An Incremental Deformation Theory for Elastic-Plastic Plate Buckling under Non-Proportional Planar Loading

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Abstract

Deformation theories are known to give solutions similar to flow theory for proportional loading only. However, if buckling is involved, which implies non proportional stresses, deformation theories generally give better agreement with experiments than the flow theory. In this paper the constitutive relationships of the deformation theory by Stowell have been recast in an incremental form which, with the incorporation of special iterative techniques permits predicting the elasto-plastic buckling load for plates subject to non-proportional planar loading. Comparison with experiments involving non-proportional pre-buckling stresses confirmed the correctness of the procedure.

1. Introduction

For several decades it has been argued that deformation theories are limited in application to structures subject to proportional loading history where the components of the stress tensor at all points grow proportionally. Plastic buckling of uniformly stressed plates is a well known example involving non-proportional stresses at buckling, yet analytical solutions incorporating deformation theories showed better agreement with experiments than when the flow theory is employed [1]. Different opinions have emerged to clarify this controversy. Shanley [2] and more recently Dubey [3,4] argued that the inaccuracy of the flow theory is attributed to its inability to account for the shear strains caused by rotation of the principal axes. Deformation theories seem to account for this effect. Dubey and Lind [5] showed that if the small rotation of the principal directions is accounted for, the plastic shear strain becomes comparable to the elastic shear strain increment and must not be neglected as implied by the classical flow theory. Lay [6] used a discontinuous stress-strain model and obtained an analytical expression for the tangent shear modulus. Lay argued that some of the slip planes that form during axial deformation can also remain active during a shear perturbation. Therefore, if shear strains are formed along the active slip planes, then the value of this shear strain must be different from their elastic value as predicted by a flow theory.

Budiansky [7] showed on a qualitative basis that deformation theories of plasticity may be used for a range of loading paths other than proportional loading without violating the general requirements for the physical soundness of a plasticity theory. The argument stems from the possibility of the existence of sharp corners in the yield surface which relax the restrictions on the direction of the plastic strain increment. Sewell drew attention to the significant sensitivity of the buckling stress to the shape of the yield surface [8] and later published a paper [9] where he constructed a plastic flow rule at a yield vertex; in doing so, the incipient shear modulus becomes less than its elastic value.

A third opinion in the literature suggests that if initial imperfections are considered in the analysis, the flow theory agrees more closely with experiments. Onat and Drucker [10] first suggested the importance of initial imperfections in the plastic buckling analysis of plates. Crisfield [11] and more recently Hutchinson and Budiansky [12] also support this opinion. This approach, in the authors' opinion, may mislead structural analysts and lead to erroneous results as it leaves the buckling stresses totally dependent on the magnitude of the prescribed imperfections which, in turn, are known to assume a random distribution.

2. Formulation

It has become by now physically accepted that plastic strains are stress history dependent and that unloading from a plastic state causes permanent residual strains. This stems from the following two inequalities [13]

$$d\sigma_{ij}\,(d\varepsilon^{e}_{ij} + d\varepsilon^{p}_{ij}) > 0 \tag{1}$$

$$d\sigma_{ij}\;d\varepsilon^{p}_{ij} \geq 0 \tag{2}$$

The first inequality indicates that positive work is done during the application of the structural disturbance $(d\sigma_{ij}, d\varepsilon_{ij})$. The second inequality implies that, if the external agency causing $(d\sigma_{ij})$ is removed, the elastic component (de^{e}_{ij}) will be completely recovered. Deformation theory formulation does not account for the second inequality and assume that there is a regular progression of plastic deformations and, therefore, no sequential "loading-unloading" of any yield mode occurs over the loading history.

