Rates of Convergence for the Distance
Between Distribution Function Estimators

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ABSTRACT

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The normed difference between "kernel" distribution function estimators $\hat{F}_n$ and the empirical distribution function F_n is investigated. Conditions on the kernel and bandwidth of $\hat{F}_n$ are given so that $a_n||\hat{F}_n-F_n|| \xrightarrow{wp1} 0$ as $n\to\infty$ for both the sup-norm $||g||_\infty = \sup_x|g(x)|$ and L_1 norm $||g||_1 = \int|g(x)|dx$. Applications include equivalence in asymptotic distribution of $T(\hat{F}_n)$ and $T(F_n)$ (to order a_n) for certain robust functionals $T(\cdot)$.

1. Introduction

The empirical distribution function $F_n(x) = n^{-1}\sum I(X_i \leq x)$ is the most widely used nonparametric distribution function estimator. However, integrals of kernel density estimators form a large class of smooth competitors. These estimators may be expressed as

$$\hat{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} K\left(\frac{x-X_i}{b_n}\right) = \int_{-\infty}^{\infty} K\left(\frac{x-y}{b_n}\right) dF_n(y) , \tag{1.1}$$

where K is a distribution function on $(-\infty,\infty)$ and $b_n > 0$ is the "bandwidth." Winter (1973,1979), Yamato (1973), and Azzalini (1980) have studied the convergence of $\hat{F}_n$ to F, the distribution function of the observations. The purpose of this present note is to give sufficient conditions on F, K, and b_n so that $a_n||\hat{F}_n-F_n|| \xrightarrow{wp1} 0$ for several norms $||\cdot||$. Of particular interest is the sup-norm $||\hat{F}_n-F_n||_\infty = \sup_{-\infty<x<\infty}|\hat{F}_n(x)-F_n(x)|$ and $a_n = n^{\frac{1}{2}}$. Then $n^{\frac{1}{2}}||\hat{F}_n-F_n||_\infty \xrightarrow{wp1} 0$ implies weak convergence of $n^{\frac{1}{2}}[\hat{F}_n(F^{-1}(t))-t]$, the Chung-Smirnov property (law of the iterated logarithm for $||\hat{F}_n-F||_\infty$) of Winter (1979), and the asymptotic equivalence in distribution of $n^{\frac{1}{2}}[T(\hat{F}_n)-T(F)]$ and $n^{\frac{1}{2}}[T(F_n)-T(F)]$ for the many robust functionals $T(\cdot)$ which are Lipschitz with respect to $||\cdot||_\infty$, i.e., $|T(G)-T(H)| \leq C||G-H||_\infty$. More details and applications are given in Section 3. The main results proved in Section 2 are simple and rely on the usual integration by parts representation of $\hat{F}_n$ and Serfling's (1981) generalization of a theorem of Sen and Ghosh (1971) on the uniform convergence of $F_n(x+t) - F(x+t) - [F_n(x)-F(x)]$ as $t \to 0$.

2. Main Results

For functions G and H on $(-\infty,\infty)$ define $||G-H||_\infty = \sup_{-\infty<x<\infty}|G(x)-H(x)|$ and $||G-H||_1 = \int_{-\infty}^{\infty}|G(x)-H(x)|dx$. The sampling situation is

(S) $X_1,\ldots,X_n$ are independent real-valued random variables having the distribution function F.

The first two theorems require

(C) K has support in a compact interval $[c,d]$, $-\infty<c<d<\infty$.

