SOME NEW RESULTS IN THE THEORY OF RECURRENT EVENTS
A PRELIMINARY REPORT

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Institute of Statistics Mimeo Series No. 1091

November, 1976

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* Research supported by the Office of Naval Research under N00014-76-C-0550.

ABSTRACT

The main focus of the work is to obtain approximate formulae for u_n, the expected number of occurrences of an aperiodic recurrent event at time n. The formulae are all of the type $u_n = \phi_n + \rho_n$, say, where ϕ_n is explicitly calculable in terms of known features of the model and ρ_n is a remainder term tending to zero as $n\to\infty$. The thrust of the work lies in showing the way in which assumptions about the underlying model are reflected by ρ_n. In particular, suppose r_n to be the probability the interval between successive occurrences of the event shall exceed the integer n. Suppose $T(n) \to \infty$ and is what is called in the paper either a *moment function* or a *tail function*. Then four kinds of results are obtained, of the kinds: (1) $\sum_1^\infty T(n) r_n < \infty$ implies $\sum_1^\infty T(n)|u_n - \phi_n| < \infty$; (2) $r_n = O(1/T(n))$ implies $u_n - \phi_n = O(1/T(n))$; (3) $r_n = o(1/T(n))$ implies $u_n - \phi_n = o(1/T(n))$; (4) $r_n \sim \rho/T(n)$, as $n\to\infty$, for some $0<\rho<\infty$, then $u_n - \phi_n \sim \rho'/T(n)$, say, where ρ' is given in terms of known quantities. The conditions and details are too complicated to be listed in this abstract and are given in §1. The main tool for the proving of these results is a many-sided version of a famous theorem of Wiener and Lévy, which can also be used in treating renewal theory. It is explained, however, that slightly stronger results could be obtained (and have been, by other authors) if use were made of the theory of commutative Banach algebras.

Key Words and Phrases: Recurrent events, limit theorems, absolutely convergent fourier series, functions of regular variation.

§1. Introduction

The theory of recurrent events, introduced by Feller (1949), has found a wide range of applications and led to a considerable amount of further research. We may introduce the basic ideas as follows. Let $\{X_j\}_{j=1}^{\infty}$ be a sequence of iid random variables taking integer values and let $f_k = P\{X_j=k\}$ for $k = 0,\pm1,\pm2,\ldots$, and so on. Form the partial sums $S_n = X_1 + X_2 + \ldots + X_n$, letting $S_0 = 0$, and suppose some *recurrent event* E takes place at "times" $S_0,S_1,S_2,\ldots$, and so on. When $P\{X_n \le 0\} = 0$ then, almost surely, E cannot occur more than once at a given time; otherwise multiple occurrences have a positive probability of occurring. Let $M(n)$ be the number of occurrences of E at time n, and define $u_n = EM(n)$. It is well known that if $E|X_j| < \infty$ and $EX_j \ne 0$ then $u_n < \infty$. In Feller (1949) it was supposed that $P\{X_j < 0\} = 0$, which is certainly the case in most applications of his theory; in much of later work this assumption of almost sure non-negativity has been dropped, and we shall adopt the more general model in the present paper.

For simplicity in the discussion that follows we shall assume, with no loss of generality, that there is no integer $k>1$ such that X_j/k is almost surely an integer; we thus have an *aperiodic* process.

When the corresponding absolute moments exist we write $\mu_k = E(X_j)^k$. A large part of the research on the theory of recurrent events has been devoted to the study of the asymptotic behavior of u_n as $n \to \pm\infty$ and to the influence on this behavior of the moments μ_k which happen to be finite. A fundamental result, under the assumption of aperiodicity, is that when $\mu_1 > 0$, $u_n \to \mu_1^{-1}$ as $n\to\infty$. This was proved first under the positivity assumption $P\{X_j < 0\} = 0$ by Erdös, Feller, and Pollard (1949), although their result is implicit in a theorem of Kolmogorov (1937) on Markov Chains. It was freed of the positivity assumption by Chung and Pollard (1952) and by Chung and Wolfowitz (1952). Using this basic result Feller showed, for instance, that $\sum_0^n (u_j - \mu_1^{-1}) \to (\mu_2 - \mu_1)/2\mu_1^2$ as $n\to\infty$ provided μ_2 exists; he also showed that the existence of μ_2 ensured $\sum_0^\infty |u_j - \mu_1^{-1}| < \infty$. A number of extensions and generalizations of these results have been given; we mention a few of them. Gel'fond (1964) by a careful analysis of certain contour integrals showed that when $\mu_k < \infty$ for integer $k\geq 1$, one has

$$(1.1) \qquad u_n = \frac{1}{\mu 1} + \frac{1}{\mu_1^2}\sum_{n+1}^{\infty} f_j + O\left(\frac{\log n}{n^\nu}\right) .$$

Borokov (1964) developed a useful modification of the famous Lévy-Wiener theorem on absolutely convergent fourier series (Lévy (1933)); his result depended on a number of somewhat complicated conditions, involving slowly varying functions, being met. From this basic tool he obtained various results concerning u_n ; they are typically like (1.1) but with an improved estimate of the error term. The price to be paid for such an improvement lies in various alternative assumptions on the sequence $\{f_n\}$; it would take too much space to give exact details here, but they typically involve f_n being dominated by sequences like $\{L(n)/n^\alpha\}$, $\alpha>1$, where $L(n)$ is a slowly varying function.

Parallel to the theory of recurrent events is the theory of renewals; the model is essentially the same except the $\{X_j\}$ are freed from their restriction to integer values. There is a great similarity between the two theories and a theorem in one usually has a fair facsimile in the other. In renewal theory u_n corresponds to $U(I_t)$, say, the expected number of visits of the random walk $\{S_n\}$ to the interval $I_t \equiv (t, t+1]$. There seems to have been even more research devoted to estimates of $U(I_t)$ than to u_n ; these papers usually include the observation that theorems, similar to those obtained, can be derived for u_n by obvious changes in the proofs given. It would take us too far afield to give a full account of work done in renewal theory on the estimation of $U(I_t)$, we must content ourselves with mentioning a few papers on this subject. Stone (1965a), (1965b), (1966) considers various related matters, too many to detail here, but including situations in which $P\{X_j > n\}$ decreases geometrically fast as $n\to\infty$; the results are derived by various complicated arguments employing fourier analysis. Smith (1966) considered renewal theoretic matters and not merely questions concerning the expected numbers of "renewals" in a time interval; he developed techniques for treating expectations of arbitrary (non-integral) powers of the number of renewals. To achieve these ends he developed a suitable modification of a famous theorem on functions of Fourier-Stieltjes transforms of functions of bounded variation. In the context of recurrent events his work provided conditions for the absolute convergence of series like $\sum_0^\infty M(n)|u_n - \mu_1^{-1}|$, where $M(n)$ belongs to a fairly general class of functions.

In tackling questions about branching processes Chistyakov (1964) introduced a useful and interesting class of distributions; it has recently received an interesting examination by Teugels (1974). The Chistyakov class depends on the requirement that $P\{S_n > x\} \sim nP\{X_1 > x\}$ as $x\to\infty$, for each fixed $n = 2,3,\ldots$; Teugels describes various consequences of this requirement. These same ideas are basic to the work of Chover, Ney, and Wainger (1973). By means of sophisticated functional analysis and the introduction of various Banach algebras they develop a further modification of the Wiener-Lévy theorem; their version enables one to make statements about the asymptotic equivalence of two measures, one of which is a suitably analytic function of a given measure with appropriate asymptotic behavior. Their applications of these results are principally to questions in branching processes.

Methods of Banach algebra are also used by Essén (1973) in his treatment of various questions in the theories of renewal and recurrent events. Essén's work concentrates on theorems which give formulae like (1.1), that is, approximations to u_n in which the remainder term is shown to be of a specified "O" or "o" size, depending on assumptions made concerning the $\{f_k\}$. His results are no less powerful, and usually more powerful, than any previously published results of this type.

The present report has developed from the preparation of a forthcoming book on renewal theory. It represents an attempt, not wholly successful (for reasons given later in this section), to develop a unified theory of recurrent events from "first principles", that is, not depending on profound results in the theory of commutative Banach algebra. The hope is that in this way the theory will be accessible to a somewhat wider audience including, one hopes, those who may find use of the results in applications.

We shall derive a number of results in the theory of recurrent events in the broad category exemplified by (1.1). Many of these results seem new, and some are more-or-less equivalent to results obtained in papers we have mentioned. Exact comparisons are difficult because, roughly speaking, each paper employs a slightly different family of monotone functions as its measure of rate of growth of the tails of the various probability distributions which arise. All the results we obtain can, with certain well-known provisos, be translated into renewal theorems; we defer this translation for a later report.

Much of the present report is devoted to the proof of a suitable version of the Wiener-Pitt-Lévy Theorem (Theorem 3.1, below) which embraces all the different modifications utilized in the various papers mentioned above. In harmony with our program the proof is developed entirely from first principles and these notes are completely self-contained; the basic method of proof of Theorem 3.1 is the one used for the proof of a similar result in Smith (1965).

Before we can describe our theorems on recurrent-events it is necessary to say something about various classes of monotone functions that enter the discussion. Suppose $T(x)$, for $x \geq 1$, to be given by the asymptotic formula

$$T(x) \sim \exp\{\int_1^x \frac{\alpha(u)}{u}\, du\}, \text{ as } x \to \infty. \tag{1.2}$$

in which $\alpha(u)$ is non-negative and bounded in every *finite* interval $1 \leq u \leq R < \infty$. Plainly $T(x)$ is a non-decreasing function of $x \geq 1$. We say $T(x)$ belongs to:

Class I : if $\alpha(u) \to 0$ as $u \to \infty$.

Class II : if $\alpha(u) \to \rho$ as $u \to \infty$, where ρ is a non-zero, finite, limit.

Class III: if $\alpha(u) \to \infty$ as $u \to \infty$.

The functions in Class I are more usually refereed to as *functions of slow growth*, (fsg), or as slowly varying functions. Such functions were originally studied by Karamata (1930), (1933), who derived a canonical form for them which is now well-known. The functions of Class II are more usually referred to as *functions of regular variation* (frv), with index ρ; for a full discussion of such functions we refer to Feller (1971). The functions of Class III seem to have escaped being named in the literature although they might well be called *super-power functions* (spf) since they represent functions which grow faster than any power.

If $\alpha(u)/u \to 0$ as $u\to\infty$ ($\alpha(u)$ not necessarily non-negative), we shall say $T(x)$ is a *function of moderate growth* (fmg); a necessary and sufficient condition for $T(x)$ to be a fmg is that, for every fixed $c>0$, $T(x+c) \sim T(x)$ as $x\to\infty$. We shall call a non-decreasing fmg (with $\alpha(u)$ non-negative) a *tail function*.

If $M(x) \equiv 1$ for $x<0$, if $M(x)$ is non-decreasing for $x\geq 0$, and if $M(x+y) \leq M(x)M(y)$ for all finite x,y, then we call $M(x)$ a *right moment function* (rmf). Further discussion of these useful functions will be given in §2 below; they played a vital role in Smith (1966), see also Smith (1969). For $x\geq 1$, the function $T(x)$ of (1.2) will be a moment function if $\alpha(u)/u$ decreases to zero as $u\to\infty$.

Through all this report it is supposed that tail functions and moment functions satisfy what we shall call the *Umbrella condition* U; this amounts to requiring that

$$\int_1^\infty \frac{\alpha(u)}{u^2}\,du < \infty. \tag{1.3}$$

Notice that this rules out the possibility of exponential growth rate for our tail and moment functions.

Corresponding to a given tail function we shall require a moment function $G(\circ)$, to be called a *gauge function*. It must have the property that, for every $r>0$,

$$\sup_{x\geq 2r} \frac{T(x)}{T(x-r)} \leq CG(r), \tag{1.4}$$

where C is a finite constant, independent of x and r.

If $\underline{a} = \{a_n\}$ is a sequence and M is a rmf we define $||\underline{a}||_M = \sum_{-\infty}^{+\infty} M(n)|a_n|$; then $S(M)$ is the class of all sequences $\underline{a}$ such that $||\underline{a}||_M < \infty$. In dealing with a tail function $T(x)$, say, we often require a sequence to belong to $S(G)$, where G is a corresponding gauge function. In many important applications this requirement is automatically fulfilled. If ν is any positive real we write $T_\nu(x)$ for any tail function such that $x^\nu T(x) \sim T_\nu(x)$ as $x\to\infty$. The same gauge function will do for both T and T_ν. But if $T(x)$ is also a moment function (which, by the way, appears to be the case for the scale functions used by Essén) then we may take $G(x) = T(x)$ for all $x\geq 0$. Suppose $a_n \sim 1/T_\nu(n)$, as $n\to\infty$. Then $T(n)a_n \sim 1/n^\nu$, as $n\to\infty$. Thus, if $\nu>1$, it is plain that $\underline{a} \in S(G)$. This argument should be borne in mind in considering several of the theorems given below; if in a given application the tail function is also a moment function then the references to $S(G)$ can usually be ignored. In this connection it should also be noticed that if T is in Classes I or II then $G(x)$ may be taken as identically equal to unity; the requirement $\underline{a} \in S(G)$ then merely amounts to asking that $\sum_{-\infty}^{+\infty} |a_n| < \infty$, which will always be the case. Thus references to $S(G)$ can also be ignored when T is in Classes I or II, i.e. when T is a fsg or a frv.

Let us henceforth suppose $\mu_1 = EX_j$ to be finite and non-zero and set

$$r_n = \frac{1}{\mu_1} \sum_{n+1}^{\infty} f_j \qquad \text{for } n \geq 0$$
$$= \frac{1}{\mu_1} \sum_{-\infty}^{n} f_j \qquad \text{for } n < 0.$$

Then $\sum_{-\infty}^{+\infty} r_n = 1$. Write $\underline{r}$ for the sequence $\{r_n\}$. Also write $v_n = u_n - u_{n-1}$, all n, and set $\underline{v} \equiv \{v_n\}$.

For any sequence $\underline{a} \equiv \{a_n\}$, say, we shall write: $\underline{a} \in V(T)$ if $a_n = O(1/T(n))$ as $n \to \infty$; $\underline{a} \in V_0(T)$ if $a_n = o(1/T(n))$ as $n \to \infty$; $\underline{a} \in W(T)$ if $a_n T(n)$ tends to a finite limit as $n \to \infty$. In the latter case we write $\langle \underline{a} \rangle_T$ for the limit. Then we can state our first theorem.

THEOREM 1.1. *Let* M *be a rmf,* T *a tail function,* G *a corresponding gauge function.*

(R1) $\underline{v} \in S(M)$ *if and only if* $\underline{r} \in S(M)$.

(R2) $\underline{v} \in V(T) \cap S(G)$ *if and only if* $\underline{r} \in V(T) \cap S(G)$.

(R3) $\underline{v} \in V_0(T) \cap S(G)$ *if and only if* $\underline{r} \in V_0(T) \cap S(G)$.

(R4) $\underline{v} \in W(T) \cap S(G)$ *if and only if* $\underline{r} \in W(T) \cap S(G)$ *and, in this case,*

$$\langle \underline{v} \rangle_T = -\mu_1^{-1} \langle \underline{r} \rangle_T$$

implying

$$u_n - u_{n-1} \sim -\frac{r_n}{\mu_1 T(n)} \quad \text{as } n \to \infty.$$

Let us set

$$\left. \begin{aligned} s_n &= \sum_{n+1}^{\infty} r_j \quad \text{for } n \geq 0 \\ &= \sum_{-\infty}^{n} r_j \quad \text{for } n < 0 \end{aligned} \right\}$$

Let us also set

$$\left.\begin{aligned}\tilde{\omega} &= \mu_1^{-1} \quad \text{if} \quad \mu_1 > 0 \\ &= 0 \quad\quad \text{if} \quad \mu_1 < 0\end{aligned}\right\} .$$

An immediate consequence of Theorem 1.1 is then

Corollary 1.1.1. *Under (R4) above,*

$$u_n - \tilde{\omega} \sim \frac{s_n}{\mu_1} \quad \textit{as} \quad n\to\infty.$$

Corollary 1.1.2. *Suppose in Theorem 1.1 the tail function* $T(n) = n^{1+\nu}L(n)$, *where* $\nu>0$ *and* $L(n)$ *is a fsg. Then the references to* $S(G)$ *in the theorem may be ignored and*

(i) *Under (R2),*

$$u_n - \tilde{\omega} = O(1/n^{\nu}L(n)).$$

(ii) *Under (R3),*

$$u_n - \tilde{\omega} = o(1/n^{\nu}L(n)).$$

(iii) *Under (R4),*

$$u_n - \tilde{\omega} \sim \frac{\langle \underline{r} \rangle_T}{\nu\mu_1 n^{\nu} L(n)}, \quad \textit{as} \quad n\to\infty.$$

Corollary 1.1.3. *Under (R4) of Theorem 1.1, if* $T(n) \sim e^{cn^{\alpha}} n^{\beta} L(n)$, *as* $n\to\infty$, *where* $0<\alpha<1$, $c>0$, *and* $L(n)$ *is a fsg, then, as* $n\to\infty$,

$$u_n - \tilde{\omega} \sim \frac{e^{-cn^{\alpha}} n^{\beta-\alpha+1} L(n)}{\mu_1 c\alpha} .$$

Similar results hold under (R2) and (R3).

We say the recurrent event is *positive* if $P\{X_j < 0\} = 0$. In such a case, if we set $U_n = u_0 + u_1 + \dots + u_n$, then it is well-known that U_n is the expected number of occurrences of E at times $t \in [0,n]$. Concerning U_n we have:

Corollary 1.1.4. *Suppose* $T(n) = n^{1+\nu}L(n)$ *as in Corollary 1.1.2 and that the recurrent event process is positive. Then*

(i) Under (R2), $U_n - \mu_1^{-1}(n+1) = O(n^{1-\nu}/L(n))$;

(ii) Under (R3), $U_n - \mu_1^{-1}(n+1) = o(n^{1-\nu}/L(n))$;

(iii) Under (R4), with $0 < \nu < 1$,

$$U_n - \mu_1^{-1}(n+1) \sim \frac{n^{1-\nu}\langle \underline{r}\rangle_T}{\mu_1 \nu(1-\nu)L(n)}, \quad n\to\infty;$$

but if $\nu=1$,

$$U_n - \mu_1^{-1}(n+1) \sim \frac{\langle \underline{r}\rangle_T}{\mu_1}\int_1^n \frac{dx}{xL(x)}.$$

Part (iii) of Corollaries 1.1.2 and 1.1.4 can both be extended to cover the case $\nu=0$; we leave this to the reader, however.

When $\mu_2 < \infty$ we can prove more explicit results than those so far given. To explain these we need further notation.

If $M(n)$ is a rmf we can define a further rmf, called $M_I(n)$, by setting $M_I(0) = 1$ and $M_I(n) = M(0) + M(1) + \dots + M(n-1)$, $n\ge1$. Plainly $M_I(n) \ge n$ so, for example, the requirement $\underline{r} \in S(M_I)$ implies $\sum_1^\infty n|r_n| < \infty$. Similarly, when necessary, we define $M_{II}(0) = 1$ and

$$M_{II}(n) = M_I(0) + M_I(1) + \dots + M_I(n-1), \quad n\ge1.$$

Then, for example, $\underline{r} \in S(M_{II})$ implies $\sum_1^\infty n^2|r_n| < \infty$.

Theorem 1.2. *Let* $\mu_2 < \infty$, *and* M *be a rmf. If* $\sum_1^\infty M_I(n)|r_n| < \infty$ *then*

$$\sum_1^\infty M_I(n)|u_n - \tilde{\omega} - \frac{s_n}{\mu_1}| < \infty,$$

where $\tilde{\omega} = \mu_1^{-1}$ *if* $\mu_1 > 0$, $\tilde{\omega}=0$ *if* $\mu_1 < 0$.

To avoid a certain amount of repetitiveness let us write X to denote either V, or V_0 , or W; it is to be understood, of course, that X shall retain a given meaning throughout any theorem.

Theorem 1.3. *Let* $\mu_2 < \infty$, *let* T *be a tail function and let* G *be an associated gauge function. For* $\mu_1 > 0$ *define*

$$\begin{aligned} g_n &= u_n - \frac{1}{\mu_1} - \frac{s_n}{\mu_1}, \quad n\geq 0, \\ &= u_n - \frac{s_n}{\mu_1} \qquad\quad , \quad n<0 . \end{aligned}$$

For $\mu_1 < 0$ *make the obvious modification to this definition of* g_n . *Then if* $\underline{r} \in X(T_2) \cap S(G_I)$ *it follows that* $\underline{g} \in X(T_2) \cap S(G_I)$, *and, in the case when* $\underline{r} \in W(T_2) \cap S(G_I)$ *we find*

$$n^2T(n)g_n \to -\left(\frac{\mu_2-\mu_1}{\mu_1^2}\right)\langle\underline{r}\rangle_{T_2} , \text{ as } n\to\infty.$$

In the theorems that follow we set $U_n = u_0 + u_1 + \ldots + u_n$, as earlier, but do not necessarily suppose the recurrent event process to be positive.

Theorem 1.4. *Assume the same hypotheses as Theorem 1.2.*

(a) *If* $\mu_1 > 0$ *then, for* $n\geq 0$, *define*

$$g_n = U_n - \frac{(n+1)}{\mu_1} - \left(\frac{\mu_2-\mu_1}{2\mu_1^2}\right) + \frac{1}{\mu_1}\sum_{n+1}^{\infty} s_j - \frac{1}{\mu_1}\sum_{j=-\infty}^{+\infty} s_{n-j}s_j .$$

It follows that $\sum_1^\infty M_I(n)|g_n| < \infty$.

