

Bayesian analysis of the extreme value of a time history

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1. INTRODUCTION

This study has emerged from the contradiction between the current seismic design practices according to which time-history analyses are performed with only one, or exceptionally two statically independent accelerograms [1] and the observation that the maximum response of secondary systems may vary considerably between various accelerograms [2]. This paper considers the problem of extracting information about the extreme value of a random process from the analysis of a single record. This is done by using the information contained in the successive extrema of the record.

Consider a finite duration sample of a zero mean, stationary, ergodic, gaussian process. The duration T is assumed to be too short to allow a reliable estimate of its power spectral density function. Successive extrema are ordered by decreasing magnitude and, for each of them, the conditional probability density function of the peak factor is developed, using the assumption of Poisson threshold crossings. Next, these density functions are combined assuming that the successive extrema have been observed independently.

The above assumptions are adequate for wide band processes. The results are improved for narrow band processes by introducing an exclusion zone which account for the correlation between successive extrema.

2. POISSON CROSSINGS

The assumption that the threshold crossings occur according to a Poisson process is acceptable for wide band processes and asymptotically correct for large values of the threshold. Accordingly, the probability of exactly n crossings of the normalized threshold $\eta = b/\sigma$, in $[0, T]$, is

$$(1) \quad P_{\eta} \left\{ n \text{ crossings} \right\} = e^{-2 \nu_{\eta} T} \frac{(2 \nu_{\eta} T)^n}{n!}$$

where ν_{η} is the average rate of up-crossings of the threshold η , connected to the central frequency by

$$(2) \quad v_{\eta} = v_0 e^{-\eta^2/2}$$

If we consider the identical events

$$\left\{ \begin{array}{l} \text{The largest extreme value is} \\ \text{lower than or equal to } \eta \end{array} \right\} \equiv \left\{ \begin{array}{l} \text{There is no crossing of} \\ \text{the threshold } \eta \end{array} \right\}$$

the probability distribution function of the largest extreme value (peak factor) can be obtained from equations (1) and (2) :

$$(3) \quad W_1(\eta, N) = P_{\eta} \{ 0 \} = \exp \left[- N e^{-\eta^2/2} \right]$$

where $N = 2 v_0 T$ is the average number of half-cycles. This result is well known. The maximum likelihood u of this distribution is given approximately by

$$(4) \quad u \cong (2 \ln N)^{1/2}$$

Similarly, the identity between the events

$$\left\{ \begin{array}{l} \text{The } (i+1)^{\text{th}} \text{ largest} \\ \text{extreme value is} \\ \text{lower than or} \\ \text{equal to } \eta \end{array} \right\} \equiv \left\{ \begin{array}{l} \text{The } i^{\text{th}} \text{ largest} \\ \text{extreme value is} \\ \text{lower than or} \\ \text{equal to } \eta \end{array} \right\} \cup \left\{ \begin{array}{l} \text{There are exact-} \\ \text{ly } i \text{ crossings} \\ \text{of the thres-} \\ \text{hold } \eta \end{array} \right\}$$

leads to the following recurrence relationship for the probability distribution function of the $(i+1)^{\text{th}}$ extreme value :

$$(5) \quad \begin{aligned} W_{i+1}(\eta, N) &= W_i(\eta, N) + p_{\eta} \{ i \} \\ &= W_i(\eta, N) + \exp \left[- N e^{-\eta^2/2} \right] \cdot \frac{(N e^{-\eta^2/2})^i}{i!} \end{aligned} \quad (i > 1)$$

Upon substituting N from equation (4), equations (3) and (5) become respectively

$$(6) \quad W_1(\eta, u) = \exp \left[- e^{\frac{u^2 - \eta^2}{2}} \right]$$

$$(7) \quad \begin{aligned} W_{i+1}(\eta, u) &= W_i(\eta, u) + \exp \left[- e^{\frac{u^2 - \eta^2}{2}} \right] \cdot \frac{e^{\frac{u^2 - \eta^2}{2}}}{i!} \end{aligned} \quad (i > 1)$$

If the j^{th} largest extreme value observed is η_j , the probability density function

$$(8) \quad p_j(\eta_j, u) = \left[\frac{\partial W_j(\eta, u)}{\partial \eta} \right]_{\eta = \eta_j}$$

considered as a function of u , can be seen as the relative likelihood of u , given the observed η_j .

