An Energy-Based Methods for Stress and Strain Calculation at Notches

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Abstract

In this paper a method is proposed whereby stresses and strains at notches can be calculated based on the similarity condition in the strain energy density at the notch root. The method relies on the constancy of the J-integral along any path surrounding the notch tip.

Predictions of the proposed method are compared with the available data for several materials and different notch shapes. A number of stress concentration methods advocated by other investigators are shown to be particular cases of the one developed herein.

1. Introduction

It is generally agreed that the crack initiation and subsequent propagation at the notch roots are primarily controlled by the local strain and stress fields.

From a design point of view, it is desirable to relate the nominal stresses, σ_n, and/or nominal strains, ε_n, to the local strains and stresses at the notch root. Once the local strains and stresses have been determined, the crack initiation life can be estimated from the smooth bar fatigue data [1]. For the crack propagation beyond the notch controlled field, the fracture mechanics approach can be used.

2. Methods of Estimating the Maximum Stress/Strain at Notch Roots

The traditional approach to estimate the local notch stress/strain fields of an elastic material, is based on defining a nominal stress (or strain) and the corresponding theoretical stress concentration factor, K_T. The product $\sigma_n K_T$ (or $\varepsilon_n K_T$) is the maximum "theoretical" principal stress, σ_T (or strain, ε_T) at the notch root. However, localized plastic yielding generally occurs at notch roots in which case the stress concentration starts to decrease whereas the corresponding strain concentration increases.

The values of $\sigma_n K_T$ and $\varepsilon_n K_T$ previously described now become "fictitious" local stresses and strains, respectively. A number of attempts have been made to relate the elastic-plastic stress and strain concentration factors, $K_\sigma = \sigma_{max}/\sigma_n$ and $K_\varepsilon = \varepsilon_{max}/\varepsilon_n$, to the known K_T, and the uniaxial stress-strain curve of the material. For example, Neuber's formula [2],

$$K_T^2 = K_\sigma K_\varepsilon \tag{1}$$

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postulates that during plastic deformation the product of stress and strain concentration factors remains constant and equal to K_T^2.

If the nominal stress is elastic, eq. (1) can be written as,

$$(\sigma_n K_T)^2 = E\sigma_{max}\,\varepsilon_{max}. \qquad (2)$$

Stowell's approximate formula [3] was modified by Hardrath and Ohman [4] in the following form:

$$K_\sigma = 1 + (K_T - 1)\frac{E_s}{E_{ns}} \qquad (3)$$

where $E_s = \sigma_{max}/\varepsilon_{max}$ is the secant modulus at the point of maximum local notch strain, and $E_{ns} = \sigma_n/\varepsilon_n$ is the nominal secant modulus. Equation (2) or (3) can be combined with the uniaxial monotonic or cyclic stress-strain relation to obtain the maximum stress and strain values. The cyclic or monotonic stress-strain curve for metals can be represented by the following relationship,

$$\varepsilon = \varepsilon^e + \varepsilon^p = \frac{\sigma}{E} + \left(\frac{\sigma}{K'}\right)^{1/n'} \qquad (4)$$

where n' is the cyclic (or monotonic) strain hardening exponent and K' is the strength coefficient.

It is known that Neuber's, and Hardrath and Ohman's formulae cited above tend to overpredict notch root strains [5,6], and may not be accurate enough for fatigue life predictions. Based on an energy interpretation of the Neuber's rule, Molski and Glinka [7] recently proposed a relationship for the elastic-plastic notch root stress and strain in the form of:

$$W_n K_T^2 = W_a^{max} \qquad (5)$$

where $W_n = \sigma_n^2/2E$, is the nominal elastic strain energy density, and

$$W_a^{max} = \int_0^{\varepsilon_{max}} \sigma_{max}\, d\varepsilon_{max} = \frac{\sigma_{max}^2}{2E} + \frac{1}{1+n'}\,\sigma_{max}\,\varepsilon^p_{max} \qquad (6)$$

is the "actual" maximum strain energy density at the notch root. In deriving eq. (6) it was assumed that the relationship between σ_{max} and ε_{max} is similar to that of eq. (4). From eq. (4) and (5) the local stress/strain at a notch root can be determined. However, when σ_n approaches the yield strength of the material, σ_y, the method tends to underestimate the notch root strain [7].

