ANALYSIS OF VISCOELASTOPLASTIC STRUCTURAL BEHAVIOR OF ANISOTROPIC SHELLS BY THE FINITE ELEMENT METHOD

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ABSTRACT

The finite element method of analyzing an anisotropic viscoelastoplastic shell is described herein. A relaxation kernel in the form of exponential series is used and conveniently expanded into a numerical time integration process. At each integration time increment, viscoelastic displacement and stress fields are determined; and these are subsequently used to iterate through incremental plastic loadings based on the Huber-Mises yield criterion and Prandtl-Reuss flow rule. For the successive time increments, the identical procedure is repeated with each step dependent on the results of the previous time increment. The anisotropic parameters are calculated and introduced into the yield function. These parameters are to be considered constant within an increment but may vary from one increment to another. Yielding of various points through the thickness of a shell is checked so that an appropriate treatment for the transition and unloading elements may be provided. Both perfect plastic and strain hardening behavior based on the isotropic hardening theory is considered in the analysis. It is demonstrated that the finite element technique is efficient and convenient for not only obtaining the necessary equations but also for solving iteratively and incrementally these equations with matrix arrays. Shells of various configurations, axisymmetrical and nonaxisymmetrical, subjected to arbitrary loadings, distributed and concentrated can be treated. To demonstrate the theory and procedures, a numerical example of a spherical cap under uniformly distributed loads is presented.

1. INTRODUCTION

The mechanical behavior of meterials with memory has attracted a considerable attention in recent years. Notably, Green and Rivlin [1] and Green, Rivlin, and Spencer [2] developed an isothermal theory of viscoelasticity based on the approximation of the stress functional. Coleman [3], [4] presented the thermodynamic theory of materials with memory derived in terms of general functionals.

Recently, Chang, White, Oden, Malone, and Cost [5], [6], [7], [8], and [9] among others used the finite element method to solve problems in viscoelasticity. Scattered attempts have been made to deal with the subject of viscoplasticity. However, rigorous numerical applications to solve complex structural systems with viscoelastoplastic behavior have not received much attention. The present paper proposes one such technique. Specifically, the finite element method is utilized to analyze viscoelastoplastic response of an arbitrary anisotropic shell.

In the present study, a relaxation kernel in the form of exponential series is used and conveniently expanded into a numerical time integration process. At each integration time increment, viscoelastic displacement and stress fields are determined and these are subsequently used to iterate through incremental plastic loadings based on the Huber-Mises yield criterion and Prandtl-Reuss flow rule. For the successive time increments, the identical procedure is repeated with each step dependent on the results of the previous time increment.

The anisotropic parameters are calculated and introduced into the yield function. These parameters are to be considered constant within an increment but may vary from one increment to another. Yielding of various points through the thickness of a shell is checked so that an appropriate treatment for the transition and unloading elements may be provided. Both perfect plastic and strain hardening behavior based on the isotropic hardening theory is considered in the analysis.

The numerical examples include a uniformly loaded axisymmetric shell undergoing viscoelastoplastic deformations.

2. THE FINITE ELEMENT EQUATIONS OF AN ANISOTROPIC ELASTOPLASTIC SHELL

Applications of the finite element method to structural mechanics are well known (see Zienkiewicz and Cheung, and Oden [10], [11]). Following the usual procedure, we approximate the local displacement fields over the element, so that the i^{th} generalized displacement coordinate within the element may be given by

$$\theta_i = \psi_{iN}(x)\theta^N \tag{1}$$

where $\psi_{iN}(x)$ is the normalized local interpolation function (Chung and Bandy [15], and θ^N is the component of generalized displacements at all nodes of the element, N being summed from 1 to N_e, the total number of element nodal generalized coordinates.