Stowell [14] assumes a nonlinear relationship in the plastic range between total stress and total strain in the form

$$\{\sigma\} = [D\{\sigma\}]\{\varepsilon\} \tag{3}$$

Equation (3) implies nonlinear elastic behavior due to the dependency of [D] on $\{\sigma\}$ which is yet to be found. It was decided to obtain the incremental version of (3) in order to define a tangent element stiffness matrix. Following the procedure in [15] it is possible to obtain the following equation

$$\{d\sigma\} = [D_{ep}]\{d\varepsilon\} \tag{4}$$

where,

$$[D_{ep}] = E_s \begin{bmatrix} \frac{4}{3} + \eta\frac{\sigma_x^2}{\bar{\sigma}^2} & \frac{2}{3} + \eta\frac{\sigma_x \cdot \sigma_y}{\bar{\sigma}^2} & \eta\frac{\tau \cdot \sigma_x}{\bar{\sigma}^2} \\ & \frac{4}{3} + \eta\frac{\sigma_y^2}{\bar{\sigma}^2} & \eta\frac{\tau \cdot \sigma_y}{\bar{\sigma}^2} \\ \text{sym} & & \frac{1}{3} + \eta\frac{\tau^2}{\bar{\sigma}^2} \end{bmatrix} \tag{5}$$

and $$\eta = \frac{E_t}{E_s} - 1 \tag{6}$$

E_t is the tangent modulus and $\bar{\sigma}$ is defined as

$$\bar{\sigma} = (\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau^2)^{1/2} \tag{7}$$

The element planar tangent stiffness matrix takes the form

$$[k_t]^{p\ell} = \int_{V_e} [B]^T [D_{ep}] [B] \, dV_e \tag{8}$$

where [B] is the element planar strain-displacement matrix. The solution strategy can be described as multi-stage analysis and starts by dividing the external loads into small increments according to the actual history of load application. Within each increment, past the elastic range, the plasticity problem is treated as nonlinear elasticity where (4) describes the incremental stress-strain relationship. Within the single load increment no unloading is allowed and progressive development of the yield surface takes place. The Modified Newton-Raphson technique is generally used to obtain the planar stress distribution if the external loads indicate loading everywhere in the structure [16]. The following recurrence equation is used when the structure is loading

$$[K_t]^{p\ell}_{j-1} \{\delta\Delta\}^r_j = \{\delta R\}^{r-1}_j \tag{9}$$

where $[K_t]$ is the global structural tangent stiffness matrix while j and r refer to the load increment and iteration number, respectively. $\{\delta\Delta\}$ is the displacement increment and $\{\delta R\}$ is either the external load vector (r=1) or the unbalanced nodal load vector (r>1).

If unloading is detected at any point in the structure after the first iteration for a particular load increment, the program ceases to adopt $[K_t]$ as the direction of travel and re-starts the analysis for that load increment using the global structural elastic stiffness matrix $[K_o]$, such that

$$[K_o] \{\delta\Delta\}^r_j = \{\delta R\}^{r-1}_j \tag{10}$$

Therefore unloading is enforced to occur along an elastic path as implied in (2). The program resumes using $[K_t]$ for the next load increment unless unloading is detected in which case it continues using (10) for the analysis.

The stability criterion employed here is that the plate is in equilibrium in the planar state (trivial) and in an infinitesimally close bent configuration (non-trivial). The plate is perturbed while in the bent configuration and the principle of Virtual Work is applied equating the internal strain energy due to induced infinitesimal moments $\{\delta M\}$ multiplied by virtual curvatures $\{\delta\tilde{C}\}$ and the work done by the projection of the finite planar stress resultant $(N_x, N_y, N_{yx}, N_{xy})$ multiplied by the angles of virtual rotations $(\delta\tilde{W}_{,x}, \delta\tilde{W}_{,y})$. The resulting expression reads

$$\int_A \{\delta\tilde{C}\}\{\delta M\} dA = \int_A [\delta\tilde{W}_{,x} \;\; \delta\tilde{W}_{,y}] \begin{bmatrix} N_x & N_{yx} \\ N_{xy} & N_y \end{bmatrix} \begin{Bmatrix} \delta W_{,x} \\ \delta W_{,y} \end{Bmatrix} dA \tag{11}$$

By adopting a tangent modulus approach [17] for the plastic elements (i.e., assuming no unloading at buckling for the plastic zones), the uniqueness criterion in (11) can be cast as

$$\{[K^b] + [K^g]\}\{\delta W\} = \{0\} \tag{12}$$

where $[K^b]$ is the structural bending stiffness matrix

$[K^g]$ is the structural geometric stiffness matrix

and $\{\delta W\}$ is the global out of plane infinitesimal displacements.

