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ON GENERALIZED INVERSES IN A LINEAR ASSOCIATIVE ALGEBRA AND
THEIR APPLICATIONS IN THE ANALYSIS OF A CLASS OF DESIGNS*

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1. INTRODUCTION

1.1. The association matrices B_0, B_1, \dots, B_m of an m -class association scheme on v objects are defined [3] by

$$(1.1) \quad B_0 = I_{v \times v}, \quad B_i = ((b_{\alpha\beta}^i)), \quad i = 1, 2, \dots, m,$$

where

$$b_{\alpha\beta}^i = \begin{cases} 1 & \text{if objects } \alpha \text{ and } \beta \text{ are } i\text{-th associates,} \\ 0 & \text{otherwise.} \end{cases}$$

$$(1.2) \quad B_0 + B_1 + \dots + B_m = J_{v \times v},$$

where $J_{v \times v}$ is the matrix with 1 as element everywhere. Clearly, B_i is a symmetric matrix with all row and column sums equal to n_i . Further,

$$(1.3) \quad B_j B_k = p_{jk}^0 B_0 + p_{jk}^1 B_1 + \dots + p_{jk}^m B_m.$$

The commutative and associative laws hold for the multiplication of these matrices. It has been shown [3] that the linear forms $c_0 B_0 + c_1 B_1 + \dots + c_m B_m$ form a linear associative and commutative algebra with a unit element, if the coefficients c_0, c_1, \dots, c_m belong to a field. In this article, c_i 's will be considered as reals. We note that B_0, B_1, \dots, B_m form a basis of this algebra.

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It is also known [3] that the $(m+1) \times (m+1)$ matrices $P_0 = I_{(m+1) \times (m+1)}$, P_1, \dots, P_m defined by

$$(1.4) \quad P_k = ((p_{ik}^j)), \quad i, j = 0, 1, \dots, m, \quad k = 0, 1, \dots, m$$

provide a regular representation of the algebra defined by the matrices B_0, B_1, \dots, B_m . The matrices P_0, P_1, \dots, P_m are linearly independent and provide a basis for the vector space generated by the linear forms $c_0 P_0 + c_1 P_1 + \dots + c_m P_m$ where c_0, c_1, \dots, c_m belong to a field. P_i 's are not, necessarily, symmetric.

1.2. We state here, without proof, a result which we shall need later. For proof, see [3].

Lemma 1.1. The two matrices $B = c_0 B_0 + \dots + c_m B_m$ and $P = c_0 P_0 + \dots + c_m P_m$ have the same minimal polynomial and hence the same distinct characteristic roots.

Let z_{ui} , $u = 0, 1, \dots, m$ denote the $m+1$ characteristic roots of P_i , $i = 0, 1, \dots, m$. It is known that z_{ui} , $u = 0, 1, \dots, m$, $i = 0, 1, \dots, m$ are all real and that the $(m+1) \times (m+1)$ matrix

$$(1.5) \quad Z = ((z_{ui})), \quad u = 0, 1, \dots, m, \quad i = 0, 1, \dots, m$$

is non-singular [3]. Further, for a suitable ordering of z_{ui} for each i , the characteristic roots of the matrix $P = \sum_{i=0}^m c_i P_i$ are given by

$$(1.6) \quad \theta_u = \sum_{i=0}^m c_i z_{ui}, \quad u = 0, 1, \dots, m.$$

The distinct characteristic roots of $B = \sum_{i=0}^m c_i B_i$ are, therefore, to be found among θ_u given by (1.6).

Lemma 1.2. For fixed $u = 0, \dots, m$, the roots z_{ui} satisfy the relations

$$(1.7) \quad z_{uj} z_{uk} = \sum_{i=0}^m p_{jk}^i z_{ui}. \quad \text{For proof, see [3].}$$

This lemma helps us to determine the ordering of the roots z_{ui} for a given i .

