\overrightarrow{H}$	$(i\pi\,(2z_1+1))\,.\omega_{\lambda_3}.\overrightarrow{H}$
$1\leq q\leq p=3$	λ_2 $\left\{0,\frac{1}{2},\frac{1}{2},1,1,1,0\right\}$	ω_{λ_2} $\left\{1,\frac{3}{2},2,3,\frac{5}{2},2,1\right\}$		
	λ_3 $\{0,0,0,0,0,0,1\}$	ω_{λ_3} $\left\{1,\frac{3}{2},2,3,\frac{5}{2},2,\frac{3}{2}\right\}$		
$E_7^{4,4}(VI,VI,\varepsilon_i)$	λ_1 $\{1,0,0,0,0,0,0\}$	ω_{λ_1} $\{2,2,3,4,3,2,1\}$	$(2i\pi z_1)\,.\omega_{\lambda_1}.\overrightarrow{H}$	$(i\pi\,(2z_1+1))\,.\omega_{\lambda_1}.\overrightarrow{H}$
	λ_2 $\{0,0,1,0,0,0,0\}$	ω_{λ_2} $\{3,4,6,8,6,4,2\}$	$(2i\pi z_1)\,.\omega_{\lambda_2}.\overrightarrow{H}$	$(i\pi\,(2z_1+1))\,.\omega_{\lambda_2}.\overrightarrow{H}$
	λ_3 $\left\{0,\frac{1}{2},0,1,\frac{1}{2},0,0\right\}$	ω_{λ_3} $\left\{2,3,4,6,\frac{9}{2},3,\frac{3}{2}\right\}$	$(4i\pi z_1)\,.\omega_{\lambda_3}.\overrightarrow{H}$	$(2i\pi\,(2z_1+1))\,.\omega_{\lambda_3}.\overrightarrow{H}$
	λ_4 $\left\{0,0,0,0,\frac{1}{2},1,\frac{1}{2}\right\}$	ω_{λ_4} $\left\{1,\frac{3}{2},2,3,\frac{5}{2},2,1\right\}$	$(4i\pi z_1)\,.\omega_{\lambda_4}.\overrightarrow{H}$	$(2i\pi\,(2z_1+1))\,.\omega_{\lambda_4}.\overrightarrow{H}$
$E_7^{4,3}(VI,VII,\varepsilon_i)$	λ_1 $\left\{1,\frac{1}{2},1,1,\frac{1}{2},0,0\right\}$	ω_{λ_1} $\left\{1,1,\frac{3}{2},2,\frac{3}{2},1,\frac{1}{2}\right\}$	$(4i\pi z_1)\,.\omega_{\lambda_1}.\overrightarrow{H}$	$(2i\pi\,(2z_1+1))\,.\omega_{\lambda_1}.\overrightarrow{H}$
	λ_2 $\left\{0,\frac{1}{2},\frac{1}{2},1,1,1,\frac{1}{2}\right\}$	ω_{λ_2} $\left\{\frac{1}{2},\frac{3}{4},1,\frac{3}{2},\frac{5}{4},1,\frac{1}{2}\right\}$	$(4i\pi z_1)\,.\omega_{\lambda_2}.\overrightarrow{H}$	

Table 9.12: Continued

(σ,θ)-type	restricted basis		fundamental weights		Standard Correction	Non-Standard Correction
$E_7^{3,3}(VII,VII,\varepsilon_i)$	λ_1	$\left\{1,\frac{1}{2},1,1,\frac{1}{2},0,0\right\}$	ω_{λ_1}	$\left\{1,1,\frac{3}{2},2,\frac{3}{2},1,\frac{1}{2}\right\}$	$(4i\pi z_1)\,.\omega_{\lambda_1}.\overrightarrow{H}$	$(2i\pi\,(2z_1+1))\,.\omega_{\lambda_1}.\overrightarrow{H}$
	λ_2	$\left\{0,\frac{1}{2},\frac{1}{2},1,1,1,0\right\}$	ω_{λ_2}	$\left\{1,\frac{3}{2},2,3,\frac{5}{2},2,1\right\}$	$(4i\pi z_1)\,.\omega_{\lambda_2}.\overrightarrow{H}$	$(2i\pi\,(2z_1+1))\,.\omega_{\lambda_2}.\overrightarrow{H}$
	λ_3	$\{0,0,0,0,0,0,1\}$	ω_{λ_3}	$\left\{1,\frac{3}{2},2,3,\frac{5}{2},2,\frac{3}{2}\right\}$	$(2i\pi z_1)\,.\omega_{\lambda_3}.\overrightarrow{H}$	$(i\pi\,(2z_1+1))\,.\omega_{\lambda_3}.\overrightarrow{H}$

Nice Pairs in Type E_8

Table 9.13: Nice Pairs in Type E_8

(σ,θ)-type	res basis	fund weights	Standard	Non-Standard
$E_8^{8,8}(VIII, VIII, \varepsilon_i)$	(λ_i)	(ω_{λ_i})	$(2i\pi z_1)\,.\omega_{\lambda_q}.\overrightarrow{H}$	$(i\pi\,(2z_1+1))\,.\omega_{\lambda_p}.\overrightarrow{H}$ $(i\pi\,(2z_1+1))\,.\omega_{\lambda_q}.\overrightarrow{H}$

