A METHOD FOR FINDING THE CLOSEST STABLE AUTOREGRESSIVE POLYNOMIAL TO AN UNSTABLE ONE

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1 Introduction

The q-th order autoregressive polynomial is defined as

$$(\forall z \in \mathbf{C}) \qquad A_q(z) = z^q - \sum_{i=1}^{q} a_i z^{n-i} \tag{1}$$

Such polynomials form the denominator of the transfer function of IIR discrete systems and arise in applications such as filter design, system identification, linear prediction, and ARMA modelling. In all of these problems, methods for synthesizing or estimating the regression vector $\mathbf{a} = [a_1 \cdots a_q]^T$ do not always guarantee that the associated autoregressive polynomial $A_q(z)$ is stable[1], i.e. that none of its roots are outside the unit circle in the complex plane[2]. In such instances, one needs a stabilization procedure, i.e. a rule for assigning a stable polynomial to an unstable one. Several approaches have been proposed in the literature. A first option is to scale the regression vector by an appropriate factor. One can also replace those poles outside the unit circle by their reciprocal. This approach has the advantage of preserving the frequency response but, as any pole manipulation technique, it is computationally expensive. Another method is to truncate to unity the magnitude of the reflection coefficients. These procedures generally produce significant perturbations in the coefficients of the polynomial, which results in undesirable effects (e.g. increase in the power of the residual error in linear prediction).
Let $\mathbf{E}^q$ be the Hilbert space obtained by endowing the Cartesian space $\mathbf{R}^q$ with the distance

$$(\forall(\mathbf{a},\mathbf{b}) \in \mathbf{R}^q \times \mathbf{R}^q) \quad d(\mathbf{a},\mathbf{b}) = (\mathbf{a}-\mathbf{b})^T \mathbf{Q}\,(\mathbf{a}-\mathbf{b}) \tag{2}$$

where $\mathbf{Q}$ is a symmetric, positive definite $q \times q$ matrix. In this report, we propose a method for finding a stable regression vector which is the closest, in $\mathbf{E}^q$, to an unstable one. This procedure permits us to stabilize an autoregressive polynomial while causing the least perturbation in its coefficients. In geometrical terms, the unstable regression vector is projected onto the stability set

$$\mathcal{S}_q = \{\mathbf{a} \in \mathbf{R}^q \mid A_q(z) = 0 \quad \Rightarrow \quad |z| \leq 1\} \tag{3}$$

The geometry of the stability set is discussed in Section II, both in the regression and reflection space. The projection algorithm is presented in Section III and discussed in Section IV. For purpose of illustration, the application of the method to the stabilization of fourth order autoregressive polynomials is worked out in Section V, and numerical results are presented.

[1] Such polynomials are also called Hurwitz polynomials.

[2] This is the definition of wide sense stability. Stability in the strict sense requires that all the poles be inside the unit circle.

2 The Geometry of Stability

The reflection coefficients [6] constitute an alternative representation of a regression filter and, therefore, of an autoregressive polynomial. Mathematically, the Levinson recursion is a q-linear mapping $\mathcal{L}$ from $\mathbf{R}^q$ into $\mathbf{R}^q$ which maps a reflection vector $\mathbf{k} = [k_1 \cdots k_q]^T$ into a regression vector $\mathbf{a} = [a_1 \cdots a_q]^T$. For instance, for $q = 3$, one gets

$$a_1 = \mathcal{L}_1(\mathbf{k}) = k_1 - k_1 k_2 - k_2 k_3 \quad (4)$$

$$a_2 = \mathcal{L}_2(\mathbf{k}) = k_2 - k_1 k_3 + k_1 k_2 k_3 \quad (5)$$

$$a_3 = \mathcal{L}_3(\mathbf{k}) = k_3 \quad (6)$$

The geometry of stability in the reflection space is quite simple since the stability region is the hypercube

$$\mathcal{K}_q = \{\mathbf{k} \in \mathbf{R}^q \mid (\forall i \in \{1, ..., q\}) \ |k_i| \leq 1\} \quad (7)$$