(b) *If* $\mu_1 < 0$ *then*

$$\sum_{m=-\infty}^{-1} \left(u_m - \frac{1}{|\mu_1|}\right) = \Gamma, \textit{ say},$$

is finite; and if, for $n \geq 0$, *we define*

$$g_n = U_n + \Gamma - \left(\frac{\mu_2-\mu_1}{2\mu_1^2}\right) + \frac{1}{\mu_1}\sum_{n+1}^{\infty} s_j - \frac{1}{\mu_1}\sum_{j=-\infty}^{+\infty} s_{n-j}s_j$$

then it follows that $\sum_1^\infty M_I(n)|g_n| < \infty$.

Theorem 1.5. *Assume the same hypotheses as Theorem 1.3 but define* g_n, *depending on the sign of* μ_1, *as in Theorem 1.4. Then* $\underline{g} \in X(T_2)$ *and, when* $\underline{r} \in W(T_2) \cap S(G_I)$, *we have*

$$\langle \underline{g} \rangle_{T_2} = -\frac{3(\mu_2-\mu_1)^2}{4\mu_1^3}\langle \underline{r} \rangle_{T_2} .$$

The theorems described so far depend on the tail probabilities $\{r_n\}$ behaving suitably. If we impose conditions on the point probabilities $\{f_n\}$ then even better results are obtainable, especially if we assume finiteness of absolute third moments of $\underline{f}$.

Theorem 1.6. *Let* $\sum_{-\infty}^{+\infty} |j|^3 f_j < \infty$, *and let* M *be a rmf. If* $\sum_1^\infty n\, M_{II}(n) f_n < \infty$, *then*

$$\sum_1^\infty n\, M_{II}(n)\left|u_n - \tilde{\omega} - \frac{s_n}{\mu_1} + \left(\frac{\mu_2-\mu_1}{\mu_1^2}\right) r_n\right| < \infty,$$

where, as usual, $\tilde{\omega} = \mu_1^{-1}$ *if* $\mu_1 > 0$ *and* $\tilde{\omega}=0$ *if* $\mu_1 < 0$.

A further idea needs explanation before we can give the theorem. Suppose T is a tail function. We shall see in §5 that one can introduce a new tail function $\tilde{T}$ by the definition

$$\frac{1}{\tilde{T}(x)} = 2x^2 \int_x^\infty \frac{du}{u^3 T(u)} \; ; \quad x \geq 1$$

and that $\tilde{T}(x) \geq T(x)$. For certain results we require that

$$\lim_{x\to\infty} \frac{T(x)}{\tilde{T}(x)} = \rho, \text{ say,}$$

exists. It is well-known that if $0 < \rho < \infty$ then this limit exists if and only if T is a function of regular variation with index ρ. We shall also be interested in the possibility that $\rho=0$. It is not difficult to show that a sufficient condition for $T(x) = o(\tilde{T}(x))$ is that $T(x)$ be a super-power function (implying that $\tilde{T}(x)$ is also such a function) and a necessary condition is that $\tilde{T}(x)$ be a super-power function. We can now state

<u>Theorem 1.7</u>. *Let* T *be a tail function which is unbounded and let* G *be an associated gauge function. Suppose*

$$\sum_{-\infty}^{+\infty} |j|^3 f_j < \infty \quad \textit{and} \quad \sum_1^\infty j \, G_{II}(j) f_j < \infty \, .$$

Define, for $n \geq 0$,

$$t_n = u_n - \tilde{\omega} - \frac{s_n}{\mu_1} + \left(\frac{\mu_2 - \mu_1}{\mu_1^2}\right) r_n \, .$$

Then, as $n\to\infty$,

(a) $f_n = O(1/T_3(n))$ *implies* $t_n = O(1/T_3(n))$;

(b) $f_n = o(1/T_3(n))$ *implies* $t_n = o(1/T_3(n))$;

(c) *if the limit* $\rho = \lim_{x\to\infty} T(n)/\tilde{T}(n)$ *exists and if* $n^3 T(n) f_n \to \phi$, $n\to\infty$, *where* ϕ *is some finite limit, then*

$$n^3 T(n) t_n \to \frac{\phi}{\mu_1^3}\left\{\frac{\lambda_3}{3} - 3\mu_1\lambda^2[1 + \tfrac{1}{2}\rho]\right\}, \quad n\to\infty,$$

where $\lambda_3 = \mu_3 - 3\mu_2 + 2\mu_1$ *and* $\lambda = (\mu_2 - \mu_1)/(2\mu_1)$ *as usual.*

If we can only assume a finite second moment we can still obtain improvements on Theorem 1.3 by assuming suitable behavior of f_n for large n.

Theorem 1.8. *Let* T, G, *be as usual and suppose* $T(n) \to \infty$ *as* $n\to\infty$, $\sum_{-\infty}^{+\infty} j^2 f_j < \infty$, $\sum_1^{\infty} j\, G_I(j) f_j < \infty$. *Then if we set*

$$u_n - \tilde{\omega}_n - \frac{s_n}{\mu_1} = g_n \text{, say,}$$

and define $\tilde{T}_1(x) = 1/\{\int_x^{\infty} dy/T_2(y)\}$ *for all* $x>0$, *we have*

(a) $f_n = O(1/T_2(n))$ *implies* $g_n = O(1/\tilde{T}_1(n))$,

(b) $f_n = o(1/T_2(n))$ *implies* $g_n = o(1/\tilde{T}_1(n))$,

(c) $f_n \sim \phi/T_2(n)$ *as* $n\to\infty$, *for some finite constant* ϕ, *implies*

$$g_n \sim -\left\{\frac{\mu_2-\mu_1}{\mu_1^3}\right\}\frac{1}{\tilde{T}_1(n)}, \quad n\to\infty.$$

Finally we consider briefly what happens if r_n decreases geometrically fast. The whole story of this promises to be involved and difficult; we only consider one relatively tractable special case. Our theorem is as follows.

Theorem 1.9. *Suppose for finite constants* $\rho>1$, $\nu>0$, $c\geq 0$, $r_n \sim c\rho^n/n^{1+\nu}T(n)$ *as* $n\to\infty$; *where* T *is a tail function with an associated gauge function* G *such that* $\sum_1^\infty G(n)/n^{1+\nu}T(n)$ *converges. Let* $R(z) = \sum_{-\infty}^{+\infty} r_n z^n$ *have only* κ *(finite) zeros in the annulus* $1 < |z| \leq \rho^{-1}$ *and, for simplicity, assume these to be single zeros situated at* $\zeta_1, \zeta_2, \ldots, \zeta_\kappa$ *and none are on* $|z| = \rho^{-1}$. *Then, for* $\mu_1 > 0$,

$$u_n = \frac{1}{\mu_1} + \frac{1}{\mu_1}\sum_{s=1}^{\kappa} \frac{1}{(\zeta_s-1)R'(\zeta_s)\zeta_s^{n+1}} + t_n ,$$

where, as $n\to\infty$,

$$t_n \sim \frac{c\rho^{n+1}}{\mu_1[R(\rho^{-1})]^2(1-\rho)n^{1+\nu}T(n)} .$$

(A similar result holds if $\mu_1 < 0$ *.)*

It might be mentioned that O and o versions of this theorem can also (more easily) be proved. A challenging problem we have so far been unable to solve is that posed when $r_n \sim c\rho^n T(n)$.

Finally we close this section by mentioning an unfortunate weakness of the method we have adopted. The umbrella condition U is needed for the manufacture of smooth mutilator functions in §3; these are vital to our "elementary" method of proof. But they are unnecessary if we adopt a Banach algebra attack. As far as the theory of recurrent events is concerned, all that is needed is the inference that if the Fourier series $\hat{\underline{r}}(\theta)$ is non-vanishing and in a certain sub-algebra then so is $1/\hat{\underline{r}}(\theta)$; this could easily be dealt with by an appeal to a "well-known" result concerning maximal ideals in a commutative Banach algebra (see, e.g. Rudin (1974), Theorem 18.17, p. 395). The gain by using the more sophisticated attack is a slight broadening of the class of tail functions we could allow; condition U could be replaced by the weaker requirement $\alpha(u)/u \to 0$ as $u\to\infty$. However, if the present methods are applied to corresponding problems in renewal theory in continuous time, the use of smooth mutilator functions seems unavoidable, even if one is prepared to appeal to the theory of commutative Banach algebras. It would take us too far afield in these notes to explain what prompts this claim. Thus it seems that in renewal theory the U condition is less of a drawback.

§2. SOME PRELIMINARY LEMMAS

In this work we are largely concerned with sequences tending to limits and with the rapidity of that convergence. As a means of estimating that rapidity we shall make use of certain classes of monotone functions. This section is devoted to the defining of these classes and the discovery of various important properties of the functions in these classes.

A function $f(x)$, say, of real $x\geq 0$, is called a *function of moderate growth* (fmg) if $f(x+c) \sim f(x)$ as $x\to\infty$, for every fixed c. In a discussion in which functions are only evaluated at integers it is enough if $f(n+1) \sim f(n)$ as $n\to\infty$ through the integers.

We denote by M the class of *right moment functions* (rmf), which satisfy the following conditions.

(M1) $M(x)\equiv 1$ *for* $x<0$, $M(x)\geq 1$ *for* $x\geq 0$, $M(x)$ *is non-decreasing in* $[0,\infty)$.

(M2) $M(x) \sim M(x+c)$ *as* $x\to\infty$, *for all fixed* c.

(M3) $M(x+y) \leq M(x)M(y)$ *for all* x,y.

Notice that (M3) need only be verified for $x\geq 0$ and $y\geq 0$ since (M1) ensures its automatic fulfillment otherwise.

If $A(x)$ and $B(x)$ are functions of $x\geq 0$, we say they are *equivalent* and write $A(x) \asymp B(x)$ if $A(x)/B(x)$ and $B(x)/A(x)$ are both bounded as $x\to\infty$.

Suppose $N(x)$ is a non-decreasing function of moderate growth such that, for finite positive constants Δ_1 and Δ_2, it is true that

$N(x+y) \leq \Delta_2 N(x)N(y)$ for all $x \geq \Delta_1$ and all $y \geq \Delta_1$. By increasing Δ_2 if necessary, we can ensure that $\Delta_2 N(\Delta_1) \geq 1$. Then define

$$\begin{aligned} M(x) &\equiv 1 && \text{for } x<0, \\ &\equiv \Delta_2 N(\Delta_1) && \text{for } 0 \leq x \leq \Delta_1, \\ &\equiv \Delta_2 N(x) && \text{for } x>\Delta. \end{aligned}$$

It may be verified that $M(x)$ is indeed a rmf and $N(x) \asymp M(x)$. Because we merely use moment functions to indicate convergence properties of series, we are only concerned with asymptotic behavior of a rmf as $x \to \infty$. Thus, as will appear, it is entirely adequate to give a function like $N(x)$, above, knowing that an equivalent rmf satisfying the more convenient conditions (M1), (M2), (M3) exists. Let M^* be the class of functions like $N(x)$. Then three examples of functions in M^* are as follows: (a) $M(x) \equiv 1$; (b) any non-decreasing function which $\sim x^{\alpha} L(x)$ as $x \to \infty$, where $\alpha \geq 0$ is a constant and $L(x)$ is a function of slow growth; (c) any non-decreasing function which $\sim \exp \sqrt{x}$, as $x \to \infty$.

We denote by T the class of *tail functions* $T(x)$, defined for $x \geq 0$, which satisfy:

(T1) $T(x) \geq 1$ for all $x \geq 0$ and $T(x)$ is non-decreasing

(T2) $T(x) \sim T(x+c)$ as $x \to \infty$, for every fixed c (i.e. $T(x)$ is a fmg).

As was the case for the classes M and M^*, we are only concerned with the behavior of $T(x)$ for large x; thus, in specifying a tail function we need only give its asymptotic form as $x \to \infty$. It may be noted that examples (a), (b), (c), above, of right moment functions provide examples also of tail functions.

With any tail function $T(x)$ we associate functions called *gauge functions*. A *gauge function* is any right moment function $G(x)$, say, such that for some finite constant $C>0$:

$$\frac{T(n)}{T(n-r)} \leq CG(r) \tag{2.1}$$

for all large n and all $n \geq 2r$. It might be noted that, since $T(x)$ is non-decreasing, (2.1) is automatic for $r<0$ since, being a rmf, $G(r)\equiv 1$ when $r<0$.

By way of illustration we remark that in example (b), above, we can take $G(r)$ as a constant for $r\geq 0$. In case (c) we see that

$$\frac{e^{\sqrt{n}}}{e^{\sqrt{n-r}}} \leq e^{(\sqrt{2}-1)\sqrt{r}} .$$

Notice that in both of these examples the gauge function grows much more slowly than the "parent" tail function. This is typical for the situations we encounter but not necessarily true for an arbitrary tail function.

We denote sequences by letters $\underline{a}, \underline{b}, \underline{c}, \ldots$ as far as possible. Typically these sequences are doubly infinite, thus $\underline{a}$ has terms

$$\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots .$$

We call $\underline{a}$ a *right sequence* if $a_n = 0$ for all $n<0$, and R for the class of such sequences. We call $\underline{a}$ a *positive sequence* if $a_n \geq 0$ for all n, and P for the class of such sequences. Sometimes, when the symbol for a sequence is cumbersome, we denote its nth term as $(\underline{a})_n$, but whenever possible we write a_n, b_n, etc. The norm of $\underline{a}$ is defined as

$$||a|| = \sum_{n=-\infty}^{+\infty} |a_n|.$$

If $T(x)$ is any non-negative function on the integers, we also write

$$||\underline{a}||_T = \sum_{n=-\infty}^{+\infty} T(n)|a_n|.$$

We shall write $S(T)$ for the class of sequences $\underline{a}$ such that $||\underline{a}||_T < \infty$.

If $T(x)$ is any non-negative function of the positive integers we shall write

$$[\underline{a}]_T = \limsup_{n\to\infty} T(n)|a_n|,$$

and $V(T)$ is the class of sequences $\underline{a}$ for which $[\underline{a}]_T$ is finite.

Suppose however, $T(n)a_n$ tends to a, possibly complex, limit as $n\to\infty$. We then say $\underline{a}$ belongs to $W(T)$ and write

$$\langle\underline{a}\rangle_T = \lim_{n\to\infty} T(n)a_n .$$

Note that $W(T) \subset V(T)$ and, if $\underline{a} \in W(T)$, $[\underline{a}]_T = |\langle\underline{a}\rangle_T|$.

If $[\underline{a}]_T = 0$ we shall say $\underline{a}$ is *T-null* and write $V_0(T)$ for the class of such sequences. Obviously $V_0(T) \subset W(T)$ and if $\underline{a} \in V_0(T)$ then $\langle\underline{a}\rangle_T = 0$.

If $\underline{a}$ and $\underline{b}$ are two sequences we write $\underline{a}*\underline{b}$ for the convolution whose nth term is

$$(\underline{a}*\underline{b})_n = \sum_{m=-\infty}^{+\infty} a_m b_{n-m} .$$

It is clear that $\underline{a}*\underline{b} = \underline{b}*\underline{a}$ and that $||a*b|| \leq ||a||\cdot||b||$. In an obvious way we define sequences $\underline{a}*\underline{a}*\ldots*\underline{a}$ (k terms), more compactly written a^{*k} , and have $||a^{*k}|| \leq ||a||^k$.

In a few places it is helpful to interpret $(\underline{a}^{*r})$ as the sequence $\{\underline{\delta}_n\} = \underline{\delta}$, say, in which $\delta_0 = 1$ and $\delta_n = 0$ for all $n\neq 0$.

When $||a|| < 1$ we can introduce the sequence $\Delta[\underline{a}]$ whose nth term is

$$(\underline{a}^{*0})_n + (\underline{a}^{*1})_n + (\underline{a}^{*2})_n + \dots .$$

We find that

$$||\Delta[\underline{a}]|| \le 1/(1 - ||\underline{a}||).$$

LEMMA 2.1. *If* $\underline{a}$ *and* $\underline{b}$ *are sequences in* $S(M)$, *for some right moment function* M, *then* $||\underline{a}*\underline{b}||_M \le ||\underline{a}||_M \circ ||\underline{b}||_M$.

PROOF. Let $\underline{c} = \underline{a}*\underline{b}$. Then

$$||\underline{a}*\underline{b}||_M = \sum_{-\infty}^{+\infty} M(n)c_n$$

$$= \sum_{n_1} \sum_{n_2} M(n_1+n_2)a_{n_1}b_{n_2}$$

$$\le \sum_{n_1} \sum_{n_2} M(n_1)M(n_2)a_{n_1}b_{n_2} ,$$

by (M3). Thus the lemma follows.

LEMMA 2.2. *If* $\underline{a} \in S(M)$, *where* M *is a rmf, then there exists a sequence* $\{\varepsilon_k\}$, *decreasing to zero as* $k\to\infty$, *such that*

$$||\underline{a}^{*k}||_M \le (||a||+\varepsilon_k)^k , \quad \text{all } k\ge 1. \tag{2.2}$$

PROOF. Plainly we need only prove (2.2) for large k. To begin with suppose $\underline{a} \in P\cap R$, and write, for simplicity, $\sigma_k = ||\underline{a}^{*k}||_M$. Then, for $k>1$,

$$\sigma_k = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} M(n_1+n_2+\ldots+n_k)a_{n_1}a_{n_2}\ldots a_{n_k}$$
$$\leq \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} M(k+n_1+n_2+\ldots+n_k)a_{n_1}a_{n_2}\ldots a_{n_k} ,$$
$$= \tilde{\sigma}_k \text{ , say,}$$

since M is non-decreasing in $[0,\infty)$. Set

$$\tau_k = \sum_{n_k=0}^{\infty} \frac{M(k+n_1+n_2+\ldots+n_k)}{M(k-1+n_1+\ldots+n_{k-1})} a_{n_k} \tag{2.3}$$

$$= \sum_{r=0}^{\infty} \frac{M(k+w+r)}{M(k-1+w)} a_r ,$$

where $w = n_1+n_2+\ldots+n_{k-1} \geq 0$. However, since $M(x)$ is a fmg,

$$\frac{M(k+w+r)}{M(k-1+w)} \to 1,$$

as $k\to\infty$, for r fixed, uniformly with respect to w. But, from (M3),

$$\frac{M(k+w+r)}{M(k-1+w)} \leq M(1+r)$$

and it is trivial that $\sum_0^\infty M(1+r)a_r = C$, say, is finite. Thus dominated convergence applied to (2.3) shows that $\tau_k \to ||\underline{a}||$ as $k\to\infty$, uniformly with respect to $w\geq 0$. Thus we can set $\tau_k \leq ||a|| + \eta_k$, say, where $\eta_k \downarrow 0$ as $k\to\infty$. But $\tilde{\sigma}_k = \tau_k\tilde{\sigma}_{k-1}$ and we are evidently led to the result

$$\sigma_k \leq \tilde{\sigma}_k \leq C(||\underline{a}||+\eta_2)(||a||+\eta_3) \ \ldots \ (||a||+\eta_k).$$

It is a routine matter, now, to establish (2.2).

We shall now extend this result to a sequence in P but not necessarily in R. Let $\underline{a}$ be such a sequence, and define a new sequence $\underline{a}^R = \{a_n^R\}$ by

$$a_n^R = a_n \quad \text{for} \quad n \ge 0$$
$$= 0 \quad \text{otherwise.}$$

Let us also set $a_n^L = a_n - a_n^R$ to define the sequence $\{\underline{a}^L\}$. Then the first part of this proof applies to $\underline{a}^R$ and gives, in obvious notation,

(2.4) $$\sigma_k^R \le C(||\underline{a}^R||+\eta_2)(||\underline{a}^R||+\eta_3) \ldots (||\underline{a}^R||+\eta_k).$$

Evidently we must now consider

(2.5) $$\sigma_k = \sum_{-\infty}^{+\infty} \ldots \sum_{-\infty}^{+\infty} M(n_1+n_2+\ldots+n_k)a_{n_1}a_{n_2}\ldots a_{n_k}$$
$$= \sum_{\ell=0}^{k} \binom{k}{\ell} \sum_{n_1\ge 0} \sum_{n_2\ge 0} \ldots \sum_{n_\ell \ge 0} \sum_{n_{\ell+1}<0} \ldots \sum_{n_k<0} \quad \text{etc.}$$

But if $n_1\ge 0, n_2\ge 0, \ldots, n_\ell\ge 0, n_{\ell+1}<0, \ldots, n_k<0$, then

$$M(n_1+n_2+\ldots+n_k) \le M(n_1+n_2+\ldots+n_\ell)$$

and (2.5) gives, using (2.4)

(2.6) $$\sigma_k \le C \sum_{\ell=0}^{k} \binom{k}{\ell} ||\underline{a}^L||^{k-\ell}(||\underline{a}^R||+\eta_2) \ldots (||\underline{a}^R||+\eta_k).$$

Recall that $\eta_k \downarrow$. Let $\rho(k)\to\infty$, $\rho(k)/k \to 0$, as $k\to\infty$. Then it is not hard to see

$$\sigma_k \le C \frac{(||\underline{a}^R||+\eta_2)^{\rho(k)-2}}{(||\underline{a}^R||+\eta_{\rho(k)})^{\rho(k)-1}} \sum_{\ell=0}^{k} \binom{k}{\ell} ||\underline{a}^L||^{k-\ell}(||\underline{a}^R||+\eta_{\rho(k)})^{\ell}$$
$$\le C \frac{(||\underline{a}^R||+\eta_2)^{\rho(k)-2}}{||\underline{a}^R||^{\rho(k)-1}} (||\underline{a}^L||+||\underline{a}^R||+\eta_{\rho(k)})^k .$$

From which it follows, since $||a|| = ||a^L||+||a^R||$ and $\eta_{\rho(k)} \to 0$, that

$$\limsup_{k\to\infty} (\sigma_k)^{1/k} \le ||a||.$$

This proves the lemma since the dropping of the assumption $\underline{a}\epsilon P$ calls for only the most obvious device.