Lemma 2.1 of Winter (1979) justifies the use of integration by parts to reexpress $\hat{F}_n$ of (1.1) as

$$\hat{F}_n(x) = \int_{-\infty}^{\infty} F_n(x-b_n y)dK(y) .$$

THEOREM 1. Suppose that (S) and (C) hold and F satisfies the Lipschitz condition $|F(x)-F(y)| \leq L|x-y|$ on $(-\infty,\infty)$. If $a_n b_n \xrightarrow{\text{wp1}} 0$ and $nb_n/\log n \xrightarrow{\text{wp1}} \infty$, then

$$a_n||\hat{F}_n-F_n||_\infty \xrightarrow{\text{wp1}} 0 \quad \text{as} \quad n\to\infty . \tag{2.1}$$

PROOF.

$$
\begin{aligned}
a_n||\hat{F}_n-F_n||_\infty &\le a_n \sup_x \int_c^d |F_n(x-b_ny)-F_n(x)|dK(y) \\
&\le a_n \sup_x \int_c^d |F_n(x-b_ny)-F(x-b_ny)-[F_n(x)-F(x)]|dK(y) \qquad (2.2)\\
&+ a_n \sup_x \int_c^d |F(x-b_ny)-F(x)|dK(y) \\
&\le a_n \sup_x \sup_{|t|\le(d-c)b_n} |F_n(x+t)-F(x+t)-[F_n(x)-F(x)]| \qquad (2.3)\\
&+ a_nb_nL \int_c^d |y|dK(y) .
\end{aligned}
$$

Rewrite the first term of (2.3) as a_nQ_n . Lemma 2.2 of Serfling (1981) yields

$$\left[\frac{n}{b_n \log n}\right]^{\frac{1}{2}} Q_n = 0(1) \quad \text{wp1} \quad \text{as} \quad n\to\infty$$

for uniform random variables. Following the remark on page 194 of Sen and Ghosh (1971), this result also holds for all F which are Lipschitz. Then $a_nQ_n \xrightarrow{\text{wp1}} 0$ since $a_n[b_n \log n/n]^{\frac{1}{2}} = a_nb_n[\log n/nb_n]^{\frac{1}{2}} \xrightarrow{\text{wp1}} 0$.

REMARKS. The proof shows that the weaker result, $a_n||\hat{F}_n-F_n||_\infty = 0(1)$ wp1, holds if only $a_nb_n = 0(1)$ wp1 is assumed. The bandwidth b_n is allowed to be random since b_n must be estimated in most applications. The usual choice of a_n is $n^{\frac{1}{2}}$ (see Section 3) so that Theorem 1 allows $b_n \sim n^{-\alpha}$, $\alpha > \frac{1}{2}$.

The next theorem strengthens conditions of F and K in order to reduce conditions on b_n . In particular, if $a_n = n^{\frac{1}{2}}$ then Theorem 2 allows $b_n \sim n^{-\alpha}$, $\alpha > \frac{1}{4}$.

THEOREM 2. Suppose that (S) and (C) hold and $\int_c^d x\,dK(x) = 0$. Let F have derivatives f and f' which exist everywhere on $(-\infty,\infty)$ with $||f||_\infty < \infty$ and $||f'||_\infty < \infty$. If $a_n b_n^2 \xrightarrow{wp1} 0$ and $nb_n/\log n \xrightarrow{wp1} \infty$, then (2.1) holds.

PROOF. The proof is the same as for Theorem 1 except that the second term of (2.2) is expanded by Taylor's theorem to yield

$$a_n \sup_x \left|\int_c^d [F(x-b_n y)-F(y)]dK(y)\right| = a_n \sup_x \left|\int_c^d [-f(x)(b_n y)+\tfrac{1}{2}f'(t_n^*)(b_n y)^2]dK(y)\right|$$

$$\leq a_n b_n^2 (\tfrac{1}{2}) ||f'||_\infty \int y^2 dK(y) \ .$$

EXAMPLE. Suppose that F is the uniform distribution function on [0,1]. Then Theorem 1 applies but not Theorem 2.

Theorems 1 and 2 are similar in spirit to Theorems 3.2 and 3.3 of Winter (1979) which conclude that

$$\lim_{n\to\infty} \sup\{2n/\log\log n\}^{\frac{1}{2}} ||\hat{F}_n - F||_\infty \leq 1 \quad \text{wp1} \ . \tag{2.4}$$

Here, condition C and a slightly stronger condition on b_n than Winter required yield (2.1) which in turn yields (2.4).