In the regression space, the stability region is well known for low orders: $\mathcal{S}_1 = [-1, 1]$; $\mathcal{S}_2$ is the triangle with vertices $[-2, -1]^T$, $[0, 1]^T$, and $[2, -1]^T$ [2]. If $q > 2$, $\mathcal{S}_q$ is more complex, the most striking characteristic being that it is not convex. The pole stability condition has been expressed as constraints on the component of the regression vector[3] in [5]. For instance, $\mathcal{S}_3$ is determined by the inequalities

$$a_3(a_3 - a_1) - a_2 \leq 1 \quad (8)$$

$$a_1 + a_2 + a_3 \leq 1 \quad (9)$$

$$-a_1 + a_2 - a_3 \leq 1 \quad (10)$$

$$|a_3| \leq 1 \quad (11)$$

Of course, the sets $\mathcal{S}_q$ and $\mathcal{K}_q$ are strongly related. The restriction of $\mathcal{L}$ to the interior of $\mathcal{K}_q$ is a bijection onto the interior of $\mathcal{S}_q$ which is continuously differentiable, and so is its inverse [1]. However, $\mathcal{L}$ does not admit an inverse on the boundary of $\mathcal{S}_q$. Indeed, $\mathcal{S}_q$ possesses $q+1$ vertices which are the image under $\mathcal{L}$ of the 2^q vertices of $\mathcal{K}_q$ [8]. For instance, the vertices of $\mathcal{S}_3$ are $[-3, -3, -1]^T$, $[-1, 1, 1]^T$, $[1, 1, -1]^T$, and $[3, -3, 1]^T$ (results pertaining to the image under $\mathcal{L}$ of the edges and faces of $\mathcal{K}_q$ are given in [7]).

To understand the geometry of $\mathcal{S}_q$, it is helpful to mention the following properties, which can be found in [7]. A sufficient condition for the stability of $A_q(z)$ is $\sum_{i=1}^{q} |a_i| \leq 1$. Therefore, for any order q

$$\mathcal{D}_q = \{\mathbf{a} \in \mathbf{R}^q \mid \sum_{i=1}^{q} |a_i| \leq 1\} \quad \subset \quad \mathcal{S}_q \quad (12)$$

[3] These conditions were first derived in [9]. Indeed, it can be shown that the stability of a discrete IIR filter is equivalent to the stationarity of the output ARMA time series if the input is a white noise process.

The stable regression vector of maximal length is the vertex $\mathbf{a} = \mathcal{L}([-1 \cdots -1]^T)$ and its length is $||\mathbf{a}|| = \sqrt{C_{2q}^q}$. Thus, $\mathcal{S}_q$ has a star shape with $q+1$ legs connected to the diamond $\mathcal{D}_q$. Finally, $\mathcal{S}_q$ is symmetric in the sense that

$$[a_1 \ a_2 \ a_3 \cdots a_q]^T \in \mathcal{S}_q \quad \Rightarrow \quad [-a_1 \ a_2 \ -a_3 \cdots (-1)^q a_q]^T \in \mathcal{S}_q \tag{13}$$

3 The Projection Algorithm

Let b be an unstable regerssion vector. A projection of $\mathbf{b}$ onto $\mathcal{S}_q$ is found by minimizing, over $\mathcal{S}_q$, the objective function

$$\Psi(\mathbf{a}) = \frac{1}{2} (\mathbf{a} - \mathbf{b})^T \mathbf{Q} (\mathbf{a} - \mathbf{b}) \tag{14}$$

It is noted that $\Psi(.)$ is continuous and that $\mathcal{S}_q$ is compact. Therefore, a minimum is achieved and a solution for the problem exists. Note, however, that it may not be unique because $\mathcal{S}_q$ is not convex if $q > 2$. Moreover, any local minimum of $\Psi(.)$ is global since $\mathcal{S}_q$ is delimited by convex (or concave) hypersurfaces. Carrying the minimization out in the regression space is difficult because of the geometrical complexity of $\mathcal{S}_q$. On the other hand, since the stability set in the reflection space is much simpler, the minimization will be carried out there. By invoking the Levinson transformation, an equivalent objective function is

$$\Phi(\mathbf{k}) = \frac{1}{2} (\mathcal{L}(\mathbf{k}) - \mathbf{b})^T \mathbf{Q} (\mathcal{L}(\mathbf{k}) - \mathbf{b}) \tag{15}$$