In the following a new approach is developed for calculating the stress and strain at a notch root. It is based on the similarity condition in strain energy densities at the notch root and the remote nominal fields.

3. Energy Approach

To obtain the similarity condition in strain energy density, consider the J-integral defined by Rice [8],

$$J = \int_\Gamma \left(W\,dy - \bar{T}\frac{\partial \bar{u}}{\partial x}\,ds\right) \qquad (7)$$

which has the same value for any path Γ surrounding the notch tip as shown in Fig. 1. In eq. (7) W is the strain energy density; $\bar{T}$ is the traction vector; $\bar{u}$ is the displacement vector, and ds is an element of arc length along Γ. By taking Γ far from the notch tip, the J-integral is made dependent only on the remote nominal stress and strain fields. In particular, the path may be shrunk to the curve of smooth-ended notch tip, denoted by the arc Γ_t in Fig. 1. Since the traction vector $\bar{T} = 0$ along Γ_t, then

$$J = \int_{\Gamma_t} W_T dy = \int_{\Gamma_t} W_a \, dy \tag{8}$$

where W_T and W_a are the "fictitious equivalent" and "actual" strain energy densities along Γ_t. The relation (8) can be interpreted as an averaged similarity measure of the strain energy density along the notch boundary. A particular case of the similarity condition, eq. (8), is the one which holds locally for energy density values along Γ_t, i.e. $W_a = W_T$. It then follows that,

$$W_T^{max} = W_a^{max} \tag{9}$$

Assuming uniaxial stress condition at the notch root (an actual condition in the plane stress state) and elastic nominal stresses, i.e. $\sigma_n \leq \sigma_y$, we can calculate the "fictitious equivalent" maximum strain energy density as follows:

$$W_T^{max} = \int_0^{\varepsilon_T} \sigma_T \, d\varepsilon_T = \int_0^{\sigma_n} K_T \sigma_n \, d(K_T \frac{\sigma_n}{E}) = K_T^2 \frac{\sigma_n^2}{2E} = \frac{1}{2} K_T^2 \sigma_n \varepsilon_n^e \tag{10}$$

When the nominal stress exceeds the material yield stress, i.e. $\sigma_n \geq \sigma_y$, then both elastic and plastic components of nominal strain will contribute towards defining the "fictitious equivalent" maximum strain energy density. This equivalent energy density is evaluated as follows.

The contribution of the nominal elastic strain, $\varepsilon_n^e = \varepsilon_n - \varepsilon_n^p$, can be calculated from eq. (10). As a first approximation, the contribution of the nominal plastic strain, ε_n^p, to the "fictitious equivalent" strain energy density can be assumed to be equal to nominal plastic strain energy density, W_n^p. Using the cyclic stress-strain relation, eq. (4), we have

$$W_T^p = W_n^p = \int_0^{\varepsilon_n^p} \sigma_n \, d(\varepsilon_n^p) = \frac{1}{1+n'} \sigma_n \varepsilon_n^p. \tag{11}$$

Substituting eq. (6), (10) and (11) into (9), we get,

$$K_T^2 \frac{\sigma_n^2}{2E} + \frac{1}{1+n'} \sigma_n \varepsilon_n^p = \frac{\sigma_{max}^2}{2E} + \frac{1}{1+n'} \sigma_{max} \varepsilon_{max}^p \quad . \tag{12}$$

For the cyclic loading eq. (4) and (12) should be written in terms of the representative strain and stress ranges, i.e.