$(\sigma,\,\theta)$-type	restricted basis		fundamental weights	
$E_8^{8,8}(VIII, VIII, \varepsilon_i)$	λ_1	$\{1,0,0,0,0,0,0,0\}$	ω_{λ_1}	$\{4,5,7,10,8,6,4,2\}$
	λ_2	$\{0,1,0,0,0,0,0,0\}$	ω_{λ_2}	$\{5,8,10,15,12,9,6,3\}$
	λ_3	$\{0,0,1,0,0,0,0,0\}$	ω_{λ_3}	$\{7,10,14,20,16,12,8,4\}$
	λ_4	$\{0,0,0,1,0,0,0,0\}$	ω_{λ_4}	$\{10,15,20,30,24,18,12,6\}$
	λ_5	$\{0,0,0,0,1,0,0,0\}$	ω_{λ_5}	$\{8,12,16,24,20,15,10,5\}$
	λ_6	$\{0,0,0,0,0,1,0,0\}$	ω_{λ_6}	$\{6,9,12,18,15,12,8,4\}$
	λ_7	$\{0,0,0,0,0,0,1,0\}$	ω_{λ_7}	$\{4,6,8,12,10,8,6,3\}$
	λ_8	$\{0,0,0,0,0,0,0,1\}$	ω_{λ_8}	$\{2,3,4,6,5,4,3,2\}$

Table 9.14: Continued

(σ,θ)-type	res basis	fund weights	Standard	Non-Standard
$E_8^{8,4}(VIII,IX,\varepsilon_i)$	λ_1 $\left\{1,\frac{1}{2},1,1,\frac{1}{2},0,0,0\right\}$	ω_{λ_1} $\left\{2,\frac{5}{2},\frac{7}{2},5,4,3,2,1\right\}$	$(2i\pi z_1).\omega_{\lambda_1}.\overrightarrow{H}$	
	λ_2 $\left\{0,\frac{1}{2},\frac{1}{2},1,1,1,0,0\right\}$	ω_{λ_2} $\left\{3,\frac{9}{2},6,9,\frac{15}{2},6,4,2\right\}$	$(2i\pi z_1).\omega_{\lambda_2}.\overrightarrow{H}$	$(i\pi(2z_1+1)).\omega_{\lambda_4}.\overrightarrow{H}$
	λ_3 $\{0,0,0,0,0,0,1,0\}$	ω_{λ_3} $\{4,6,8,12,10,8,6,3\}$	$(2i\pi z_1).\omega_{\lambda_3}.\overrightarrow{H}$	$(i\pi(2z_1+1)).\omega_{\lambda_3}.\overrightarrow{H}$
	λ_4 $\{0,0,0,0,0,0,0,1\}$	ω_{λ_4} $\{2,3,4,6,5,4,3,2\}$	$(2i\pi z_1).\omega_{\lambda_4}.\overrightarrow{H}$	
$E_8^{4,4}(IX,IX,\varepsilon_i)$	λ_1 $\left\{1,\frac{1}{2},1,1,\frac{1}{2},0,0,0\right\}$	ω_{λ_1} $\left\{2,\frac{5}{2},\frac{7}{2},5,4,3,2,1\right\}$	$(4i\pi z_1).\omega_{\lambda_1}.\overrightarrow{H}$	$(2i\pi(2z_1+1)).\omega_{\lambda_1}.\overrightarrow{H}$
	λ_2 $\left\{0,\frac{1}{2},\frac{1}{2},1,1,1,0,0\right\}$	ω_{λ_2} $\left\{3,\frac{9}{2},6,9,\frac{15}{2},6,4,2\right\}$	$(4i\pi z_1).\omega_{\lambda_2}.\overrightarrow{H}$	$(2i\pi(2z_1+1)).\omega_{\lambda_2}.\overrightarrow{H}$
	λ_3 $\{0,0,0,0,0,0,1,0\}$	ω_{λ_3} $\{4,6,8,12,10,8,6,3\}$	$(2i\pi z_1).\omega_{\lambda_4}.\overrightarrow{H}$	$(i\pi(2z_1+1)).\omega_{\lambda_4}.\overrightarrow{H}$
	λ_4 $\{0,0,0,0,0,0,0,1\}$	ω_{λ_4} $\{2,3,4,6,5,4,3,2\}$	$(2i\pi z_1).\omega_{\lambda_3}.\overrightarrow{H}$	$(i\pi(2z_1+1)).\omega_{\lambda_3}.\overrightarrow{H}$

9.1.10 Types F & G

All admissible pairs are nice. See Tables 9.15 and 9.17 below.