The equilibrium equation of a shell may be given by

$$\int_A \sigma^{\alpha\beta} \frac{\partial \gamma_{\alpha\beta}}{\partial \theta^N} \, dA = P_N \tag{2}$$

where P_N represents the generalized forces at node N, and the total stress tensor $\sigma^{\alpha\beta}$ is given by

$$\sigma^{\alpha\beta} = hE^{\alpha\beta\lambda\mu} e_{\lambda\mu} + \frac{h^3}{12} E^{\alpha\beta\lambda\mu} \chi_{\lambda\mu} \tag{3}$$

Here h is the thickness, $e_{\alpha\beta}$ and $\chi_{\alpha\beta}$ are the membrane and bending strain tensors, respectively; $E^{\alpha\beta\lambda\mu}$ is the array of elastic constants, and the total strain tensor $\gamma_{\alpha\beta}$ is given by

$$\gamma_{\alpha\beta} = e_{\alpha\beta} + z\chi_{\alpha\beta} \tag{4}$$

where z is the thickness coordinate, and

$$e_{\alpha\beta} = \frac{1}{2}\{u_{\alpha}|_{\beta} + u_{\beta}|_{\alpha} - 2u^3 b_{\alpha\beta} + u^3_{,\alpha}u^3_{,\beta} + u^{\lambda} b_{\alpha\lambda} u\ n_{\beta\gamma}\} \tag{5}$$

$$\chi_{\alpha\beta} = -u^3|_{\alpha\beta} - u_{\lambda}|_{\beta} b^{\lambda}_{\alpha} - u_{\lambda} n^{\lambda}_{\alpha}|_{\beta} + b^{\mu}_{\alpha} u_{\mu}|_{\beta} + b^{\mu}_{\beta} b_{\mu\alpha} u^3 \tag{6}$$

Note that nonlinear terms in the bending strains equation (5) are excluded in the present study. Here standard tensor notations are used for strain-displacement relationships. Introducing equation (1) into equation (5) and equation (6), we have

$$e_{\alpha\beta} = A_{N\alpha\beta}\theta^N + C_{NM\alpha\beta}\theta^N\theta^M \tag{7}$$

$$\chi_{\alpha\beta} = B_{N\alpha\beta}\theta^N \tag{8}$$

Here $A_{N\alpha\beta}$, $C_{NM\alpha\beta}$ represent results of the differentiation implied in equations (5) and (6).

At this point we induce a perturbation or an increment, $\delta\theta^N = \Theta^N$

$$\delta\gamma_{\alpha\beta} = (A_{N\alpha\beta} + C_{NM\alpha\beta}\theta^M + zB_{N\alpha\beta})\Theta^N$$

so that

$$\delta\left(\frac{\partial\gamma_{\alpha\beta}}{\partial\theta^N}\right) = C_{NM\alpha\beta}\Theta^N$$

With these results, the incremental form of the finite element equations of equilibrium becomes

$$\delta\left(\int_A \sigma^{\alpha\beta}\frac{\partial\gamma_{\alpha\beta}}{\partial\theta^N}\,dA\right) = \delta P_N$$

or

$$\int_A \sigma^{\alpha\beta} C_{NM\alpha\beta}\,dA\ \Theta^M + \int_A \delta\sigma^{\alpha\beta}(A_{N\alpha\beta} + C_{NM\alpha\beta}\theta^M + zB_{N\alpha\beta})\,dA = \delta P_N \tag{9}$$

Here $\delta\sigma^{\alpha\beta}$ may be related to elastoplastic behavior such that [12]

$$\delta\sigma^{\alpha\beta} = H^{\alpha\beta\lambda\mu}\delta\gamma_{\lambda\mu} \tag{10}$$

$$H^{\alpha\beta\lambda\mu} = E^{\alpha\beta\lambda\mu} - \frac{A_{\gamma\theta}Z_{\delta\eta}E^{\alpha\beta\gamma\theta}E^{\lambda\mu\delta\eta}}{E_{(p)} + Z_{\omega\kappa}Z_{\xi\zeta}E^{\omega\kappa\xi\zeta}} \tag{11}$$

in which $E_{(p)}=d\bar{\sigma}/d\bar{\epsilon}_{(p)}$ is the instantaneous plastic tangent modulus, $d\bar{\sigma}$ represents the incremental equivalent yield stress, and $Z_{\alpha\beta}$ is related by the Prandtl-Reuss flow rule [12]

$$\delta\gamma^{(p)}_{\alpha\beta} = \frac{\partial f}{\partial\sigma^{\alpha\beta}}\,\delta\lambda = Z_{\alpha\beta}\,d\bar{\epsilon}_{(p)} \tag{12}$$

where $f=f(\sigma^{\alpha\beta})$ is the Huber-Mises yield function and λ is the normality parameter, and $d\bar{\epsilon}_{(p)}$ is the incremental equivalent yield strain.