Thus, a solution to the problem is a vector $\bar{\mathbf{a}} = \mathcal{L}(\bar{\mathbf{k}})$, where

$$\Phi(\bar{\mathbf{k}}) = \min \{\Phi(\mathbf{k}) \mid \mathbf{k} \in \mathcal{K}_q\} \tag{16}$$

A constrained steepest descent can be used to solve this problem. Let $\mathcal{P}$ denote the projection operator onto $\mathcal{K}_q$, which enforces the stability constraint, i.e.

$$(\forall \mathbf{k} \in \mathbf{R}^q) \quad [\mathcal{P}(\mathbf{k})]_i = \begin{cases} -1 & \text{if } k_i \leq -1 \\ k_i & \text{if } -1 \leq k_i \leq 1 \\ 1 & \text{if } k_i \geq 1 \end{cases} \tag{17}$$

Given an initial vector $\mathbf{k}_0$ in $\mathcal{K}_q$, one performs the recursion

$$\mathbf{k}_{n+1} = \mathbf{k}_n - \alpha_n \nabla \Phi(\mathbf{k}_n) \tag{18}$$

where $\{\alpha_n\}_{n \geq 0}$ is a sequence of positive real numbers determining the step sizes at each iteration and where the gradient of Φ with respect to $\mathbf{k}$ is given by

$$\nabla \Phi(\mathbf{k}) = [M_{ij}]_{q \times q} \mathbf{Q} (\mathcal{L}(\mathbf{k}) - \mathbf{b}) \qquad \text{with} \qquad M_{ij} = \frac{\partial \mathcal{L}_j(\mathbf{k})}{\partial k_i} \tag{19}$$

Theoretically, the sequence $\{\alpha_n\}_{n\geq 0}$ should be chosen so as to optimize the speed of convergence, i.e so that α_n minimizes $\Phi(\mathbf{k}_{n+1})$. In our practical problem, a constant parameter α can be picked *ad hoc*, which is small enough to ensure that the iterations will not oscillate but sufficiently large to provide a good speed of convergence. In determining the initial point $\mathbf{k}_0$, one seeks to start the algorithm near the solution to reduce the execution time. A possible choice is $\mathbf{k}_0 = \mathcal{P}(\mathcal{L}^{-1}(\mathbf{b}))$. One should, however, beware of the fact that the operator $\mathcal{L}^{-1}$ is not defined on the hypersurfaces delimiting $\mathcal{S}_q$ (if $\mathbf{a}$ is on such an hypersurface, one of the reflection coefficients will have magnitude one and the inverse Levinson recursion will break down). The algorithm is as follows.

1. Fix $\mathbf{k}_0$, α, and ϵ. Set $n = 0$.
2. Compute $\mathbf{k}_{n+1} = \mathcal{P}\left(\mathbf{k}_n - \alpha \nabla \Phi(\mathbf{k}_n)\right)$.
3. If $\Phi(\mathbf{k}_n) - \Phi(\mathbf{k}_{n+1}) > \epsilon$, set $n = n + 1$ and return to 2.
4. $\overline{\mathbf{a}} = \mathcal{L}(\mathbf{k}_{n+1})$. Stop.

The parameter ϵ is used as a stopping criterion. It should be chosen in terms of the degree of accuracy required and the computation time constraints.

4 Discussion

In the stabilization method presented above, the choice of the metric, i.e. of the matrix $\mathbf{Q}$, depends on the application. For instance, the stability set may be one of several property sets used in set theoretic estimation to determine $\mathbf{a}$. Then, to obtain a set theoretic estimate by the successive projections algorithm of [4], one needs to determine a projection of an unstable vector onto $\mathcal{S}_q$ at each iteration. For computational reasons, it is best to use the Euclidean metric, that is $\mathbf{Q} = \mathbf{I}$. Let us now consider linear prediction. It is well known that if $\mathbf{a}$ is the vector of prediction coefficients, the power of the prediction error is given by

$$\zeta(\mathbf{a}) = r(0) - 2\mathbf{a}^T\mathbf{r} + \mathbf{a}^T\mathbf{R}\mathbf{a} \tag{20}$$

where $r(i)$ is the correlation of the data process at lag i, $\mathbf{r} = [r(1)\cdots r(q)]^T$, and $\mathbf{R} = [r(i-j)]_{q\times q}$. The residual power for the optimal predictor $\mathbf{b} = \mathbf{R}^{-1}\mathbf{r}$ is

$$\zeta(\mathbf{b}) = r(0) - \mathbf{b}^T\mathbf{r} \tag{21}$$

It follows easily that the excess residual power induced by the suboptimal filter $\mathbf{a}$ is

$$\zeta(\mathbf{a}) - \zeta(\mathbf{b}) = (\mathbf{a}-\mathbf{b})^T\mathbf{R}(\mathbf{a}-\mathbf{b}) \tag{22}$$

Therefore, in order to stabilize while minimizing the excess error, one should choose $\mathbf{Q} = \mathbf{R}$.