Nice Pairs in Type F_4

Table 9.15: Nice Pairs in Type F_4

(σ,θ)-type	res. basis	fund. weights
$F_4^{4,4}(I, I, \varepsilon_i)$	λ_1 $\{1,0,0,0\}$	ω_{λ_1} $\{2,3,4,2\}$
	λ_2 $\{0,1,0,0\}$	ω_{λ_2} $\{3,6,8,4\}$
	λ_3 $\{0,0,1,0\}$	ω_{λ_3} $\{2,4,6,3\}$
	λ_4 $\{0,0,0,1\}$	ω_{λ_4} $\{1,2,3,2\}$
$F_4^{4,1}(I, II, \varepsilon_i)$	λ_1 $\left\{\frac{1}{2},1,\frac{3}{2},1\right\}$	ω_{λ_1} $\left\{\frac{1}{4},\frac{1}{2},\frac{3}{4},\frac{1}{2}\right\}$
$F_4^{1,1}(II, II, \varepsilon_i)$	λ_1 $\left\{\frac{1}{2},1,\frac{3}{2},1\right\}$	ω_{λ_1} $\left\{\frac{1}{4},\frac{1}{2},\frac{3}{4},\frac{1}{2}\right\}$

(σ,θ)-type	Standard	Non-Standard
$F_4^{4,4}(I, I, \varepsilon_i)$ $1 \le q \le 4$	$(2i\pi z_1).\omega_{\lambda_q}.\overrightarrow{H}$	$(i\pi(2z_1+1)).\omega_{\lambda_q}.\overrightarrow{H}$
$F_4^{4,1}(I, II, \varepsilon_i)$	$(4i\pi z_1).\omega_{\lambda_1}.\overrightarrow{H}$	$\{\}$
$F_4^{1,1}(II, II, \varepsilon_i)$	$(4i\pi z_1).\omega_{\lambda_1}.\overrightarrow{H}$	$\{\}$

Nice Pairs in Type G

All admissible pairs are nice. (The one and only.)

Table 9.17: Nice Pairs in Type G_2

(σ,θ)-type	res. basis	fund. weights	Standard	Non-Standard
$G_2^{2,2}(\varepsilon_i)$	λ_1 $\{1,0\}$	ω_{λ_1} $\{2,1\}$	$(2i\pi z_1).\omega_{\lambda_1}.\overrightarrow{H}$	$(i\pi(2z_1+1)).\omega_{\lambda_1}.\overrightarrow{H}$
	λ_2 $\{0,1\}$	ω_{λ_2} $\{3,2\}$	$(2i\pi z_1).\omega_{\lambda_2}.\overrightarrow{H}$	$(i\pi(2z_1+1)).\omega_{\lambda_2}.\overrightarrow{H}$

9.2 Better-than-Okay: σ_Δ and $\theta_\Delta \operatorname{Int}(h)$ commute and are a standard pair.

9.2.1 Type D

These would be nice except that θ_Δ for θ of type $D_\ell^{(\ell-1)/2} IIIb$ must be corrected to an involution:

$$\theta_\Delta \operatorname{Int}\left(\exp\left(\frac{-i\pi}{2} \sum_{\alpha_j \in \Delta_0(\theta)} (4z_j + 1)H_j\right)\right).$$

Better-than-Okay Pairs in Type $D_{\ell=2k-1}$

Table 9.18: Better-than-Okay Pairs in Type D_{2k-1}

(σ,θ)-type	restricted basis	fundamental weights	Standard Correction	Non-Standard Correction
$D_\ell^{2p,(\ell-1)/2}(Ia, IIIb, \varepsilon_i)$ $1 \leq q < p$	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(2i\pi z_1)\,.\omega_{\lambda_q}.\vec{H}$ $(4i\pi z_1)\,.\omega_{\lambda_p}.\vec{H}$	$\{\}$
$D_\ell^{\ell,(\ell-1)/2}(Ib, IIIb, \varepsilon_i)$ $1 \leq q < p = (\ell-1)/2$	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(2i\pi z_1)\,.\omega_{\lambda_q}.\vec{H}$ $(4i\pi z_1)\,.\omega_{\lambda_p}.\vec{H}$	$\{\}$
$D_\ell^{(\ell-1)/2,(\ell-1)/2}(IIIb, IIIb, \varepsilon_i)$ $1 \leq q < p = (\ell-1)/2$	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(4i\pi z_1)\,.\omega_{\lambda_q}.\vec{H}$ $(8i\pi z_1)\,.\omega_{\lambda_p}.\vec{H}$	$(2i\pi\,(2z_1+1))\,.\omega_{\lambda_q}.\vec{H}$ $(4i\pi\,(2z_1+1))\,.\omega_{\lambda_p}.\vec{H}$

9.3 Decent Pairs : σ_Δ and θ_Δ commute but $(\sigma_\Delta, \theta_\Delta)$ is not standard.

9.3.1 Decent Pairs in Type A

Odd Rank $\ell = 3 + 4k$: $A_{\ell=3+4k}^{(\ell-1)/2,(\ell+1)/2}(\mathrm{II}, \mathrm{IIIb}, \varepsilon_i)$

Decent Pairs in Type $A_{\ell=3+4k}$

In the following table, for $1 \le q < p$

restricted basis	fundamental weights
$\lambda_q = \frac{1}{4}\sum_{j=2q-1}^{2q}(\alpha_j + \alpha_{j+1} + \alpha_{\ell-j} + \alpha_{\ell-j+1})$	$\omega_{\lambda_q} = \sum_{j=1}^{2q}\left(\frac{1}{4}\sum_{i=j}^{\ell-j+1}\alpha_i\right)$
$\lambda_p = \frac{1}{2}\sum_{j=2p-1}^{2p}(\sum_{i=j}^{\ell-j+1}\alpha_i)$	$\omega_{\lambda_p} = \frac{1}{2}\sum_{j=1}^{2p}\left(\frac{1}{2}\sum_{i=j}^{\ell-j+1}\alpha_i\right)$

