# Dynamic Analysis of Two Dimensional Elastic-Plastic Structures Using Finite Element Method

W.-H. Chen, W.-Y. Lin

*Department of Power Mechanical Engineering,*
*National Tsing Hua University, 855 Kuang Fu Road, Hsinchu, Taiwan 300, Republic of China*

Summary

In structural modeling and optimum design consideration, dynamic stress analyses of realistic ductile structures are becoming increasingly important. However, analytic methods are very difficult or impractical to deal with such problems because of their irregular geometries and complicated boundary conditions. The aim of this work is thus to develop an efficient and accurate numerical proecdure which is applicable to the problem when inertia effects interact with nonlinear material, geometry and/or boundary conditions.

The transient response analysis of complex nonlinear structural systems usually involves the discrete modeling of the structure by finite element method. Based on the Hamilton's principle and a continuously updated Lagrangean coordinate formulation, the mass and stiffness matrices are derived in this work. As a result of the progress made in the static solution procedures of nonlinear structures, a dynamic finite element incremental procedure has been developed here to handle both geometric and material nonlinearities. The high-order isoparametric elements which can provide smoother yield zones are employed and the optimum size of these elements is determined via empirical criteria. The factors to be considered in the selection of an appropriate numerical method for structural dynamic system are accuracy, stability and ease of computer implementation. The direct time integration technique is thus taken in this work and Newmark method is considered.

To account for changes of geometry at each load level of various time steps, tangent modulus method appears most appropriate among several incremental numerical methods. Together with tangent modulus numerical procedure, an "equilibrium check" is employed at any point in the present work to reduce the cumulative error in nodal point equilibrium. The constitutive relation between the incremental Truesdell stress and the incremental Green strain is derived based on: (1) Huber-Mises-Hencky yield criterion, (2) Drucker's normality condition and (3) Prager-Ziegler's kinematic hardening law.

To demonstrate the applicability of the solution technique discussed above, several example problems subjected to various dynamic loadings have been solved. These loading conditions include: (a) impact of longitudinal load, (b) trapezoidal impulse and (c) triangular impulse. For want of space, only the results of case (a) are presented here. The dynamic spreading of elastic-plastic boundaries at various time steps is examined. Then, the transient response of deformation profiles and the distribution of inelastic stress and strain curves for each time step are displayed in detail respectively.

These studies are of practical use for estimating safety of nuclear reactor operation.

## 1. Introduction

Although several finite element approaches have been devoted to the analysis of dynamically loaded beam-type structures which undergo large-deflection, elastic-plastic transient response behavior [1,2], the work which is done on solving two dimensional ductile structures with finite deformation, dynamic response behavior is still limited [3]. Recently, one of the authors has successfully established a rigorous finite deformation, elastic-plastic finite element incremental procedure to analyze two dimensional or three dimensional static problems [4-6]. Therefore, interest in extending the procedure developed to deal with those ductile problems subjected to dynamic loadings has arisen as a natural generalization of the application to static problems. A contibution to this area is the subject of this study.

Presented herein is a finite element analysis of the elastic-plastic dynamic problem where the formulation is based on the Hamilton's principle and a continuously updated Lagrangean coordinate system. The incremental constitutive relationship whish is derived in [7] is again employed here. The material is modeled as being rate-independent elastic-pleastic material with arbitrary strain hardening. The resulting nonlinear equations of motion are solved numerically in a timewise step-by-step manner using the modified Newmark procedure [8]. After completion of each time step analysis, the geometry of the structure is updated to produce a new geometry. For this reason, tangent modulus incremental numerical method is taken in this work.

To demonstrate the solution techinque, an illustrative example is presented.