Let $\Psi(z)$, a function of complex z, be analytic in the disc $|z| \leq r$, for some $r>0$. Then $\Psi(z)$ has a unique Taylor expansion

$$\Psi(z) = \sum_{m=0}^{\infty} \psi_m z^m \text{ , say,}$$

and the radius of convergence of this series is at least r. Now let $\underline{a}$ be a sequence in $S(M)$, where M is some moment function. We can unambiguously define a new sequence $\Psi(\underline{a})$ whose nth term is

$$\sum_{m=0}^{\infty} \psi_m (\underline{a}^{*m})_n \text{ .}$$

<u>LEMMA 2.3.</u> *If* $\Psi(z)$ *is analytic in the disc* $|z| \leq r$, *if* $\underline{a} \in S(M)$, *and if* $||a|| < r$, *then* $\Psi(\underline{a})$ *also belongs to* $S(M)$.

<u>PROOF.</u> We can find $\varepsilon>0$ such that $||a||+\varepsilon < r$, and then $k_0(\varepsilon)$ such that for all $m\geq k$,

$$||\underline{a}^{*m}||_M \leq (||\underline{a}||+\varepsilon)^m \text{ .}$$

Thus, if

$$\Psi_{k_0}(\underline{a}) = \sum_{m=k_0}^{\infty} \psi_m(\underline{a}^{*m})$$

then

$$\begin{aligned} ||\Psi_{k_0}(\underline{a})||_M &\leq \sum_{m=k_0}^{\infty} |\psi_m|(||a||+\varepsilon)^m \\ &< \infty, \end{aligned}$$

since $\sum_0^\infty \psi_m z^m$ is absolutely convergent within its radius of convergence. But

$$\begin{aligned} ||\Psi(\underline{a})-\Psi_{k_0}(\underline{a})||_M &\leq \sum_{m=0}^{k_0-1} |\psi_m| \ ||a^{*m}||_M \\ &< \infty, \end{aligned}$$

by finitely many applications of Lemma 2.1. Thus the lemma is proved.

In what follows $T(x)$ will be a tail function and $G(r)$ a corresponding gauge function.

<u>LEMMA 2.4</u>. *If* $\underline{a}$ *and* $\underline{b}$ *are sequences in* $V(T)$ *and* $S(G)$ *then*

$$[\underline{a}*\underline{b}]_T \le ||b||[\underline{a}]_T + ||a||[\underline{b}]_T . \tag{2.7}$$

Furthermore, if $\underline{a}$ *and* $\underline{b}$ *are also in* $W(T)$ *then so is* $\underline{a}*\underline{b}$ *and*

$$\langle\underline{a}*\underline{b}\rangle_T = \langle\underline{a}\rangle_T \sum_{-\infty}^{+\infty} b_n + \langle\underline{b}\rangle_T \sum_{-\infty}^{+\infty} a_n .$$

<u>PROOF</u>. Let us set $c_n = (\underline{a}*\underline{b})_n$; then for n large and positive

$$c_n = \sum_{r\le\frac{1}{2}n} a_{n-r}b_r + \sum_{r<\frac{1}{2}n} b_{n-r}a_r$$

$$= c_n' + c_n'' , \text{ say.}$$

Then we observe that

$$T(n)c_n' = \sum_{r\le\frac{1}{2}n} \frac{T(n)}{T(n-r)} \{T(n-r)a_{n-r}\}b_r .$$

For all $r \le \frac{1}{2}n$ we have

$$\frac{T(n)}{T(n-r)} \le CG(r),$$

where $G(r) \equiv 1$ for $r<0$. If we suppose $\underline{a} \in V(T)$ then

$$T(n-r)|a_{n-r}| \le [\underline{a}]_T + \varepsilon_n , \text{ say,}$$

where $\varepsilon_n \to 0$, uniformly with respect to $r \le \frac{1}{2}n$. Since $T(n) \sim T(n-1)$ as

$n\to\infty$, and since, if $\underline{b} \in S(G)$,

$$\sum_{-\infty}^{-1} |b_r| + \sum_{0}^{\infty} G(r)|b_r| < \infty,$$

we can appeal to dominated convergence to infer that

$$\limsup_{n\to\infty} T(n)|c_n'| \le [\underline{a}]_T \,||\underline{b}||.$$

On the other hand, if $\underline{a} \in W(T)$ the same dominated convergence will show that, as $n\to\infty$,

$$T(n)c_n' \to \langle\underline{a}\rangle_T \sum_{-\infty}^{+\infty} b_r \,.$$

In a similar way we can deal with c_n'' and complete the proof of the lemma.

Suppose $\underline{a}^{(1)},\underline{a}^{(2)},\ldots,\underline{a}^{(k)}$ are k sequences in $W(T) \cap S(G)$. Let us define, for $r=1,2,\ldots,k$,

$$\underline{c}^{(r)} = \underline{a}^{(1)} * \underline{a}^{(2)} * \ldots * \underline{a}^{(r)} \,.$$

Lemma 2.1 shows, by repetitive applications, that $\underline{c}^{(r)} \in S(G)$ and one can also easily obtain the equation:

$$\sum_{n=-\infty}^{+\infty} c_n^{(r)} = \prod_{j=1}^{r} \left(\sum_{n=-\infty}^{+\infty} a_n^{(j)}\right) .$$

Lemma 2.3 shows that (assuming $k\ge 2$):

$$[\underline{c}^{(2)}]_T \le [\underline{a}^{(1)}]_T||\underline{a}^{(2)}|| + [\underline{a}^{(2)}]_T||\underline{a}^{(1)}||$$

and (if $k\ge 3$) one can then deduce from the same lemma the result:

$$[\underline{c}^{(3)}]_T \le [\underline{c}^{(2)}]_T||\underline{a}^{(3)}|| + [\underline{a}^{(3)}]_T||\underline{c}^{(2)}||.$$

Since $||\underline{c}^{(2)}|| \le ||\underline{a}^{(1)}|| \circ ||\underline{a}^{(2)}||$ this gives

$$[\underline{c}^{(3)}]_T \le \sum_{j=1}^{3} [\underline{a}^{(j)}]_T \prod_{r\neq j} ||\underline{a}^{(r)}||.$$

The generalization of this should be plain, as should be the way by which Lemma 2.3 can apply to $\underline{c}^{(r)}$ if every sequence $\underline{a}^{(j)}$ belongs to $W(T)$. Thus we have the following

LEMMA 2.5. *In the notation just established, if every* $\underline{a}^{(j)}$ *belongs to* $V(T) \cap S(G)$ *then* $\underline{c}^{(k)}$ *also belongs to* $V(T) \cap S(G)$ *and*

$$[c^{(k)}]_T \le \sum_{j=1}^{k} [\underline{a}^{(j)}]_T \prod_{r\neq j} ||\underline{a}^{(r)}||.$$

Furthermore, if every $\underline{a}^{(j)} \in W(T) \cap S(G)$ *then* $\underline{c}^{(k)}$ *belongs to* $W(T) \cap S(G)$ *and*

$$\langle\underline{c}^{(k)}\rangle_T = \sum_{j=1}^{k} \langle\underline{a}^{(j)}\rangle_T \prod_{r\neq j} \left\{\sum_{n=-\infty}^{+\infty} a_n^{(r)}\right\}.$$

An immediate and important consequence of this lemma is the following.

LEMMA 2.6. *If* $\underline{a}$ *is a sequence in* $V(T) \cap S(G)$ *then, for every integer* $k\ge 1$, $\underline{a}^{*k}$ *is in* $V(T) \cap S(G)$ *and*

$$[\underline{a}^{*k}]_T \le k[\underline{a}]_T ||\underline{a}||^{k-1} .$$

If, moreover, $\underline{a}$ *is in* $W(T) \cap S(G)$ *then so is* $\underline{a}^{*k}$ *and*

$$\langle\underline{a}^{*k}\rangle_T = k\langle\underline{a}\rangle_T \{\sum_{-\infty}^{+\infty} a_n\}^{k-1} .$$

Suppose now that $\underline{a}$ is any sequence in $P \cap R$ such that $||\underline{a}|| = \sum_0^\infty a_n < 1$. We shall sometimes write $\Delta(z)$ for the function $1/(1-z)$, which is analytic in any circle $|z| \le r$ with $0<r<1$. Thus $\Delta(\underline{a})$ is also a sequence in $P \cap R$, and $||\Delta(\underline{a})|| < \infty$ by Lemma 2.3. For ease let us write $\underline{g}$ for $\Delta(\underline{a})$. Thus

$$g_n = a_n + (a^{*2})_n + (a^{*3})_n + \dots \tag{2.8}$$

and an easy calculation shows

$$||\underline{g}|| = \textstyle\sum_0^\infty g_n = \frac{||\underline{a}||}{1-||\underline{a}||} . \tag{2.9}$$

LEMMA 2.7. *Let* $T(x)$ *be a tail function and* $G(x)$ *a corresponding gauge function. Let* $\underline{a} \in P \cap R$ *and suppose* $||\underline{a}|| < 1$. *Then if* $\underline{a} \in S(G)$ *it follows that* $\Delta(\underline{a}) \in S(G)$. *Furthermore:*

(A) If $\underline{a} \in V(T) \cap S(G)$ *then*

$$[\Delta(\underline{a})]_T \le \frac{[\underline{a}]_T}{(1-||\underline{a}||)^2} . \tag{2.10}$$

(B) If $\underline{a} \in W(T) \cap S(G)$ *then* $\Delta(\underline{a})$ *also belongs to* $W(T) \cap S(G)$ *and*

$$\langle \Delta(\underline{a}) \rangle_T = \frac{\langle \underline{a} \rangle_T}{(1-||\underline{a}||)^2} . \tag{2.11}$$

PROOF. The fact that $\Delta(\underline{a}) \in S(G)$ follows at once from Lemma 2.3. Evidently

$$g_n = a_n + \sum_{r=0}^{\infty} a_r g_{n-r} . \tag{2.12}$$

Next suppose $\underline{a} \in V(T) \cap S(G)$, and rewrite (2.12), as follows:

$$g_n = a_n + \sum_{0 \le r \le \frac{1}{2}n} a_r g_{n-r} + \sum_{0 \le r < \frac{1}{2}n} g_r a_{n-r} . \tag{2.13}$$

Let us then set

$$c_n' = \sum_{0 \le r < \frac{1}{2}n} g_r a_{n-r} . \tag{2.14}$$

Then, as in the proof of Lemma 2.4, we argue:

$$T(n)|c_n'| \le \sum_{0\le r<\frac{1}{2}n} \frac{T(n)}{T(n-r)} g_r\{T(n-r)a_r\}. \tag{2.15}$$

We have established $||\underline{g}||_G < \infty$ and so the dominated convergence argument will go just as before, and show

$$[\underline{c}']_T \le [\underline{a}]_T||\underline{g}||.$$

Our major difficulty, however, is concerned with

$$\sum_{0\le r\le\frac{1}{2}n} a_r g_{n-r} = c_n'' \text{ , say.}$$

Let us set $T(n)g_n = \tau(n)$, and suppose that $\tau(n)$ is unbounded as $n\to\infty$. Then it is possible to find a sequence of integers $n_1<n_2<\ldots$ such that $\tau(n_j) \to \infty$ and $\tau(n) < \tau(n_j)$ for all $n < n_j$. We can then deduce from (2.13) that

$$\tau(n_j) \le T(n_j)a_{n_j} + T(n_j)c_{n_j}' + \tau(n_j) \sum_{0\le r\le\frac{1}{2}n} \frac{T(n_j)}{T(n_j-r)} a_r . \tag{2.16}$$

Hence

$$\tau(n_j) \le \frac{T(n_j)(a_{n_j}+c_{n_j}')}{1-\sum_{0\le r\le\frac{1}{2}n} \frac{T(n_j)}{T(n_j-r)} a_r} . \tag{2.17}$$

Since $||\underline{a}||_G < \infty$, a dominated convergence argument will show

$$\sum_{0\le r\le\frac{1}{2}n} \frac{T(n_j)}{T(n_j-r)} a_r \to \sum_0^\infty a_r \text{ , as } n_j \to \infty. \tag{2.18}$$

Thus (2.17) coupled with (2.18) and the asymptotic behavior of a_n and c_n' gives

$$\limsup_{j\to\infty} \tau(n_j) \le \frac{[\underline{a}]_T(1+||g||)}{1-||a||} . \tag{2.19}$$

This contradicts the hypothesis that $\tau(n)$ is unbounded. Thus $\tau(n)$, as $n\to\infty$, has a finite upper limit Λ, say, and this alone establishes that $\underline{g} \in V(T)$. However, let $\{n_j\}$ now be an increasing sequence of integers such that $\tau(n_j) \to \Lambda$ as $j\to\infty$. For all $n \geq \frac{1}{2}n_j$ we may claim that

$$\tau(n) \leq \tau(n_j) + \varepsilon_j ,$$

where $\varepsilon_j \to 0$ as $j\to\infty$. Thus (2.13) yields, on lines similar to (2.16),

$$\tau(n_j) \leq T(n_j)a_{n_j} + \{\tau(n_j)+\varepsilon_j\} \sum_{0\leq r\leq\frac{1}{2}n_j} \frac{T(n_j)}{T(n_j-r)} a_r + T(n_j)c'_{n_j} .$$

Hence, letting $j\to\infty$ we infer

$$\Lambda \leq [\underline{a}]_T\{1+||g||\} + \Lambda \circ ||a||,$$

which tells us that

$$[\underline{g}]_T \leq \frac{[\underline{a}]_T \; 1+||\underline{g}||}{1-||\underline{a}||} . \tag{2.20}$$

If we note that $\{1+||\underline{g}||\} = 1/\{1-||\underline{a}||\}$ then (2.20) implies (2.11) in the lemma.

Finally, suppose $\underline{a} \in W(T)\cap S(G)$. This implies $\underline{a} \in V(T)$, of course, so the preceding results are all valid. Further, (2.14) shows

$$T(n)c'_n = \sum_{0\leq r\leq\frac{1}{2}n} \frac{T(n)}{T(n-r)} g_r\{T(n-r)a_r\}$$

and a, by now familiar, dominated convergence argument gives

$$T(n)c'_n \to \langle\underline{a}\rangle_T||\underline{g}||. \tag{2.21}$$

Let us write

$$\Gamma = \liminf_{n\to\infty} \tau(n).$$

Then we can claim that for all $r \le \frac{1}{2}n$

$$\tau(n-r) \ge \Gamma - \varepsilon_n ,$$

where $\varepsilon_n \to 0$ as $n\to\infty$. From (2.13) we may therefore write

$$(2.22) \qquad \tau(n) \ge T(n)\, a_n + c_n' + (\Gamma-\varepsilon_n) \sum_{0\le r<\frac{1}{2}n} \frac{T(n)}{T(n-r)} g_r .$$

Dominated convergence shows

$$\sum_{0\le r<\frac{1}{2}n} \frac{T(n)}{T(n-r)} g_r \to ||\underline{g}||, \text{ as } n\to\infty,$$

and so (2.22) yields (if we let $n\to\infty$ through a subsequence such that $\tau(n) \to \Gamma$)

$$\Gamma \ge \langle\underline{a}\rangle_T[1+||\underline{g}||] + \Gamma \cdot ||\underline{g}||.$$

Thus

$$\Gamma \ge \frac{\langle\underline{a}\rangle_T[1+||\underline{g}||]}{1-||\underline{a}||},$$

showing that $\Gamma = \Lambda$, i.e. $T(n)g_n$ tends to a limit $\langle\underline{g}\rangle_T$, as $n\to\infty$, and

$$\langle\underline{g}\rangle_T = \frac{\langle\underline{a}\rangle_T[1+||\underline{g}||]}{1-||\underline{a}||} .$$

Thus the lemma is proved.

However, we now wish to extend Lemma 2.7 to deal with sequences in P but not necessarily in R. Our argument parallels those arising in the study of ladder variables in random walk theory.

Suppose, therefore, that $\underline{a}\in P$ and that $\sum_{-\infty}^{+\infty} a_n = ||\underline{a}|| < 1$.

For $n=-1,-2,\ldots$ define

(2.23) $$b_n = \sum_{k=1}^{\infty} \{ \sum_{J(n,k)} a_{r_1} a_{r_2} \dots a_{r_k} \},$$

where $J(n,k)$ is the set of suffices $\{r_1, r_2, \dots, r_k\}$ such that:

$$\begin{cases} r_1 < 0 \\ r_1 + r_2 < 0 \\ \dots . \\ r_1 + r_2 + \dots + r_{k-1} < 0 \\ r_1 + r_2 + \dots + r_k = n. \end{cases}$$

For $n \geq 0$, define $b_n = 0$.

As before, we shall write $\Delta(\underline{a}) = \underline{g}$, for ease. It follows from Lemma 2.3 that $\underline{g} \in S(G)$ and one has without difficulty the result:

$$||\underline{g}|| = \frac{||\underline{a}||}{1-||\underline{a}||} < \infty.$$

Since $b_n \leq g_n$ for all n and, since $b_n = 0$ for $n \geq 0$, one has:

$$||\underline{b}||_G = ||b|| \leq ||\underline{g}|| < \infty.$$

Next define a sequence $\underline{\ell} \equiv \{\ell_n\} \in P \cap R$ as follows:

If $n<0$ then, of course, $\ell_n = 0$.

If $n \geq 0$ then

(2.24) $$\ell_n = a_n + \sum_{m=-\infty}^{-1} a_{n-m} b_m .$$

We have immediately, from (2.24) and Lemma 2.1, that

(2.25) $$||\underline{\ell}|| = ||\underline{a}||\{1 + ||\underline{b}||)$$

and

(2.26) $$||\underline{\ell}||_G \leq ||g||_G\{1 + ||\underline{b}||\}.$$

If we now assume $\underline{a} \in V(T)$ then a familiar argument (which does not need the assumption $\underline{a} \in S(G)$) yields from (2.24)

$$[\underline{\ell}]_T \leq [\underline{a}]_T(1 + ||\underline{b}||). \tag{2.27}$$

Alternatively, if $\underline{a} \in W(T)$, an almost identical argument shows

$$<\underline{\ell}>_T = <\underline{a}>_T(1 + ||\underline{b}||). \tag{2.28}$$

Now, evidently,

$$g_n = \sum_{k=1}^{\infty} \sum_{I(n,k)} a_{r_1} a_{r_2} \ldots a_{r_k} , \tag{2.29}$$

where $I(n,k)$ is the set of suffices $(r_1,r_2,\ldots,r_k)$ such that $r_1+r_2+\ldots+r_k = n$. A careful dissection of the set

$$\sum_{k=1}^{\infty} I(n,k)$$

leads to the following important relation, valid for all n:

$$g_n = b_n + \ell_n + (\underline{\ell}*\underline{g})_n . \tag{2.30}$$

(This equation is inspired by the work on ladder variables, already mentioned.)

An immediate deduction from (2.30), by summation, is that

$$||\underline{g}|| = ||\underline{b}|| + ||\underline{\ell}|| + ||\underline{\ell}|| \cdot ||\underline{g}||$$

and hence, that

$$||\underline{g}|| = \frac{||\underline{b}||+||\underline{\ell}||}{1-||\underline{\ell}||} . \tag{2.31}$$

Further, if we set $\underline{k} \equiv \Delta(\underline{\ell})$, it follows from (2.30) that

(2.32) $$\underline{g} = \underline{k} + \underline{b} + \underline{k}*\underline{b}.$$

However, $\underline{\ell} \in P \cap R$, so we can appeal to Lemma 2.7 and (2.27) or (2.28). If we suppose $\underline{a} \in V(T)$ then

(2.33) $$[\underline{k}]_T \leq \frac{[\underline{a}]_T(1+||\underline{b}||)}{(1-||\underline{\ell}||)^2} .$$

If $\underline{a} \in W(T)$ then $k \in W(T)$ and

(2.34) $$<\underline{k}>_T = \frac{<\underline{a}>_T(1+||\underline{b}||)}{(1-||\underline{\ell}||)^2} .$$

If we note that $\underline{b}$ is a T-null sequence then we can deduce from Lemma 2.4 the further results:

If $\underline{a} \in V(T)$ then

(2.35) $$[\underline{b}*\underline{k}]_T \leq \frac{[\underline{a}]_T(1+||\underline{b}||)||\underline{b}||}{(1-||\underline{\ell}||)^2} .$$

If $\underline{a} \in W(T)$ then

(2.36) $$<\underline{b}*\underline{k}>_T = \frac{<\underline{a}>_T(1+||\underline{b}||)||\underline{b}||}{(1-||\underline{\ell}||)^2} .$$

It follows from (2.32) and either (2.35) or (2.36) that

If $\underline{a} \in V(T)$ then

(2.37) $$[\underline{g}]_T \leq [\underline{a}]_T \frac{(1+||\underline{b}||)^2}{(1-||\underline{\ell}||)^2} .$$

If $\underline{a} \in W(T)$ then

(2.38) $$<\underline{g}>_T = <\underline{a}>_T \frac{(1+||\underline{b}||)^2}{(1-||\underline{\ell}||)^2} .$$

But (2.31) tells us that

$$1 + ||\underline{g}|| = \frac{1+||\underline{b}||}{1-||\underline{\ell}||}$$

and, since $||\underline{g}|| = ||\underline{a}||/(1-||\underline{a}||)$, we deduce

$$\frac{1+||\underline{b}||}{1-||\underline{\ell}||} = \frac{1}{1-||\underline{a}||} .$$

This result, used in (2.37) and (2.38), completes the proof of the lemma extended to sequences $\underline{a}$ not necessarily in R:

LEMMA 2.8. *The conclusions of Lemma 2.7 are valid without the requirement* $\underline{a} \in R$.

If $\Psi(z)$, as earlier, is a function regular in $|z| < r$ (>0), and if

$$\Psi(z) \equiv \sum_0^\infty \psi_r z^r , \quad |z| < r,$$

then define

$$|\Psi|(z) \equiv \sum_0^\infty |\psi_r| z^r .$$

Evidently $|\Psi|(z)$ is also regular in $|z| < r$.