A non-unique solution (bifurcation) exists if the determinant of the sum of $\{[K^b] + [K^g]\}$ vanishes. Notice that $[K^b]$ depends on the elastic plastic properties, i.e., not constant, and therefore a direct eigenvalue analysis is not possible and the solution has to proceed incrementally [18]. In order to check the uniqueness condition in (12) the procedure explained in [18] based on using a scaled inverse iterative version of the Power Method may be effectively employed.

3. Experimental Verification

In Fig. (1) an (HEA 200) section, used as a column (test A), was subjected to a prescribed vertical stress (0.25 of the yield stress). Horizontal forces were then gradually increased until the column web buckled. The agreement between the analytical and experimental prediction for both buckling load and mode, Fig. (2), is satisfactory. In test B (HEA 200 section), Fig. (3), 0.8 of the yield stress was first applied axially followed by horizontal forces which were increased until buckling occurred. The extent of the plastic zones at buckling shows a longitudinal yielded strip along the column center line due to the combined effect of residual stresses (compression at center of the web) and the applied column axial load. The photo in Fig. (4) confirms the occurrence of this early yielding. The general pattern of the yielded zones resembles, to a fair extent, those observed experimentally. Fig. (5) demonstrates the capability of the analytical procedure to handle loading-unloading and reloading in the various elements using a deformation theory. Early yielding occurs near the column center line as a result of the combined effects of the compressive axial loads and the assumed compressive residual stress there, Fig. (5a). As the horizontal compressive force (C) is applied a complementary tension field is created in the vertical direction (Poisson's effect) causing unloading of some of the previously yielded elements along the column center line, Fig. (5b). As more horizontal load (C) is applied the increasing compressive stresses in the X direction cause some of the unloaded elements to reload again as seen in Fig. (5c).

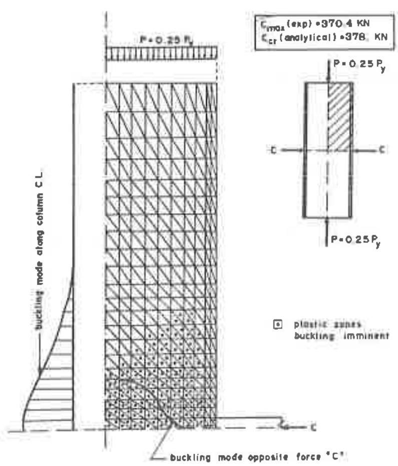

Fig. 1 – Analysis of column A
(1 KN = 0.225 kip)