For a two-class association scheme ($m=2$), the matrix Z of ordered characteristic roots of association matrices is given by

$$(1.8) \quad Z = \begin{bmatrix} 1 & n_1 & n_2 \\ 1 & \frac{\gamma-1+\sqrt{\Delta}}{2} & \frac{-\gamma-1-\sqrt{\Delta}}{2} \\ 1 & \frac{\gamma-1-\sqrt{\Delta}}{2} & \frac{-\gamma-1+\sqrt{\Delta}}{2} \end{bmatrix}$$

where

$$\gamma = p_{12}^2 - p_{12}^1, \quad \beta = p_{12}^2 + p_{12}^1, \quad \Delta = \gamma^2 + 2\beta + 1.$$

The multiplicities $\alpha_0, \alpha_1, \alpha_2$, are given by

$$(1.9) \quad \alpha_0 = 1, \quad \alpha_1 = \frac{n_1+n_2}{2} - \frac{(n_1-n_2)+\gamma(n_1+n_2)}{2\sqrt{\Delta}},$$

$$\alpha_2 = \frac{n_1+n_2}{2} + \frac{(n_1-n_2)+\gamma(n_1+n_2)}{2\sqrt{\Delta}}.$$

2. GENERALIZED INVERSE OF A MATRIX IN A LINEAR ASSOCIATIVE ALGEBRA

2.1. Consider a matrix $B = c_0 B_0 + c_1 B_1 + c_2 B_2$ in the linear associative algebra generated by B_0, B_1 and B_2 . Then $P = c_0 P_0 + c_1 P_1 + c_2 P_2$ is the image of B in the regular representation by the algebra generated by P_0, P_1 and P_2 which are 3×3 matrices. B and P have the same minimal polynomial $h(x)$ of degree at most three. Suppose that the distinct characteristic roots of B (and hence of P) are $0, \theta_1$ and θ_2 . The minimal polynomial $h(x)$ is then [5],

$$(2.1) \quad h(x) = x(x-\theta_1)(x-\theta_2).$$

Since every matrix satisfies its minimal polynomial, we have

$$(2.2) \quad B(B-\theta_1 B_0)(B-\theta_2 B_0) = 0.$$

We define

$$(2.3) \quad S = -\frac{1}{\theta_1 \theta_2} [B - (\theta_1 + \theta_2) B_0].$$

Then

$$(2.4) \quad BSB = B.$$

This implies that $S = B^-$ is a generalized inverse of B . In the same way we find that

$$(2.5) \quad P^- = -\frac{1}{\theta_1 \theta_2} [P - (\theta_1 + \theta_2) P_0]$$

is a generalized inverse of P . It is easy to verify that the characteristic roots of P^- are $(1/\theta_1 + 1/\theta_2)$, $1/\theta_1$ and $1/\theta_2$ corresponding to the roots 0 , θ_1 and θ_2 of P . The same is true of the distinct characteristic roots of B^- .

If B has the characteristic roots 0 , θ_1 and θ_2 , B^2 has the characteristic roots 0 , θ_1^2 and θ_2^2 and the minimal polynomial of B^2 is

$$(2.6) \quad g(x) = x(x-\theta_1^2)(x-\theta_2^2).$$

Let

$$(2.7) \quad M = \frac{1}{\theta_1^2 \theta_2^2} [-(\theta_1^2 + \theta_2^2) B_0 + B^2]$$

and

$$(2.8) \quad G = MB = \frac{1}{\theta_1^2 \theta_2^2} [-(\theta_1^2 + \theta_2^2) B + B^3].$$

Then we have

$$(2.9) \quad BCB = B, \quad GBG = G, \quad (BG)' = BG, \quad (GB)' = GB.$$

Hence $G = B^+$ is the Moore-Penrose inverse of B ([7],[8]). Similarly,

$$(2.10) \quad T = P^+ = -\frac{1}{\theta_1^2 \theta_2^2} [-(\theta_1^2 + \theta_2^2)P + P^3]$$

is the Moore-Penrose inverse of P . We note that the distinct characteristic roots of B^+ are $0, 1/\theta_1$ and $1/\theta_2$. The same is true of P^+ . Further, B^+ and P^+ belong to the subalgebras generated by B and P respectively.

3. GENERALIZED INVERSE OF A PARTITIONED MATRIX

We consider a matrix

$$(3.1) \quad F = \left[\begin{array}{c|c} \overline{NN'} & \overline{K} \\ \hline \overline{K'} & \overline{O} \end{array} \right] = \left[\begin{array}{c|c} \overline{B} & \overline{K} \\ \hline \overline{K'} & \overline{O} \end{array} \right]$$

where N, K, K' are real matrices, and K' is the transpose of K . We assume that the column-vectors of K belong to the vectorspace generated by the column-vectors of NN' . That is, $K = BL$. Note that F is not necessarily *non-negative*.