Table 9.19: Decent Pairs in Type A_{3+4k}

(σ,θ)-type	restricted basis	fundamental weights	Standard	Non-Standard
$A_\ell^{(\ell-1)/2,(\ell+1)/2}(II, IIIb, \varepsilon_i)$ $\ell = 3 + 4k$ $p = (\ell+1)/4$ $1 \le q < p$	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(2i\pi\,(2z_1+1))\,.\omega_{\lambda_p}.\overrightarrow{H}$	$(4i\pi z_1)\,.\omega_{\lambda_q}.\overrightarrow{H}$ $(4i\pi z_1)\,.\omega_{\lambda_p}.\overrightarrow{H}$

9.3.2 Decent Pairs in Type B

Decent Pairs in Type B_ℓ

In the following table $z_1 \in \mathbb{Z}$ as usual and we use $x_1 \in \mathbb{Z}$ to alert the reader to a special case.

Table 9.20: Decent Pairs in Type B_ℓ

(σ,θ)-type	restricted basis	fundamental weights	Standard Correction	Non-Standard Correction
$B^{p,n}_{\ell=2k-1}(I,I,\varepsilon_i)$ $1 \leq p < n \neq \ell - 1$ $1 \leq q < p$ p odd and n even, or p even and n odd	$\lambda_q{=}\alpha_q$ $\lambda_p{=}\sum_p^\ell \alpha_i$	$\omega_{\lambda_q}{=}\sum_{j=1}^q(\sum_{i=j}^\ell \alpha_i)$ $\omega_{\lambda_p}{=}\frac{1}{2}\sum_{j=1}^p(\sum_{i=j}^\ell \alpha_i)$	$(2i\pi\,(2z_1+1))\,.\omega_{\lambda_p}.\overrightarrow{H}$	$(i\pi z_1)\,.\omega_{\lambda_q}.\overrightarrow{H}$ $(4i\pi z_1)\,.\omega_{\lambda_p}.\overrightarrow{H}$
$B^{p,\ell-1}_{\ell=2k-1}(I,I,\varepsilon_i)$ $1 \leq p < \ell - 1,\ p$ odd $1 \leq q < p$	$(\lambda_i)_1^p$ as above	$(\omega_{\lambda_i})_1^p$ as above	$\left(\frac{1}{2}i\pi\,(2x_1+1)\right).\omega_{\alpha_\ell}.\overrightarrow{H}$	$(i\pi z_1)\,.\omega_{\lambda_q}.\overrightarrow{H}$ $(2i\pi z_1)\,.\omega_{\lambda_p}.\overrightarrow{H}$
$B^{p,n}_{\ell=2k}(I,I,\varepsilon_i)$ $1 \leq p < n \leq \ell$ $1 \leq q < p$ p odd and n even, or p even and n odd	$(\lambda_i)_1^p$ as above	$(\omega_{\lambda_i})_1^p$ as above	$(2i\pi\,(2z_1+1))\,.\omega_{\lambda_p}.\overrightarrow{H}$	$(i\pi z_1)\,.\omega_{\lambda_q}.\overrightarrow{H}$ $(4i\pi z_1)\,.\omega_{\lambda_p}.\overrightarrow{H}$

9.3.3 Type D

Decent Pairs in Type D_ℓ

In the following table, for $1 \leq q < p$

restricted basis	fundamental weights
$\lambda_q{=}\alpha_q$ $\lambda_p{=}\sum_p^{\ell-2} \alpha_i + \frac{1}{2}(\alpha_{\ell-1} + \alpha_\ell)$	$\omega_{\lambda_q}{=}\sum_{j=1}^{q} \left(\sum_{i=j}^{\ell-2} \alpha_i + \frac{1}{2}(\alpha_{\ell-1} + \alpha_\ell)\right)$ $\omega_{\lambda_p}{=}\frac{1}{2} \sum_{j=1}^{p} \left(\sum_{i=j}^{\ell-2} \alpha_i + \frac{1}{2}(\alpha_{\ell-1} + \alpha_\ell)\right)$

Table 9.21: Decent Pairs in Type D_ℓ

(σ,θ)-type	restricted basis	fundamental weights	Standard Correction	Non-Standard Correction
$D_\ell^{p,n}(Ia, Ia, \varepsilon_i)$ $1 \leq p < n \leq \ell - 1$ $1 \leq q < p$ p odd and n even, or p even and n odd	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(2i\pi\,(2z_1 + 1))\,.\omega_{\lambda_p}.\overrightarrow{H}$	$(i\pi z_1)\,.\omega_{\lambda_q}.\overrightarrow{H}$ $(4i\pi z_1)\,.\omega_{\lambda_p}.\overrightarrow{H}$

9.4 Feisty Pairs: σ_Δ and θ_Δ do not commute.

9.4.1 Type A

Odd Rank: $A_\ell^p IIIa$ with AI or AII (if admissible).