LEMMA 2.9. *Let* Ψ, $|\Psi|$, r, *be as above. Let* T *be a tail function with gauge function* G. *Let* $\underline{a} \in S(G)$ *and suppose* $||\underline{a}|| < r$. *Then*

(A) If $\underline{a}$ *belongs to* $V(T)$ *it follows that* $\Psi(\underline{a})$ *also belongs to* $V(T)$ *and*

$$[\Psi(\underline{a})]_T \le [\underline{a}]_T |\Psi|'(||\underline{a}||).$$

(B) If $\underline{a}$ *belongs to* $W(T)$ *it follows that* $\Psi(\underline{a})$ *also belongs to* $W(T)$ *and*

$$\langle\Psi(\underline{a})\rangle_T = \langle\underline{a}\rangle_T \Psi'(\textstyle\sum_{-\infty}^{+\infty} a_n).$$

PROOF. Let us write $\tilde{\underline{a}}$ for the sequence $\{|a_n|\}$.

Thus $||\tilde{\underline{a}}|| = ||\underline{a}|| < r$. For ease in writing let us also denote the sequence $\Psi(\underline{a})$ by $\underline{p} \equiv \{p_n\}$.

Lemma 2.3 assures us that $\underline{p} \in S(G)$.

By a well-known property of regular functions $\Psi'(z)$ and $|\Psi|'(z)$ will also be regular in $|z| < r$ and, indeed,

$$\sum_{1}^{\infty} r\psi_r z^{r-1} \tag{2.39}$$

is absolutely convergent for all $|z| < r$.

Since $|\sum_{-\infty}^{+\infty} a_n| \le ||\underline{a}|| < r$, the series (2.39) is absolutely convergent at $\sum_{-\infty}^{+\infty} a_n = \zeta$, say.

Thus, given $\varepsilon>0$, there exists $N(\varepsilon)$ such that

$$\left|\sum_{N+1}^{\infty} r|\psi_r|\ |\zeta|^{r-1}\right| < \varepsilon. \tag{2.40}$$

Let us define two sequences, depending on N fixed,

$$^{N}\underline{p} = \sum_{r=0}^{N} \psi_r \underline{a}^{*r}$$

$$\underline{p}^{N} = \sum_{r=N+1}^{\infty} \psi_r \underline{a}^{*r} .$$

It follows from Lemma 2.6 that:

If $\underline{a} \in V(T)$,

$$[^{N}\underline{p}]_T \le [\underline{a}]_T \sum_{r=1}^{N} r|\psi_r|\ ||\underline{a}||^{r-1} . \tag{2.41}$$

If $\underline{a} \in W(T)$, (recall $\zeta = \sum_{-\infty}^{+\infty} a_n$)

$$<^{N}\underline{p}>_T = <\underline{a}>_T \sum_{r=1}^{N} r\psi_r \zeta^{r-1} . \tag{2.42}$$

Let us choose ρ such that $||\underline{a}|| < \rho < r$. Then $\psi_r \rho^r \to 0$ as $r \to \infty$ and we can therefore assume $N(\varepsilon)$ chosen so large that $|\psi_r|\rho^r \leq 1$ for all $r \geq N$. Then define a sequence $\underline{b} = \{b_n\}$ by setting $b_n = \tilde{a}_n/\rho$, for all n. Notice then that $||\underline{b}|| = ||\underline{a}||/\rho < 1$ and, moreover, if $\underline{a} \in V(T)$ then $\underline{b} \in V(T)$ also, and

$$[\underline{b}]_T = \rho^{-1}[\underline{a}]_T \ .$$

Evidently

(2.43)
$$\begin{aligned} |p_n^N| &\leq \sum_{r=N+1}^{\infty} |\psi_r|(\tilde{a}^{*r})_n \\ &\leq \sum_{r=N+1}^{\infty} \rho^{-r}(\tilde{a}^{*r})_n \\ &= \sum_{r=N+1}^{\infty} (b^{*r})_n \\ &= \left((\underline{b}^{*(N+1)})*\Delta(\underline{b})\right)_n \ . \end{aligned}$$

But if $\underline{b} \in V(T)$, since $||\underline{b}|| < 1$, we can infer from Lemma 2.8 that

(2.44)
$$[\Delta(\underline{b})]_T \leq \frac{[\underline{b}]_T}{(1-||\underline{b}||)^2} \ .$$

In addition, Lemma 2.6 yields:

$$[b^{*(N+1)}]_T \leq (N+1)[\underline{b}]_T||\underline{b}||^N \ .$$

This inequality, coupled with (2.44) and Lemma 2.4, enables us to deduce from (2.43) that

$$\begin{aligned} [p^N]_T &\leq (N+1)[\underline{b}]_T||\underline{b}||^N||\Delta(\underline{b})|| + \frac{[\underline{b}]_T}{(1-||\underline{b}||)^2}||\underline{b}^{*(N\_1)}|| \\ &= [\underline{b}]_T\left\{\frac{(N+1)||\underline{b}||^N}{1-||b||} + \frac{||\underline{b}||^{N+1}}{(1-||\underline{b}||)^2}\right\} \\ &= [\underline{a}]_T\left\{\frac{(N+1)}{\rho-||\underline{a}||} + \frac{||\underline{a}||}{(\rho-||\underline{a}||)^2}\right\}\left(\frac{||\underline{a}||}{\rho}\right)^N. \end{aligned}$$

Since $||\underline{a}|| < \rho$ it is apparent that we can make $[p^N]_T < \varepsilon$ by taking N large enough.

If we suppose $\underline{a} \in V(T)$ and appeal to (2.41) we see that

$$[\underline{p}]_T \le [{}^N\underline{p}]_T + \varepsilon$$
$$\le [\underline{a}]_T \sum_{r=1}^{N} r|\psi_r| \circ ||\underline{a}||^{r-1} + \varepsilon.$$

By letting $N\to\infty$ and $\varepsilon\to 0$ we plainly obtain the desired conclusion:

$$[\underline{p}]_T \le [\underline{a}]_T |\Psi|'(||\underline{a}||).$$

Next suppose that $\underline{a} \in W(T)$. We have:

$$|T(n)p_n - \langle\underline{a}\rangle_T \Psi'(\zeta)| \le |T(n)^N p_n - \langle\underline{a}\rangle_T \sum_1^N r\psi_r \zeta^{r-1}| + |T(n)p_n^N| + \langle\underline{a}\rangle_T \varepsilon, \text{ by (2.40).}$$

But $|T(n)p_n^N| < 2\varepsilon$ for all sufficiently large n, if N is taken large. Thus, in view of (2.42) we have

$$|T(n)p_n - \langle\underline{a}\rangle_T \Psi'(\zeta)| < 4\varepsilon$$

for all large n. This plainly completes the proof.

We close this section by showing that for any tail function T(x) a natural gauge function exists which is, in a sense, minimal; for this reason we shall call it the *minimal gauge function* (mgf).

Define, for $x \ge 0$,

$$\gamma(x) = \sup_{y \ge 2x} \frac{T(y)}{T(y-x)} .$$

Thus for $x_1 \ge 0$, $x_2 \ge 0$,

$$\gamma(x_1)\gamma(x_2) = \sup_{\substack{y_1 \ge 2x_1 \\ y_2 \ge 2x_2}} \frac{T(y_1)T(y_2)}{T(y_1-x_1)T(y_2-x_2)} .$$

But if $y_2 = y_1 - x_1$ and $y_1 \ge 2x_1 + 2x_2$, then $y_1 \ge 2x_1$ and $y_2 \ge 2x_2$. Thus

$$(2.45) \qquad \gamma(x_1)\gamma(x_2) \ge \sup_{y \ge 2x_1+2x_2} \frac{T(y_1)}{T(y_1-x_1-x_2)} = \gamma(x_1+x_2).$$

We may then define a non-decreasing $G(x)$ by

$$G(x) = \sup_{y \le x} \alpha(y).$$

Given any $x_1 \ge 0$ and $x_2 \ge 0$, there must be, for every $\varepsilon > 0$, a $y^* \le x_1 + x_2$, such that $G(x_1+x_2) \le \gamma(y^*) + \varepsilon$. But we must be able to find $u_1^* \le x_1$ and $u_2^* \le x_2$ such that $u_1^* + u_2^* = y^*$. Thus $G(x_1+x_2) \le \gamma(u_1^*)\gamma(u_2^*) + \varepsilon$, by (2.45). Since $\gamma(u_1^*) \le G(x_1)$, $\gamma(u_1^*) \le G(x_2)$, and since ε is arbitrary, it follows that

$$G(x_1+x_2) \le G(x_1)G(x_2).$$

Thus $G(x)$ satisfies all the conditions of a rmf if we can prove it to be a fmg. Of course, we take $G(x) \equiv 1$ for $x<0$. We shall omit the relatively easy proof that $G(x)$ is indeed a fmg, and conclude that $G(x)$ is the desired minimal gauge function.

If we restrict all discussion to integer values of the arguments, the minimality property of $G(x)$ is easily established and is as follows. If H is a gauge function such that

$$\sup_{r \le \frac{1}{2}n} \frac{T(n)}{T(n-r)} \le H(r)$$

for all sufficiently large n, say all $n \ge \Delta$, then $G(r) \le H(r)$ for all $r \ge \frac{1}{2}\Delta$. We omit the proof of this. The import of this result is that, whenever possible, we should use the minimal gauge function and that if $\underline{a} \in S(H)$ for any gauge function H then it is automatic that $\underline{a} \in S(G)$ for the mgf.

§3. ON THE ALGEBRA OF TAIL FUNCTIONS

This section is mainly devoted to proving Theorem 3.1, below, which is to be our principle tool in treating limit theorems of recurrent events. It will be noted that it is an extension of the famous Wiener-Pitt-Lévy Theorem on analytic functions of absolutely convergent fourier series. It is also similar to Theorem 2 of Smith (1966), but that theorem referred to Fourier-Stieltjes transforms rather than trigonometric series and the present theorem is considerably wider in its realm of application.

If $\underline{a} \equiv \{a_n\}$ be any sequence such that $||\underline{a}|| < \infty$ we shall write

$$(3.1) \qquad \hat{\underline{a}}(\theta) = \sum_{n=-\infty}^{+\infty} a_n e^{in\theta}, \quad -\infty<\theta<\infty.$$

The following deductions are then almost immediate: (i) $|\hat{\underline{a}}(\theta)| \leq ||\underline{a}||$, all θ; (ii) $\hat{\underline{a}}(\theta)$ has period 2π; (iii) $\hat{\underline{a}}(\theta)$ is uniformly continuous; (iv) the range of $\hat{\underline{a}}(\theta)$ is a continuous curve, to be called $C_{\underline{a}}$, in the complex plane. In view of (i) it is clear that $C_{\underline{a}}$ lies entirely within the closed disc $|z| \leq ||\underline{a}||$.

One more notion is needed before we can state our theorem of this section. If $K(x)$ is any measurable function defined on $[0,\infty)$, at least, we say it satisfies *Condition U* if

$$(3.2) \qquad \int_0^\infty \frac{|\log K(x)|}{1+x^2}\, dx < \infty.$$

<u>THEOREM 3.1.</u> *Let* $\underline{a}$ *be a sequence such that* $||\underline{a}|| < \infty$ *and let* $\Phi(z)$, *a function of complex* z, *be regular at every point of* $C_{\underline{a}}$. *Then there is another sequence* $\underline{b}$, *with* $||\underline{b}|| < \infty$, *such that* $\hat{\underline{b}}(\theta) \equiv \Phi(\hat{\underline{a}}(\theta))$. *Moreover,*

let M *be a rmf,* T *a tail function, and* G *a corresponding gauge function, and suppose* M, T, *and* G, *all satisfy Condition U, above. Then:*

(A) *If* $\underline{a} \in S(M)$ *then* $\underline{b} \in S(M)$ *also;*

(B) *If* $\underline{a} \in V(T) \cap S(G)$ *then* $\underline{b} \in V(T) \cap S(G)$*;*

(C) *If* $\underline{a} \in W(T) \cap S(G)$ *then* $\underline{b} \in W(T) \cap S(G)$, *and*

$$\langle \underline{b} \rangle_T = \langle \underline{a} \rangle_T \, \Phi' \Big(\sum_{-\infty}^{+\infty} a_n \Big).$$

Before we proceed with the proof of this theorem it will be helpful to introduce a simple extension of our notation. If $\underline{a} \in S(T)$, for example, we write $\hat{\underline{a}} \in \hat{S}(T)$, and so on. Thus we write $\hat{\underline{a}} \in \hat{V}(T)$ to mean $\underline{a} \in V(T)$.

A vital tool in proving Theorem 3.1 is the smooth mutilator function (SMF) introduced in Smith (1966). However the SMF developed in that paper is inadequate to deal with our present problems, so we must first provide a suitable improvement.

LEMMA 3.2. *Let* $K(x)$, *defined for* $x \geq 1$, *be a continuous and strictly increasing function, and suppose* $K(1) \geq 1$, *and*

$$\int_1^\infty \frac{K(x)}{x^2}\, dx < \infty. \tag{3.3}$$

Then there is a finite constant $A>0$ *and a symmetric probability density function* $p(x)$ *with a characteristic function* $\psi(\theta)$ *such that* $p(x) \equiv 0$ *for* $|x| > A$ *and*

$$|\psi(\theta)| \leq e^{-\int_1^{|\theta|} u^{-1} K(u)\, du}, \tag{3.4}$$

for all large $|\theta|$.

PROOF. $K(x)$ will have a continuous and strictly increasing inverse function $Q(x)$, say, such that $Q(K(x)) = x$ for all $x \geq 1$. Let us set $K(1) = \alpha$, so $K(Q(y)) = y$ for all $y \geq \alpha$. By (3.3) we have

$$\frac{K(x)}{x} \leq \int_x^\infty \frac{K(u)}{u^2}\,du \to 0 \quad \text{as} \quad x \to \infty.$$

Thus, if we set $x = Q(y)$ in (3.3) we may integrate by parts and get

$$\int_\alpha^\infty \frac{dy}{Q(y)} < \infty. \tag{3.5}$$

Obviously (3.5) implies

$$\sum_{r=1}^\infty \frac{1}{Q(r+\alpha-1)} < \infty. \tag{3.6}$$

Let $\{X_r\}_{r=1}^\infty$ be a sequence of mutually independent random variables such that X_r has a rectangular distribution on the interval $[-a_r, +a_r]$, where

$$a_r = \frac{1}{Q(r+\alpha-1)}, \quad r=1,2,\dots .$$

Then because of (3.6) it follows that $\sum_1^\infty X_r = Z$, say is a proper random variable such that $|Z| \leq A$, almost surely, where A is the sum (3.6). If we let $p(x)$ be the pdf of Z then the symmetry and compact support of $p(x)$ are plain. However, the characteristic function of Z is

$$\psi(\theta) = \prod_{r=1}^\infty \left(\frac{\sin a_r\theta}{a_r\theta}\right).$$

Thus

$$|\psi(\theta)| \leq \prod_{|a_r\theta| \geq 1} |a_r\theta|^{-1} \tag{3.7}$$

$$= \exp\{-\textstyle\sum(\theta)\}, \text{ say,}$$

where

$$\sum(\theta) = \sum_{Q(r)\le|\theta|} \ell n|\theta/Q(a+r-1)|.$$

Since $Q(r)$ is non-decreasing,

$$\sum(\theta) \ge \int_{\alpha}^{K(|\theta|)-1} \ell n|\theta/Q(y)|dy \tag{3.8}$$

$$\ge [K(|\theta|)-1-\alpha]\ell n|\theta| - \int_{\alpha}^{K(|\theta|)} \ell n Q(y) \circ dy.$$

The substitution $y = K(x)$ shows

$$\int_{\alpha}^{K(|\theta|)} \ell n Q(y)dy = \int_{1}^{|\theta|} (\ell n\ x)dK(x),$$

and an integration by parts shows this integral equals

$$(\ell n|\theta|)K(|\theta|) - \int_{1}^{|\theta|} \frac{K(x)}{x}\,dx.$$

Thus, from (3.8), we find

$$\sum(\theta) \ge -(1+\alpha)\ell n|\theta| + \int_{1}^{|\theta|} \frac{K(x)}{x}\,dx.$$

If we use this inequality in (3.7) we obtain

$$|\psi(\theta)| \le |\theta|^{(1+\alpha)} \exp\{-\int_{1}^{|\theta|} u^{-1}K(u)du\}. \tag{3.9}$$

This result is not exactly in the form (3.4) we have claimed. However, we can replace $K(x)$ in our proof by $(1+\alpha) + K(x) = K_1(x)$, say. All the conditions assumed for $K(x)$ will hold for $K_1(x)$ and we end, in place of (3.9), with the inequality

$$|\psi(\theta)| \le |\theta|^{(1+\alpha)} \exp\{-\int_{1}^{|\theta|} u^{-1}[(1+\alpha)+K(u)]du\}$$

$$= \exp\{-\int_{1}^{|\theta|} u^{-1}K(u)du\},$$

as required.

In our use of the non-decreasing functions $M(x)$, $T(x)$, and $G(x)$, it is only their values at integer values of their arguments that matter. Thus we may, with no loss of generality, assume them to be continuous functions.

LEMMA 3.3. *Let* $T(x)$ *be a tail function which satisfies Condition U, above. Then there is a finite* $A>0$ *and a symmetric probability density* $p(x)$, *with characteristic function* $\psi(\theta)$, *such that* $p(x) \equiv 0$ *for all* $|x| > A$ *and*

$$\psi(\theta) = 0\left(\frac{1}{[|\theta|^2 T(\tfrac{1}{2}|\theta|)]^\Delta}\right), \tag{3.10}$$

for every large Δ.

PROOF. Since $T(x)$ satisfies (3.2) we can introduce the strictly decreasing function

$$R(x) = \int_x^\infty y^{-2}\{\ell n\ T(y)\}dy.$$

Let us then define

$$K(x) = \frac{C\ \ell n\ T(x)}{\sqrt{\{R(x)\}}},$$

where C is chosen to make $K(1) = 1$. We then find

$$\int_1^\infty \frac{K(x)}{x^2}\,dx = -C\int_1^\infty \frac{dR(x)}{\sqrt{\{R(x)\}}}$$

$$= 2C\sqrt{\{R(1)\}} < \infty.$$

Thus $K(x)$ satisfies the conditions of Lemma 3.2 and we may conclude a suitable symmetric pdf exists with a cf $\psi(\theta)$ satisfying (3.4).

$$\int_1^x u^{-1}K(u)du > \int_{x-2}^x \frac{C\ \ell n\ T(y)}{y\surd\{R(y)\}}\,dy$$

$$> \frac{C\ \ell n\ T(x/2)}{\surd\{R(x/2)\}}\int_{x-2}^x \frac{dy}{y}$$

$$= \frac{C(\ell n\ 2)\ell n\ T(x/2)}{\surd\{R(x/2)\}}\ .$$

Since $R(x) \to 0$ as $x\to\infty$, for every large $\Delta>0$ it is true that

$$\int_1^x u^{-1}K(u)du > \Delta\ \ell n\ T(x/2),$$

for all sufficiently large x. Thus we have from Lemma 3.2 that

$$\text{(3.11)} \qquad \psi(\theta) = 0\left(\{T(|\theta|/2)\}^{-\Delta}\right), \quad |\theta| \to \infty.$$

However, if $T(x)$ satisfies (3.2) then so does $x^2T(x)$ and the derivation of (3.11) works equally well; thus we can obtain in place of (3.11) the conclusion (3.10).

Suppose now that $T(x)$ is a given tail function and $G(x)$ an associated gauge function and that both these functions satisfy (3.2). Then $T(x)G(x)$ is a tail function satisfying (3.2) and Lemma 3.3 assures us there exists a symmetric probability density with compact support and characteristic function $\psi(\theta)$ such that, as $|\theta| \to \infty$,

$$\text{(3.12)} \qquad \psi(\theta) = 0\left(\frac{1}{|\theta|^{2\Delta}\{T(\tfrac{1}{2}|\theta|)G(\tfrac{1}{2}|\theta|)\}^{\Delta}}\right).$$

Let λ be any positive constant and let k be the least positive integer such that $k\lambda \geq 2$ (to avoid triviality we may suppose $\lambda<2$). Then, for large positive θ,

$$\begin{aligned} T(\theta) &\le T(\tfrac{1}{2}k\lambda\theta) \\ &\le G(\tfrac{1}{2}\lambda\theta)T\big(\tfrac{1}{2}(k-1)\lambda\theta\big) \\ &\le \{G(\tfrac{1}{2}\lambda\theta)\}^{k-1}T(\tfrac{1}{2}\lambda\theta). \end{aligned}$$

By choosing Δ large enough it then follows from (3.12) that

$$T(\theta)\psi(\lambda\theta) \to 0, \text{ as } \theta\to\infty. \tag{3.13}$$

Next let $M(x)$ be a moment function satisfying (3.2) and let $\psi(\theta)$ be the appropriate cf provided by Lemma 3.3. For any large integer N we can evidently show, by taking Δ sufficiently large, that

$$\sum_{n=1}^{\infty} \{M(\lambda n/2)\}^N |\psi(\lambda n)| < \infty. \tag{3.14}$$

But $M(n) \le \{M(\tfrac{1}{2}k\lambda n)\} \le \{M(\tfrac{1}{2}\lambda n)\}^k$. Thus, by taking $N=k$ in (3.14), we have

$$\sum_{n=1}^{\infty} M(n)|\psi(\lambda n)| < \infty. \tag{3.15}$$

Suppose now that we are dealing with a problem involving a moment function M, say, and that $p(x)$ is the probability density provided by Lemma 3.3. Suppose further we are given $\alpha<\beta<\gamma<\delta$ as four points on the real axis. We define $q^+(x;\alpha,\beta,\gamma,\delta)$ as the SMF based on these points thus:

$$q^+(x;\alpha,\beta,\gamma,\delta) = \int_{-\infty}^{x} \left\{\frac{1}{(\beta-\alpha)}\, p\left(\frac{y-\frac{1}{2}(\alpha+\beta)}{\beta-\alpha}\right) - \frac{1}{(\delta-\gamma)}\, p\left(\frac{y-\frac{1}{2}(\gamma+\delta)}{\delta-\gamma}\right)\right\}dy . \tag{3.16}$$

The SMF is identically zero when $x\ge\alpha$ or $x\ge\delta$ and identically equal to unity when $\beta\le x\le\gamma$. It also has the convenient property that if $\delta<\gamma'<\delta'$ then

(3.17) $$q^{+}(x;\ \alpha,\beta,\gamma,\delta) + q^{+}(x;\ \gamma,\delta,\gamma',\delta') = q^{+}(x;\ \alpha,\beta,\gamma',\delta').$$

Most important of all, if we set

$$q(\theta;\ \alpha,\beta,\gamma,\delta) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\theta x}\, q^{+}(x;\ \alpha,\beta,\gamma,\delta)dx$$

then

$$q(\theta;\ \alpha,\beta,\gamma,\delta) = \frac{1}{i\theta}\left\{(\delta-\gamma)e^{\frac{1}{2}i\theta(\gamma+\delta)}\ \psi\big(\theta(\delta-\gamma)\big) - (\beta-\alpha)e^{\frac{1}{2}i\theta(\alpha+\beta)}\ \psi\big(\theta(\beta-\alpha)\big)\right\}.$$

If $M(x)$ is a rmf satisfying Condition U then we may suppose, by (3.15), that the sequence

(3.18) $$\{q(n\lambda;\ \alpha,\beta,\gamma,\delta)\}_{n=1}^{\infty}$$

belongs to $S(M)$ for every fixed $\lambda>0$. Similarly for a gauge function $G(x)$ satisfying Condition U. However, if the associated tail function $T(x)$ also satisfies Condition U then (3.13) shows the sequence (3.18) to be T-null. Notice particularly that these remarks are valid for every $\lambda>0$.