If the m largest extreme values η_1, \dots, η_m have been observed independently on the same record, the likelihood function of u , given the observed values η_j , is

$$(9) \quad L_m(u/\eta_1, \dots, \eta_m) = \prod_{j=1}^m p_j(\eta_j, u)$$

where the probability density functions are given by equation (8). According to Bayes theorem, if the prior distribution of u is $p'(u)$, the posterior distribution is

$$(10) \quad p''(u) = C \cdot L_m(u/\eta_1, \dots, \eta_m) \cdot p'(u)$$

where C is a normalizing constant insuring that $p''(u)$ represents a probability density function. Typical likelihood $p_j(\eta_j, u)$ are displayed in figure 1. The normalized likelihood functions $L_m(u/\eta_1, \dots, \eta_m)$ are represented in figure 2 ; as m increases, more information becomes available, and more peaked is the likelihood function L_m . The efficacy of the estimator as a function of m can be further assessed from figure 3, which shows the evolution of the relative value of the standard deviation of the distribution as a function of the number m of the extreme values considered in the analysis.

3. IMPROVEMENT FOR NARROW BAND PROCESSES

The assumption of Poisson crossings is consistent with the assumption that the amplitudes of successive extrema are independent. This assumption is inadequate for narrow band processes because of the slow variation of the envelope. This tends to make the η_i to occur in adjacent cycles. As a result, the foregoing model tends to overestimate u . This bias can be alleviated as illustrated in figure 4. In selecting the extrema, each time an extremum η_k at t_k is taken, an exclusion zone $\pm \Delta$ is excluded from the record in the search for subsequent extreme values. The effect of Δ on the bias of the estimates is illustrated in figure 5 for various values of the bandwidth of the process. A reasonable choice for Δ is [3]

$$(11) \quad \Delta = \frac{\langle CS \rangle}{2 \nu_0}$$

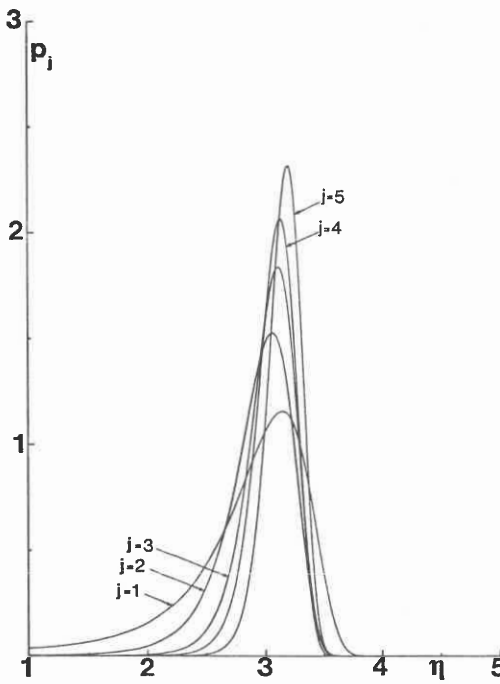


Figure 1

Typical partial likelihood $p_j(\eta_j, u)$

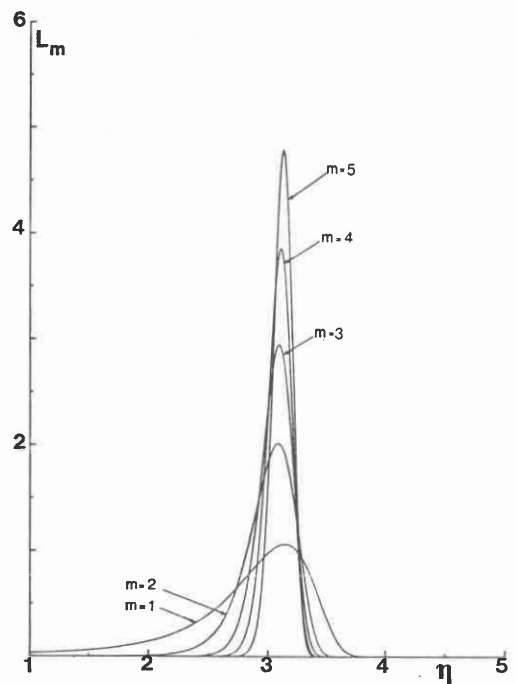


Figure 2

Normalized likelihood functions $L_m(u/\eta_1, \dots, \eta_m)$

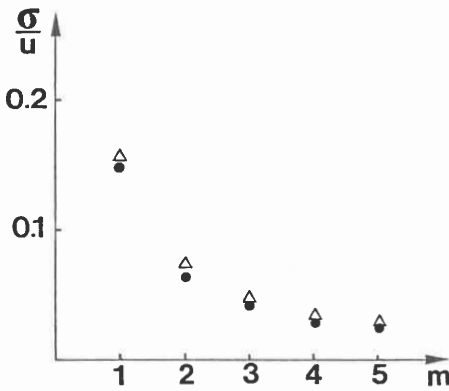


Figure 3

Standard deviation of the normalized likelihood function as a function of the number of extrema considered in the analysis. The symbols Δ and \bullet refer to different samples