Substituting equation (1) and equation (8) into equation (9) gives

$$\begin{aligned} K_{NM}\Theta^{M} &= \delta P_{N} \\ K_{NM} &= K^{(I)}_{NM} + K^{(II)}_{NM} + K^{(III)}_{NM} \end{aligned} \tag{13}$$

Here $K^{(I)}_{NM}$ is the linear stiffness matrix, $K^{(II)}_{NM}$ is the geometric stiffness matrix, and $K^{(III)}_{NM}$ is the incremental plastic tangent stiffness matrix, and δP_N is the applied load,

$$K^{(I)}_{NM} = \int_A E^{\alpha\beta\lambda\mu}A_{N\alpha\beta}A_{\lambda\mu M}\,dA + \int_A z^2E^{\alpha\beta\lambda\mu}B_{N\alpha\beta}B_{\lambda\mu M}\,dA \tag{14}$$

$$\begin{aligned} K^{(II)}_{NM} = &\int_A E^{\alpha\beta\lambda\mu}C_{Nq\,\alpha\beta}A_{M\lambda\mu}\theta^{q}\,dA + h\int_A E^{\alpha\beta\lambda\mu}C_{Np\alpha\beta}C_{Mq\,\lambda\mu}\theta^{p}\theta^{q}\,dA \\ &+ \frac{h^3}{12}\int_A E^{\alpha\beta\lambda\mu}C_{Np\alpha\beta}B_{M\lambda\mu}\theta^{p}\,dA \end{aligned} \tag{15}$$

$$K^{(III)}_{NM} = -\int_A \frac{Z_{\gamma\theta}Z_{\delta\eta}E^{\alpha\beta\gamma\theta}E^{\delta\eta\lambda\mu}}{E_{(p)} + Z_{\omega\kappa}Z_{\xi\eta}E^{\omega\kappa\xi\eta}}\left(A_{N\alpha\beta}A_{M\lambda\mu} + z^2B_{N\alpha\beta}B_{M\alpha\beta}\right)dA \tag{16}$$

in which all terms associated with $C_{NM\alpha\beta}$ in $K^{(I)}_{NM}$ and $K^{(III)}_{NM}$ are neglected.

In matrix notation, (13) may be written

$$\underset{\sim}{K}\,\underset{\sim}{\Theta} = \underset{\sim}{P} \tag{17}$$

If the shell is anisotropic, we simply modify the form of Huber-Mises yield function,

$$f(\bar{\sigma}^{\alpha\beta}) = \bar{\sigma}^2 = \frac{1}{2}\,\underset{\sim}{\sigma}^{T}\,\underset{\sim}{A}\,\underset{\sim}{\sigma}$$

where $\underset{\sim}{A}$ is an array of anisotropic parameters determined in terms of yield stresses (see

Chung and Eidson [12]).

3. VISCOELASTOPLASTIC BEHAVIOR

The use of relaxation kernel in the exponential series appears to be most convenient for an application of the finite element method to the viscoelastoplastic problems (Chang, White, Malone [5], [6], [8]). The constitutive relationship based on such model may be of the form

$$\sigma^{\alpha\beta} = E^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu} + \eta^{\alpha\beta\lambda\mu}\dot{\gamma}_{\lambda\mu} + \sum_{i=1}^{n} E_{(i)}^{\alpha\beta\gamma\mu} q_{\lambda\mu}^{i} \tag{18}$$

where $\eta^{\alpha\beta\lambda\mu}$ and $E_{(i)}^{\alpha\beta\lambda\mu}$ are arrays of viscous and spring constants, respectively, and

$$q_{\lambda\mu}^{i} = \int_0^t e^{-(t-\tau)/T_i}\dot{\gamma}_{\lambda\mu}\, d\tau \tag{19}$$