## 2. Brief Description of Dynamic Finite Element Formulation for Finite Deformation, Elastoplastic Problems

Assuming the loadint path of the nonlinear transient problem can be divided into a number of dynamic equilibrium states between time $t = t_1$ and $t = t_2$, say, $C_0, C_1, C_2 \ldots C_N \ldots$ where $C_0$ is the initial state at $t = t_1$, while $C_N$ is an arbitrary intermediate state between $t = t_1$ and $t = t_2$. It is assumed that all the "dynamic" state variables, such as stresses, strains, displacements, yeild surface translations [9] and the loading history, are known up to the $C_N$ state. The interest is then to formulate an updated Lagrangean incremental theory [10] for determining all the dynamic state variables in the current state between $t = t_1$ and $t = t_2$, say $C_{N+1}$ state, under an assumption that the $C_{N+1}$ state is incrementally close to the $C_N$ state. Neglecting the terms of higher order product of the incremental displacements, the Hamilton's incremental principle which governs the equilibrium of finite deformation, elastic-plastic transient response of two dimensional structures in $C_{N+1}$ state between $t = t_1$ and $t = t_2$ can be shown to be $\delta\Delta\pi = 0$, where

$$\Delta\pi(\Delta u_i) = \int_{t_1}^{t_2} \sum_m \{ \int_{Am} (\frac{1}{2}\Delta S_{ij}\Delta\varepsilon_{ij} + \frac{1}{2}\tau_{ij}^{(N)}\Delta U_{k,i}\Delta U_{k,j} - \Delta\bar{F}_i\Delta U_i - \frac{1}{2}\rho\Delta\dot{U}_i\Delta\dot{U}_i)dA$$

$$- \int_{s\sigma m} \Delta\bar{T}_i\Delta U_i ds - \varepsilon_m^* \}dt \qquad (1)$$

and

$$\varepsilon_m^* = \int_{Am}(-\tau_{ij}^{(N)}\Delta U_{i,j} + \bar{F}_i^{(N)}\Delta U_i)dA + \int_{s\sigma m} \bar{T}_i^{(N)}\Delta U_i ds$$

In the above, Am is the area of mth element in $C_N$ state between $t = t_1$ and $t = t_2$, $\partial_{Am}$ is

the boundary of Am, and $s_{\sigma m}$ is the portion of $\partial_{Am}$ where the surface tractions $\bar{T}_i^{(N)}$ and incremental surface tractions $\Delta\bar{T}_i$ are prescribed. $\Delta S_{ij}$ is the incremental Truesdell stress tensor and $\Delta\varepsilon_{ij}$ is the incremental Green strain tensor [10] from $C_N$ to $C_{N+1}$ state. The rate-independent constitutive relationship between $\Delta S_{ij}$ and $\Delta\varepsilon_{ij}$ can be written as (in matrix notation)

$$\{\Delta S\} = [E]^t\{\Delta\varepsilon\} \tag{2}$$

which is also given and discussed in detail in [7]. $[E]^t$ is the current constitutive tensor and is a function of $\tau_{ij}^{(N)}$. $\tau_{ij}^{(N)}$ is the true stress in $C_N$ state, measured per unit area in $C_N$ state and in the fixed Cartesian metric. $\bar{F}_i$ and $\Delta\bar{F}_i$ are body forces and incremental body forces, respectively. $\rho$ is the density. $\varepsilon_m^{*i}$ is the correction term to "check" the dynamic equilibrium of initial stress state in the $C_N$ state. The dynamic incremental displacements $\Delta U_i(t)$ can be assumed in the finite element approximations as

$$\{\Delta U(t)\} = [D]\ \{\Delta q(t)\}, \tag{3}$$

[D] is a matrix of shape function and $\{\Delta q(t)\}$ is a column matrix of undetermined element dynamic incremental nodal displacements. From eq.(3), several matrices can be deduced:

$$\{\varepsilon\} = [B]\ \{\Delta q\} \tag{4}$$

$$\{\Delta\dot{U}\} = [D]\ \{\Delta\dot{q}\} \tag{5}$$

$$\begin{Bmatrix} \Delta U_{1,1} \\ \Delta U_{1,2} \end{Bmatrix} = [W_1]\ \{\Delta q\} \tag{6}$$

and

$$\begin{Bmatrix} \Delta U_{2,1} \\ \Delta U_{2,2} \end{Bmatrix} = [W_2]\{\Delta q\} \tag{7}$$