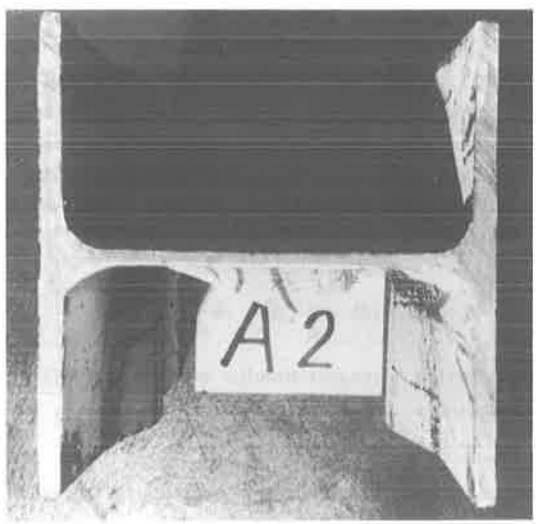

Fig. 2

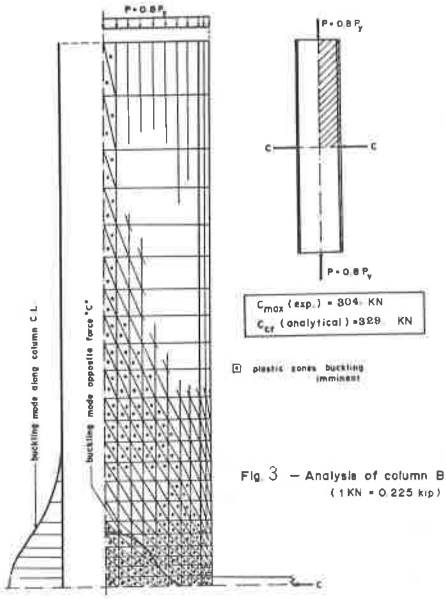

Fig 3 — Analysis of column B
(1 KN = 0.225 kip)

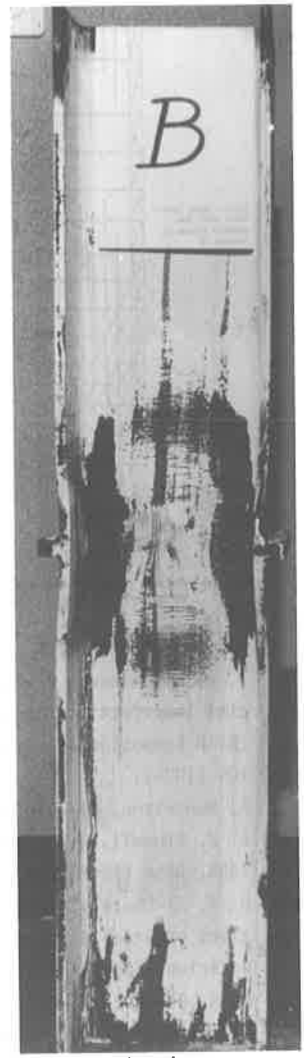

Fig. 4

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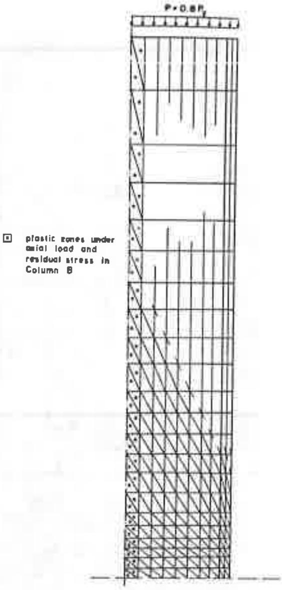

Fig. 5-a Plastic zones in column B under zero horizontal force "C"

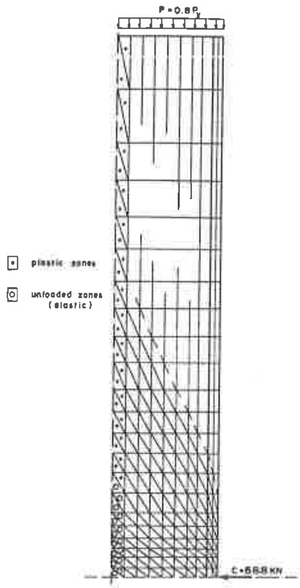

Fig 5 -b Unloading in column B (1 KN = 0.225 kip)

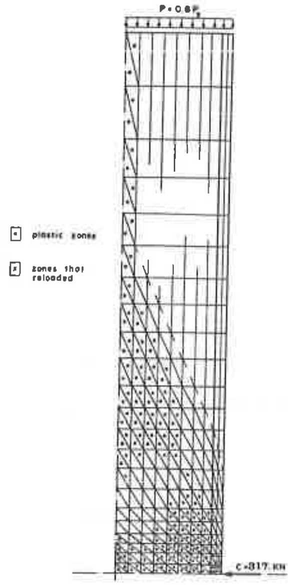

Fig 5 -c Reloading in column B (1 KN = 0.225 kip)

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