Here T_i is the relaxation time, and $\dot{\gamma}_{\lambda\mu}$ may be considered to vary linearly between the discrete time increment, Δt, so that

$$\dot{\gamma}_{\lambda\mu} = \dot{\gamma}_{\lambda\mu}(t-\Delta t) + \frac{\tau-(t-\tau t)}{\Delta t}[\dot{\gamma}_{\lambda\mu}(t) - \dot{\gamma}_{\lambda\mu}(t-\Delta t)] \tag{20}$$

Substituting equation (16) into equation (15) gives

$$q_{\lambda\mu}^{(i)}(t) = A^{(i)}X_{\lambda\mu}^{(i)}(t,t-\Delta t) + B^{(i)}\dot{\gamma}_{\lambda\mu}(t-\Delta t) + C^{(i)}\dot{\gamma}_{\lambda\mu}(t) \tag{21}$$

where

$$A^{(i)} = e^{-\frac{\Delta t}{T_{(i)}}}$$

$$B^{(i)} = T_{(i)}[-A^{(i)} + \frac{T_{(i)}}{\Delta t}(1-A^{(i)})]$$

$$C^{(i)} = T_{(i)}[1 - \frac{T_{(i)}}{\Delta t}(1-A^{(i)})]$$

$$X_{\lambda\mu}^{(i)}(t,t-\Delta t) = \sum_{m-1}^{m\Delta t-1} a_m b_m[\dot{\gamma}_{\lambda\mu}\{(m-1)\Delta t\} - m\dot{\gamma}_{\lambda\mu}(m\Delta t) - m\dot{\gamma}_{\lambda\mu}\{(m-1)\Delta t\}] + \sum_{m-1}^{m\Delta t} \frac{a_m c_m}{\Delta t}[\dot{\gamma}_{\lambda\mu}(m\Delta t) - \dot{\gamma}_{\lambda\mu}\{(m-1)\Delta t\}]$$

$$a_m = e^{-\frac{m\Delta t}{T_i}}, \qquad b_m = T_i\, e^{\frac{m\Delta t}{T_i}}(1-e^{-\frac{\Delta t}{T_i}})$$

$$c_m = T_i^2\, e^{\frac{m\Delta t}{T_i}}[(\frac{m\Delta t}{T_i} - 1)(1 - e^{-\frac{\Delta t}{T_i}}) + \frac{\Delta t}{T_i}\, e^{-\frac{\Delta t}{T_i}}$$

Here m represents the increment number.

Substituting equation (18) and equation (21) into equation (2) gives the finite element analogue of the anisotropic viscoelastoplastic shell in the form

$$\underset{\sim}{C}\,\dot{\underset{\sim}{\theta}} + \underset{\sim}{K}\,\underset{\sim}{\theta} = \underset{\sim}{P} \tag{22}$$

Here

$$\underset{\sim}{C} = \underset{\sim}{C}^{(I)} + \underset{\sim}{C}^{(II)}$$

in which $\underset{\sim}{C}^{(I)}$ and $\underset{\sim}{C}^{(II)}$ are identical to equation (14) and equation (15), respectively, except that $E^{\alpha\beta\lambda\mu}$ is replaced by

$$\eta^{\alpha\beta\lambda\mu} + \sum_{i=1}^{n} E^{\alpha\beta\lambda\mu}$$

$$\underset{\sim}{P} = \underset{\sim}{F} + \underset{\sim}{L}$$

where

$$\underset{\sim}{L} = L_N = \int_A A_{N\alpha\beta}(L^{\alpha\beta}_{(1)} + L^{\alpha\beta}_{(2)})\,dA$$

$$L^{\alpha\beta}_{(1)} = \sum_{i=1}^{n} E^{\alpha\beta\lambda\mu}_{(i)} A^{(i)} X^{(i)}_{\lambda\mu}(t, t-\Delta)$$

$$L^{\alpha\beta}_{(2)} = \sum_{i=1}^{n} E^{\alpha\beta\lambda\mu}_{(i)} B^{(i)} \dot{\gamma}_{\lambda\mu}(t-\Delta t)$$