Let us now suppose $\delta-\alpha \leq 2\pi$. Then we can construct a periodic SMF (abbreviated hereafter as SPMF):

$$\tilde{q}^{+}(\theta;\ \alpha,\beta,\gamma,\delta) = \sum_{n=-\infty}^{+\infty} q^{+}(\theta+n\pi;\ \alpha,\beta,\gamma,\delta).$$

Evidently $\tilde{q}^{+}(\theta;\ \alpha,\beta,\gamma,\delta)$ has a Fourier series representation:

$$\tilde{q}^{+}(\theta;\ \alpha,\beta,\gamma,\delta) = \sum_{n=-\infty}^{n=+\infty} e^{ni\theta}K_n(\alpha,\beta,\gamma,\delta),\ \text{say}.$$

The Fourier coefficients are given by

$$(3.19)\qquad K_n(\alpha,\beta,\gamma,\delta) = \frac{1}{2\pi}\int_0^{2\pi} \tilde{q}^+(\theta;\ \gamma,\beta,\gamma,\delta)e^{-in\theta}d\theta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} q^+(\theta;\ \alpha,\beta,\gamma,\delta)e^{-in\theta}d\theta$$

$$= q(n;\ \alpha,\beta,\gamma,\delta).$$

In view of (3.19) we may claim that the sequence $\{K_n(\alpha,\beta,\gamma,\delta)\}$ belongs to $S(M)$, $S(G)$, and $V_0(T)$.

Let us simply write $q^+(\theta)$ for $q^+(\theta;\ -2,-1,1,2)$. Then, for any $\lambda>0$, $q^+(\theta/\lambda) = q^+(\theta;\ -2\lambda,-\lambda,\lambda,2\lambda)$; this fact will be useful shortly. For small $\lambda>0$ we write simply $\sigma_\lambda(\theta)$ for the PSMF correctly denoted $q^+(\theta;\ -2\lambda,-\lambda,\lambda,2\lambda)$. We then write K_n^λ for the Fourier coefficients of $\sigma_\lambda(\theta)$:

$$(3.20)\qquad \sigma_\lambda(\theta) = \sum_{-\infty}^{+\infty} e^{in\theta}K_n^\lambda\ .$$

We now turn to the proof of Theorem 3.1. If θ_0 be any fixed real, then $\Phi(z)$ must be regular at $\hat{a}(\theta_0) = z_0$, say, by the assumptions of the theorem. Thus for all z for which $|z-z_0|$ is sufficiently small we must have an expansion:

$$(3.21)\qquad \Phi(z) = \Phi(z_0) + \sum_{r=1}^{\infty} c_r(z-z_0)^r\ .$$

Let ρ_0 be the radius of convergence of the series in (3.21); we must have $\rho_0 > 0$. Since $\hat{a}(\theta)$ is continuous there will be a δ_0, say, such that $|\hat{a}(\theta)-\hat{a}(\theta_0)| \le \rho_0$ for all $|\theta-\theta_0| \le \delta_0$. Thus, for $|\theta-\theta_0| \le \rho_0$, we have from (3.21):

$$(3.22)\qquad \Phi(\hat{a}(\theta)) = \Phi(\hat{a}(\theta_0)) + \sum_{r=1}^{\infty} c_r\{\hat{a}(\theta)-\hat{a}(\theta_0)\}^r .$$

Define, for some suitably small $\lambda>0$,

$$(3.23)\qquad \Omega_\lambda(\theta) = \sigma_{\frac{1}{2}\lambda}(\theta-\theta_0)\Phi(\hat{a}(\theta)).$$

For any integer $n\geq 0$,

$$\sigma_{\frac{1}{2}\lambda}(\theta-\theta_0) = \sigma_{\frac{1}{2}\lambda}(\theta-\theta_0)\{\sigma_\lambda(\theta-\theta_0)\}^n .$$

Thus (3.22) and (3.23) yield

$$(3.24)\quad \Omega_\lambda(\theta) = \sigma_{\frac{1}{2}\lambda}(\theta-\theta_0)\Phi(\hat{a}(\theta_0)) + \sum_{r=1}^{\infty} c_r\sigma_{\frac{1}{2}\lambda}(\theta-\theta_0)\{\sigma_\lambda(\theta-\theta_0)[\hat{a}(\theta)-\hat{a}(\theta_0)]\}^r .$$

However,

$$\sigma_\lambda(\theta-\theta_0) = \sum_{-\infty}^{+\infty} e^{in\theta}\{e^{-in\theta_0}K_n^\lambda\},$$

so $\sigma_\lambda(\theta-\theta_0)\hat{a}(\theta)$ is a periodic function with Fourier coefficients $\{b_n'\}$, say, where

$$(3.25)\qquad b_n' = \sum_{r=-\infty}^{+\infty} a_r e^{-i(n-r)\theta_0}K_{n-r}^\lambda .$$

Thus, if we set

$$(3.26)\qquad \hat{b}^\lambda(\theta) = \sigma_\lambda(\theta-\theta_0)[\hat{a}(\theta)-\hat{a}(\theta_0)]$$

then $\hat{b}^\lambda(\theta) = \sum_{-\infty}^{+\infty} e^{in\theta}b_n^\lambda$, say, where

$$(3.27)\qquad \begin{aligned} b_n^\lambda &= b_n' - \hat{a}(\theta_0)e^{-in\theta_0}K_n^\lambda \\ &= \sum_{r=-\infty}^{+\infty} \{K_{n-r}^\lambda - K_n^\lambda\}e^{-i(n-r)\theta_0}a_r . \end{aligned}$$

Suppose $\underline{a} \in S(M)$. We see from (3.25) that $|b_n'|$ does not exceed

$$\sum_{r=-\infty}^{+\infty} |a_r| \, |K_{n-r}^{\lambda}| \; .$$

But $\{K_n^{\lambda}\} \in S(M)$ and so, by Lemma 2.1, it follows that $\{b_n'\} \in S(M)$. Also, we note from (3.27) and the fact that $|\hat{a}(\theta)| \leq ||\underline{a}||$, the inequality

$$|b_n^{\lambda}-b_n'| \leq ||\underline{a}|| \; |K_n^{\lambda}|.$$

Thus, if we write $\underline{b}^{\lambda} \equiv \{b_n^{\lambda}\}$, we have $\underline{b}^{\lambda} \in S(M)$.

However, (3.27) yields:

$$\text{(3.28)} \qquad ||\underline{b}^{\lambda}|| \leq \sum_{r=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} |K_{n-r}^{\lambda} - K_n^{\lambda}| \; |a_r|.$$

But

$$K_n^{\lambda} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-in\theta} \; q^{+}(\theta/\lambda)d\theta$$

$$= \frac{\lambda}{2\pi} \int_{-\infty}^{+\infty} e^{-in\lambda u} \; q^{+}(u)du.$$

Hence, if we write $q(x)$ for the Fourier transform of $q^{+}(u)$,

$$K_n^{\lambda} = \lambda q(n\lambda).$$

Thus

$$\text{(3.29)} \qquad \sum_{n=-\infty}^{+\infty} |K_{n-r}^{\lambda} - K_n| \leq \lambda \sum_{n=-\infty}^{+\infty} |q(n\lambda-r\lambda) - q(n\lambda)|.$$

Since $q(x) = O(|x|^{-N})$ for large x, with N fixed, but arbitrarily large, and since $q(x)$ and all its derivatives are bounded and uniformly continuous, it is a matter of routine analysis to show that, as $\lambda \to 0$

$$\text{(3.30)} \qquad \lambda \sum_{-\infty}^{+\infty} |q(n\lambda)| \to \int_{-\infty}^{+\infty} |q(x)|dx,$$

and, for r fixed,

(3.31) $$\lambda \sum_{-\infty}^{+\infty} |q(n\lambda - r\lambda) - q(n\lambda)| \to 0.$$

Furthermore (3.30) ensures that the convergence (3.31) takes place boundedly. Thus, from (3.29), for r fixed, as $\lambda \to 0$:

$$\sum_{n=-\infty}^{+\infty} |K_{n-r}^{\lambda} - K_n^{\lambda}| \to 0, \text{ boundedly.}$$

If this result is used in (3.28) we see that by choosing λ small enough we can make $||\underline{b}^{\lambda}||$ less than any prescribed positive constant. But we have already seen that $||\underline{b}||_M < \infty$.

Let us now, temporarily, set

(3.32) $$\hat{\underline{p}}(\theta) = \sum_{r=1}^{\infty} c_r\{\sigma_\lambda(\theta-\theta_0)[\hat{a}(\theta)-a(\theta_0)]\}^r$$

$$= \sum_{r=1}^{\infty} c_r\{\hat{\underline{b}}^{\lambda}(\theta)\}^r .$$

Then $\hat{\underline{p}}(\theta) = \sum e^{in\theta} p_n$, say, where p_n is the nth term in the sequence

$$\underline{p} \equiv \sum_{r=1}^{\infty} c_r\left((\underline{b}^{\lambda})^{*r}\right).$$

Since $||\underline{b}^{\lambda}||$ is arbitrarily small, by choice of λ, we may suppose $||\underline{b}^{\lambda}||' < \rho_0$ and appeal to Lemma 2.3 for the conclusion $\underline{p} \in S(M)$. But we must check that $\underline{b}^{\lambda} \in S(M)$. We do this by referring to (3.26); $\hat{a}(\theta) - \hat{a}(\theta_0) \in \hat{S}(M)$, by hypothesis; $\sigma_\lambda(\theta-\theta_0) \in \hat{S}(M)$, by the construction of $q^+(\theta/\lambda)$; thus $\hat{\underline{b}}^{\lambda}(\theta) \in \hat{S}(M)$, as required. Thus we have shown that under hypothesis (A) of the theorem, $\underline{b}^{\lambda} \in S(M)$.

Suppose next that (B) or (C) applies. Then the previous logic shows $\underline{b}^{\lambda} \in S(G)$.

From (3.20) we see that $\sigma_\lambda(\theta-\theta_0)$ is represented by a Fourier series based on the sequence $\{e^{-in\theta_0}K_n^\lambda\}$, say, and the norm of this sequence is

$$||\{e^{-in\theta_0}K_n^\lambda\}|| = \sum_{n=-\infty}^{+\infty} |K_n^\lambda|.$$

Now under (B) or (C) we have $\sigma_\lambda(\theta-\theta_0) \in V_0^n(T)$. Thus, from (3.26) and Lemma 2.4, we can infer that $\underline{b}^\lambda \in V(T)$ and

(3.33) $$[\underline{b}^\lambda]_T \leq [\underline{a}]_T \sum_{-\infty}^{+\infty} |K_n^\lambda|,$$

if (B) holds; and that $\underline{b}^\lambda \in W(T)$ and

(3.34) $$\langle\underline{b}^\lambda\rangle_T = \langle\underline{a}\rangle_T \sum_{-\infty}^{+\infty} e^{-in\theta_0}K_n^\lambda ,$$

if (C) holds. But the sum on the right side of (3.34) is $\sigma_\lambda(-\theta_0)$. Thus, if (C) holds:

(3.35) $$\langle\underline{b}^\lambda\rangle_T = \langle\underline{a}\rangle_T\sigma_\lambda(-\theta_0).$$

Bearing in mind that $||\underline{b}^\lambda|| < \rho_0$, we may apply Lemma 2.9 to draw the conclusions:

If (B) holds then $\underline{p} \in V(T)$ and

$$[\underline{p}]_T \leq [\underline{b}^\lambda]_T \sum_{-\infty}^{+\infty} |c_r| \cdot ||\underline{b}^\lambda||^r .$$

If (C) holds then $\underline{p} \in W(T)$ and

(3.36) $$\langle\underline{p}\rangle_T = \langle\underline{b}^\lambda\rangle_T \sum_{r=1}^{\infty} rc_r\Big(\sum_{-\infty}^{+\infty} b_n^\lambda\Big)^{r-1}$$

$$= \langle\underline{a}\rangle_T\sigma_\lambda(-\theta_0)\Phi'(z_0 + \sum_{-\infty}^{+\infty} b_n^\lambda),$$

by (3.35).

If we restrict θ_0 to the closed interval $[-\pi,\pi]$ then it can be shown that $\sigma_\lambda(-\theta_0) = q^+(\theta_0/\lambda)$. If we choose λ so that $\lambda \le \frac{1}{2}|\theta_0|$, it follows that $\sigma_\lambda(-\theta_0) = 0$. Thus we have the intermediate result:

<u>RESULT I.</u> *If* $2\lambda \le |\theta_0| \le \pi$, *and (C) holds, then* $\underline{p} \in V_0(T)$.

On the other hand, let us see what happens in the special case $\theta_0 = 0$. We find the following:

$$\text{(i)} \quad \sum_{-\infty}^{+\infty} b_n^\lambda = \hat{b}^\lambda(0) = 0, \text{ by (3.26)}.$$

$$\text{(ii)} \qquad z_0 = \hat{a}(0) = \sum_{-\infty}^{+\infty} a_n \,.$$

$$\text{(iii)} \quad \sigma_\lambda(0) = q^+(0) = 1.$$

Thus, from (3.36), we have

<u>RESULT II.</u> *For* λ *sufficiently small and* $\theta_0 = 0$, $p \in W(T)$ *and*

$$\langle \underline{p} \rangle_T = \langle \underline{a} \rangle_T \Phi'\Big(\sum_{-\infty}^{+\infty} a_n\Big).$$

Now (3.24) and (3.32) show that

$$\Omega_\lambda(\theta) = \sigma_{\frac{1}{2}\lambda}(\theta-\theta_0)\Phi\big(\hat{\underline{a}}(\theta_0)\big) + \sigma_{\frac{1}{2}\lambda}(\theta-\theta_0)\hat{\underline{p}}(\theta).$$

Because $\sigma_{\frac{1}{2}\lambda}(\theta-\theta_0)$ is the sum of a Fourier series whose coefficients form a sequence in $S(M)$, $S(G)$, and $V_0(T)$, it should be evident at this stage that what we have succeeded in showing so far is that for every real θ_0 we can construct a function $\Omega_\lambda(\theta)$ which is identically equal to $\Phi(\hat{a}(\theta))$ in the interval $[\theta_0-\frac{1}{2}\lambda, \theta_0+\frac{1}{2}\lambda]$.

Provided λ is chosen sufficiently small we also may claim: (i) Under (A), $\Omega_\lambda(\theta) \in \hat{S}(M)$; (ii) Under (B), $\Omega_\lambda(\theta) \in \hat{V}(T)$; (iii) Under (C), $\Omega_\lambda(\theta) \in \hat{V}(T)$.

If (C) holds and we choose $\lambda < \frac{1}{2}|\theta_0|$, for $\theta_0 \neq 0$, then $\Omega_\lambda(\theta) \in \hat{V}_0(T)$.

Let us write $I(\theta_0)$ for the interval $[\theta_0-\frac{1}{2}\lambda, \theta_0+\frac{1}{2}\lambda]$, and bear in mind that the choice of λ may well vary with θ_0. Let us call the restriction $\lambda \leq \frac{1}{2}|\theta_0|$: Condition A. In the argument which follows we must indicate the dependence of λ and $\Omega_\lambda(\theta)$ on θ_0. We shall therefore henceforth denote them as $\lambda(\theta_0)$ and $\Omega(\theta; \theta_0)$ respectively.

Thus every point θ^*, say, is the center of a closed interval $I(\theta^*)$ throughout which $\Omega(\theta; \theta^*) \equiv \Phi(\hat{\theta}(\theta))$. Let us write $J(\theta^*)$ for the open interval obtained from $I(\theta^*)$ by deleting its end-points.

Define $I_L \equiv [-\pi, -\pi+\frac{1}{2}\lambda(\pi)]$ and $I_R \equiv [\pi-\frac{1}{2}\lambda(\pi), \pi]$. Then, because of periodicity, $\Omega(\theta;\pi) \equiv \Phi(\hat{a}(\theta))$ throughout I_L and I_R. By the Heine-Borel theorem we can select finitely-many points $\theta_1 < \theta_2 < \ldots < \theta_k$, say, in the *open* interval $(-\pi+\frac{1}{2}\lambda(\pi), \pi-\frac{1}{2}\lambda(\pi))$ such that the closure of that interval is covered by the union of the open intervals $J(\theta_1) \cup J(\theta_2) \cup \ldots \cup J(\theta_k)$. It is not hard to see that one of the points $\{\theta_j\}$ will have to be the origin because of Condition A. Let $J(\theta_j) \equiv (\beta_j, \gamma_j)$. Then we may suppose

$$-\pi < \beta_1 < -\pi+\tfrac{1}{2}\lambda(\pi) < \beta_2 < \gamma_1 < \beta_3 < \gamma_2 < \ldots < \beta_k < \gamma_{k-1} < \pi-\tfrac{1}{2}\lambda(\pi) < \gamma_k < \pi.$$

It will be helpful to write

$$\beta_0 = -\pi$$

$$\gamma_0 = -\pi+\tfrac{1}{2}\lambda(\pi)$$

$$\beta_{k+1} = \pi-\tfrac{1}{2}\lambda(\pi)$$

$$\gamma_{k+1} = \pi.$$

Consider the two overlapping closed intervals $I(\theta_1)$ and $I(\theta_2)$. If we refer to the property (3.17) of SMF's based on overlapping intervals we see the function

$$(3.37)\quad \tilde{q}(\theta;\ \beta_0,\beta_1,\beta_2,\gamma_1)\Omega(\theta;\ \theta_1) + \tilde{q}(\theta;\ \beta_2,\gamma_1,\gamma_2,\beta_4)\Omega(\theta;\ \theta_2)$$

$$\equiv \Omega(\theta;\ \theta_1,\theta_2),\quad \text{say},$$

is identically equal to $\Phi(\hat{a}(\theta))$, in the larger interval $I(\theta_1)\cup I(\theta_2) \equiv [\beta_1,\gamma_2]$.

By construction, any PSMF we use may be assumed to belong to $\hat{S}(M)$, $\hat{S}(G)$, and $\hat{V}_0(T)$, as necessary. It follows from (3.37) that:

(i) Under hypothesis (A), $\Omega(\theta;\ \theta_1,\theta_2) \in \hat{S}(M)$.

(ii) Under hypothesis (B), $\Omega(\theta;\ \theta_1,\theta_2) \in \hat{S}(G) \cap \hat{V}(T)$.

(iii) Under hypothesis (C), $\Omega(\theta;\ \theta_1,\theta_2) \in \hat{S}(G)\cap\hat{V}_0(T)$, unless θ_1 or θ_2 happens to be 0. In the latter case $\Omega(\theta;\ \theta_1,\theta_2) \in \hat{S}(G)\cap\hat{W}(T)$, and $\Omega(\theta;\ \theta_1,\theta_2) = \sum e^{in\theta}t_n$, say, where

$$\langle \underline{t}_1\rangle_T = \langle \underline{a}\rangle_T\Phi'(\sum a_n).$$

In claiming (i) we have appealed to Lemma 2.1; for (ii) and (iii) we have appealed to Lemma 2.4.

It will be apparent that this process can be continued, yielding $\Omega(\theta;\ \theta_1,\theta_2,\theta_3)$, $\Omega(\theta;\ \theta_1,\theta_2,\theta_3,\theta_4)$, and so on, until we reach

$$\Omega(\theta;\ \theta_1,\theta_2,\ldots,\theta_k) \equiv \Omega^*(\theta),\ \text{say.}$$

It will follow that $\Omega^*(\theta)$ belongs to the appropriate classes, depending on our choice of hypothesis. Furthermore $\Omega^*(\theta) \equiv \Phi\left(\hat{\underline{a}}(\theta)\right)$ for θ in the interval $[\beta_1,\gamma_k]$. Notice that, under hypothesis (C), since we will have perforce included $J(0)$ among the intervals, $\Omega^*(\theta) \in \mathcal{W}(T)$ and

$$\Omega^*(\theta) = \sum_{-\infty}^{+\infty} e^{in\theta} t_n^*\ ,\ \text{say,}$$

where $\langle \underline{t}^* \rangle_T = \langle \underline{a} \rangle_T \Phi'(\sum a_n)$.