Substituting eq. (2)-eq.(7) into eq.(1) and neglecting the body force terms, closely following the procedure presented in the work of Chen et al. [11], one can reach the following final simultaneous ordinary differential equations:

$$[M^*]\{\Delta\ddot{q}^*\}_N^{N+1} + ([K_g^*] + [K_m^*])\{\Delta q^*\}_N^{N+1} = \{\Delta Q^*\} + \{\Delta Q_c^*\} \tag{8}$$

where

$$[M^*] = \sum_m \int_{Am} \rho[D]^T[D]\ dA$$

$$[K_g^*] = \sum_m \int_{Am} ([W_1]^T\{\tau_{ij}^{(N)}\}\ [W_1] + [W_2]^T\{\tau_{ij}^{(N)}\}[W_2])\ dA$$

$$[K_m^*] = \sum_m \int_{Am} [B]^T\ [E]^t\ [B]\ dA$$

$$\{\Delta Q^*\} = \sum_m \int_{s\sigma m} [D]^T \{\Delta \bar{T}_i\}\, ds$$

and

$$\{\Delta Q_c^*\} = \sum_m \{-\int_{Am} [B]^T \{\tau_{ij}^{(N)}\}\, dA + \int_{s\sigma m} [D]^T \{\bar{T}_i^{(N)}\}\, ds\}$$

$[\,]^T$ is the transpose of $[\,]$. $[M^*]$, $[K_g^*]$, $[K_m^*]$, $\{\Delta Q^*\}$ and $\{\Delta Q_c^*\}$ are the global matrices which are assembled by the corresponding element properties. $[M^*]$ is the mass matrix accounting for inertia effects. $([K_g^*] + [K_m^*])$ represents the updated tangent stiffness matrix from $C_N$ state to $C_{N+1}$ state between $t = t_1$ and $t = t_2$, which includes the effect of finite deformations $[K_g^*]$ and prior plastic yielding $[K_m^*]$. $\{\Delta Q^*\}$ is the dynamic incremental load vector, and $\{\Delta Q_c^*\}$ corresponds to the dynamic nodal force imbalance term to check the dynamic equilibrium of $C_N$ state. All these matrices can be evaluated numerically by using the Gaussian quadrature formulae. $\{\Delta q^*\}_N^{N+1}$ and $\{\Delta \ddot{q}^*\}_N^{N+1}$ denote the values of independent generalized global dynamic nodal displacements and accelerations from $C_N$ state to $C_{N+1}$ state, respectively.

## 3. Numerical Calculation Procedure

As mentioned before, the solution to eq. (8) can be accomplished by using the incremental iterative procedure [4-6] together with the timewise Newmark method. One reason for choosing Newmark method is the flexibility which is available with the parameters $\delta$ and $\alpha$ [8], with which the analyst can control the stability requirements on the time step $\Delta t$. The results which are presented in this work are calculated with $\delta = 0.8$ and $\alpha = 0.4225$. In order to insure a numerically stable solution for dynamic problems, the time-step size, $\Delta t$, must be small enough. In this work, use is made of the criterion defined by the linear problem as a guideline for the selection of an appropriate time step $\Delta t$, i.e., $\Delta t$ is bounded by the smallest traversal time of the wave across an element.

Without loss of generality, as shown in Fig.1, the calculation procedure can be roughly described as follows:

(i) To start with, the linear elastic dynamic computations are first carried out. In these steps, as represented by state A shown in Fig.1, for example, $\{\Delta Q_c^*\}$ in eq.(8) is taken as zero while $([K_g^*] + [K_m^*])$ is considered as constant. Hence, only the Newmark integration procedure is required.