$$\frac{\Delta\varepsilon}{2} = \frac{\Delta\varepsilon^e}{2} + \frac{\Delta\varepsilon^p}{2} = \frac{\Delta\sigma}{2E} \left(\frac{\Delta\sigma}{2K'}\right)^{1/n'} \tag{13}$$

and

$$K_T^2 \frac{\Delta\sigma_n^2}{2E} + \frac{1}{1+n'} \Delta\sigma_n \Delta\varepsilon_n^P = \frac{\Delta\sigma^2}{2E} + \frac{1}{1+n'} \Delta\sigma \Delta\varepsilon^P \quad (14)$$

where $\Delta\sigma$ and $\Delta\varepsilon^P$ are the notch root stress and plastic strain ranges, respectively.

Equations (4) and (12), or (13) and (14) can now be used to calculate the local elastic-plastic strain and stress for a given nominal stress/strain field and theoretical stress concentration factor K_T.

4. Comparison with Available Results

Elastic notch stress amplitudes $K_T \sigma_n$, versus the maximum notch root strain amplitudes, ε, for a circumferentially notched round bar are shown in Fig. 2. In this figure, the finite element results of Ref. [5] for two values of K_T are compared with the prediction of the present model. Also shown are predictions of Neuber's rule, eq. (1), and Molski and Glinka approach, eq. (5). It is seen that the latter two methods given unique relationship between $K_T \sigma_n$ and the maximum notch strain for different notch geometries. In contrast finite element results [5] indicate a significant variation due to change in the notch sharpness. It is to be noted that the proposed model indicates a similar trend and is in close agreement with the finite element results, see Fig. 2.

Wilson's results [6] of the plane strain finite element solution for bluntly notched compact specimens made of 304 stainless steel are depicted in Fig. 3, together with Neuber plane strain predictions. Three different notch root radii were used to obtain a wide range of theoretical stress concentration factors: 8.32, 4 and 2.09. Shown also in Fig. 3 are the results obtained from the present approach. Equations (4) and (12) were modified, as described in Ref. [1], to reflect the plane strain assumption used in the finite element analysis. The uniaxial cyclic stress-strain curve of the material is depicted in Fig. 3 on the same scales as $K_T \sigma_n$ and notch strain ε.

Note that the proposed method agrees fairly well with the finite element results for the higher K_T values. However for the $K_T = 2.09$, the finite element results predict lower strains then those of the proposed method.

Flate plate specimens made of A-285 Gr.C steel weakened by a central circular, elliptical, and oblique circular cylindrical apertures have been investigated by Ellyin [8]. For monotonically increasing tensile load, the strains and strain increment distributions around the apertures were measured experimentally. The maximum stresses were then calculated by four different methods, i.e. "incremental" and "deformation" theories of plasticity, and Neuber and Hardrath-Ohman formulae. The results reported by Ellyin [8] and those calculated using the present model are shown in Fig. 4. In this figure, the maximum stress concentrations are plotted against nominal to yield stress ratios.

The monotonic stress-strain relation similar to eq. (4) was used in all the calculations. It is seen that Hardrath-Ohman and present model predict results which are reasonably close to those given by the incremental theory of plasticity.

5. Discussion

Let us now consider special cases of eq. (12). For the case of the nominal plastic deformation $\varepsilon_n^P \approx 0$, we obtain eq. (5) proposed in Ref. [7]. Furthermore, eq. (12) can be written in the form of:

$$K_T^{\,2}\,\sigma_n\,\varepsilon_n = \sigma_{max}\left\{\varepsilon^e_{max} + \frac{2}{1+n'}\,\varepsilon^P_{max}\right\} \qquad (15)$$

when $\varepsilon_n^P = 0$. In addition if $2/(1+n') \approx 1$, the term in parenthesis in the R.H.S. of eq.(15) becomes equal to ε_{max}, and eq. (15) reduces to,

$$K_T^2 = K_\sigma\,K_\varepsilon \qquad (16)$$

which is the Neuber's product formula, eq. (1). Therefore, it is evident that for $\sigma_n < \sigma_y$, the model developed here, and the method proposed by Molski and Glinka [7] will give similar results.