Remark 9.4.2. Any involution σ of type AI or AII has lifting constants $c_\alpha^{\sigma_\Delta} = (-1)^{\mathrm{ht}\,\alpha - 1}$ for any $\alpha \in \Phi$ and θ of type $AIIIa$ has lifting constants $c_\alpha^{\theta_\Delta} = 1$ for all $\alpha \in \Phi$.

Example 9.4.3 (Admissible $A_\ell^\ell I$ with $A_\ell^p IIIa$)**.** Note the connected components in $\Delta_0(\sigma, \theta)$ are contained in either $\Delta_0(\sigma)$ or $\Delta_0(\theta)$ (or in both):

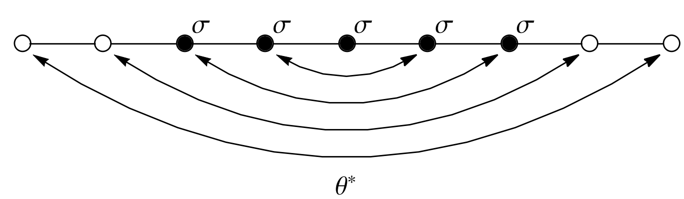


Figure 9.1: The (σ, θ)-index of type $A_9^{9,2}(I, IIIa, \varepsilon_i)$

Example 9.4.4 (Admissible $A_\ell^{(\ell-1)/2} II$ with $A_\ell^p IIIa$)**.** Note the connected components in $\Delta_0(\sigma, \theta)$ are contained in either $\Delta_0(\sigma)$ or $\Delta_0(\theta)$ (or in both):

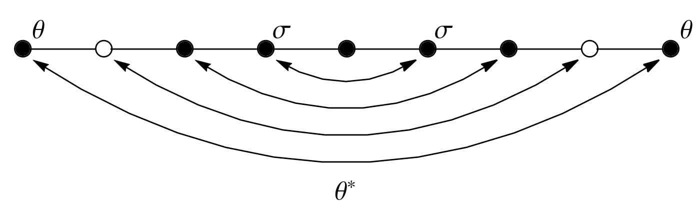


Figure 9.2: The (σ, θ)-index of type $A_9^{4,2}(II, IIIa, \varepsilon_i)$

In rank $\ell = 7, 11, \ldots, 3 + 4k, k \geq 0$, the (σ, θ)-pair of type $A_\ell^{(\ell-1)/2, 2m-1}(\mathrm{II}, \mathrm{IIIa}, \varepsilon_i)$ is not admissible because the connected component in the center of the diagram that is in $\Delta_0(\sigma, \theta)$ is contained in neither $\Delta_0(\sigma)$ nor $\Delta_0(\theta)$.

Non-Example 9.4.5 (Non-admissible $A_\ell^{2m-1}IIIa$ with AII)**.** Note that when $p = 2m-1$ is odd the connected component in the center of the diagram which is in $\Delta_0(\sigma,\theta)$ is contained in neither $\Delta_0(\sigma)$ nor $\Delta_0(\theta)$:

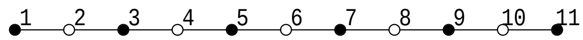


Figure 9.3: σ-index

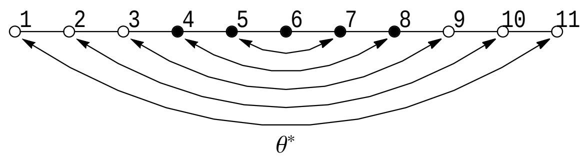


Figure 9.4: θ-index

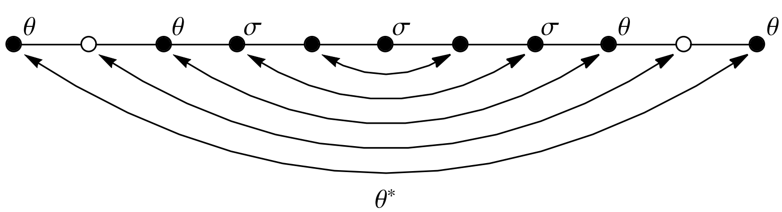


Figure 9.5: The (σ,θ)-index of type $A_{11}^{5,3}(II, IIIa, \varepsilon_i)$

Feisty Pairs in Type $A_{\ell=2k-1}$

In the following table, for $1 \leq q < p$

(σ,θ)-type	restricted basis	fundamental weights
$A_\ell^{\ell,p}(I, IIIa, \varepsilon_i)$	$\lambda_q = \frac{1}{2}\left(\alpha_q + \alpha_{\ell-q+1}\right)$ $\lambda_p = \frac{1}{2}\sum_p^{\ell-p+1} \alpha_i$	$\omega_{\lambda_q} = \sum_{j=1}^{q}\left(\frac{1}{2}\sum_{i=j}^{\ell-j+1} \alpha_i\right)$ $\omega_{\lambda_p} = \frac{1}{2}\sum_{j=1}^{p}\left(\frac{1}{2}\sum_{i=j}^{\ell-j+1} \alpha_i\right)$
$A_\ell^{(\ell-1)/2,2p}(II, IIIa, \varepsilon_i)$	$\lambda_q = \frac{1}{4}\sum_{j=2q-1}^{2q}\left(\alpha_j + \alpha_{j+1} + \alpha_{\ell-j} + \alpha_{\ell-j+1}\right)$ $\lambda_p = \frac{1}{4}\sum_{j=2p-1}^{2p}(\sum_{i=j}^{\ell-j+1} \alpha_i)$	$\omega_{\lambda_q} = \sum_{j=1}^{2q}\left(\frac{1}{4}\sum_{i=j}^{\ell-j+1} \alpha_i\right)$ $\omega_{\lambda_p} = \frac{1}{2}\sum_{j=1}^{2p}\left(\frac{1}{4}\sum_{i=j}^{\ell-j+1} \alpha_i\right)$