The last step left is the somewhat awkward one of combining $\Omega^*(\theta)$ with $\Omega(\theta;\pi)$. However, because of the periodicity of our functions; it may be verified that

$$\tilde{q}(\theta;\ \beta_1,\gamma_0,\beta_{k+1},\gamma_k)q^*(\theta) + \tilde{q}(\theta;\ \beta_{k+1},\gamma_k,2\pi+\beta_1,2\pi+\gamma_0)\Omega(\theta,\pi)$$

is identically equal to $\Phi\left(\hat{\underline{a}}(\theta)\right)$ for *all* θ, and belongs to appropriate classes, according to our choice of hypothesis. Thus $\Phi\left(\hat{\underline{a}}(\theta)\right)$ has an absolutely convergent fourier series with coefficients which behave as claimed in the theorem. This completes the proof.

§4 PROOFS OF THEOREMS ON RECURRENT EVENTS

As in §1 we write $\underline{f} \equiv \{f_n\}$ for the probability distribution of the aperiodic recurrent event, and suppose $\mu_1 \neq 0$. Let

$$\mu_1 r_n = \sum_{n+1}^{\infty} f_j \ , \quad n \geq 0,$$
$$= -\sum_{-\infty}^{n} f_j \ , \quad n < 0.$$

Then $\underline{r} \equiv \{r_n\}$ is such that $||\underline{r}|| < \infty$ and $\sum_{-\infty}^{+\infty} r_n = 1$; furthermore it is easy to establish that

$$\hat{\underline{r}}(\theta) = \frac{1-\hat{\underline{f}}(\theta)}{\mu_1(1-e^{i\theta})} \ . \tag{4.1}$$

If $S(x) = x^{-1} \sin x$ then there are constants $\alpha > 0$ and $\beta > 0$ such that $S(x) \geq \alpha$ for all $|x| \leq \beta$, and $S(x) < \alpha$ for all $|x| > \beta$. Thus, since

$$I\ \hat{\underline{f}}(\theta) = \sum_{|n\theta| \leq \beta} f_n \sin n\theta + \sum_{|n\theta| > \beta} f_n \sin n\theta,$$

we have

$$|I\ \hat{\underline{f}}(\theta)| \geq \alpha |\theta| \{|\textstyle\sum_1(\theta)| - |\textstyle\sum_2(\theta)|\}, \text{ say},$$

where

$$\textstyle\sum_1(\theta) = \displaystyle\sum_{|n\theta| \leq \beta} n\, f_n \ ,$$

and

$$\textstyle\sum_2(\theta) = \displaystyle\sum_{|n\theta| > \beta} n\, f_n \ .$$

But $\sum_1(\theta) \to \mu_1 \neq 0$ and $\sum_2(\theta) \to 0$, as $|\theta| \to 0$. Thus we have proved:

<u>LEMMA 4.1.</u> *There is a finite* $\gamma > 0$ *such that* $|I\ \hat{\underline{f}}(\theta)| \geq \gamma |\theta|$ *for all small* $|\theta|$.

For $0<\rho<1$ define a sequence ${}^{\rho}\underline{u}$ by the relation:

$$ {}^{\rho}u_n = 1 + \rho f_n + \rho^2(\underline{f}^{*2})_n + \rho^3(\underline{f}^{*3})_n + \dots . $$

Then, as $\rho\uparrow 1$, ${}^{\rho}u_n \uparrow u_n$. But it is easy to see that

$$ {}^{\rho}\hat{\underline{u}}(\theta) = \frac{1}{1-\rho\hat{\underline{f}}(\theta)} , $$

and so

$$ (4.2) \qquad {}^{\rho}u_n = \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-in\pi} \frac{d\theta}{1-\rho\hat{\underline{f}}(\theta)} . $$

This integral is absolutely convergent, since $|\hat{\underline{f}}(\theta)| \le 1$ for all θ. We then deduce from (4.2) that

$$ (4.3) \qquad {}^{\rho}u_n - {}^{\rho}u_{n-1} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-i\ \theta} \left\{\frac{1-e^{i\theta}}{1-\rho\hat{\underline{f}}(\theta)}\right\} d\theta . $$

Now for all $|\theta| \le \delta$, say, $|1-e^{i\theta}| \le 2|\theta|$ and $|1-\rho\hat{\underline{f}}(\theta)| \ge \rho|I\ \hat{\underline{f}}(\theta)| \ge \rho\gamma|\theta|$, by Lemma 4.1. Thus for all $\rho\ge\frac{1}{2}$ and $|\theta| \le \delta$,

$$ \left|\frac{1-e^{i\theta}}{1-\rho\hat{\underline{f}}(\theta)}\right| \le \frac{4}{\gamma} . $$

On the other hand, since $\hat{\underline{f}}(\theta)$ is continuous and may assume the value 1 in neither of the closed intervals $[-\pi,-\delta]$, $[\delta,\pi]$, it is an easy matter to conclude that

$$ \left|\frac{1-e^{i\theta}}{1-\rho\hat{\underline{f}}(\theta)}\right| $$

is uniformly bounded for $\frac{1}{2}\le\rho\le 1$ and all θ. Thus by bounded convergence we can infer from (4.3) the result

$$ u_n - u_{n-1} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-in\theta} \left\{\frac{1-e^{i\theta}}{1-\hat{\underline{f}}(\theta)}\right\} d\theta $$

$$ = \frac{1}{2\pi\mu_1}\int_{-\pi}^{+\pi} e^{-in\theta} \frac{d\theta}{\hat{\underline{r}}(\theta)} . $$

Therefore, if we set $v_n = u_n - u_{n-1}$, and $\underline{v} \equiv \{v_n\}$, we have:

(4.4) $$\hat{\underline{v}}(\theta) = \frac{1}{\mu_1 \hat{\underline{r}}(\theta)} .$$

This result, allied to Theorem 3.1, is fundamental to the proof of all the theorems announced in §1.

Proof of Theorem 1.1. The characteristic function $\hat{\underline{r}}(\theta)$ is continuous and assumes the value unity when $\theta = 0 \bmod 2\pi$; it cannot vanish for other values of θ for this would conflict with the aperiodicity assumption. Thus $\hat{\underline{r}}(\theta)$ has no zeros and Theorem 3.1 allows us to draw various inferences about $1/\hat{\underline{r}}(\theta)$.

(a) If $\underline{r} \in S(M)$ it follows that $\underline{v} \in S(M)$ also. Conversely, since $\hat{\underline{r}}(\theta)$ is bounded (being a characteristic function), (4.4) shows $\hat{v}(\theta)$ can have no zeros either. Thus we can apply Theorem 3.1 to $1/\hat{v}(\theta)$ and deduce that $\underline{v} \in S(M)$ implies $\underline{r} \in S(M)$.

(b) If (R2) holds, i.e. $\underline{r} \in V(T) \cap S(G)$, then it follows from Theorem 3.1 (B) that $\underline{v} \in V(T) \cap S(G)$ also. Plainly the converse follows, as in (a).

(c) It is obvious that the proofs of cases (R3) and (R4) follow a similar simple argument. All that needs calculation is the limit value $\langle \underline{v} \rangle_T$. Since $\sum_{-\infty}^{+\infty} r_j = 1$, it follows from Theorem 3.1 (C) that this limit value is $-\rho/\mu_1$.

Proof of Corollary 1.1.2. If the tail function $T(n) \sim \frac{1}{n^{1+\nu} L(n)}$ we need only the well-known fact that, when $\nu > 0$,

(4.5) $$\sum_{j=n+1}^{\infty} \frac{1}{j^{1+\nu} L(j)} \sim \frac{1}{\nu n^{\nu} L(n)}$$

to deduce this corollary from Theorem 1.1. A proof of (4.5) may be found in, for example, Feller (1971). However, we need to show that the condition $\underline{r} \in S(G)$ is trivial, or, more correctly, that under present circumstances $S(G) \equiv S(I)$. We may assume $x^{1+\nu}L(x)$ is non-decreasing. Thus, for fixed large $x>0$,

$$\sup_{x\geq 2r} \frac{x^{1+\nu}L(x)}{(x-r)^{1+\nu}L(x-r)} = \frac{2^{1+\nu}L(x)}{L(x/2)} . \tag{4.6}$$

Since $L(x) \sim L(x/2)$ as $x\to\infty$, it is clear the right-hand of (4.6) is bounded. This is enough to establish our claim about G. The case $\nu=0$ needs a brief special comment. Of course, the argument about the gauge function goes as for the case when $\nu>0$. But the special case $\nu=0$ requires the introduction of a new function of slow growth $L_J(x)$, where

$$L_J(x) \sim \int_n^\infty \frac{du}{uL(u)} \sim \sum_{n+1}^\infty \frac{1}{jL(j)} .$$

It is an easy exercise to verify that this is, indeed, a fsg. Thus Corollary 1.1.2 will actually extend to the case $\nu=0$ with the introduction of L_J .

Proof of Corollary 1.1.3. Here $T(n) \sim e^{cn^\alpha} n^\beta L(n)$. Thus, for fixed r,

$$\begin{aligned} G(r) &= \sup_{n\geq 2r} \frac{e^{cn^\alpha} n^\beta L(n)}{e^{c(n-r)^\alpha}(n-r)^\beta L(n)} \\ &= O\left(\sup_{n\geq 2r} \exp[cn^\alpha - c(n-r)^\alpha]\right) \\ &= O(\exp \lambda cn^\alpha), \end{aligned}$$

where $\lambda = 1 - 2^{-\alpha} < 1$. Thus, if $\underline{r} \in V(T)$, then $\underline{r} \in S(G)$ if

$$\sum_1^\infty \frac{e^{\lambda cn^\alpha}}{T(n)} < \infty.$$

This series does converge, as desired, because $\lambda<1$. Thus when T grows as fast as it does in the present case, the gauge-function condition is automatically satisfied. All we need do to complete the proof of the corollary is to estimate

$$\sum_{n+1}^{\infty} \frac{1}{T(j)} \sim \sum_{n+1}^{\infty} \frac{e^{-cj^{\alpha}} j^{-\beta}}{L(j)}, \text{ as } n\to\infty.$$

Since, as is easily seen, and as it should be, the present $T(x)$ is a function of moderate growth, one has that

$$\sum_{n+1}^{\infty} \frac{1}{T(j)} \sim \int_{n}^{\infty} \frac{dx}{T(x)} .$$

A routine computation then shows that, as $n\to\infty$,

$$\int_{n}^{\infty} \frac{e^{-cx^{\alpha}}}{x^{\beta}L(x)}\,dx \sim \frac{1}{\alpha}\int_{(\exp n^{\alpha})}^{\infty} \frac{dy}{y^{1+c}(\ell n\ y)^{w}L\left((\ell n\ y)^{1/\alpha}\right)},$$

where we have written $w = (\alpha+\beta-1)/\alpha$. But familiar behavior of such integrals involving functions of slow growth shows this last integral

$$\sim \frac{1}{\alpha c}\,\frac{e^{-cn^{\alpha}}}{n^{(\alpha+\beta-1)}L(n)} .$$

Thus the corollary is proved.

<u>Proof of Corollary 1.1.4</u>. This is an immediate deduction from Corollary 1.1.2. We merely need to use the fact that when $0<\nu<1$,

$$\sum_{1}^{n} \frac{1}{j^{\nu}L(j)} \sim \frac{n^{1-\nu}}{(1-\nu)L(n)}, \text{ as } n\to\infty.$$

For the case $\nu=0$ we use the relation

$$\sum_{1}^{n} L_J(j) \sim nL_J(n), \text{ as } n\to\infty.$$

For the case $\nu=1$ we write

$$L_I(x) = \int_1^x \frac{du}{uL(u)} .$$

It is easy to show $L_J(x)$ is a fsg and that

$$\sum_1^n \frac{1}{jL(j)} \sim L_I(n), \text{ as } n\to\infty.$$

These asymptotic relations are sufficient to prove the corollary, and complete the first section of these proofs.

We now turn to problems when $\mu_2 < \infty$. In this case $\sum_{-\infty}^{+\infty} |n| r_n < \infty$, and one can introduce $\underline{s} = \{s_n\}$, where

$$s_n = (r_{n+1} + r_{n+2} + \ldots), \qquad \text{for } n\geq 0,$$
$$= -(r_n + r_{n-1} + r_{n-2} + \ldots), \text{ for } n<0.$$

It is not hard to show that $||\underline{s}|| < \infty$, that

(4.7) $$\hat{\underline{s}}(\theta) = \frac{1-\hat{\underline{r}}(\theta)}{1-e^{i\theta}},$$

and that

(4.8) $$\sum_{j=-\infty}^{+\infty} s_j = \frac{\mu_2-\mu_1}{2\mu_1} .$$

Let $M(n)$ be a rmf and, for integer $n\geq 1$, define

(4.9) $$M_I(n) = M(0) + M(1) + \ldots + M(n-1).$$

We can set $M_I(n) \equiv 1$ for $n\leq 0$ and make $M_I(x)$ an increasing continuous function of $x>0$, if we wish, agreeing with (4.9) when x is integer-valued. Then $M_I(n)$ is strictly increasing and, for any arbitrarily large N, when $n>N$ we have

$$\frac{M_I(n+N)}{M_I(n)} \le 1 + \frac{M(n)}{NM(n-N)} .$$

Since $M(n) \sim M(n-N)$ as $n \to \infty$,

$$\limsup_{n\to\infty} \frac{M_I(n+N)}{M_I(n)} \le 1 + \frac{1}{N} .$$

The arbitrariness of N allows the inference $M_I(n+N) \sim M_I(n)$, i.e. $M_I(n)$ is a fmg. Moreover (4.9) shows that for all integers $k_1 \ge 1$ and $k_2 \ge 1$,

$$M_I(k_1+k_2) \le M_I(k_1) + M(k_1)M_I(k_2). \tag{4.10}$$

But $M(k_1) = o\left(M_I(k_1)\right)$ as $k \to \infty$ and $M_I(k_2) \ge 1$; thus (4.10) implies

$$M_I(k_1+k_2) \le 2M_I(k_1)M_I(k_2) \tag{4.11}$$

for all large k_1 and k_2 . It follows that $M_I(n)$ is equivalent to a rmf, i.e. $M_I(n) \in \mathcal{M}^*$.

<u>LEMMA 4.2.</u> *Suppose* $\mu_2 < \infty$ *and* $\underline{r} \in S(M_I)$, *then* $\underline{s} \in S(M)$.

<u>Proof</u>. Since $\sum_1^\infty M_I(n)|r_n| < \infty$ it follows that $M_I(n)|s_n| \to 0$. However $|r_n| = |s_{n-1}| - |s_n|$, $n \ge 1$, so

$$\sum_1^\infty \{M(0) + M(1) + \ldots + M(n-1)\}(|s_{n-1}| - |s_n|) < \infty.$$

If we use the result $M_I(n)|s_n| \to 0$, a rearrangement of this series is legitimate and produces the result

$$\sum_1^\infty M(n)|s_n| < \infty,$$

as required. The fact that $\sum_{-\infty}^0 |s_n| < \infty$ follows, of course, because $\mu_2 < \infty$.

LEMMA 4.3. *Let* $\underline{a} \in S(M_I)$, $\sum_{-\infty}^{+\infty} |na_n| < \infty$, $\sum_{-\infty}^{+\infty} a_n = 0$. *Suppose* $\mu_2 < \infty$ *and* $\underline{r} \in S(M_I)$. *Then* $\underline{s}*\underline{a} \in S(M_I)$ *and* $\sum_{-\infty}^{+\infty} |n|\ |(\underline{s}*\underline{a})_n| < \infty$.

Proof. By Lemma 4.2 we have $\underline{s} \in S(M)$. Since $\sum_{-\infty}^{+\infty} a_n = 0$,

$$(4.12) \qquad (\underline{s}*\underline{a})_n = \sum_{j\le\frac{1}{2}n} (s_{n-j}-s_n)a_j + \sum_{j>\frac{1}{2}n} s_{n-j}a_j - s_n \sum_{j>\frac{1}{2}n} a_j ,$$

$$= b_n' + b_n'' - b_n''' , \text{ say.}$$

Now

$$|b_n'| \le \sum_{1\le j\le\frac{1}{2}n} (|r_{n-j+1}| + |r_{n-j+2}| + \dots + |r_n|)|a_j| + \sum_{j\ge 1} (|r_{n+1}| + |r_{n+2}| + \dots + |r_{n+j}|)|a_{-j}|,$$

$$= c_n' + c_n'' , \text{ say,}$$

where the sum c_n' is void if $n\le 1$. Let us consider c_n' first; we have, after a rearrangement,

$$\sum_{n=2}^{\infty} M_I(n)c_n' = \sum_{j=1}^{\infty} |a_j| \sum_{s=1} \{\sum_{k\ge j} M_I(k+j)|r_{k+s}|\}$$

$$\le \sum_{j=1}^{\infty} |a_j| \sum_{s=1}^{j} \{\sum_{k\ge j} [M_I(j-s) + M(j-s)M_I(k+s)]|r_{k+s}|\},$$

when we use (4.10). But

$$\sum_{s=1}^{j} \sum_{k\ge j} M_I(j-s)|r_{k+s}| \le M_I(j) \sum_{s=1}^{j} \sum_{k\ge j} |r_{k+s}|$$

$$\le M_I(j) \sum_{k\ge j} k|r_k|$$

$$= O(M_I(j)).$$

On the other hand

$$\sum_{s=1}^{j}\sum_{k\geq j} M(j-s)M_I(k+s)|r_{k+s}| \leq \sum_{k\geq j+1}\{\sum_{s=1}^{j} M(j-s)\}M_I(k)|r_k|,$$

$$= M_I(j)\sum_{k\geq j+1} M_I(k)|r_k|,$$

$$= O(M_I(j)), \text{ also.}$$

Thus $\sum_2^\infty M_I(n)c_n' < \infty$, and we must next consider

$$\sum_{n=1}^{\infty} M_I(n)c_n'' = \sum_{j\geq 1}|a_{-j}|\{\sum_{n=1}^{\infty}\sum_{s=1}^{j} M_J(n)|r_{n+s}|\}.$$

However, for $j\geq 2$,

$$\sum_{n=1}^{\infty}\sum_{s=1}^{j} M_I(n)|r_{n+s}| = \sum_{n=2}^{j}\{\sum_{s=1}^{n-1} M_I(s)\}|r_n| + \sum_{n\geq j+1}\{\sum_{n-j}^{n-1} M_I(s)\}|r_n|$$

$$\leq j\sum_{n=2}^{\infty} M_I(n)|r_n|$$

$$= O(j).$$

A slightly simpler argument will dispose of the case $j=1$ and show therefore, since $\sum_1^\infty j|a_{-j}| < \infty$, that $\sum_1^\infty M_I(n)c_n'' < \infty$. Thus $\sum_1^\infty M_I(n)|b_n'| < \infty$.

Next we consider

$$\sum_{n=1}^{\infty} M_I(n)|b_n''| \leq \sum_{j=1}^{\infty}|a_j|\sum_{1\leq n<j} M_I(n)|s_{n-j}| + \sum_{j=1}^{\infty}|a_j|\sum_{j\leq n<2j} M_I(n)|s_{n-j}|,$$

$$\leq \sum_{j=1}^{\infty} M_I(j)|a_j|\sum_{r\leq n\leq j}|s_{n-j}| + \sum_{j=1}^{\infty}|a_j|\sum_{j<n<2j}\{M_I(n-j) + M(n-j)M_I(j)\}|s_{n-j}|,$$

$$\leq \sum_{j=1}^{\infty} M_I(j)|a_j|\sum_{-\infty}^{\infty}|s_k| + \sum_{j=1}^{\infty}|a_j|M_I(j)\sum_{1}^{\infty} M_I(k)|s_k|,$$

and the convergence of $\sum_1^\infty M_I(n)|b_n''|$ follows from hypothesis.

Finally we deal with b_n'''. We observe that, since $\sum |ja_j| < \infty$, $\sum_{j>\frac{1}{2}n} |a_j| = o(n^{-1})$ as $n\to\infty$. Thus all we need to show is that $\sum_1^\infty n^{-1}M_I(n)|s_n| < \infty$. But $n^{-1}M_I(n) \le M(n)$, so our desired conclusion follows from the fact that $\underline{s} \in S(M)$. Thus we have shown $\sum_1^\infty M_I(n)|(\underline{s}*\underline{a})_n| < \infty$. Since $M_I(n) \ge n$, it follows that $\sum_1^\infty n|(\underline{s}*\underline{a})_n| < \infty$. An argument such as we have just given, but on the left instead of the right tail of the relevant sequences, and with $M_I(n)$ replaced by $|n|$, will obviously conclude the proof. With possible future applications of this lemma in mind we draw attention to the fact that its proof does not use the monotonicity of r_n.

<u>LEMMA 4.4</u>. *Let* $\underline{a}$ *be such that* $\sum_{-\infty}^{+\infty} a_n = 0$ *and* $\sum_{-\infty}^{+\infty} |na_n| < \infty$. *Let* $X(T_2)$ *represent either* $V(T_2)$, $V_0(T_2)$, *or* $W(T_2)$. *Suppose* $\mu_2 < \infty$. *Then, if both* $\underline{a}$ *and* $\underline{r}$ *belong to* $X(T_2) \cap S(G_I)$, *it follows that* $\underline{s}*\underline{a} \in X(T_2) \cap S(G_I)$ *also. In case* $X(T_2) \equiv W(T_2)$ *it follows that* $T(x)$ *must be unbounded, and*

$$<\underline{s}*\underline{a}>_{T_2} = <\underline{r}>_{T_2} \sum_{-\infty}^{+\infty} ja_j + \left(\frac{\mu_2-\mu_1}{2\mu_1}\right)<\underline{a}>_{T_2} .$$

<u>Proof</u>. Lemma 4.3 assures us that $\underline{s}*\underline{a} \in S(G_I)$. We shall prove the case $X(T_2) \equiv W(T_2)$ of the present lemma; the other cases (except for one point noted later) are proved very similarly, indeed, more straightforwardly.