The above algorithm produces a wide sense stable autoregressive polynomial[4]. It can easily be modified to produce a strictly stable polynomial by using a smaller hypercube within which to carry the iterations in the reflection space. To do so, it suffices to modify the projection operator $\mathcal{P}$ of (17) by replacing '1' by '1-δ', where $0 < \delta < 1$.

5 Example and Numerical Results

To show an example of implementation of the method, let us consider a fourth order case in the Euclidean space, i.e. $\mathbf{Q} = \mathbf{I}$. The regression polynomial is given by

$$(\forall z \in \mathbf{C}) \qquad A_4(z) = z^4 - a_1 z^3 - a_2 z^2 - a_3 z - a_4 \tag{23}$$

The Levinson recursion gives

$$\mathbf{a} = \mathcal{L}(\mathbf{k}) = \begin{bmatrix} k_1 - k_1k_2 - k_2k_3 - k_3k_4 \\ k_2 - k_1k_3 - k_2k_4 + k_1k_2k_3 + k_1k_3k_4 - k_1k_2k_3k_4 \\ k_3 - k_1k_4 + k_1k_2k_4 + k_2k_3k_4 \\ k_4 \end{bmatrix} \tag{24}$$

Let $\mathbf{a} = \mathcal{L}(\mathbf{k})$. It follows at once from (19) that the expression of $\nabla\Phi(\mathbf{k})$ is

$$\begin{bmatrix} (1-k_2)(a_1-b_1+k_3(k_4-1)(a_2-b_2)-k_4(a_3-b_3)) \\ (k_1+k_3)(b_1-a_1+k_4(a_3-b_3))+(1+k_1k_3)(1-k_4)(a_2-b_2) \\ (k_2+k_4)(b_1-a_1)+k_1(1-k_4)(k_2-1)(a_2-b_2)+(1+k_2k_4)(a_3-b_3) \\ k_3(b_1-a_1)+(k_1k_3(1-k_2)-k_2)(a_2-b_2)+(k_1(k_2-1)+k_2k_3)(a_3-b_3)+a_4-b_4 \end{bmatrix} \tag{25}$$

The algorithm is implemented with $\alpha = 0.005$ and $\epsilon = 0.001$. The unstable autoregressive polynomial is

$$B_4(z) = z^4 - 1.100z^3 - 0.500z^2 + 1.600z - 0.600 \tag{26}$$

The magnitudes of its poles are $|Z_1| = 0.500$, $|Z_2| = |Z_3| = 1.042$, and $|Z_4| = 1.106$. Using the initial point $\mathbf{k_0} = \mathcal{P}(\mathcal{L}^{-1}(\mathbf{b}))$, the stabilized polynomial

$$A_4(z) = z^4 - 1.210z^3 - 0.248z^2 + 1.366z - 0.596 \tag{27}$$

was obtained in 8 iterations and $\Psi(\mathbf{a}) = 0.361$. The magnitudes of its poles are $|Z_1| = 0.596$, $|Z_2| = |Z_3| = |Z_4| = 1.000$.

[4]As discussed in [3], the condition $\mathbf{k} \in \mathcal{K}_q$ is a sufficient but not a necessary condition for wide sense stability.

6 Conclusions

A method has been proposed to stabilize an autoregressive polynomial by finding a stable polynomial whose coefficients deviate the least from the unstable one, in terms of a general quadratic distance. The method is equivalent to projecting the regression vector associated with the polynomial onto the stability set. The projection is performed in the reflection space by constrained steepest descent. Applications will be found in several areas of digital signal processing, including ARMA system identification, rational spectral estimation, adaptive filtering, linear prediction, and filter design.

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