In equation (22), $\underset{\sim}{C}$ may be regarded as the viscosity matrix. As noted earlier $\underset{\sim}{K}$ contains the linear elastic stiffness matrix, geometric stiffness matrix, and the plastic tangent stiffness matrix. We also note that $\underset{\sim}{P}$ consists of both applied loads and viscosity loads which are related to the viscoelastic stress histories of the previous time step. It is convenient to place the geometric stiffness matrix and the plastic tangent stiffness matrix to the right hand side of the equation and treat them as load vectors by multiplying the displacements obtained in the previous time step. Thus, we have

$$\underset{\sim}{C}\,\dot{\underset{\sim}{\theta}} + \underset{\sim}{K}^{(I)}\underset{\sim}{\theta} = \underset{\sim}{P}^{(T)} \tag{23}$$

where $\underset{\sim}{P}^{(T)}$ is given by

$$\underset{\sim}{P}^{(T)} = \underset{\sim}{P} + \underset{\sim}{K}^{(III)}\,\underset{\sim}{\theta}_{t-1} + \underset{\sim}{K}^{(III)}\,\underset{\sim}{\theta}_{t-1} \tag{24}$$

Here t-1 indicates displacements in the previous time step.

4. STEP-BY-STEP TIME INTEGRATION

The viscoelastoplastic shell equilibrium equation (23) may be solved most efficiently by the step-by-step time integration method. Assuming a linear variation of displacements within the time increment, we can show that a recurrence formula takes the form (Chung and

Bhutwala [14],

$$\underset{\sim}{R}\,\underset{\sim}{\theta}_{t+\frac{\Delta t}{2}} = \underset{\sim}{W} \tag{25}$$

where

$$\underset{\sim}{R} = \frac{2}{\Delta t}\,\underset{\sim}{C} + \underset{\sim}{K} \tag{26}$$

$$\underset{\sim}{W} = \underset{\sim}{P}^{(T)} + \frac{2}{\Delta t}\,\underset{\sim}{C}\,\underset{\sim}{\theta}_t \tag{27}$$

and $\underset{\sim}{\theta}_{t+\frac{\Delta t}{2}}$ is the displacement vector at the mid-interval.

Initially, θ_t is zero. Thus, equation (24) becomes

$$\underset{\sim}{R}\,\underset{\sim}{\theta}_{t+\frac{\Delta t}{2}} = \underset{\sim}{P} \tag{28}$$

which can easily be solved for $\underset{\sim}{\theta}_{t+\frac{\Delta t}{2}}$. This result will then be used to replace θ_t in equation (27) for the following time step. This will permit calculation of the second term of equation (27) and the last two terms of equation (24) so that the solution of equation (25) can be carried out.

In each time step, which advances half interval at a time, a complete viscoelastoplastic solution is obtained. The viscoelastic displacements are first calculated without the plastic effect, $\underset{\sim}{K}^{(III)}$. With these displacements and corresponding viscoelastic stresses, we calculate $\underset{\sim}{K}^{(III)}$ according to the yield criterion and associated flow fule. This requires determination of equivalent yield stresses at various points through the shell thickness (4 to 8 layers), and any contribution to the second term of $H^{\alpha\beta\lambda\mu}$ equation (11) is determined. Unloading layers and elements are distinguished and accordingly assigned elastic properties.

In general, the treatment of the plastic behavior within the time increment requires a several iteration cycles before convergence. Upon satisfactory convergence, the process is moved to the next time step and identical procedures are repeated.

In this way, the viscoelastoplastic histories are carried throughout all time increments with mechanical properties of the material properly reflected in each time step of direct integration of the incremental equilibrium equations.

The computer program flow chart is given in Appendix.