Equations (15) and (16) indicate that Neuber's formula can be expected to predict good results for materials with higher value of exponent n'. It is for this reason that Neuber's formula gives better results for the plane strain condition then that of plane stress, see Fig. 2 and 3. However, it should be pointed out that both methods, i.e. Neuber, and Molski and Glina would predict results which may deviate considerably from actual values if nominal stresses are close or greater than the yield stress, σ_y (see Fig. 2 and 3).

6. Conclusions

A method has been developed which relates the "theoretical" stress concentration factor, K_T, to the maximum notch root stress/strain, and the far field nominal stress/strain. Predictions of the proposed method are not restricted to the small scale yielding, and are applicable when $\sigma_n \gtrsim \sigma_y$. The results are also in agreement with the experimental data.

A number of stress concentration methods proposed by other investigators, are shown to be particular cases of the one developed herein.

7. Acknowledgement

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8. References

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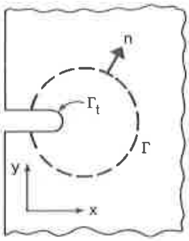

Fig. 1. A smooth-ended notch in a flat specimen.

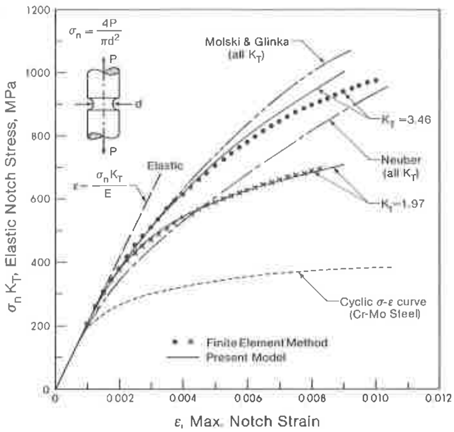

Fig. 2. Elastic notch stress vs. notch strain for circumferentially notched round bars loaded axially [5].

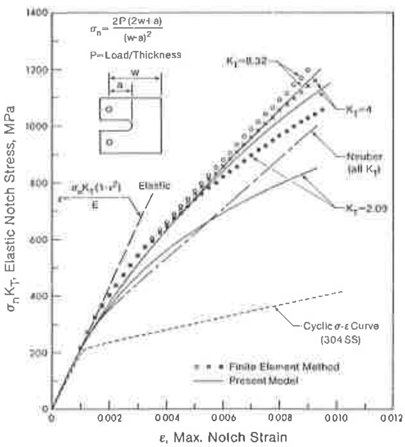

Fig. 3 Elastic notch stress vs. notch strain for blunt notched compact specimens under plane strain [6].

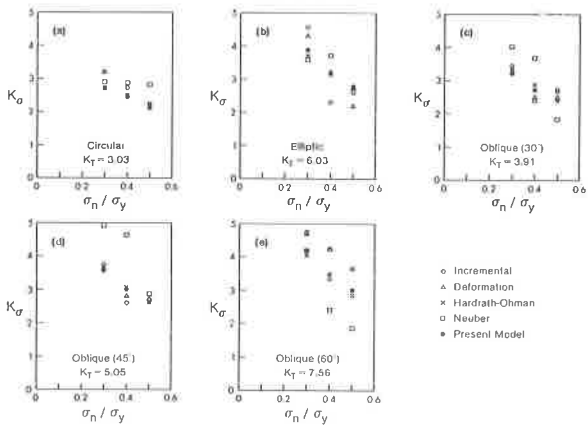

Fig. 4 Stress concentration vs. nominal to yield stress ratio for flat plate weakened by circular, elliptical and oblique circular cylindrical apertures [8].