Table 9.22: Feisty Pairs in Type $A_{\ell=2k-1}$

(σ,θ)-type	restricted basis	fundamental weights	Standard Correction	Non-Standard Correction
$A_\ell^{\ell,p}(I, IIIa, \varepsilon_i)$	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(2i\pi\,(2z_1+1))\,.\omega_{\lambda_p}.\overrightarrow{H}$	$\{\}$
$A_\ell^{(\ell-1)/2,2p}(II, IIIa, \varepsilon_i)$	$(\lambda_i)_1^p$	$(\omega_{\lambda_i})_1^p$	$(4i\pi\,(2z_1+1))\,.\omega_{\lambda_p}.\overrightarrow{H}$	$\{\}$

9.4.6 Type E

Fesity Pair $E_6^{6,2}(I, III, \varepsilon_i)$

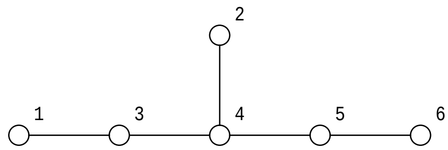


Figure 9.6: σ of type $E_6^6 I$

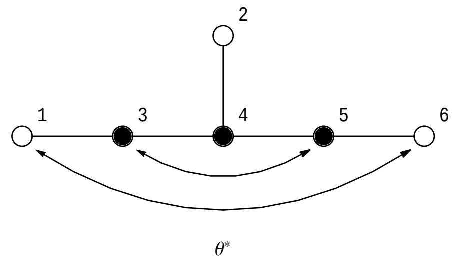


Figure 9.7: θ of type $E_6^2 III$

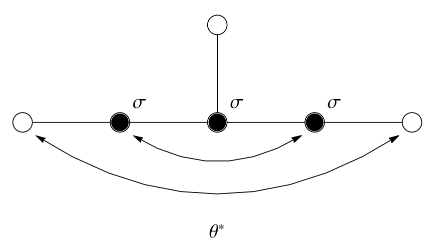


Figure 9.8: $(\sigma, \theta\,\mathrm{Int}(\varepsilon_i))$ of type $E_6^{6,2}(I, III, \varepsilon_i)$

Feisty Pair $E_6^{4,2}(II, IV, \varepsilon_i)$

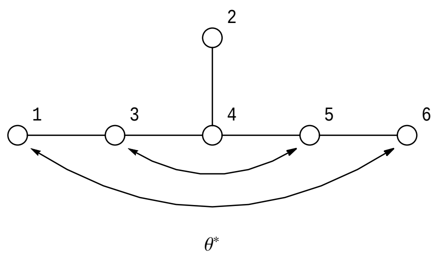


Figure 9.9: σ of type $E_6^4 II$

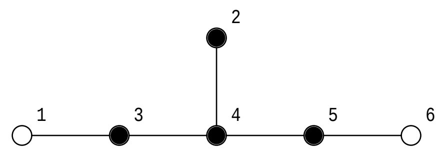


Figure 9.10: θ of type $E_6^2 IV$

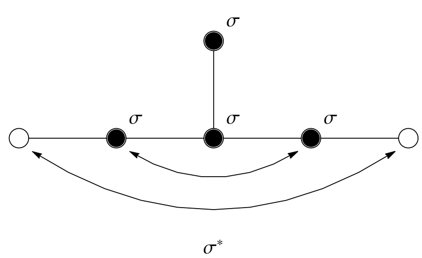


Figure 9.11: $(\sigma, \theta\,\mathrm{Int}(\varepsilon_i))$ of type $E_6^{4,2}(II, IV, \varepsilon_i)$

9.4.7 Feisty Pairs in Type E_6

Table 9.23: Feisty Pairs in Type E_6

(σ,θ)-type	restricted basis	fundamental weights	Standard Correction	Non-Standard Correction
$E_6^{6,2}(I,III,\varepsilon_i)$	λ_1 $\left\{\frac{1}{2},0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\}$ λ_2 $\left\{0,1,\frac{1}{2},1,\frac{1}{2},0\right\}$	ω_{λ_1} $\left\{\frac{1}{2},\frac{1}{2},\frac{3}{4},1,\frac{3}{4},\frac{1}{2}\right\}$ ω_{λ_2} $\left\{\frac{1}{2},1,1,\frac{3}{2},1,\frac{1}{2}\right\}$	$(2i\pi\,(2z_1+1))\,.\omega_{\lambda_1}.\vec{H}$	$\{\}$
$E_6^{4,2}(II,IV,\varepsilon_i)$	λ_1 $\left\{\frac{1}{2},\frac{1}{2},\frac{3}{4},1,\frac{3}{4},\frac{1}{2}\right\}$	ω_{λ_1} $\left\{\frac{1}{4},\frac{1}{4},\frac{3}{8},\frac{1}{2},\frac{3}{8},\frac{1}{4}\right\}$	$(4i\pi\,(2z_1+1))\,.\omega_{\lambda_1}.\vec{H}$	$\{\}$

References

[Bourbaki(1981)] N. Bourbaki. *Groupes et algébres de Lie, Éléments de Mathématique*, volume Chapitres 4, 5 et 6 of *Éléments de Mathématique*. Masson, Paris, 1981.