5. APPLICATIONS

Although the theory and procedures described in the previous sections apply to shells of any geometric configurations, we selected a spherical cap as an example. The viscoelastic behavior is based on the Voigt model of equation (18). The material properties are:

Modulus of elasticity, $E=10.5^6$ psi

Poisson's ratio, $\nu=.3$

Voigt spring constant, $D_1=10.5\times10^6$ psi

Voigt viscosity constant, $\eta_1=2.625\times10^5$ psi

Relaxation time, $T_1=\eta_1/D_1=.025$ sec.

Plastic tangent modulus, $E_{(p)}=21,000$ psi

Yield stress, $\sigma_{yield}=24,000$ psi

The geometric data are shown in Figure 1. The integration time increment was chosen 4×10^{-3} seconds. For simplicity the anisotropic effects are neglected in this analysis. The spherical cap is subjected to a uniformly distributed transverse load of 1,000 psi. The displacements at apex by the viscoelastic analysis and by the viscoelastoplastic analysis are compared in Figure 2. It is interesting to note that both viscoelastic displacement and viscoelastoplastic displacement rise to the linear elastic displacement with a lapse of time, but the viscoelastoplastic displacement becomes larger and the viscoelastic displacement smaller than the linear elastic displacement. The meridional stresses at apex are also plotted (Fig. 3) and decay of viscoelastic stress with time are clearly noted. The elastoplastic stress tends to approach the linear elastic stress and combined total stress or the viscoelastoplastic stress tends to decay from above the linear elastic stress.

6. CONCLUDING REMARKS

Extremely complicated numerical manipulations have been dealt with in this analysis of a viscoelastoplastic shell. A relaxation kernel in the form of exponential series is expanded into a numerical time integration procedure. This scheme permitted easy adaptation of the existing computer codes for the finite element elastoplastic analysis. Although the anisotropy was removed from the example problem for simplicity, such effect can easily be taken into account as indicated in Section 2. The preliminary results indicate that the theory and procedures described herein are valid and efficient. However, further studies are required in regard to the choice of adequate time increments, particularly when extended to dynamic loadings. Patterns of occurrence of yielded regions through the shell thickness with and without the vixcous effects are of interest, and results of complete studies will be reported elsewhere.