From (4.12) we see that

$$(4.13)\quad T_2(n)b_n' = \sum_{1\le j\le\frac{1}{2}n} a_j\left\{\sum_{s=0}^{j-1} \frac{T_2(n)}{T_2(n-s)}\,[T_2(n-s)r_{n-s}]\right\} + \sum_{j<1} a_{-j}\left\{\sum_{s=1}^{j} \frac{T_2(n)}{T_2(n+s)}\,[T_2(n+s)r_{n+s}]\right\}.$$

If we take formal limits in (4.13) we see that, since $T_2(n)$ is a fmg, as $n \to \infty$

(4.14) $$T_2(n)b_n' \to \sum_{j\geq 1} ja_j \langle \underline{r} \rangle_{T_2} + \sum_{j\geq 1} ja_j \langle \underline{r} \rangle_{T_2}$$
$$= \langle \underline{r} \rangle_{T_2} \sum_{-\infty}^{+\infty} ja_j \ .$$

However, we must justify this limiting procedure by showing the convergence to be suitably dominated. Evidently we may assume $T_2(n)r_n$ to be bounded, all $n \geq 0$. Thus the terms on the right of (4.13) are dominated by the corresponding terms of

$$\sum_{j\geq 1} |a_j| \{ \sum_{s=0}^{j-1} G(s) \} + \sum_{j\geq 1} j|a_{-j}| ;$$

both of these series are convergent, by hypothesis.

Next we note that

(4.15) $$T_2(n)b_n'' = \sum_{j<\frac{1}{2}n} \frac{T_2(n)}{T_2(n-j)} \{T_2(n-j)a_{n-j}\}s_j \ .$$

Since $T_2(n) \sim T_2(n-j)$ and $T_2(n-j)a_{n-j} \to \langle \underline{a} \rangle_{T_2}$ as $n \to \infty$, for j fixed, we have from (4.15) in a formal way that

$$T_2(n)b_n'' \to \langle \underline{a} \rangle_{T_2} \sum_{-\infty}^{+\infty} s_j = \langle \underline{a} \rangle_{T_2} \left(\frac{\mu_2 - \mu_1}{2\mu_1} \right).$$

To justify this taking of limits we observe that $T_2(n)a_n$ must be bounded, so the terms in the sum on the right of (4.15) are dominated by some fixed multiple of terms in

$$\sum_{j<\frac{1}{2}n} G(j)|s_j| \ .$$

Since $\underline{s} \in S(G)$, by Lemma 4.2, the domination is established.

Suppose $T(n)$ is unbounded. One can quickly show that $s_n = 0(1/nT(n))$ and that $\sum_{j>\frac{1}{2}n} a_j = o(1/nT(\frac{1}{2}n))$. Thus $T_2(n)b_n''' = 0(1/T(\frac{1}{2}n)) \to 0$. This completes the proof when $T(n) \to \infty$. On the other hand, if $\tau = \lim_{x\to\infty} T(x) < \infty$, we see

$$s_n \sim \frac{\underline{<r>} T_2}{\tau} \sum_{n+1}^{\infty} \frac{1}{j^2} \sim \frac{\underline{<r>} T_2}{n\tau} .$$

But this result contradicts the fact that $\sum_1^\infty |s_n| < \infty$, consequent upon the finiteness of μ_2. Thus when $X(T_2) \equiv \mathcal{W}(T_2)$ the tail function $T(x)$ *must* be unbounded. However, it is possible to encounter a bounded $T(x)$ in dealing with the alternative hypotheses. Consider the case $X(T_2) \equiv V_0(T_2)$. In this one can easily prove that $s_n = o(1/n)$ and that $\sum_{j>\frac{1}{2}n} a_j = o(1/n)$. Thus $n^2 T(n) b_n''' \to 0$, as desired. The case $X(T_2) \equiv V(T_2)$ can be treated equally simply.

Define $\underline{a}^{(0)} = \underline{\delta}\text{-}\underline{r}$ and, for $k = 1,2,\ldots$, define

$$\underline{a}^{(k)} = \underline{s}^{*k} - \underline{s}^{*k} * \underline{r} = \underline{s} * \underline{a}^{(k-1)} . \tag{4.16}$$

Then $\hat{\underline{a}}^{(k)}(\theta) = \{\hat{\underline{s}}(\theta)\}^k \{1-\hat{\underline{r}}(\theta)\}$ and it is easy to compute that

$$\sum_{n=-\infty}^{+\infty} a_n^{(k)} = \hat{\underline{a}}^{(k)}(0) = 0$$

and, if $\lambda = (\mu_2-\mu_1)/(2\mu_1)$,

$$\sum_{n=-\infty}^{+\infty} n a_n^{(k)} = -i\hat{\underline{a}}'(0) = i\hat{\underline{r}}'(0)\{\hat{\underline{s}}(0)\}^k = -\lambda^{k+1} .$$

It will also be apparent that repetitive use of Lemma 4.3 with $M \equiv I$ shows that

$$\sum_{n=-\infty}^{+\infty} |n a_n^{(k)}| < \infty, \text{ for every } k.$$

Thus we can deduce from that same lemma that, when $\mu_2 < \infty$, if $\underline{r} \in S(M_I)$ then $\underline{a}^{(k)} \in S(M_I)$ for every k. Similarly $\underline{r} \in S(G_I)$ implies $\underline{a}^{(k)} \in (G_I)$ for every k.

We can now make repetitive use of Lemma 4.4 to infer that if $\underline{r} \in X(T_2) \cap S(G_I)$ then $\underline{a}^{(k)} \in X(T_2) \cap S(G_I)$. In particular, if $\underline{r} \in W(T_2)$, a routine calculation will yield the result $<\underline{a}^{(k)}>_{T_2} = -(k+1)\lambda^k <\underline{r}>_{T_2}$. Thus we have

<u>LEMMA 4.5</u>. *Assume* $\mu_2 < \infty$ *and define the sequences* $\underline{a}^{(k)}$ *as in (4.16). Then if* $\underline{r} \in S(M_I)$ *it follows that* $\underline{a}^{(k)} \in S(M_I)$ *for every* $k = 1,2,\ldots$. *On the other hand, if* $\underline{r} \in X(T_2) \cap S(G_I)$, *where* X *is either* V, *or* V_0, *or* W, *then it follows that* $\underline{a}^{(k)} \in X(T_2) \cap S(G_I)$ *for every* $k = 1,2,\ldots$. *In the case when* $\underline{r} \in W(T_2)$,

$$<\underline{a}^{(k)}>_{T_2} = -(k+1)\left(\frac{\mu_2-\mu_1}{2\mu_1}\right)^k <\underline{r}>_{T_2}$$

<u>Proof of Theorem 1.2</u>. If $\underline{r} \in S(M_I)$ then by an argument now familiar, $\{\hat{\underline{r}}(\theta)\}^{-1} \in \hat{S}(M_I)$. But Lemma 4.5 shows that $\hat{\underline{a}}^{(1)}(\theta) = \hat{\underline{s}}(\theta)\{1 - \hat{\underline{r}}(\theta)\} \in \hat{S}(M_I)$. Thus Lemma 2.1 assures us that

$$\frac{\hat{\underline{s}}(\theta)\{1-\hat{\underline{r}}(\theta)\}}{\mu_1\hat{\underline{r}}(\theta)} = \hat{\underline{g}}(\theta), \text{ say,}$$

also belongs to $\hat{S}(M_I)$, where $\underline{g} = \{g_n\}$. If we use (4.7) we find

$$\frac{\{1-\hat{\underline{r}}(\theta)\}^2}{\mu_1\hat{\underline{r}}(\theta)} = (1-e^{i\theta})\hat{\underline{g}}(\theta),$$

and so, since $\hat{\underline{v}}(\theta) = 1/\{\mu_1\hat{\underline{r}}(\theta)\}$,

$$\hat{\underline{v}}(\theta) - \frac{2}{\mu_1} + \frac{\hat{\underline{r}}(\theta)}{\mu_1} = (1-e^{i\theta})\hat{\underline{g}}(\theta).$$

But $v_n = u_n - u_{n-1}$, so we can infer, (4.17) $$u_n - u_{n-1} - \frac{2\delta_n}{\mu_1} + \frac{r_n}{\mu_1} = g_n - g_{n-1} .$$

We note that $g_n \to 0$ as $|n| \to \infty$, since $\underline{g} \in S(M_I)$. Let m be any non-negative integer, and sum (4.17) with respect to n from $-\infty$ to m.

(a) If $\mu_1 > 0$, so $u_n \to 0$ as $n \to -\infty$,

(4.18) $$u_m - \frac{2}{\mu_1} + \frac{1}{\mu_1}\sum_{-\infty}^{m} r_n = g_m , \quad m \geq 0.$$

(b) If $\mu_1 < 0$, so $u_n \to -\mu_1^{-1}$ as $n \to -\infty$,

(4.19) $$u_m - \frac{1}{\mu_1} + \frac{1}{\mu_1}\sum_{-\infty}^{m} r_n = g_m , \quad m \geq 0.$$

The theorem follows at once, since $s_m = 1 - \sum_{-\infty}^{m} r_n$.

Proof of Theorem 1.3. If $\underline{r} \in S(G_I)$ then the previous proof shows $\underline{g} \in S(G_I)$, which establishes that $g_n \to 0$ as $|n| \to \infty$, and allows the derivation of the alternative equations (4.18) when $\mu_1 > 0$ and (4.19) when $\mu_1 < 0$. From Theorem 3.1 we see that $1/\hat{\underline{r}}(\theta)$ belongs to $X(T_2) \cap S(G_I)$. From Lemma 4.1 we see that $\hat{\underline{s}}(\theta)\{1 - \hat{\underline{r}}(\theta)\} \in X(T_2) \cap S(G_I)$. Thus by Lemma 2.4 we can infer that $\underline{g} \in X(T_2)$ as required. This proves the theorem except for the evaluation of the limit if $\underline{r} \in W(T_2)$. Since $\sum_{-\infty}^{+\infty} a_n^{(1)} = 0$ it follows from Lemma 2.4 that

$$\langle \underline{g} \rangle_{T_2} = \langle \hat{\underline{a}}^{(1)} \rangle_{T_2} \sum_{-\infty}^{+\infty} v_n ,$$

where we use the fact that $\hat{v}(\theta) = 1/\{\mu_1 \hat{\underline{r}}(\theta)\}$. But $\sum_{-\infty}^{+\infty} v_n = \mu_1^{-1}$, so the limit is as claimed because Lemma 4.5 shows

$$\langle \hat{\underline{a}}^{(1)} \rangle_{T_2} = -2\left(\frac{\mu_2 - \mu_1}{2\mu_1}\right)\langle \underline{r} \rangle_{T_2} .$$

Proof of Theorem 1.4. By Lemma 4.5, under the conditions of the present theorem, we have that $\{\hat{\underline{s}}(\theta)\}^2\{1 - \hat{\underline{r}}(\theta)\} \in S(M_I)$. Thus, if we define

$$\hat{\underline{g}}(\theta) = \frac{\{\hat{\underline{s}}(\theta)\}^2\{1-\hat{\underline{r}}(\theta)\}}{\mu_1\hat{\underline{r}}(\theta)}, \tag{4.20}$$

then an argument similar to that in the proof of Theorem 1.2 will show $\hat{\underline{g}}(\theta) \in S(M_I)$. But, since $\hat{\underline{s}}(\theta) = \{1 - \hat{\underline{r}}(\theta)\}/\{1 - e^{i\theta}\}$, we can obtain from (4.20) the result

$$\hat{\underline{v}}(\theta) - \frac{3}{\mu_1} + \frac{3}{\mu_1}\hat{\underline{r}}(\theta) - \frac{\{\hat{\underline{r}}(\theta)\}^2}{\mu_1} = (1-e^{i\theta})^2\hat{\underline{g}}(\theta).$$

The substitution $\hat{\underline{r}}(\theta) = 1 - (1-e^{i\theta})\hat{\underline{s}}(\theta)$ then shows

$$\hat{\underline{v}}(\theta) - \frac{1}{\mu_1} - \frac{(1-e^{i\theta})\hat{\underline{s}}(\theta)}{\mu_1} - \frac{(1-e^{i\theta})^2\{\hat{\underline{s}}(\theta)\}^2}{\mu_1} = (1-e^{i\theta})^2\hat{\underline{g}}(\theta),$$

from which it follows that

$$u_n - u_{n-1} - \frac{\delta_n}{\mu_1} - \frac{1}{\mu_1}(s_n - s_{n-1}) - \frac{1}{\mu_1}(s_n^{*2} - 2s_{n-1}^{*2} + s_{n-2}^{*2}) \tag{4.21}$$
$$= g_n - 2g_{n-1} + g_{n-2}.$$

Let us write
$$\left.\begin{aligned}\Delta_n &= 1 \quad \text{for} \quad n \geq 0\\ &= 0 \quad \text{for} \quad n < 0\end{aligned}\right\}.$$

Then if we sum (4.21) with respect to n from $-\infty$ to m, and use the fact that $g_n \to 0$ as $|n| \to \infty$, we obtain:

(a) If $\mu_1 > 0$, so $u_n \to 0$ as $n \to -\infty$,

$$u_m - \frac{\Delta_m}{\mu_1} - \frac{s_m}{\mu_1} - \frac{(s_m^{*2} - s_{m-1}^{*2})}{\mu_1} = g_m - g_{m-1}. \tag{4.21}$$

If we let $p \geq 0$ and sum (4.21) with respect to m from $-\infty$ to p we see that $\sum_{-\infty}^{p} u_m = U_p$ must be finite and that

(4.22) $$U_p - \frac{(p+1)}{\mu_1} - \frac{1}{\mu_1}\sum_{-\infty}^{p} s_m - \frac{1}{\mu_1} s_p^{*2} = g_p .$$

Since $\sum_{-\infty}^{+\infty} s_m = (\mu_2-\mu_1)/(2\mu_1)$, we can rewrite (4.22) in the form

$$U_p - \frac{(p+1)}{\mu_1} - \left(\frac{\mu_2-\mu_1}{2\mu_1^2}\right) + \frac{1}{\mu_1}\sum_{p+1}^{\infty} s_m - \frac{1}{\mu_1} s_p^{*2} = g_p ,$$

which is the result announced in the theorem.

(b) If $\mu_1 < 0$, so $u_n \to 1/|\mu_1|$ as $n \to -\infty$, we obtain by summing (4.21) the result

(4.23) $$u_m - \frac{1}{|\mu_1|} - \frac{\Delta_m}{\mu_1} - \frac{(s_m^{*2} - s_{m-1}^{*2})}{\mu_1} = g_m - g_{m-1} .$$

It follows that

$$r = \sum_{m=-\infty}^{-1} \left(u_m - \frac{1}{|\mu_1|}\right)$$

is convergent, and this part of the theorem follows on summing (4.23) in the same way as was done in the case $\mu_1 > 0$.

Proof of Theorem 1.5. The arguments required should be obvious at this stage. The only point needing comment is the evaluation of $\langle \underline{g} \rangle_{T_2}$. But this can be done on lines similar to those followed in the proof of Theorem 1.3. We find in the present context that

$$\langle \underline{g} \rangle_{T_2} = \langle \hat{\underline{a}}^{(2)} \rangle_{T_2} \sum_{-\infty}^{+\infty} v_n$$

$$= -3\left(\frac{\mu_2-\mu_1}{2\mu_1}\right)^2 \cdot \frac{\langle \underline{r} \rangle_{T_2}}{\mu_1} ,$$

which agrees with the result claimed in the enunciation.

§5 PROOF OF THEOREMS WHEN CONDITIONS ARE ON $\underline{f}$

From a given rmf $M(n)$ we construct $M_I(n)$ and $M_{II}(n)$; we note that† $n \prec M_I(n)$, $n^2 \prec M_{II}(n)$, $M_{II}(n) \prec nM_I(n) \prec n^2M(n)$. Suppose we are given that

$$\sum_1^\infty nM_{II}(n)f_n < \infty, \tag{5.1}$$

(which implies $\sum_1^\infty n^3f_n < \infty$). Then it follows that $nM_{II}(n)r_n \to 0$ as $n\to\infty$ and we can replace f_n by $(r_{n-1} - r_n)$ and legitimately rearrange (5.1) to produce

$$\sum_1^\infty \{(n+1)M_{II}(n+1) - nM_{II}(n)\}r_n < \infty. \tag{5.2}$$

But the expression in braces equals $nM_I(n) + M_{II}(n+1) \prec nM_I(n)$. Thus (5.2) is equivalent to

$$\sum_1^\infty nM_I(n)r_n < \infty, \tag{5.3}$$

This implies $nM_I(n)s_n \to 0$ as $n\to\infty$, and a similar manoeuvre produces the result

$$\sum_1^\infty nM(n)s_n < \infty. \tag{5.4}$$

In particular (5.3) implies that $\underline{r} \in S(M_{II})$ and (5.4) implies $\underline{s} \in S(M_I)$. Let $\tilde{w}_n = \mu_1^{-1}$ for $n\ge 0$, $\tilde{w}_n = 0$ for $n<0$. Then when $E|X|^3 < \infty$ it follows from Theorem 1.2 that, when $\mu_1 > 0$,

$$\sum_{-\infty}^{+\infty} n^2\left|u_n - \tilde{w}_n - \frac{s_n}{\mu_1}\right| < \infty. \tag{5.5}$$

But when the third absolute moment is finite, $\sum_{-\infty}^{+\infty} n|s_n| < \infty$, so (5.5) implies the weaker result $\sum_{-\infty}^{+\infty} n|u_n - \tilde{w}_n| < \infty$. Thus the Fourier series

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† We write, e.g., $a_n \prec b_n$ to mean $a_n = O(b_n)$ as $n\to\infty$.

$\sum_{-\infty}^{+\infty} e^{in\theta}(u_n - \tilde{w}_n) = \phi(\theta)$, say, is absolutely convergent, as is the series obtained by term by term differentiation. To lessen confusion we shall deal solely with the case $\mu_1 > 0$; the case $\mu_1 < 0$ can be treated similarly. It is fairly easy to deduce from (4.4) that

$$\phi(\theta) = \frac{1}{1-\hat{\underline{f}}(\theta)} - \frac{1}{\mu_1(1-e^{i\theta})},$$

and hence that

$$\phi'(\theta) = \frac{\hat{\underline{f}}'(\theta)}{[1-\hat{\underline{r}}(\theta)]^2} - \frac{ie^{i\theta}}{\mu_1(1-e^{i\theta})^2}. \tag{5.6}$$

A fair amount of routine calculation based on (5.6) will yield the equation

$$-i\mu_1\phi'(\theta) = \chi_1(\theta) + \chi_2(\theta) + \chi_3(\theta), \tag{5.7}$$

where

$$\chi_1(\theta) = \frac{e^{i\theta}\hat{\underline{s}}(\theta)+i\underline{r}'(\theta)}{(1-e^{i\theta})},$$

$$\chi_2(\theta) = -i\frac{d}{d\theta}\hat{\underline{s}}(\theta)[1-\hat{\underline{r}}(\theta)],$$

$$\chi_3(\theta) = \frac{e^{i\theta}\hat{\underline{a}}^{(2)}(\theta)}{\hat{\underline{r}}(\theta)} + \frac{i\hat{\underline{r}}'(\theta)[1+2\hat{\underline{r}}(\theta)]\hat{a}^{(1)}(\theta)}{[\hat{\underline{r}}(\theta)]^2}. \tag{5.8}$$

Proof of Theorem 1.6. If (5.1) holds, in view of what has been proved, it is routine to show that $\chi_3(\theta) \in \hat{S}(M_{II})$. The one point calling for comment is the term involving $\hat{\underline{r}}'(\theta)$. However $\hat{\underline{r}}'(\theta)\hat{\underline{a}}^{(1)}(\theta) = [\hat{\underline{s}}(\theta)]^2\hat{\underline{b}}(\theta)$, say, where $\mu_1\hat{\underline{b}}(\theta) = i\mu_1 e^{i\theta}\hat{\underline{r}}(\theta) - \hat{\underline{f}}'(\theta)$. It is easy to verify that $\underline{b} \in S(M_{II})$ and that $\sum_{-\infty}^{+\infty} b_n = 0$. Thus we can infer from Lemma 4.3 that $[\hat{\underline{s}}(\theta)]^2\hat{\underline{b}}(\theta) \in \hat{S}(M_{II})$, as desired.

A straightforward calculation based on (5.8) shows that

$$\chi_1(\theta) = \sum_{-\infty}^{+\infty} e^{in\theta}(ns_n). \tag{5.9}$$

Thus the real difficulty rests in the interpretation of $\chi_2(\theta)$. If we set $\hat{s}(\theta)[1 - \hat{r}(\theta)] = \hat{\underline{c}}(\theta)$, say, then $\chi_2(\theta) = \sum_{-\infty}^{+\infty} nc_n e^{in\theta}$. We can deduce from Lemma 4.3 that $\hat{\underline{c}}(\theta) \in S(M_{II})$, but this is an inadequate result for the present theorem. Set

$$c_n = s_n \sum_{j>\frac{1}{2}n} r_j - r_n \sum_{j<\frac{1}{2}n} s_j - \sum_{j\le\frac{1}{2}n} (s_{n-j} - s_n)r_j - \sum_{j<\frac{1}{2}n} (r_{n-j} - r_n)s_j$$
$$= c_n^{(1)} - c_m^{(2)} - c_n^{(3)} - c_n^{(4)} \text{, say.}$$

Since $\sum_1^\infty j^2|r_j| < \infty$ it follows that $|n^2 M_I(n)c_n^{(1)}| \prec M_I(n)|s_n|$. Thus $\sum_1^\infty nM_{II}(n)c_n^{(1)} < \infty$.

Next we notice that

$$c_n^{(2)} = \lambda r_n - r_n \sum_{j\ge\frac{1}{2}n} s_j .$$

But $\sum_1^\infty j|s_j| < \infty$, so $|c_n^{(2)} - \lambda r_n| \prec n^{-1} r_n$ and we find $\sum_1^\infty nM_{II}(n)|c_n^{(2)} - \lambda r_n| < \infty$.