[Daniel and Helminck(2008)] Jennifer R. Daniel and Aloysius G. Helminck. Algorithms for computations in local symmetric spaces. *Comm. Algebra*, 36(5):1758–1788, 2008.

[Helgason(1978)] Sigurdur Helgason. *Differential geometry, Lie groups and symmetric spaces*, volume XII of *Pure and Applied mathematics*. Academic Press, New York, 1978.

[Helminck(2000)] A. G. Helminck. On the classification of k-involutions I. *Adv. Math.*, 153(1): 1–117, 2000.

[Helminck and Wang(1993)] A. G. Helminck and S. P. Wang. On rationality properties of involutions of reductive groups. *Adv. in Math.*, 99:26–96, 1993.

[Helminck(1988)] Aloysius G. Helminck. Algebraic groups with a commuting pair of involutions and semisimple symmetric spaces. *Advances in Mathematics*, 71(1):21–91, 1988.

[Klein(2009)] Sebastian Klein. Reconstructing the geometric structure of a Riemannian symmetric space from its Satake diagram. *Geometriae Dedicata*, 138(1):25–50, October 2009.

[Knapp(2005)] A.W. Knapp. *Lie Groups Beyond an Introduction.* Progress in Mathematics. Birkhäuser Boston, 2 edition, 2005.

[Satake(1971)] I. Satake. Classification theory of semisimple algebraic groups. In *Lecture Notes in Pure and Appl. Math.* Dekker, Berlin, 1971.

[Steinberg(1968)] R. Steinberg. *Endomorphisms of Linear Algebraic Groups: Number 80.* Memoirs of the American Mathematical Society. American Mathematical Society, 1968.

[Tits(1966)] J. Tits. Classification of algebraic semisimple groups. In A. Borel and G.D. Mostow, editors, *Algebraic Groups and Discontinuous Subgroups*, volume 9, pages 33–62, Providence, Rhode Island, 1966. American Mathematical Society, American Mathematical Society.

[Watson(2010)] Robert Loyd Watson. *Lifting Automorphisms from Root Systems to Lie Algebras.* PhD thesis, North Carolina State University, April 30 2010. URL `http://www.lib.ncsu.edu/resolver/1840.16/6181`.

Appendix

Appendix A: Useful Reference Tables

Table A.1: Notation

Symbol	:	Definition
G	:	a semisimple (usually simple) Lie group
T	:	a fixed maximal torus (i.e. maximal connected diagonalizable subgroup) of G
$\mathfrak{g}$	:	the simple Lie algebra of G
$\mathfrak{t}$	:	a maximal toral subalgebra of $\mathfrak{g}$
$\mathfrak{a}$	:	a toral subalgebra $\subset \mathfrak{t}$
Φ	:	$\Phi(T)$ the root system with respect to T
X	:	$\mathbb{Z}\Phi$ - an integer lattice of Φ, may be a root or weight lattice, ($X^*(T)$ characters)
W	:	$W(T)$ Weyl group with respect to T
Ψ	:	$(X^*(T)\ ,\ \Phi(T)\ ,\ X_*(T)\ ,\ \Phi^\vee(T))$, the root datum corresponding to T
$N_{\alpha,\beta}$	:	Chevalley constants
$c_\alpha^{\sigma_\Delta}, c_\alpha^{\theta_\Delta}$	:	lifting constants of σ_Δ and θ_Δ respectively
$c_\alpha^{\sigma\theta}$	:	lifting constants of the involution lifted from $\sigma\theta = \theta\sigma$
$\langle \alpha_i, \alpha_j \rangle$	:	The product defined by the Killing form on $\mathfrak{g}$ that relates the root system and its dual.
T_σ^-, T_σ^+	:	the σ-split and σ-stable tori respectively, likewise for $T_\theta^\pm$
$T_{(\sigma,\theta)}^-, T_{(\sigma,\theta)}^+$	:	the (σ,θ)-split and (σ,θ)-stable tori respectively
$T_{\sigma\theta}^-, T_{\sigma\theta}^+$	:	the $\sigma\theta$-split and $\sigma\theta$-stable tori respectively
$\mathfrak{g}(A,\lambda)$	:	For $\lambda \in \Phi(A)$ let $\mathfrak{g}(A,\lambda) = \{X \in \mathfrak{g} \mid [H,X] = \lambda(H)X\ \forall H \in \mathfrak{a}\}$ be the corresponding root space. Since $\sigma\theta(\lambda) = \lambda$, we have $\sigma\theta(\mathfrak{g}(A,\lambda)) = \mathfrak{g}(A,\lambda)$.
$\mathfrak{g}(A,\lambda)_{\sigma\theta}^\pm$	:	$\{X \in \mathfrak{g}(A,\lambda) \mid \sigma\theta(X) = \pm X\}$
$m^\pm(\lambda,\sigma\theta)$	:	$\dim_{\Bbbk} \mathfrak{g}(A,\lambda)_{\sigma\theta}^\pm$
$\Phi(T,\lambda)$	:	$\{\alpha \in \Phi(T) \mid \alpha\|A = \lambda\}$
$m_{\text{split}}(\lambda,\sigma\theta)$	:	$\|\Phi_{\text{split}}(\lambda,\sigma\theta)\|$ such that $\Phi_{\text{split}}(\lambda,\sigma\theta) = \{\alpha \in \Phi(T,\lambda) \mid \sigma\theta(\alpha) \neq \alpha\}$
$m_{\text{check}}(\lambda,\sigma\theta)$	:	$\|\Phi_{\text{check}}(\lambda,\sigma\theta)\|$ such that $\Phi_{\text{check}}(\lambda,\sigma\theta) = \{\alpha \in \Phi(T,\lambda) \mid \sigma\theta(\alpha) = \alpha\}$
$m(\lambda)$	:	$\begin{aligned} \dim_{\Bbbk} \mathfrak{g}(A,\lambda) &= m^+(\lambda,\sigma\theta) + m^-(\lambda,\sigma\theta) = \lvert\Phi(T,\lambda)\rvert \\ &= m_{\text{split}}(\lambda,\sigma\theta) + m_{\text{check}}(\lambda,\sigma\theta) \end{aligned}$