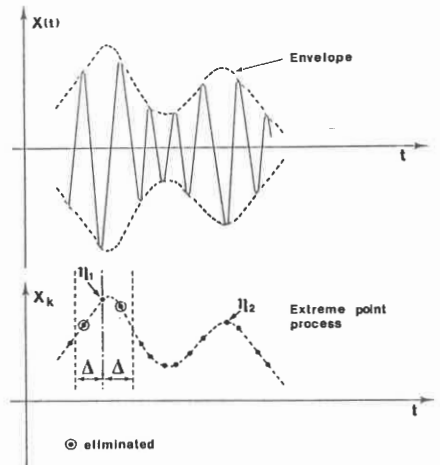


Figure 4

Effect of the exclusion zone $\pm \Delta$ on the search of extrema

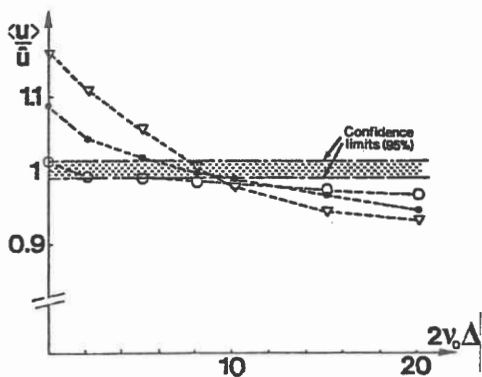


Figure 5

Effect of the duration of the exclusion zone on the bias of the estimator

▽ ideal band-pass $\Delta\omega = 0.1 \omega_c$
 ● linear oscillator $\xi = 0.01$
 ○ linear oscillator $\xi = 0.2$

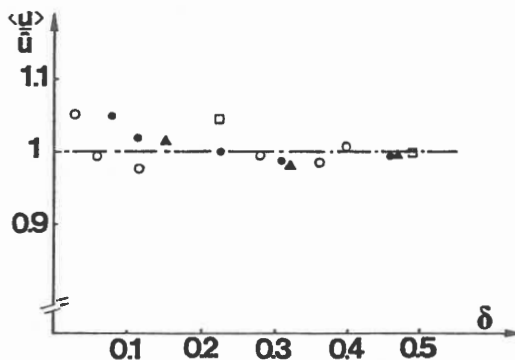


Figure 6

Bias of the procedure for various spectral shapes

● linear oscillator
 ○ ideal band-pass
 ▲ bimodal $\alpha = 0.1$
 □ bimodal $\alpha = 0.2$

where $\langle CS \rangle$ is the average clump size for the threshold η_1 . A crude estimate of $\langle CS \rangle$ can be obtained from the record itself. The bias of the procedure is illustrated in figure 6 for various spectral shapes ; δ corresponds to 400 simulations.

4. CONCLUSION

A method has been proposed, for the estimation of the most probable value of the peak factor of a stationary random process, based on the analysis of the extrema of a single sample record of short duration.

The method is based on the assumption that the threshold crossings occur according to a Poisson process, which is appropriate for wide band processes. However, for narrow band processes, the method has been extended by introducing an exclusion zone, the duration of which is related to the bandwidth of the process.

The efficiency of the estimator is an increasing function of the number m of extrema considered in the analysis, as illustrated in figure 3. The marginal improvement, however, is a decreasing function of m .

For wide band processes, the present estimator is unbiased. For narrow band processes, using an exclusion zone Δ related to the average clumpsize leads to a bias which is always smaller than 5 %, for all the cases investigated.

REFERENCES

- [1] Defalque, M. et al. (1983). "Comparison of the Methods of Seismic Analysis Applicable to Fast Reactors in the EEC Countries". BEL-GONUCLEAIRE Contract Report. n° RAP-020 B, BN 8309-07.