We set

$$c_n^{(3)} = \sum_{1\le j\le\frac{1}{2}n} (s_{n-j} - s_n)r_j - \sum_{j\ge 1} (s_n - s_{n+j})r_{-j}$$
$$= c_n^{(31)} - c_n^{(32)} \text{, say.}$$

Routine manoeuvres then show

$$\sum_{n=1}^{\infty} nM_{II}(n)\left|c_n^{(31)} - r_n \sum_{1\le j\le\frac{1}{2}n} jr_j\right| \le \sum_{j=1}^{\infty} |r_j| \sum_{s=2}^{j} (s-1) \sum_{k=j}^{\infty} (k+j)M_{II}(k+j)f_{k+s} .$$

However, one can show

$$(5.10) \qquad M_{II}(k+j) \le M(j-s)\{(j-s)^2 + M_{II}(k+s)\},$$

and, in the range of the summations, we have trivially that $k+j < 2(k+s)$ and

$(j-s) < (k+s)$. Thus to establish the convergence of the last series we consider

$$\sum_{j=1}^{\infty} |r_j| \sum_{s=2}^{j} (s-1)M(j-s) \sum_{k=j}^{\infty} \{(k+s)^3 + (k+s)M_{II}(k+s)\}f_{k+s} .$$

This is convergent because $\sum_{s=2}^{j} (s-1)M(j-1) = M_{II}(j)$ and $\underline{r} \in S(M_{II})$.

In a similar way we find

$$\sum_{n=1}^{\infty} nM_{II}(n)|c_n^{(32)} - r_n \sum_1^{\infty} jr_{-j}| \le \sum_{j=1}^{\infty} |r_{-j}| \sum_{s=1}^{j} (j-s+1) \sum_{n=1}^{\infty} nM_{II}(n)f_{n+s}$$

$$\le \{\tfrac{1}{2} \sum_{j=1}^{\infty} j(j+1)|r_{-j}|\}\{\sum_{n=s+1}^{\infty} nM_{II}(n)f_n\},$$

which is finite since $\sum_{-\infty}^{+\infty} j^2|r_j| < \infty$. However $r_n \sum_{j>\frac{1}{2}n} jr_j \prec n^{-1}r_n$, so

$$\sum_{n=1}^{\infty} nM_{II}(n)|r_n \sum_{j>\frac{1}{2}n} jr_j| < \infty.$$

Since $\sum_{-\infty}^{+\infty} jr_j = \lambda$ we can combine this result with those on $c_n^{(31)}$ and $c_n^{(32)}$ to infer

$$\sum_{n=1}^{\infty} nM_{II}(n)|c_n^3 - r_n\lambda| < \infty. \tag{5.11}$$

Finally we set

$$c_n^{(4)} = \sum_{1\le j<\frac{1}{2}n} (r_{n-j} - r_n)s_j - \sum_{j=1}^{\infty} (r_n - r_{n+j})s_{-j}$$

$$= c_n^{(41)} - c_n^{(42)} , \text{ say.}$$

In a familiar way we can show

$$\sum_{n=1}^{\infty} nM_{II}(n)|c_n^{(41)}| \le \sum_{j=1}^{\infty} j|s_j| \sum_1^{\infty} \{k^2 + kM_{II}(k)\}f_k ,$$

and

$$\sum_{n=1}^{\infty} nM_{II}(n)|c_n^{(42)}| \le \sum_{j=1}^{\infty} |s_{-j}| \sum_{s=1}^{j} \sum_{n=s+1}^{\infty} nM_{II}(n)f_n ,$$

both of which series are convergent.

Thus we have proved that

$$\sum_{n=1}^{\infty} nM_{II}(n)|c_n + 2\lambda r_n| < \infty. \tag{5.12}$$

However, we know that

$$\mu_1 n(u_n - \tilde{w}_n) = ns_n + nc_n + g_n \text{ , say,}$$

where $\hat{g}(\theta) = \chi_3(\theta)$ and so $g \in S(M_{II})$. Thus

$$\mu_1(u_n - \tilde{w}_n) - s_n + 2\lambda r_n = (c_n + 2\lambda r_n) + n^{-1}g_n \text{ .} \tag{5.13}$$

In view of (5.12) we have therefore proved:

$$\sum_{n=1}^{\infty} nM_{II}(n)|u_n - \mu_1^{-1} - \frac{s_n}{\mu_1} + \frac{2\lambda r_n}{\mu_1}| < \infty,$$

and, on noting that $2\lambda = (\mu_2 - \mu_1)/\mu_1$, this completes the proof of Theorem 1.6.

<u>Proof of Theorem 1.7</u>. Given a tail function $T(x)$ with gauge function $G(x)$, define $\tilde{T}(x)$ for all $x > 0$ by the equation

$$\frac{1}{\tilde{T}(x)} = 2x^2 \int_x^{\infty} \frac{dy}{y^3 T(y)} \text{ .} \tag{5.14}$$

Then it is fairly easy to show that $\tilde{T}(x) \geq T(x)$, that $\tilde{T}(x)$ is non-decreasing, and, by using the relation $[(y+1)^3 T(y+1)]^{-1} \sim [y^3 T(y)]^{-1}$ as $y \to \infty$, one can even infer that $\tilde{T}(x)$ is a fmg. Thus $\tilde{T}(x)$ is a tail function. Moreover, let us fix $r \geq 1$ and take $x \geq 2r$. Then

$$\int_{x-r}^{\infty} \frac{dy}{y^3 T(y)} = \int_x^{\infty} \left\{ \left(\frac{y}{y-r}\right)^3 \frac{T(y)}{T(y-r)} \right\} \frac{dy}{y^3 T(y)}$$

$$\leq 8G(r) \int_x^{\infty} \frac{dy}{y^3 T(y)} \text{ .}$$

Thus we can infer that

$$\sup_{x\geq 2r} \frac{\tilde{T}(x)}{\tilde{T}(x-r)} \leq 2G(r)$$

and so, if $\tilde{G}(x)$ is a minimal gauge function for $\tilde{T}(x)$, then $\tilde{G}(x) \prec G(x)$. Consequently, for example, $S(\tilde{G}_I) \subset S(\tilde{G}_I)$.

In what follows we shall write $\tilde{T}_2$ for what should strictly be $(\tilde{T})_2$, i.e. we intend $\tilde{T}_2(x) \sim x^2\tilde{T}(x)$ as $x\to\infty$.

Let us concentrate first on the proof of Part (c) of the theorem. In this case, as $n\to\infty$,

(5.15) $$r_n \sim \frac{\phi}{\mu_1}\int_n^\infty \frac{dx}{x^3 T(x)} = \frac{\phi}{2\mu_1 n^2 \tilde{T}(n)} .$$

Thus $\underline{r} \in W(\tilde{T}_2)$ and $\langle\underline{r}\rangle_{\tilde{T}_2} = \phi/(2\mu_1)$. Let us rewrite (5.8), setting $\hat{\underline{g}}(\theta) = \chi_3(\theta)$, as follows:

$$\begin{aligned}\hat{\underline{g}}(\theta) &= \hat{\underline{g}}^{(1)}(\theta) + \frac{i\hat{\underline{r}}'(\theta)[1+2\hat{\underline{r}}(\theta)]\hat{\underline{a}}^{(1)}(\theta)}{[\hat{\underline{r}}(\theta)]^2} \\ &= \hat{\underline{g}}^{(1)}(\theta) + \hat{\underline{g}}^{(2)}(\theta), \text{ say.}\end{aligned}$$

Note that under the present hypotheses, $\sum_1^\infty n\tilde{G}_{II}(n)f_n$, $\sum_1^\infty \tilde{G}_{II}(n)|r_n|$, $\sum_1^\infty \tilde{G}_I(n)|s_n|$ are all convergent. Familiar arguments will then show $\hat{\underline{g}}^{(1)} \in \hat{W}(T_2) \cap \hat{S}(G_{II})$.

However,

$$\hat{\underline{g}}^{(2)}(\theta) = \hat{\underline{\ell}}(\theta)\hat{\underline{s}}(\theta)[\hat{\underline{r}}'(\theta)\{1 - \hat{\underline{r}}(\theta)\}], \text{ say,}$$

where $\hat{\underline{\ell}}(\theta) \in \hat{W}(\tilde{T}_2) \cap \hat{S}(\tilde{G}_{II})$, also.

Let us temporarily set

$$\hat{\underline{r}}'(\theta)\{1 - \hat{\underline{r}}(\theta)\} = i\hat{\underline{h}}(\theta), \text{ say.}$$

Then $h_n = nr_n - \sum_{-\infty}^{+\infty} r_{n-j}\, jr_j$. By arguments similar to those employed in earlier proofs we can establish the following.

LEMMA 5.1. *Under the hypotheses of Theorem 1.7,* $\underline{h} \in S(G_{II})$ *and: (a)* $\underline{f} \in V(T_3)$ *implies* $\underline{h} \in V(T_2)$*; (b)* $\underline{f} \in V_0(T_3)$ *implies* $\underline{h} \in V_0(T_2)$*; (c)* $\underline{f} \in W(T_3)$ *implies* $\underline{h} \in W(T_2)$ *and*

$$<\underline{h}>_{T_2} = -\frac{\phi\lambda}{\mu_1} .$$

Since $\sum_{-\infty}^{+\infty} h_n = 0$ we can let $\underline{h}$ play the role of $\underline{a}$ in Lemma 4.4.

First suppose $T(n) = o(\tilde{T}(n))$ as $n\to\infty$. Then $W(\tilde{T}_2) \subset V_0(T_2)$, implying that $\underline{g}^{(1)}$ and $\underline{\ell}$ both belong to $V_0(T_2)$. Furthermore $<\underline{r}>_{T_2}$ must be zero, and Lemma 4.4 tells us that $\underline{s}*\underline{h} \in W(T_2)$ and

$$<\underline{s}*\underline{h}>_{T_2} = \lambda<\underline{h}>_{T_2} = -\frac{\phi\lambda^2}{\mu_1} .$$

Thus, if we note that $\hat{\underline{g}}^{(2)}(\theta) = i\hat{\underline{\ell}}(\theta)\hat{\underline{s}}(\theta)\hat{\underline{h}}(\theta)$, we have $\underline{g}^{(2)} \in W(T_2)$. But $\hat{\underline{\ell}}(0) = 3i$, so we can deduce

$$<\underline{g}^{(2)}>_{T_2} = -3<\underline{s}*\underline{h}>_{T_2} = -3\lambda^2\phi/\mu_1 .$$

Thus $\underline{g} \in W(T_2)$ and

$$<\underline{g}>_{T_2} = -\frac{3\lambda^2\phi}{\mu_1} . \tag{5.16}$$

Next suppose $T(n) \sim \rho\tilde{T}(n)$, where $\rho\neq 0$. This will happen if and only if $T(x)$ is a function of regular variation. From (5.15) we have $\underline{r} \in W(T_2)$ and $<\underline{r}>_{T_2} = \rho\phi/(2\mu_1)$. Thus we see easily that $\underline{g}^{(1)} \in W(T_2)$, and a straightforward computation gives $<\underline{g}^{(1)}>_{T_2} = -3\lambda^2\rho\phi/(2\mu_1)$. One can also obtain, as for the case when $T(n) = o(\tilde{T}(n))$, the result

$<\underline{g}^{(2)}>_{T_2} = -3\lambda^2\phi/\mu_1$. Thus we find in this case:

(5.17) $$<\underline{g}>_{T_2} = -\frac{3\lambda^2(1+\frac{1}{2}\rho)\phi}{\mu_1} .$$

Notice that setting $\rho=0$ in (5.14) gives a result agreeing with (5.16).

Next we must discuss $\chi_2(\theta)$. In the proof of Theorem 1.6 we introduced $\hat{\underline{c}}(\theta) = \hat{\underline{s}}(\theta)[1 - \hat{\underline{r}}(\theta)]$, and expressed $c_n = c_m^{(1)} - c_n^{(2)} - c_n^{(3)} - c_n^{(4)}$. We treat each of these four terms separately.

Because $\sum|j|^3 f_j < \infty$ it follows that $s_n = o(n^{-2})$ and $\sum_n^\infty s_j = o(n^{-1})$. Also, since $\tilde{T}_2(n)r_n \to \phi/(2\mu_1)$ it follows that $\tilde{T}_1(n)s_n$ is bounded as $n\to\infty$. Thus, remembering $\tilde{T}(n) \geq T(n)$, it is easy to show both $c_n^{(1)}$ and $c_n^{(2)} - \lambda r_n$ are $o(1/T_3(n))$, as $n\to\infty$. We content ourselves with merely giving examples of how to deal with the remaining terms. Consider $c_n^{(31)}$; we find

$$\mu_1\left(c_n^{(31)} - r_n \sum_{1\leq j\leq\frac{1}{2}n} jr_j\right) = \sum_{1\leq j\leq\frac{1}{2}n} \left\{\sum_{s=2}^{j} (s-1)f_{n-j+s}\right\}r_j .$$

Thus, in a formal way, we infer that as $n\to\infty$,

(5.18) $$\mu_1 T_3(n)\left(c_n^{(31)} - r_n \sum_{1\leq j\leq\frac{1}{2}n} jr_j\right) \to \phi \sum_1^\infty \{\sum_2^j (s-1)\}r_j$$
$$= \tfrac{1}{2}\phi \sum_1^\infty j(j-1)r_j .$$

The justification of this formal taking of the limit can be seen as follows. Since we may suppose $T_3(n)f_n$ to be bounded,

$$T_3(n) \sum_{s=2}^{j} (s-1)f_{n-j+s} = \sum_{s=2}^{j} (s-1)\left\{\frac{T_3(n)}{T_3(n-j+s)}\right\} \left\{T_3(n-j+s)f_{n-j+s}\right\}$$
$$\prec \sum_{s=2}^{j} (s-1)G(j-s) = G_{II}(j).$$

But, under the hypotheses of the theorem, $\sum_1^\infty G_{II}(j)|r_j| < \infty$; thus the thus the convergence is suitably dominated. However, since $\sum_n^\infty jr_j = o(1/n)$ implies $r_n \sum_{j>\frac{1}{2}n} jr_j = o(1/T_3(n))$, we may deduce from (5.18) the result

$$\text{(5.19)} \qquad \mu_1 T_3(n)\left(c_n^{(31)} - r_n \sum_1^\infty jr_j\right) \to \tfrac{1}{2}\phi \sum_1^\infty j(j-1)r_j \;.$$

One can also show

$$\text{(5.20)} \qquad \mu_1 T_3(n)\left(c_n^{(32)} - r_n \sum_{-\infty}^{-1} jr_j\right) \to \tfrac{1}{2}\phi \sum_{-\infty}^{-1} j(j-1)r_j \;,$$

and justify the convergence adduced by the fact that $\sum_{-\infty}^{-1} j^2|r_j| < \infty$.

It is possible to show $3\mu_1 \sum_{-\infty}^{+\infty} j(j-1)r_j = \mu_3 - 3\mu_2 + 2\mu_1$ and so, combining (5.19) and (5.20), infer

$$\text{(5.21)} \qquad T_3(n)[c_n^{(3)} - \lambda r_n] \to \frac{\phi(\mu_3 - 3\mu_2 + 2\mu_1)}{6\mu_1^2}, \text{ as } n\to\infty.$$

Similarly one can show

$$T_3(n)c_n^{(4)} \to \frac{\phi(\mu_3 - 3\mu_2 + 2\mu_1)}{6\mu_1^2}, \text{ as } n\to\infty,$$

and on combining this result with (5.21) and the earlier comments on $c_n^{(1)}$ and $c_n^{(2)}$ we see that

$$\text{(5.22)} \qquad T_3(n)[c_n + 2\lambda r_n] \to \frac{\phi(\mu_3 - 3\mu_2 + 2\mu_1)}{3\mu_1^2}, \text{ as } n\to\infty.$$

At this point we deduce from (5.13) the equation

$$\mu_1 T_3(n)(u_n - \tilde{w}_n - \mu_1^{-1}s_n + 2\mu_1^{-1}\lambda r_n) = T_3(n)(c_n + 2\lambda r_n) + T_2(n)g_n \;.$$

Part (c) of the theorem now follows from (5.16) and (5.22). The proof of Parts (a) and (b) rely on the facts that $V(\tilde{T}) \subset V(T)$ and that $V_0(\tilde{T}) \subset V_0(T)$, and are simpler than that of Part (a) although they run on similar lines; we shall therefore omit these parts of the complete proof of the theorem.

PROOF OF THEOREM 1.8. The differentiation of $\hat{\underline{r}}(\theta)$ is justified if $\mu_2 < \infty$. Thus an integration by parts based on (4.4) yields

$$n\mu_1 v_n = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-in\theta}\left\{\frac{i\hat{\underline{r}}'(\theta)}{[\hat{\underline{r}}(\theta)]^2}\right\}d\theta. \tag{5.23}$$

Let us, temporarily, set $\hat{\underline{g}}(\theta) = i\hat{\underline{r}}'(\theta)/[\hat{\underline{r}}(\theta)]^2$. Evidently,

$$\hat{\underline{g}}(\theta) = i\hat{\underline{r}}'(\theta) + \frac{i\hat{\underline{r}}'(\theta)[1-\hat{\underline{r}}(\theta)][1+\hat{\underline{r}}(\theta)]}{[\hat{\underline{r}}(\theta)]^2} \tag{5.24}$$

$$= \hat{\underline{g}}^{(1)}(\theta) + \hat{\underline{g}}^{(2)}(\theta), \text{ say.}$$

We have, for Lemma 5.1, defined $i\hat{\underline{h}}(\theta) = \hat{\underline{r}}'(\theta)[1 - \hat{\underline{r}}(\theta)]$ and can prove by methods amply delineated:

Lemma 5.2. *If* $T(x) \to \infty$, $\mu_2 < \infty$, *and* $\sum_1^\infty j\, G_I(j) f_j < \infty$, *then* $\underline{h} \in S(G_I)$ *and:* *(a)* $\underline{f} \in V(T_2)$ *implies* $\underline{h} \in V(T_1)$;

(b) $\underline{f} \in V_0(T_2)$ *implies* $\underline{h} \in V_\theta(T_1)$;

(c) $\underline{f} \in W(T_2)$ *implies* $\underline{h} \in W(T_1)$ *and*

$$\langle\underline{h}\rangle_{T_1} = -\frac{\lambda}{\mu_1}\langle\underline{f}\rangle_{T_2}.$$

Thus, if we concentrate on the most awkward case, to prove: $\underline{f} \in W(T_2)$, we see in a familiar way that $\underline{g}^{(2)} \in W(T_1)$ and

$$\langle\underline{g}^{(2)}\rangle_{T_1} = \frac{2\lambda}{\mu_1}\langle\underline{f}\rangle_{T_2} = \gamma, \text{ say.}$$

Thus from (5.23) and (5.24) we conclude

$$n\mu_1(u_n - u_{n-1}) + nr_n \sim \frac{\gamma}{T_1(n)},$$

i.e.

$$u_n - u_{n-1} + \frac{r_n}{\mu_1} \sim \frac{\gamma}{\mu_1 n T_1(n)} .$$

Hence, when $\mu_1 > 0$, by summation we have

$$u_n - \frac{1}{\mu_1} - \frac{s_n}{\mu_1} \sim - \frac{\gamma}{\mu_1} \sum_{r=n+1}^{\infty} \frac{1}{T_2(r)} .$$

The remainder of the proof follows familiar lines.

<u>PROOF OF THEOREM 1.9</u>. From (4.4) we have

$$\mu_1 v_n = \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{-in\theta} \frac{d\theta}{R(e^{i\theta})}$$

$$= \frac{1}{2\pi i} \oint \frac{dz}{z^{n+1} R(z)} ,$$

where the contour for the latter integral is the circle $|z| = 1$. It is easy to see that $R(z)$ must be analytic in $1 < |z| < \rho^{-1}$ and continuous and bounded in $1 \le |z| \le \rho^{-1}$. Thus routine change of contours coupled with bounded convergence gives

$$\mu_1 v_n = \sum_{s=1}^{\kappa} \frac{1}{\zeta_s^{n+1} R'(\zeta_s)} + \frac{\rho^n}{2\pi} \int_{-\pi}^{+\pi} \frac{e^{\pm in\theta} d\theta}{R(\rho^{-1} e^{i\theta})} .$$

However $R(\rho^{-1}e^{i\theta}) = \sum_{-\infty}^{+\infty} \rho^{-n} r_n e^{ni\theta}$ and we have $\sum_{-\infty}^{+\infty} |\rho^{-n} r_n| G(n) < \infty$ and $\rho^{-n} r_n n^{1+\nu} T(n) \to c$ as $n \to \infty$. Thus we can infer (since $R(z)$ is assumed not to vanish on $|z| = \rho^{-1}$):

$$\mu_1 v_n - \sum_{s=1}^{\kappa} \frac{1}{\zeta_s^{n+1} R'(\zeta_s)} \sim - \frac{c\rho^n}{n^{1+\nu} T(n) [R(\rho^{-1})]^2} .$$

Thus

$$u_{n-1} - u_n + \frac{1}{\mu_1} \sum_{s=1}^{\kappa} \frac{1}{\zeta_s^{n+1} R'(\zeta_s)} \sim \frac{c\rho^n}{\mu_1 n^{1+\nu} T(n) [R(\rho^{-1})]^2} .$$

Hence, when $\mu_1 > 0$ for example,

$$u_n - \frac{1}{\mu_1} + \frac{1}{\mu_1}\sum_{s=1}^{\kappa} \frac{1}{(\zeta_s - 1)\zeta_s^{n+1} R'(\zeta_s)} \sim \frac{c\rho^n}{\mu_1[R(\rho^{-1})]^2} \sum_{r=n+1}^{\infty} \frac{\rho^{r-n}}{r^{1+\nu}T(r)} .$$

Now

$$n^{1+\nu}T(n) \sum_{r=n+1}^{\infty} \frac{\rho^r}{r^{1+\nu}T(r)} = \sum_{r=1}^{\infty} \frac{\rho^r n^{1+\nu}T(n)}{(n+r)^{1+\nu}T(n+r)}$$

$$\rightarrow \sum_{r=1}^{\infty} \rho^r , \quad \text{as } n\to\infty,$$

by bounded convergence. Thus the theorem is proved.

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