Table A.2: Properties of Lie Maps

$\Psi : G \to \mathrm{Aut}(G)$	$\Psi_g : G \to G$
Lie group homomorphism: $\Psi_{gh} = \Psi_g \Psi_h$	Lie group automorphism: $\Psi_g(ab) = \Psi_g(a)\Psi_g(b)$ $(\Psi_g)^{-1} = \Psi_{g^{-1}}$
$\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$	$\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$
Lie group homomorphism: $\mathrm{Ad}\, gh = \mathrm{Ad}\, g\, \mathrm{Ad}\, h$	Lie group automorphism: Ad_g is linear $(\mathrm{Ad}_g)^{-1} = \mathrm{Ad}_{g^{-1}}$ $\mathrm{Ad}_g\, [x\ ,\ y] = [\mathrm{Ad}_g\, x\ ,\ \mathrm{Ad}_g\, y]$
$\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$	$\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$
Lie algebra homomorphism: ad is linear ad_x is linear	Lie algebra derivation: $\mathrm{ad}_{[x\ ,\ y]} = [\mathrm{ad}_x\ ,\ \mathrm{ad}_y]$ $\mathrm{ad}_x\, [y\ ,\ z] = [\mathrm{ad}_x\, y\ ,\ z] + [y\ ,\ \mathrm{ad}_x\, z]$

Table A.3: Constants Formulas and Relations

Symbol	=	Relation	Condition
$N_{-\alpha,-\beta}$	=	$-N_{\alpha,\beta}$	
$N_{\beta,\alpha}$	=	$-N_{\alpha,\beta}$	
${N_{\alpha,\beta}}^2$	=	$\frac{q(1+p)\|\alpha\|^2}{2}$	where $\beta+n\alpha$, with $-p \leq n \leq q$, is the α-string through β.
$N_{\alpha,\beta}$	=	$N_{\beta,\gamma} = N_{\gamma,\alpha}$	if $\alpha,\beta,\gamma \in \Phi$ and $\alpha+\beta+\gamma=0$
$N_{-\alpha,\alpha+\beta}$	=	$N_{-\beta,-\alpha} = -N_{-\alpha,-\beta}$	if $\alpha,\beta,\alpha+\beta \in \Phi$
$N_{\alpha,\beta}$	=	$-N_{-\alpha,-\beta}$	Combining the above relations.
	=	$N_{-\alpha,\alpha+\beta}$	
	=	$-N_{\alpha,-\alpha-\beta}$	
	=	$N_{-\alpha-\beta,\alpha}$	
$\dfrac{N_{\alpha_1+\alpha_2,\alpha_3}}{N_{\alpha_2,\alpha_3}}$	=	$\frac{N_{\alpha_1,\alpha_2+\alpha_3}}{N_{\alpha_1,\alpha_2}}$	$\alpha_1,\alpha_2,\alpha_3 \in \Phi$ form a connected subsystem such that α_1 is orthogonal to α_3, i.e. $\alpha_1+\alpha_3 \notin \Phi$
$c^{\tilde{\theta}}_{\alpha+\beta}$	=	$c^{\tilde{\theta}}_{\alpha}\, c^{\tilde{\theta}}_{\beta}\, \frac{N_{\theta(\alpha),\theta(\beta)}}{N_{\alpha,\beta}}$	for any $\tilde{\theta} \in \mathrm{Aut}(\mathfrak{g},\mathfrak{t})$, such that $\tilde{\theta}\|_{\mathfrak{t}^\vee} = \theta$
$c^{\tilde{\theta}}_{-\alpha}$	=	$\dfrac{1}{c^{\tilde{\theta}}_{\alpha}}$	for any $\tilde{\theta} \in \mathrm{Aut}(\mathfrak{g},\mathfrak{t})$, such that $\tilde{\theta}\|_{\mathfrak{t}^\vee} = \theta$