Let B^- be a generalized inverse of B . Define $Q = -K'B^-K$ and let Q^- be a generalized inverse of Q . Now $Q = -K'B^-K = -L'BB^-BL = -L'BL$. Consider the matrix

$$(3.2) \quad H = \left[\begin{array}{c|c} \overline{B^- + B^-KQ^-K'B^-} & \overline{-B^-KQ^-} \\ \hline \overline{-Q^-K'B^-} & \overline{Q^-} \end{array} \right].$$

Then it is easy to verify that

$$(3.3) \quad FHF = F,$$

and hence $H = F^-$ is a generalized inverse of F . An alternative expression for F^- is

$$(3.4) \quad F^{-} = \left[\begin{array}{c|c} B^{-} - LR^{-}L' & LR^{-} \\ \hline R^{-}L' & -R^{-} \end{array} \right]$$

where $R = L'BL$ and $K = BL$. The expression for a generalized inverse of F when it is *non-negative* Hermitian is given in [10]. For F non-singular, its inverse is given in [6].

4. LINEAR MODEL AND SOLUTION OF NORMAL EQUATIONS. APPLICATIONS.

4.1. Consider the linear model

$$(4.1) \quad E(\underline{y}) = A'\underline{\mu},$$

where \underline{y} is a column-vector of n random variables y_1, y_2, \dots, y_n which are uncorrelated and have the same variance σ^2 . It is known ([1],[9]) that the minimum variance unbiased linear estimator of an estimable linear function $\underline{1}'\underline{\mu}$ is given by $\underline{1}'\hat{\underline{\mu}}$ where $\hat{\underline{\mu}}$ is a solution of the normal equations

$$(4.2) \quad AA'\underline{\mu} = A\underline{y}.$$

If $(AA')^{-}$ is a generalized inverse of AA' , then $\hat{\underline{\mu}} = (AA')^{-}A\underline{y}$ is a solution of (4.2).

4.2. Consider a (connected) partially balanced design based on a two-class association scheme. Then the coefficient matrix C in the normal equations for estimating the treatment effects after adjusting for the block effects [2] is

$$(4.3) \quad C = r(1 - \frac{1}{k})B_0 - \frac{\lambda_1}{k}B_1 - \frac{\lambda_2}{k}B_2.$$

The representation of C is

$$(4.4) \quad P = r(1 - \frac{1}{k})P_0 - \frac{\lambda_1}{k} P_1 - \frac{\lambda_2}{k} P_2.$$

The characteristic roots of P are given by

$$(4.5) \quad Z\underline{u}, \quad \text{where } Z \text{ is as defined by (1.10) and } \underline{u}' = (u_0, u_1, u_2) \text{ is given by}$$

$$(4.6) \quad u_0 = r(1 - \frac{1}{k}), \quad u_1 = -\frac{\lambda_1}{k}, \quad u_2 = -\frac{\lambda_2}{k}.$$

Using the relation

$$(4.7) \quad r(k-1) = n_1\lambda_1 + n_2\lambda_2,$$

for a partially balanced design, it is easy to verify that 0 is a root of P and the other two roots θ_1 and θ_2 must be distinct if $\lambda_1 \neq \lambda_2$. Thus 0 , θ_1 , θ_2 will also be the distinct roots of C , 0 being a simple root and θ_1 and θ_2 will occur with multiplicities α_1 and α_2 given by (1.9). Now, it is easy to show that for solving the linear equations

$$(4.8) \quad C\underline{\mu} = \underline{Q},$$

it is enough if one solves

$$(4.9) \quad P\underline{\beta} = \underline{R},$$

where $\underline{\beta}' = (\beta_0, \beta_1, \beta_2)$ and $\underline{R}' = (R_0, R_1, R_2)$ are defined as follows:

$$(4.10) \quad \begin{aligned} \beta_0 &= \mu_1 \text{ (say), } \beta_1 = \text{the first element of } B_{1\underline{\mu}} = \text{sum of the parameters for those treatments which are first associates of the first treatment, } \\ \beta_2 &= \text{the first element of } B_{2\underline{\mu}} = \text{sum of the parameters for those treatments which are second associates of the first treatment.} \end{aligned}$$

R_0, R_1 and R_2 are similarly defined in terms of Q . Using results (2.3) through (2.10), we can find a generalized inverse for C and P and use it for solving the equations (4.8) and (4.9) respectively.

4.3. A partially balanced weighing design based on two-association classes has been defined [11] as an arrangement of v objects in b blocks such that each object occurs in r blocks and each block is of size $2p$ and every block can be divided into two subblocks of size p such that

- (i) any two objects which are first associates occur together in the same half block λ_{11} times and in different half blocks of the same block λ_{21} times,
- (ii) any two objects which are second associates occur together in the same half block λ_{12} times and in different half blocks of the same block λ_{22} times.