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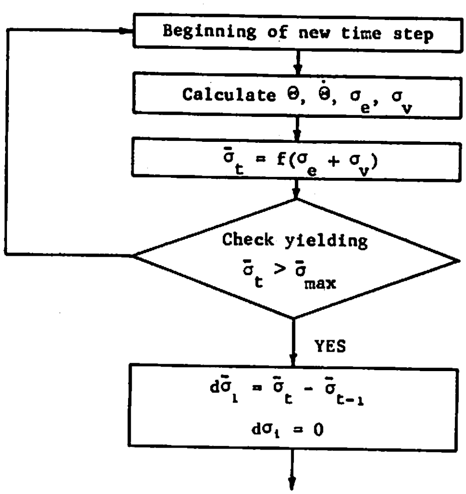
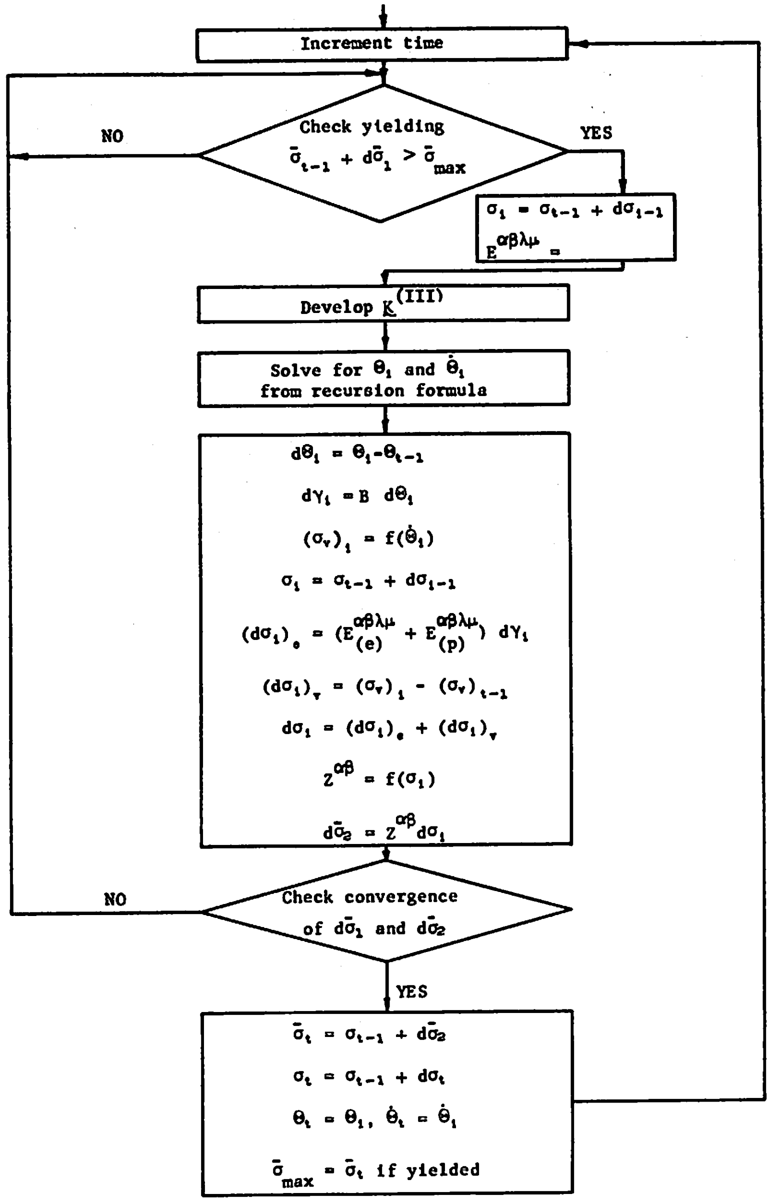
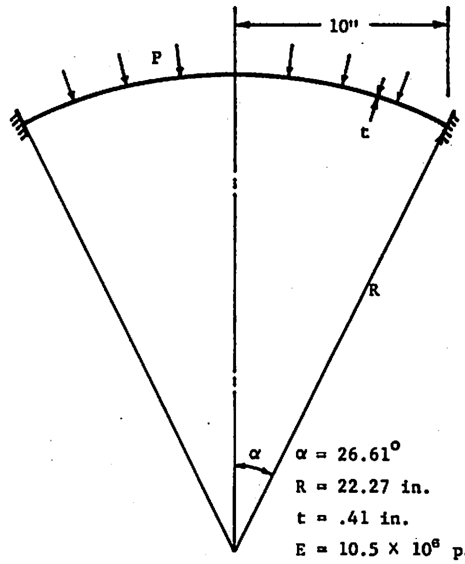
APPENDIX

Computer Program Flow Chart




Increment time
Check yielding
$\bar{\sigma}_{t-1} + d\bar{\sigma}_i > \bar{\sigma}_{max}$
NO
YES
$\sigma_i = \sigma_{t-1} + d\sigma_{i-1}$
$E^{\alpha\beta\lambda\mu} =$
Develop $\underline{K}^{(III)}$
Solve for θ_i and $\dot{\theta}_i$
from recursion formula
$d\theta_i = \theta_i - \theta_{t-1}$
$d\gamma_i = B\ d\theta_i$
$(\sigma_v)_i = f(\dot{\theta}_i)$
$\sigma_i = \sigma_{t-1} + d\sigma_{i-1}$
$(d\sigma_i)_e = (E^{\alpha\beta\lambda\mu}_{(e)} + E^{\alpha\beta\lambda\mu}_{(p)})\ d\gamma_i$
$(d\sigma_i)_v = (\sigma_v)_i - (\sigma_v)_{t-1}$
$d\sigma_i = (d\sigma_i)_e + (d\sigma_i)_v$
$Z^{\alpha\beta} = f(\sigma_i)$
$d\bar{\sigma}_2 = Z^{\alpha\beta} d\sigma_i$
Check convergence
of $d\bar{\sigma}_1$ and $d\bar{\sigma}_2$
NO
YES
$\bar{\sigma}_t = \sigma_{t-1} + d\bar{\sigma}_2$
$\sigma_t = \sigma_{t-1} + d\sigma_t$
$\theta_t = \theta_i,\ \dot{\theta}_t = \dot{\theta}_i$
$\bar{\sigma}_{max} = \bar{\sigma}_t$ if yielded



$\alpha = 26.61^{\circ}$

$R = 22.27$ in.