(ii) For the state after A, say, state B, once the stress point has reached yielding in the structure, the solution to the fictitious state B‴ need be first evaluated without any load increments applied (yet inertia effects and $\{\Delta Q_c^*\}$ are included). From state B‴ to the true state B, the applied loadings are divided into finite increments and the important nonlinear effects should be considered. At the end of each load increment, $\{\Delta Q_c^*\}$ should converge to a small quantity. Otherwise, the current $\{\Delta Q_c^*\}$ is reloaded as new applied load and the iteration procedure proceeds.

(iii) Repeat step (ii) until the whole loading procedure is finished.

## 4. Results and Discussions

To demonstrate the analysis procedure, a finite length plate subjected to a Heaviside-function loading on one end is selected as a testing example. The geometry of the plate is

shown in Fig.2. By symmetry, it is sufficient to solve the problem for the shaded half whose optimum finite element mesh is also shown in Fig.2. Eight-node isoparametric quadrilateral elements are employed. The material properties are: Young's modulus E = 206, 840 $N/mm^2$, yield stress $\sigma_y$ = 827 $N/mm^2$, Poisson's ratio $\nu$ = 0.3 and density $\rho$ = 7.867 $g/cm^3$. Plane strain assumptions are invoked. The longitudinal elastic wave speed $C_L$ = 0.596 cm/μsec.

Fig.3 shows the development of the directly computed yield zones at various time steps. As seen in Fig.3, plastic wave starts at t = 5 μsec and propagates with uniform speed until t = 17 μsec. After that time, the propagating speed is altered due to the variation of the geometry and to the reflection of elastic waves. To delineate the dynamic geometric changes of the structure, Fig.4 shows the deformation response history of the present finite element predictions.

To provide an overall state of stress and strain in the structure, Fig.5 and Fig.6 display the distribution of the effective stress and effective strain at various time steps respectively. Also shown in Fig.5, it can be seen that the effective stress distribution varies from a linear curve in pure elastic state at t = 3 μsec to a uniform value at t = 20 μsec. As seen in Fig.6, however, the effective strain curves usually reach their maximum values between x/L = 0.2 and x/L = 0.3.

The solution obtained in this work can be verified further by comparing it with the experimental data which are to be presented in the subsequent report.

## 5. Conclusions

Using the timewise Newmark integration scheme, this paper presents a rigorous finite deformation, elastic-plastic finite element incremental procedure to deal with two dimensional ductile problems subjected to dynamic loadings. The transient finite element predictions of deformation profile, yield zone, stress and strain state at various time steps are examined in detail respectively. To access further the accuracy of the present analysis, however, relevant experimental work need be done.

## Acknowledgement

The authors would like to thank National Science Council of China for financial aid. The help from Miss Shu-Fen Lin for typing the manuscript of this work is also acknowledged.

## References

[1] Salus, W.L., Jr., C., and Van Derlinden, J.W., "Design Considerations of Elastic-Plastic Structures Subjected to Dynamic Loads" AIAA/ASME 11th Structures, Structural Dynamics, and Materials Conference, AIAA, New York, pp.145-153 (1970).

[2] Wu, W.H. and Witmer, E.A., "Finite-Element Analysis of Large Elastic-Plastic Transient Deformations of Simple Structures", AIAA Journal 9, pp.1719-1724 (1971).

[3] Biffle, J.H. and Becker, E.B., "Finite Element Stress Formulation for Dynamic Elastic-Plastic Analysis", Computer Methods in Applied Mechanics and Engineering 6, pp.101-119 (1975).

[4] Chen, E.H., "Elastic-Plastic Stress Analysis of Perforated Thin Strip", Proceedings of the 2nd National Conference on Theoretical and Applied Mechanics, R.O.C., pp.150-164 (1978).

[5] Chen, W.H., "Finite Deformation, Elastoplastic Incremental Finite Element Analysis of Ductile Metal Structures", Proceedings of the 5th Intl. Conf. on Structural Mechanics in Reactor Technology, Berlin, West Germany, Paper L3/5, pp.1-8 (1979).