The last theorem in this section applies to $||\cdot||_1$.

THEOREM 3. Suppose that (S) holds and $\int |x| dK(x) < \infty$. If $a_n b_n \xrightarrow{wp1} 0$, then

$$a_n ||\hat{F}_n - F_n||_1 \xrightarrow{wp1} 0 \quad \text{as} \quad n\to\infty \ . \tag{2.5}$$

PROOF.

$$a_n \int_{-\infty}^{\infty} |\hat{F}_n(x) - F_n(x)|dx \le a_n \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F_n(x-b_n y) - F_n(x)|dK(y)dx$$

$$= a_n \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F_n(x-b_n y) - F_n(x)|dxdK(y)$$

$$= a_n|b_n| \int_{\infty}^{\infty} |y|dK(y) \ .$$

The interchange of integrals is justified by Fubini's theorem, and the last step follows since $\int_{-\infty}^{\infty}[G(x+a)-G(x)]dx = a$ for any distribution function G and constant $a \ge 0$ (see Chung (1974), p. 49, prob. 16).

3. Applications

A) Weak convergence of $n^{\frac{1}{2}}[\hat{F}_n(F^{-1}(t))-t]$ to W^o, the Brownian bridge. Choose $a_n = n^{\frac{1}{2}}$. If K is continuous, then the convergence may be carried out in $C[0,1]$ using Theorems 4.1 and 13.1 of Billingsley (1968). If K is not continuous, then the space $D[0,1]$ is appropriate and Theorems 4.1 and 16.4 of Billingsley yield the result. (In verifying the latter it helps to note that the uniform metric $\rho(x,y) = ||x-y||_\infty$ dominates either of the Skorohod metrics given by Billingsley.)

B) Theorem 4.2 of Sen and Ghosh (1971) mentioned in the proof of Theorem 1 yields for F Lipschitz

$$\sup_{-\infty<x<\infty} \ \sup_{|t|\le d_n} n^{\frac{1}{2}}|F_n(x+t) - F(x+t) - [F_n(x) - F(x)]| \xrightarrow{wp1} 0 \ , \qquad (3.1)$$

where $n^{\frac{1}{2}}d_n$ increases at a rate not slower than that of $\log n$ but not faster than that of n^{α}, $\alpha < \frac{1}{4}$. The results of Theorems 1 and 2 with $a_n = n^{\frac{1}{2}}$ allow (3.1) to hold with F_n replaced by $\hat{F}_n$.

C) Statistical functions $T(F)$. If $|T(\hat{F}_n) - T(F_n)| \leq C_n||\hat{F}_n - F_n||_\infty$ and $C_n = O_p(1)$, then $a_n[T(\hat{F}_n) - T(F_n)] \xrightarrow{p} 0$ and $T(\hat{F}_n)$ and $T(F_n)$ have the same asymptotic distribution up to order a_n. A trivial extension is to replace X_i by the perturbed random variable $X_i + Y_n$ (e.g., Pitman location alternatives). Some specific $T(\cdot)$ are given below.

1. Quantile estimation, $T(F) = F^{-1}(p) = \inf\{x : F(x) \geq p\}$. Suppose that $F'(F^{-1}(p)) > 0$. For distribution functions G and H and $||G-H||_\infty$ sufficiently small, one can verify that

$$|G^{-1}(p) - H^{-1}(p)| \leq \left(\varepsilon + \frac{1}{H'(H^{-1}(p))}\right) ||G-H||_\infty . \tag{3.1}$$