$t = .41$ in.

$E = 10.5 \times 10^{6}$ psi

$\nu = .3$

$D_1 = 10.5 \times 10^{6}$ psi

$\eta_1 = 2.625 \times 10^{5}$ psi - sec/in

$T_1 = .025$ sec.

$E_{(p)} = 21,000$ psi

$\sigma_{yield} = 24,000$ psi

$P = 1,000$ psi

Figure 1. Spherical Cap Subjected to Uniform Impulsive Loads of Infinite Duration

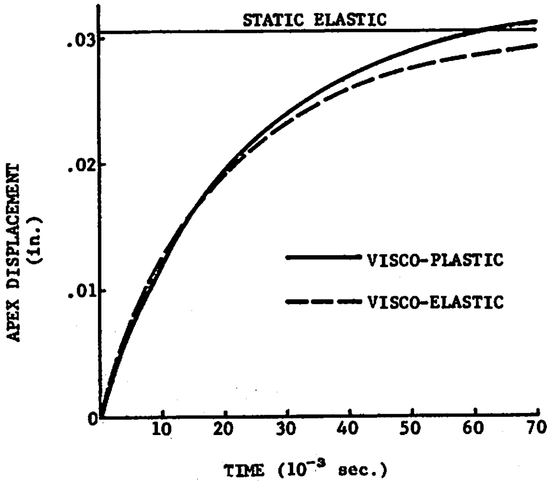


Figure 2. Transverse Displacement at Apex vs. Time

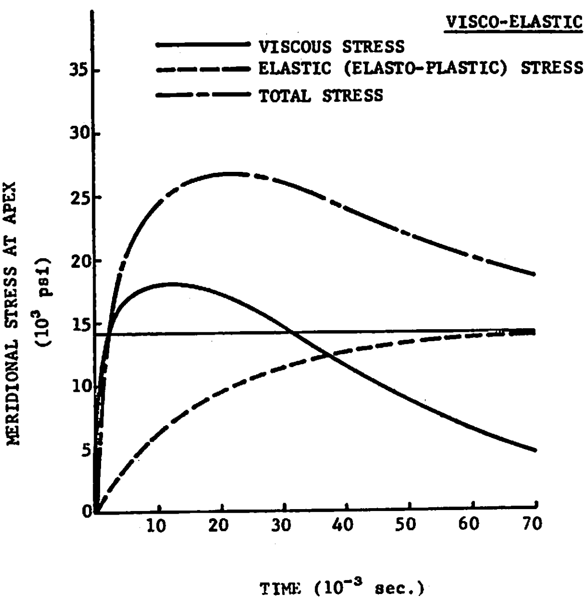


Figure 3. Meridional Stress at Apex vs. Time

DISCUSSION

Q Y. R. RASHID, U. S. A.

What is the basis for the separability of the stress into so-called "viscoelastic" and "plastic" components ?

The usual procedure in formulating the coupled viscoelastic-plastic problem is to assume that the strains are separable.

I believe if one uses the creep compliance kernel in the "strain-stress" relations instead of the relaxation modulus in the "stress-strain" relations one avoids having to introduce the fictitious subdivision of the stress components into viscoelastic and plastic parts.

A T. J. CHUNG, U. S. A.

1. A piecewise linearization from time increment to time increment permits superposition of viscoelastic stress and stresses due to material nonlinearity.
2. Use of compliance kernel would have made the present formulation different. However, the present scheme is preferred because of easiness in numerical manipulation in the computer program.