[6] Chen, W.H., "Three Dimensional, Finite Deformation, Elastic-Plastic Finite Element Analysis of Ductile Structures" to appear in Engineering Structures (1981).

[7] Miyamoto, H. and Shiratori, M., "Elastic-Plastic Response of Kinematically Hardening Cracked Sheet Under Cyclic Loading", Proceedings of the 21st National Congress for Applied Mechanics (1971).

[8] Bathe, K.J. and Wilson, E.L., Numerical Methods in Finite Element Analysis, Prentice-Hall Inc., Englewood Cliffs, N.J. (1976).

[9] Ziegler, H., "A Modification of Prager's Hardening Rule", Quarterly of Applied Mechanics, Vol.17, pp.55-65 (1959).

[10] Washizu, K., Variational Principles in Elasticity and Plasticity, 2nd edition, Pergamon Press, Oxford (1975).

[11] Chen, W.H. and Wang, C.H., "An Assumed Hybrid Displacement Finite Element Model for Elastodynamic Cracked Problems", Proceedings of XVth Int. Congress of Theo. and Appl. Mech., Toronto, Canada (1980) (Also to appear in Engineering Structures, 1981)

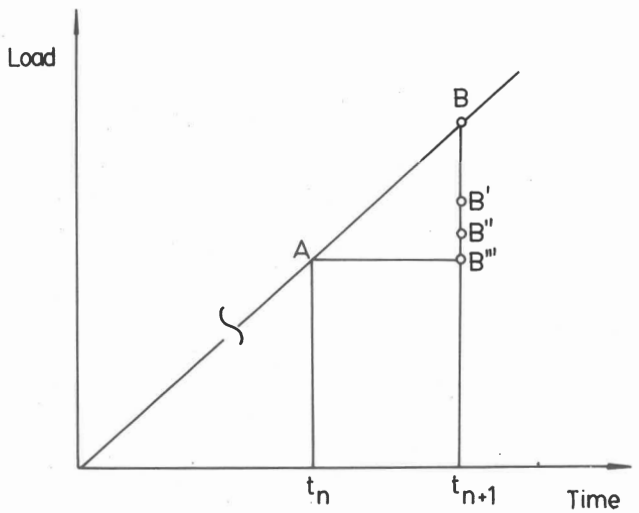


Fig.1. Numerical Calculation Procedure

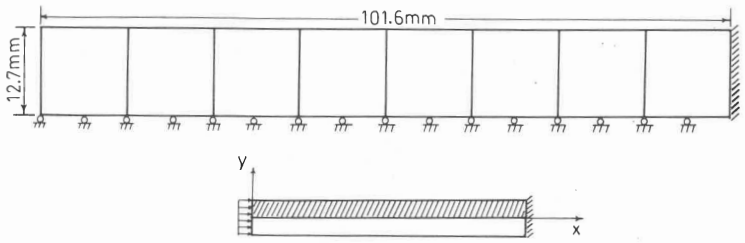


Fig.2. Finite Element Model of a Finite Plate Subjected to a Heaviside-Function Loading