Letting $G = \hat{F}_n$ and $H = F$ in (3.1) and $a_n = 1$, we get $\hat{F}_n^{-1}(p) \xrightarrow{wp1} F^{-1}(p)$. Under the conditions of Theorem A of Silverman (1978) which include uniform continuity of F' and K', we have $\hat{F}_n'(\hat{F}_n^{-1}(p)) = F'(\hat{F}_n^{-1}(p)) + [\hat{F}_n'(\hat{F}_n^{-1}(p)) - F'(\hat{F}_n^{-1}(p))]$ $\xrightarrow{wp1} F'(F^{-1}(p))$. Thus applying (3.1) with $G = F_n$ and $H = \hat{F}_n$ and setting $a_n = n^{\frac{1}{2}}$ yields $n^{\frac{1}{2}}[T(\hat{F}_n) - T(F_n)] \xrightarrow{p} 0$. Azzalini (1980) has apparently shown this result directly and also obtained the optimal rate for b_n of the order $n^{-\frac{1}{4}}$. Theorem 2 requires $b_n < C_1 n^{-\frac{1}{4}}$.

2. L-estimators with smooth score function, $T(F) = \int F^{-1}(t)J(t)dt$. Boos (1979a), Theorem 1, showed that $T(\cdot)$ has a Frechet differential with respect to $||\cdot||_\infty$ under certain conditions on J and F.

Similarly, it can be shown that if J is bounded and integrable on $[0,1]$ and equal to zero in neighborhoods of 0 and 1, then

$$|T(\hat{F}_n) - T(F_n)| \leq C||\hat{F}_n - F_n||_\infty .$$

Alternatively, if J is continuous on $[0,1]$ (and thus bounded), then

$$|T(\hat{F}_n) - T(F_n)| \leq ||J||_\infty ||\hat{F}_n - F_n||_1 .$$

Each of these bounds follows directly from the representation $T(G) - T(H) = \int_{-\infty}^{\infty}[S(H(x)) - S(G(x))]dx$ where $S(t) = \int_0^t J(u)du$.

3. Other examples include classes of R-estimators, minimum distance estimators, and one-step M-estimators.

4. For certain cases $T(\hat{F}_n)$ may be easier to handle than $T(F_n)$ because of the smoothness of $\hat{F}_n$. In Boos (1979b) the minimum value of

$$d_{F_n}(\theta) = \int_{-\infty}^{\infty} [1-F_n(\theta+x) - F_n(\theta-x)]^2 dF_n(\theta+x)$$

was proposed as a test statistic for detecting asymmetry in F. The limiting distribution of the *location* of the minimum $\theta(F_n)$ as well as that of the minimum $T(F_n) = d_{F_n}(\theta(F_n))$ is not hard to find. However, details of proof are made difficult by the fact that $d_{F_n}(\theta)$ is a step function. Consider the following approach. If H is a continuous distribution function, then simple calculations show that

$$\sup_\theta |d_G(\theta) - d_H(\theta)| \leq C||G-H||_\infty$$

and thus

$$\left|\min_{\theta} d_G(\theta) - \min_{\theta} d_H(\theta)\right| \le C\|G-H\|_\infty . \tag{3.2}$$

First let $G = F$ and $H = \hat{F}_n$. Then (3.2) and (2.1) implies that $\theta(\hat{F}_n) \xrightarrow{wp1} \theta(F)$ (see Lemma 2 of Beran (1978) for the technique of proof). Note that consistency of $\theta(F_n)$ does not follow since we have not shown $|\theta(G)-\theta(H)| \le C\|G-H\|_\infty$. However, we can proceed directly and find the limiting distribution of $nT(\hat{F}_n)$ (see Boos (1979b), Section 2, for analogous details). Then using (3.2) again with $G = F_n$ and $H = \hat{F}_n$ and choosing $a_n = n$ in (2.1), we get that $nT(\hat{F}_n)$ and $nT(F_n)$ have the same limiting distribution. Here, Theorem 2 is necessary to get useful results since $nb_n/\log n \to \infty$. Note also that only one $\hat{F}_n$ meeting the above requirements must be found since $T(F_n)$ is the statistic of interest. Koul (1980) has obtained the asymptotic distribution of $\theta(F_n)$ using linear rank methods.

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