ON THE MINIMALITY OF A BOUNDEDLY COMPLETE SUFFICIENT SUB FIELD

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This note, written for its pedagogical interest, attempts at a simplification of a proof due to R. R. Bahadur (1957) of the minimality of a boundedly complete sufficient sub-field.

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SUMMARY

This note, written for its pedagogical interest, attempts at a simplification of a proof due to R. R. Bahadur (1957) of the minimality of a boundedly complete sufficient sub-field[†].

NOTATIONS AND DEFINITIONS

Let $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ be our probability structure. That is, $\mathcal{P}$ is a family $\{P\}$ of probability measures on a σ - field $\mathcal{A}$ of sub-sets of a sample space $\mathcal{X}$.

Definition 1: The set $N \in \mathcal{A}$ is said to be $\mathcal{P}$-null if

$$P(N) = 0 \text{ for all } P \in \mathcal{P} .$$

Definition 2: (a) The two sets A and B belonging to $\mathcal{A}$ are said to be $\mathcal{P}$- equivalent if their symmetric difference $A \Delta B$ is $\mathcal{P}$ -null

(b) The two $\mathcal{A}$ -mble functions f and g are said to be $\mathcal{P}$- equivalent if the set $\{x \mid f(x) \neq g(x)\}$ is $\mathcal{P}$ -null.

Definition 3: An $\mathcal{A}$ -mble function f is said to be $\mathcal{P}$ -integrable if

$$\int |f| \, dP < \infty \text{ for all } P \in \mathcal{P} .$$

[1]This research was supported by the Mathematics Division of the Air Force Office of Scientific Research.

[†]As usual we use the term sub-field to mean a sub-σ-field.

Definition 4: A sub-field $\mathcal{A}_*$ of $\mathcal{A}$ is said to be sufficient if corresponding to each $\mathcal{P}$ - integrable, $\mathcal{A}$ -mble f there exists an $\mathcal{A}_*$-mble f_* such that, for all $B \in \mathcal{A}_*$ and $P \in \mathcal{P}$,

$$\int_B f \, dP = \int_B f_* \, dP \ .$$

The function f_* is then called the conditional expectation of f given $\mathcal{A}_*$ and is determined upto a $\mathcal{P}$ -equivalence.

Definition 5: The sub-field $\mathcal{A}_0$ is said to be boundedly complete if the only bounded $\mathcal{A}_0$-mble functions satisfying the identity

$$\int f \, dP \equiv 0 \quad \text{for all } P \in \mathcal{P}$$

are those that are $\mathcal{P}$ -equivalent to zero.

Definition 6: $\mathcal{A}_0$ is said to be a minimal sufficient sub-field if each member of $\mathcal{A}_0$ is $\mathcal{P}$ -equivalent to some member of every alternative sufficient sub-field $\mathcal{A}_*$.

Now if $\mathcal{A}_*$ be sufficient then for each $\mathcal{A}$ -mble and square $\mathcal{P}$ -integrable f the conditional expection f_* is also square $\mathcal{P}$ -integrable and we have in addition

$$\int f^2 \, dP = \int f_*^2 \, dP + \int (f-f_*)^2 dP \quad \text{for all } P \in \mathcal{P} \ .$$

In other words,

$$\int f^2 dP \geq \int f_*^2 \, dP \qquad \text{for all } P \in \mathcal{P}$$

the sign of equalty holding for all $P \in \mathcal{P}$ if and only if f and f_* are $\mathcal{P}$ - equivalent.

THE THEOREM

Theorem: If $\mathcal{A}_0$ be a boundedly complete sufficient sub-field then $\mathcal{A}_0$ is a minimal sufficient sub-field.

Proof: Let $\mathcal{A}_*$ be any alternative sufficient sub-field and let A be an arbitrary member of $\mathcal{A}_o$. We have to prove the existence of a set $B \in \mathcal{A}_*$ such that A and B are $\mathcal{P}$-equivalent.

Let f be the indicator of A and let f_* be the conditional expectation of f given $\mathcal{A}_*$ and f_{*o} the conditional expectation of f_* given $\mathcal{A}_o$. Since f is bounded we can, without any loss of generality, assume that both f_* and f_{*o} are bounded.

Now, from definition 4 we have, for each $P \in \mathcal{P}$,

$$\int f \, dP = \int f_* \, dP = \int f_{*o} \, dP$$

Thus,

$$\int (f - f_{*o}) \, dP \equiv 0 \text{ for all } P \in \mathcal{P}$$

and $f - f_{*o}$ is a bounded $\mathcal{A}_o$-mble function. From the bounded completeness of $\mathcal{A}_o$ it then follows that f and f_{*o} are $\mathcal{P}$-equivalent and hence

$$\int f^2 \, dP = \int f_{*o}^2 \, dP \quad \text{for all} \quad p \in \mathcal{P}$$

But we know that for all $P \in \mathcal{P}$

$$\int f^2 \, dP \geq \int f_*^2 \, dP \geq \int f_{*o}^2 \, dP$$

Therefore, $\int f^2 dP \equiv \int f_*^2 \, dP$ for all $P \in \mathcal{P}$. and hence f and f_* are $\mathcal{P}$-equivalent.

Thus, the set

$$A = \{x \mid f(x) = 1\}$$

is $\mathcal{P}$-equivalent to the set

$$B = \{x \mid f_*(x) = 1\}$$

and so B is the $\mathcal{A}_*$-mble set we are searching after.

REFERENCES

Bahadur, R. R. (1957): On Unbiased Estimates of Uniformly Minimum Variance, Sankhyā Vol. 18, p. 211

Lehmann, E. L. and Scheffé, H. (1950): Completeness, Similar Regions, and Unbiased Estimation, Sankhyā Vol. 10, p. 305

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344. Roberts, Charles D. An asymptotically optimal sequential design for comparing several experimental categories with a standard or control. 1963.
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363. Avi-Itzhak, B. and P. Naor. Multi-purpose service stations in queueing problems. 1963.
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365. Yniguez, A. D'B., R. J. Monroe, and T. D. Wallace. Some estimation problems of a non-linear supply model implied by a minimum decision rule. 1963.
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370. Shimi, I. N. Inventory problems concerning compound products. 1963.
371. Heathcote, C. R. A second renewal theorem for nonidentically distributed variables and its application to the theory of queues. 1963.
372. Portman, R. M., H. L. Lucas, R. C. Elston and B. G. Greenberg. Estimation of time, age, and cohort effects. 1963.
373. Bose, R. C. and J. N. Srivastava. Analysis of irregular factorial fractions. 1963.
374. Potthoff, R. F. Some Scheffe-Type tests for some Behrens-Fisher type regression problems. 1963.