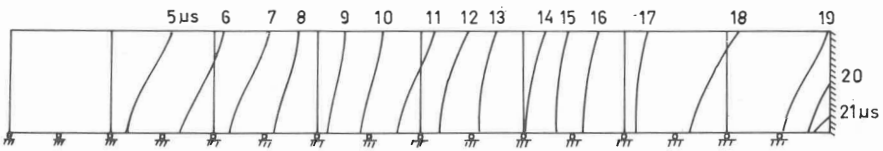


Fig.3. Distribution of Yield Zones at Various Time Steps

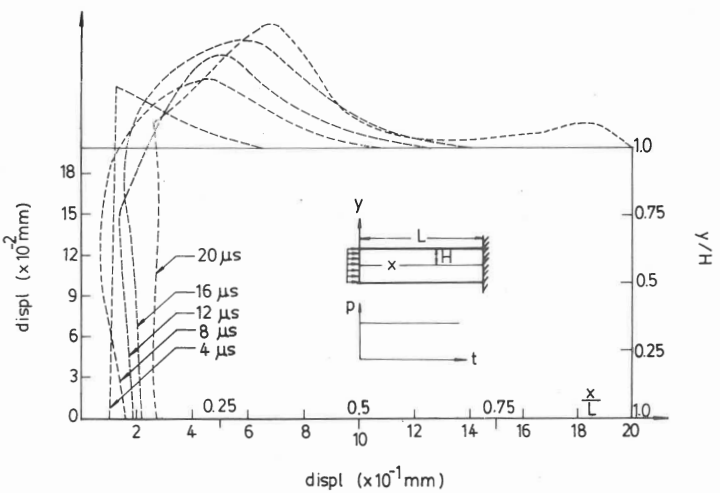


Fig.4. Deformation Response at Various Time Steps

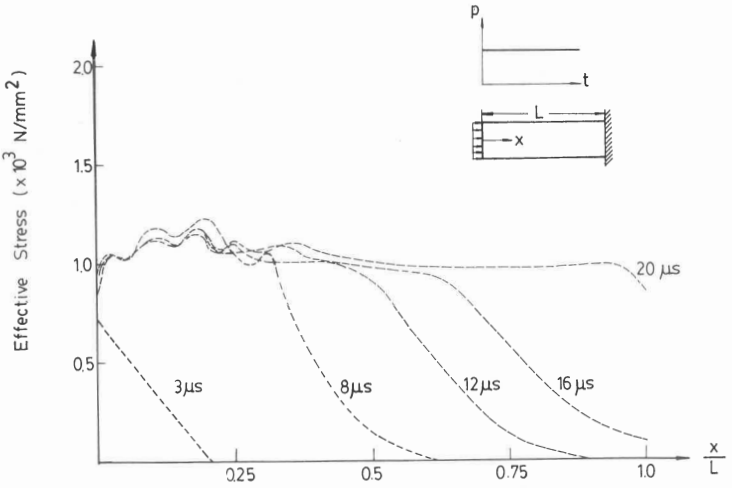


Fig.5. Distribution of Effective Stress at Various Time Steps

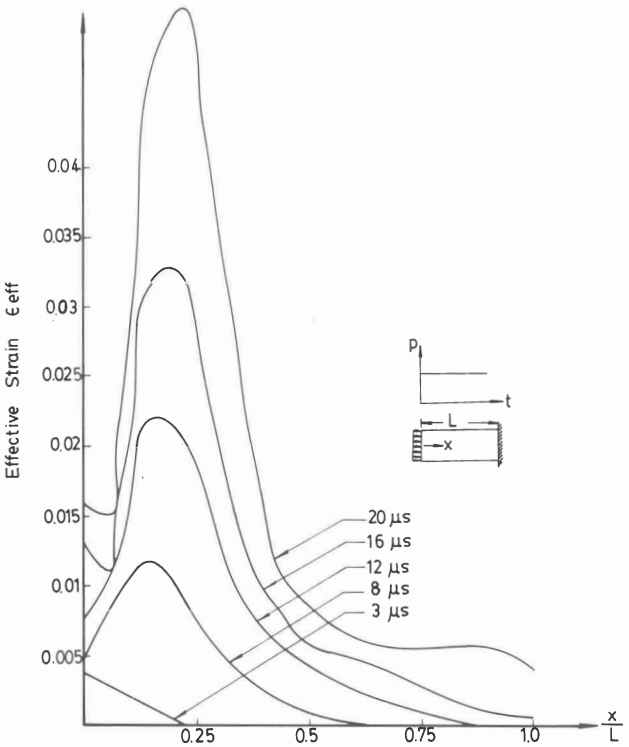


Fig.6. Distribution of Effective Strain at Various Time Steps