Let $\underline{\mu}$ be the vector of parameters for the v objects and the model

$$(4.11) \quad E(\underline{y}) = N'\underline{\mu},$$

where \underline{y} is the vector of n random variables and $N' = ((n_{ij}))$ the design matrix.

$$(4.12) \quad \begin{aligned} n_{ij} &= 1 \text{ if the } i\text{-th object is in the first half of the } j\text{-th block,} \\ &= -1 \text{ if the } i\text{-th object is in the second half of the } j\text{-th block,} \\ &= 0 \text{ if the } i\text{-th object does not occur in the } j\text{-th block.} \end{aligned}$$

Then it is easy to verify that

$$(4.13) \quad NN' = rB_0 - \lambda_1 B_1 - \lambda_2 B_2 = B \text{ (say)}$$

where $\lambda_1 = \lambda_{21} - \lambda_{11}$, $\lambda_2 = \lambda_{22} - \lambda_{12}$ and $r = n_1 \lambda_1 + n_2 \lambda_2$. The normal equations are then,

$$(4.14) \quad B\underline{\mu} = N\underline{y}.$$

It is easy to verify that B will have three distinct characteristic roots $0, \theta_1$ and θ_2 if $\lambda_1 \neq \lambda_2$ and λ_1 and λ_2 are such that θ_1 and θ_2 are not zero. Thus using results of Section 2, we can find a generalized inverse B^- given by (2.3) and a Moore-Penrose inverse B^+ given by (2.8). We can use any one of B^- and B^+ to get a solution of the equations (4.14).

4.4. For the partially balanced weighing design of 4.3., sometimes one has restrictions on the parameters. The model is

$$(4.15) \quad E(\underline{y}) = N'\underline{\mu} \quad \text{with the restrictions} \quad K'\underline{\mu} = \underline{m}.$$

In this situation, the equations to be solved [9] are

$$(4.16) \quad \left[\begin{array}{c|c} NN' & K \\ \hline K' & O \end{array} \right] \begin{bmatrix} \underline{\mu} \\ \underline{\phi} \end{bmatrix} = \begin{bmatrix} \underline{T} \\ \underline{m} \end{bmatrix}.$$

Let F denote the matrix of coefficients in (4.16). Then if K belongs to the vectorspace generated by the column vectors of $NN' = B$, a generalized inverse of F is given by H as defined in (3.2) and (3.4).

If K is such that F is non-singular, its regular inverse is given [6] by

$$(4.17) \quad F^{-1} = \left[\begin{array}{c|c} B^+ & H(K'H)^{-1} \\ \hline (H'K)^{-1}H' & O \end{array} \right]$$

where B^+ is the Moore-Penrose inverse of B and H is such that $H'B = 0$ and $H'K$ is non-singular. Such an H always exists iff F is non-singular [6].

SOMMAIRE

On considère l'algèbre linéaire associative engendrée par les matrices d'association $(v,v) B_0, B_1, \dots, B_m$ d'un schéma d'association à m classes. Cette algèbre est isomorphe à l'algèbre engendrée par les matrices $(m+1, m+1) P_0, P_1, \dots, P_m$. Dans cet isomorphisme une matrice $B = \sum_{i=0}^m c_i B_i$ et son image $P = \sum_{i=0}^m c_i P_i$ ont le même polynôme minimal. A l'aide de celui-ci on obtient une inverse généralisée et l'inverse de Moore-Penrose de P et de B . On donne une formule pour le calcul d'une inverse généralisée d'une matrice symétrique partitionnée, non nécessairement non-négative. Enfin ces inverses généralisées sont utilisées pour résoudre les équations normales et pour faire l'estimation dans les plans d'expériences construits sur des schémas d'association à 2 classes.

SUMMARY

In the regular representation of the linear association algebra generated by the $v \times v$ association matrices B_0, B_1, \dots, B_m of an m -class association scheme, in terms of the $(m+1) \times (m+1)$ matrices P_0, P_1, \dots, P_m , a matrix $B = \sum_{i=0}^m c_i B_i$ and its image $P = \sum_{i=0}^m c_i P_i$ have the same minimal polynomial. A generalized inverse and the Moore-Penrose inverse of P and B have been derived using their minimal polynomial. An expression for a generalized inverse of a symmetric partitioned matrix, *not necessarily non-negative*, is given. Applications of these generalized inverses for solving normal equations and estimation in designs based on two-class association schemes are discussed.

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